PROPER HOLOMORPHIC EMBEDDINGS INTO STEIN MANIFOLDS WITH THE DENSITY PROPERTY

By

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Abstract. We prove that a Stein manifold of dimension d admits a proper holomorphic embedding into any Stein manifold of dimension at least $2d + 1$ satisfying the holomorphic density property. This generalizes classical theorems of Remmert, Bishop and Narasimhan, pertaining to embeddings into complex euclidean spaces, as well as several other recent results.

1 Introduction

A complex manifold X is said to satisfy the **density property** if the Lie algebra generated by all the $\mathbb{C}$-complete holomorphic vector fields on X is dense in the Lie algebra of all holomorphic vector fields on X in the compact-open topology; see [24, 25] or [12, Section 4.10]. This condition holds trivially on compact manifolds, in which case every vector field is complete, but is fairly restrictive on non-compact manifolds. It is especially interesting on Stein manifolds, in which case it implies the Andersén-Lempert-Forstnerič-Rosay theorem on approximation of isotopies of injective holomorphic maps of Runge domains by holomorphic automorphisms. (See [2, 3, 14] for the case $X = \mathbb{C}^n$ and [12, Theorem 4.10.6, p. 132] or [21, Appendix] for the general case.) Similarly, one defines the **volume density property** of a Stein manifold X, endowed with a holomorphic volume form ω, by considering the Lie algebra of all holomorphic vector fields on X with vanishing ω-divergence; their flows can be approximated by ω-preserving automorphisms of X.

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Denote by $\mathscr{O}(S)$ the algebra of all holomorphic functions on the complex manifold S, endowed with the compact-open topology. A compact set K in S is said to be **$\mathscr{O}(S)$-convex** if for every point $x \in S \setminus K$ there exists $g \in \mathscr{O}(S)$ with $|g(x)| > \sup_K |g|$.

In this paper, we prove the following result.

Theorem 1.1. *Let X be a Stein manifold satisfying either the density property or the volume density property and S be a Stein manifold, $2 \dim S + 1 \le \dim X$. Then every continuous map $f : S \to X$ is homotopic to a proper holomorphic embedding $F : S \hookrightarrow X$. If, in addition, K is a compact $\mathscr{O}(S)$-convex set in S such that f is holomorphic on a neighborhood of K, and S' is a closed complex subvariety of S such that the restricted map $f|_{S'} : S' \hookrightarrow X$ is a proper holomorphic embedding of S' into X, then F can be chosen to agree with f on S' and to approximate f uniformly on K as closely as desired.*

Since a Stein manifold X with the density property is an Oka manifold (see [16] or [12, Theorem 5.5.18, p. 206]), it follows that every continuous map $S \to X$ from a Stein manifold S is homotopic to a holomorphic map; see [12, Theorem 5.4.4, p. 193]). Furthermore, the jet transversality theorem [12, Section 7.8] shows that a generic holomorphic map $S \to X$ is an injective immersion when $2 \dim S + 1 \le \dim X$.

The main new point of Theorem 1.1 is that it gives *proper* holomorphic embeddings. This is a nontrivial addition, since the Oka property for holomorphic maps need not imply the corresponding Oka property for *proper* holomorphic maps; see [8, Example 1.3]). We do not know whether Theorem 1.1 holds for every Stein Oka manifold X.

In the special case when S is a relatively compact, smoothly bounded, strongly pseudoconvex domain in another Stein manifold $\widetilde{S}$, Theorem 1.1 holds (except perhaps the interpolation condition) for every Stein manifold X without the density property assumption; see [7, 8].

We actually prove the following more precise result, showing, in particular, that the proper holomorphic embedding $F : S \hookrightarrow X$ in Theorem 1.1 can be chosen to avoid a given compact holomorphically convex subset L of the target manifold X.

Theorem 1.2. *Let X be a Stein manifold of dimension n satisfying either the density property or the volume density property, L a compact $\mathscr{O}(X)$-convex set in X, S a Stein manifold of dimension d with $2d + 1 \le n$, S' a closed complex subvariety of S, K a compact $\mathscr{O}(S)$-convex subset of S, and $f : S \to X$ a continuous map such that*

(a) *f is holomorphic on a neighborhood of K,*
(b) *the restriction $f|_{S'}: S' \to X$ is a proper holomorphic embedding, and*
(c) *$f(S \setminus \mathring{K}) \subset X \setminus L$.*

Then there exist a proper holomorphic embedding $F: S \hookrightarrow X$ and a homotopy of continuous maps $f_t: S \to X$ ($t \in [0,1]$), with $f_0 = f$ and $f_1 = F$, such that for every $t \in [0,1]$,

- *f_t is holomorphic on a neighborhood of K and uniformly close to f on K,*
- *$f_t|_{S'} = f|_{S'}$, and*
- *$f_t(S \setminus \mathring{K}) \subset X \setminus L$.*

Comparable results for mappings to $X = \mathbb{C}^n$ were obtained by Forstnerič and Ritter in [13]. We prove Theorem 1.2 in Section 3, after preparing the necessary tools in Section 2.

Applying Theorem 1.2 to maps $S \to X$ whose images do not intersect the compact subset $L \subset X$, we see that the complement $X \setminus L$ of any compact $\mathscr{O}(X)$-convex set in a Stein manifold X with the density property has the **basic Oka property with approximation** for maps from Stein manifolds S satisfying $2 \dim S + 1 \leq \dim X$.

Condition (c) on the initial map f in Theorem 1.2 can be replaced by the weaker condition that $f(bK \cup (S' \setminus K)) \subset X \setminus L$. For topological reasons, a map f satisfying the latter condition can be deformed to a map satisfying condition (c) by a homotopy that is fixed on a neighborhood of K and on S'; see [13, Remark 2].

Theorems 1.1 and 1.2 generalize classical results of Remmert [20], Bishop [5], and Narasimhan [19] concerning the existence of a proper holomorphic embedding $S \hookrightarrow \mathbb{C}^n$ of any Stein manifold S with $2 \dim S + 1 \leq n$. The corresponding result on interpolation of embeddings into $\mathbb{C}^n$ on closed complex subvarieties of S was proved by Acquistapace, Broglia and Tognoli [1]. A new proof and generalizations of these classical theorems of complex analysis were given recently by Forstnerič and Ritter in [13].

In the special case $\dim S = 1$, i.e., S an open Riemann surface, the existence of a proper holomorphic embedding in Theorem 1.1 was proved recently by Andrist and Wold [4]. They also constructed proper holomorphic immersions $S \to X$ when $\dim X = 2$. Here, we adapt their construction to the case $\dim S > 1$. Our proof follows the general strategy used in Oka theory (see, e.g., [12, Chapter 5]), but with nontrivial additions to ensure that we obtain *proper* holomorphic embeddings. The main technical step in the proof is extending a holomorphic map (by approximation) across a convex bump B attached to a strongly pseudoconvex domain A in a Stein manifold S, so that B is mapped into the complement $X \setminus L$ of a given compact $\mathscr{O}(X)$-convex set L; see Lemma 2.2. The latter property is used to obtain a proper limit map. It is not known whether such complements $X \setminus L$ are Oka manifolds when X is Oka; this is an open problem even for the complement of a ball in $\mathbb{C}^n$. Instead of using these methods, we use tools from the Andersén-Lempert theory concerning the approximation of certain isotopies of biholomorphic maps of Runge domains in X by holomorphic automorphisms of X. It seems likely that the method developed in [13] can also be used to extend a holomorphic map across a convex bump; see Remark 3.1 below.

We do not know how to prove the analogous result for proper holomorphic maps $S \to X$ when $\dim S < \dim X$ but $2\dim S + 1 > \dim X$; cf. [13] for the case $X = \mathbb{C}^n$. The reason is that our method requires that the holomorphic map in an inductive step be an injective immersion on the attaching set $A \cap B$ of the bump B. To achieve this, we appeal to a general position argument, which necessitates that $2\dim S + 1 \leq \dim X$.

A much stronger result is known for embeddings into $\mathbb{C}^n$. Every Stein manifold of dimension $d > 1$ admits a proper holomorphic embedding into $\mathbb{C}^n$ with $n = \left[\frac{3d}{2}\right] + 1$; see [9], [23]. We do not know whether this result can be generalized to the target manifolds considered in this paper.

Examples 1.3. We illustrate the scope of Theorems 1.1 and 1.2 by collecting known examples of Stein manifolds having either the density property or the volume density property.

(1) The complex euclidean space $\mathbb{C}^n$, $n \geq 1$, has the volume density property with respect to the volume form $dz_1 \wedge \cdots \wedge dz_n$; see [2].
(2) The complex euclidean space $\mathbb{C}^n$, $n \geq 2$, has the density property; see [3].
(3) The Stein manifold $(\mathbb{C}^*)^n$, $n \geq 1$, has the volume density property with respect to the volume form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$. (Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.) It is not known whether $(\mathbb{C}^*)^n$, $n > 1$, satisfies the density property.
(4) For any Stein Lie group G with an invariant Haar form ω, $G \times \mathbb{C}$ has the density property and the volume density property with respect to $\omega \wedge dz$; see [24].
(5) Let H be a closed proper reductive subgroup of a linear algebraic group G. Then the homogeneous space $X = G/H$ is a Stein manifold having the density property, except when $X = \mathbb{C}$, $X = (\mathbb{C}^*)^n$, or X is a $\mathbb{Q}$-homology plane with fundamental group $\mathbb{Z}_2$; see [6, Theorem 6]. In particular, a linear algebraic group with connected components different from $\mathbb{C}$ or $(\mathbb{C}^*)^n$ has the density property; see [16, Theorem 3].
(6) If $p : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with smooth reduced zero fiber, then the Stein manifold $X = \{(x, y, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n : xy = p(z)\}$ has the density property; see [15].

(7) A Cartesian product $X_1 \times X_2$ of two Stein manifolds X_1, X_2 having the density property also has the density property. A product $X_1 \times X_2$ of two Stein manifolds (X_1, ω_1), (X_2, ω_2), each having the volume density property, has the volume density property with respect to the volume form $\omega_1 \wedge \omega_2$: see [17, Theorem 1].

(8) Let R be a reductive subgroup of a linear algebraic group G such that the homogeneous space $X = G/R$ has a left-invariant volume form ω. Then the space (X, ω) has the volume density property; see [18, Corollary 6.2].

2 The main lemma

In this section, we develop the key analytic ingredients used in the proof of Theorem 1.1, namely, Lemma 2.2 and Proposition 2.3.

Recall that a compact subset K of a complex manifold S is said to be a **Stein compact** if it admits a basis of open Stein neighborhoods in S. If $K \subset A$ are compacts in S, we say that K is $\mathscr{O}(A)$**-convex** if there exists an open set $U \subset S$ containing A such that K is $\mathscr{O}(U)$-convex.

We recall the notions of a (special) Cartan pair and of a convex bump, adjusting slightly [12, Definition 5.10.2, p. 218]; see also [12, Figure 5.2, p. 219].

Definition 2.1. A pair of compact sets (A, B) in a complex manifold S is said to be a **Cartan pair** if

(i) $A, B, D := A \cup B$, and $C := A \cap B$ are Stein compacta, and

(ii) A, B are **separated** in the sense that $\overline{A \backslash B} \cap \overline{B \backslash A} = \varnothing$.

A Cartan pair (A, B) is said to be **special**, and B is said to be a **convex bump on** A if

(iii) A and $D = A \cup B$ are compact strongly pseudoconvex domains, and

(iv) there exists a holomorphic coordinate system on a neighborhood of B in S in which the sets B and $C = A \cap B$ are strongly convex.

In the definition on [12, p. 218], the sets B and C are not explicitly required to be convex along $bC \cap \mathring{A} = bB \cap \mathring{A}$ (it is merely required that the boundaries of A and $A \cup B$ be strongly convex in local coordinates near B). The assumption here is satisfied trivially in the context of that definition by a suitable choice of the set B.

Lemma 2.2. *Let S be a Stein manifold of dimension d and X a Stein manifold of dimension n, where $2d + 1 \le n$. Let $D = A \cup B$ be a compact strongly pseudoconvex domain in S such that (A, B) is a special Cartan pair and B is a compact convex bump attached to A along the convex set $C = A \cap B$. Let*

(a) *L be a compact $\mathscr{O}(X)$-convex subset of X,*

(b) *K be a compact subset of $\mathring{A} \setminus C$ such that $K \cup C$ is $\mathscr{O}(A)$-convex,*

(c) *$W \subset S$ be an open set containing A, and*

(d) *$f : W \hookrightarrow X$ be an injective holomorphic immersion such that $f^{-1}(L) \subset \mathring{K}$.*

If X has either the density property or the volume density property, then f can be approximated as closely as desired, uniformly on A, by an injective holomorphic immersion $\tilde{f} : \widetilde{W} \to X$ on a neighborhood $\widetilde{W}$ of $D = A \cup B$ satisfying $\tilde{f}^{-1}(L) \subset \mathring{K}$.

Proof. First consider the case in which X has the density property; the necessary modifications required for the case X has the volume density property are explained at the end of the proof.

Replacing S with a suitable Stein neighborhood of the compact strongly pseudoconvex domain $D = A \cup B$, if necessary, we may assume that A and D are $\mathscr{O}(S)$-convex. It then follows from condition (b) that C and $K \cup C$ are also $\mathscr{O}(S)$-convex.

Pick a smoothly bounded, strongly pseudoconvex Runge domain $W_0 \subset S$ such that $A \subset W_0 \Subset W$ and A is $\mathscr{O}(W_0)$-convex. By [8, Theorem 1.1], we can approximate f uniformly on A by a proper holomorphic embedding $g : W_0 \hookrightarrow X$ such that $g^{-1}(L) \subset \mathring{K}$. To see this, pick a strongly plurisubharmonic exhaustion function $\sigma : X \to \mathbb{R}$ such that $L \subset \{\sigma < 0\}$ and $\sigma > 0$ on the compact set $f(\overline{W_0 \setminus K})$. Given $\epsilon > 0$, the cited result lets us approximate f uniformly on A by a proper holomorphic map $g : W_0 \to X$ satisfying $\sigma(g(z)) > \sigma(f(z)) - \epsilon$ for all $z \in W_0$. Choosing $\epsilon > 0$ small enough, we obtain $g^{-1}(L) \subset \mathring{K}$, as claimed. The fact that g can be chosen to be an embedding follows by a general position argument, since $2d + 1 \leq n$.

The image $\Sigma := g(W_0)$ is a closed complex submanifold of X. Since $K \cup C$ is $\mathscr{O}(A)$-convex and A is $\mathscr{O}(W_0)$-convex, $K \cup C$ is also $\mathscr{O}(W_0)$-convex, and hence $g(K \cup C)$ is $\mathscr{O}(\Sigma)$-convex. Moreover, $L \cap \Sigma \subset g(K) \subset g(K \cup C)$ since $g^{-1}(L) \subset \mathring{K}$, and [10, Lemma 6.5] then shows that $L \cup g(K \cup C)$ is also $\mathscr{O}(X)$-convex. Replacing f with g and W with W_0, we may assume that f has these properties.

Set $L' = L \cup f(K)$. Then $L' \cap f(C) = \varnothing$; and L', $f(C)$, and $L' \cup f(C)$ are $\mathscr{O}(X)$-convex. Pick a compact set $P \subset X \setminus L'$ containing $f(C)$ in its interior such that $L' \cup P$ is also $\mathscr{O}(X)$-convex. The hypotheses on the pair (A, B) imply the existence of a holomorphic coordinate system $z = (z_1, \ldots, z_d) : V_0 \to \mathbb{C}^d$ on a neighborhood $V_0 \subset S$ of B such that, in these coordinates, the compact sets B and $C = A \cap B$ are geometrically convex. Recall that the embedding f is defined on a neighborhood W of A.

Choose open convex neighborhoods $U, V \subset S$ of C and B, respectively, such that $U \subset V \cap W$ and $V \subset V_0$. (More precisely, assume the sets $z(U) \subset z(V) \subset \mathbb{C}^d$

to be convex.) We can find an isotopy $r_t \colon V \to V$ of injective holomorphic self-maps, depending smoothly on the parameter $t \in [0, 1]$, such that

(1) r_0 is the identity map on V,
(2) $r_t(U) \subset U$ for all $t \in [0, 1]$, and
(3) $r_1(V) \subset U$.

In the coordinates z on V_0, we can simply choose r_t to be a family of linear contractions towards a point in U.

Since U is convex, by the Oka-Grauert principle (cf. [12, Subsection 5.3]), the normal bundle of the embedding $f \colon W \hookrightarrow X$ is holomorphically trivial over U. Hence, shrinking W around A and U around C if necessary, we have a holomorphic map $F \colon W \times \mathbb{D}^{n-d} \to X$ ($\mathbb{D}^{n-d}$ is the polydisc in $\mathbb{C}^{n-d}$) such that F is injective holomorphic on $U \times \mathbb{D}^{n-d}$ and $F(z, 0) = f(z)$ for all $z \in W$. Shrinking the neighborhood $U \supset C$ further and rescaling in the fiber variable $w \in \mathbb{D}^{n-d}$, if necessary, we may also assume that the Stein domain $\Omega := F(U \times \mathbb{D}^{n-d}) \subset P \subset X \setminus L'$ is Runge in $\mathring{P}$ and its closure $\overline{\Omega}$ is $\mathscr{O}(P)$-convex. Since $L' \cup P$ is $\mathscr{O}(X)$-convex, $L' \cup \overline{\Omega}$ is also $\mathscr{O}(X)$-convex. Hence there exists a Stein neighborhood $\Omega' \subset X$ of L' such that $\overline{\Omega} \cap \overline{\Omega}' = \varnothing$ and the union $\Omega_0 := \Omega \cup \Omega'$ is a Stein Runge domain in X.

Consider the isotopy of biholomorphic maps $\phi_t \colon V \times \mathbb{D}^{n-d} \to V \times \mathbb{D}^{n-d}$ given by

$$\phi_t(z, w) = (r_t(z), w), \quad z \in V,\ w \in \mathbb{D}^{n-d},\ t \in [0, 1]. \tag{2.1}$$

Define a smooth isotopy of injective holomorphic maps $\psi_t \colon \Omega_0 \to X$ ($t \in [0, 1]$) by

$$\psi_t = F \circ \phi_t \circ F^{-1} \quad \text{on } \Omega, \tag{2.2a}$$

$$\psi_t = \mathrm{Id} \quad \text{on } \Omega'. \tag{2.2b}$$

The map ψ_t is defined on Ω since $r_t(U) \subset U$ for all $t \in [0, 1]$. Note that ψ_0 is the identity on Ω_0, and the domain $\psi_t(\Omega_0)$ is Runge in X for all $t \in [0, 1]$. By the Andersén-Lempert-Forstnerič-Rosay Theorem [12, Theorem 4.10.6, p. 132], we can approximate the map $\psi_1 = F \circ \phi_1 \circ F^{-1} \colon \Omega_0 \to X$ uniformly on compacta in Ω_0 by holomorphic automorphisms $\Psi \in \mathrm{Aut}(X)$. Fix such Ψ and consider the injective holomorphic map

$$G = \Psi^{-1} \circ F \circ \phi_1 \colon V \times \mathbb{D}^{n-d} \to X.$$

Observe that G is indeed defined on $V \times \mathbb{D}^{n-d}$, since $\phi_1(z, w) = (r_1(z), w)$ and $r_1(V) \subset U$ (see condition (3) above); so $\phi_1(V \times \mathbb{D}^{n-d}) \subset U \times \mathbb{D}^{n-d}$.

By (2.2b), ψ_1 is the identity map on $\Omega' \supset L'$; hence Ψ can be chosen to approximate the identity as closely as desired on a neighborhood of L', and we may assume that $G(V \times \mathbb{D}^{n-d}) \subset X \setminus L'$. From (2.2a), we see that

$$G = \Psi^{-1} \circ \psi_1 \circ F \quad \text{on } U \times \mathbb{D}^{n-d}.$$

Since $\Psi^{-1} \circ \psi_1$ is close to the identity map on $F(U \times \mathbb{D}^{n-d})$ by the choice of Ψ, G is close to F on $U \times \mathbb{D}^{n-d}$. More precisely, the above argument shows that for each compact subset M of $U \times \mathbb{D}^{n-d}$, we can choose the automorphism $\Psi \in \mathrm{Aut}(X)$ such that the associated map G is as close as desired to F on M.

Assuming, as we may, that the approximation of F by G is close enough, and shrinking their domains slightly, we can glue the holomorphic maps $F\colon W \times \mathbb{D}^{n-d} \to X$ and $G\colon V \times \mathbb{D}^{n-d} \to X$ into a holomorphic map $\widetilde{F}\colon (A \cup B) \times \rho\mathbb{D}^{n-d} \to X$ for some $0 < \rho < 1$ such that $\widetilde{F}$ is close to F on $A \times \rho\mathbb{D}^{n-d}$ and is close to G on $B \times \rho\mathbb{D}^{n-d}$. (Apply the gluing lemma [11, Theorem 4.1]; see also [12, Theorem 8.7.2, p. 359].) The holomorphic map $\tilde{f} := \widetilde{F}(\cdot\,, 0)\colon A \cup B = D \to X$ then satisfies the conclusion of Lemma 2.2, except that it need not be an injective embedding. It can, however, be made into an injective embedding by a small perturbation, since $2d + 1 \leq n$. Making all the approximations close enough ensures that the preimage under $\tilde{f}$ of the compact set $L \subset X$ is close to the preimage under f of L, so $\tilde{f}^{-1}(L) \subset \mathring{K}$. This proves Lemma 2.2 when X has the density property.

For the case X satisfies the volume density property with respect to a holomorphic volume form ω, choose $\rho > 0$ small and change the definition of the isotopy $\phi_t\colon V \times \rho\mathbb{D}^{n-d} \to V \times \mathbb{D}^{n-d}$ in (2.1) to

$$\phi_t(z, w) = (r_t(z), g_t(z, w)), \quad z \in V,\ w \in \rho\mathbb{D}^{n-d},$$

where the holomorphic map $g_t(z, w)$ is chosen such that $g_0(z, w) = w$, $g_t(z, 0) = 0$, for all $z \in V$ and $t \in [0, 1]$, and such that ϕ_t preserves the volume form $F^*\omega$ on $V \times \mathbb{D}^{n-d}$; see [4] for the details. The conjugate isotopy ψ_t given by (2.2) then preserves the volume form ω in the sense that $\psi_t^*\omega = \omega$ for all $t \in [0, 1]$. Since $V \times \rho\mathbb{D}^{n-d}$ (and hence its image $F(V \times \rho\mathbb{D}^{n-d})$) is contractible, all of its cohomology groups vanish. Hence the volume-preserving version of the Andersén-Lempert-Forstnerič-Rosay theorem applies and shows that ψ_1 can be approximated by ω-preserving automorphisms Ψ of X which are close to the identity map on Ω'.[1] The rest of the proof is exactly as in the case of a manifold X having

[1] We only need the vanishing of the group $H^{n-1}(\cdot\,, \mathbb{C})$ to approximate the infinitesimal generator of the isotopy on small time intervals by flows of globally defined holomorphic vector fields on X with vanishing ω-divergence; see the proof of [12, Theorem 4.9.2, p. 125] for the case ω is the standard volume form $dz_1 \wedge \cdots \wedge dz_n$ on $X = \mathbb{C}^n$. Although this cohomology vanishing condition need not hold for the domain Ω', this is irrelevant, since we are approximating the constant isotopy $\psi_t = \mathrm{Id}$ there. Now apply [12, Proposition 4.10.4, p. 132] to obtain the approximation by ω-preserving automorphisms of X. This argument can also be found in [17, Theorem 2].

the density property. □

Using Lemma 2.2, we now deal with the noncritical case in the proof of Theorem 1.1. Let $A \subset A'$ be compact strongly pseudoconvex domains in a Stein manifold S. We say that A' is a **noncritical strongly pseudoconvex extension** of A (cf. [12, p. 218]) if there exist a strongly plurisubharmonic function ρ without critical points on a neighborhood U of $\overline{A' \setminus A}$ and a pair of real numbers $c < c'$ satisfying

$$U \cap A = \{z \in U : \rho(z) \le c\}, \quad U \cap A' = \{z \in U : \rho(z) \le c'\}. \tag{2.3}$$

Proposition 2.3. *Assume that $A \subset A'$ is a noncritical strongly pseudoconvex extension in a Stein manifold S. Let X be a Stein manifold with the density property (or the volume density property) satisfying* $2\dim S + 1 \le \dim X$*, and let $L \subset X$ be a compact $\mathscr{O}(X)$-convex set. Given an injective holomorphic immersion $f : W \hookrightarrow X$ on an open set $W \supset A$ such that $f^{-1}(L) \subset \mathring{A}$, we can approximate f as close as desired uniformly on A by an injective holomorphic immersion $f' : W' \hookrightarrow X$, defined on a small open neighborhood $W' \subset S$ of A', such that $f'(W' \setminus \mathring{A}) \subset X \setminus L$.*

Proof. Set $d = \dim S$ and $n = \dim X$, so $2d + 1 \le n$. Replacing S by a suitable Stein neighborhood of A', we may assume that A and A' are $\mathscr{O}(S)$-convex. Pick a compact set $K \subset \mathring{A}$ such that K is $\mathscr{O}(S)$-convex and $f^{-1}(L) \subset \mathring{K}$. (With ρ as in (2.3), we can simply take $K = \{z \in A : \rho(z) \le c_0\}$ for some constant $c_0 < c$ close to c.) Choose a finite open cover $\{U_1, \dots, U_l\}$ of the compact set $\overline{A' \setminus A}$ such that for each $j = 1, \dots, l$, there exists a biholomorphic map $\theta_j : U_j \to \mathbb{B}^d \subset \mathbb{C}^d$ onto the unit ball in $\mathbb{C}^d$ and such that $K \cup \overline{U}_j$ is $\mathscr{O}(S)$-convex. The latter condition can be satisfied by choosing the sets U_j small enough.

By [12, Lemma 5.10.3, p. 218], there exist compact strongly pseudoconvex domains $A = A_0 \subset A_1 \subset \cdots \subset A_m = A'$ such that for each $k = 0, 1, \dots, m-1$, $A_{k+1} = A_k \cup B_k$, where B_k is a convex bump on A_k and (A_k, B_k) is a special Cartan pair; see Definition 2.1. Furthermore, each B_i is contained in one of the sets $U_{j(i)}$, and the compact sets B_i and $C_i = A_i \cap B_i$ are geometrically convex with respect to the holomorphic coordinates $\theta_{j(i)} : U_{j(i)} \to \mathbb{B}^d$. In particular, B_i and C_i are $\mathscr{O}(U_{j(i)})$-convex. Since for each $j = 1, \dots, l$, $K \cup \overline{U}_j$ is $\mathscr{O}(S)$-convex, $K \cup B_i$ and $K \cup C_i$ are also $\mathscr{O}(S)$-convex (and hence $\mathscr{O}(A_{i+1})$-convex) for every $i = 0, 1, \dots, m-1$.

Using Lemma 2.2, we inductively find injective holomorphic immersions $f_i : W_i \hookrightarrow X$ for $i = 0, \dots, m$, $f_0 = f$, such that for every $i = 1, \dots, m$, f_i is defined on a small open neighborhood W_i of A_i, it approximates f_{i-1} as closely as desired uniformly on A_{i-1}, and it satisfies $f_i^{-1}(L) \subset \mathring{K}$. The hypotheses in

Lemma 2.2 are clearly satisfied at each step of the induction. If the aproximation is sufficiently close at every step, then the final map $f' = f_m \colon W_m \to X$, which is defined on a neighborhood $W' = W_m$ of A', satisfies the conclusion of the proposition. □

3 Proof of Theorems 1.1 and 1.2

We focus on Theorem 1.1; it will be clear that the same construction gives the more precise statement in Theorem 1.2. The proof amounts to an inductive application of Proposition 2.3 and an additional argument applied to critical points of an exhaustion function on S. The procedure is similar to that used in Oka theory (cf. [12, Chapter 5]), but contains additional ingredients to ensure that the limit map is proper.

We begin with the case $S' = \varnothing$, i.e, without the interpolation condition. By a general position argument for holomorphic maps (see, e.g., [12, Section 7.8]), we may assume that the initial map $f_0 = f$ is an injective holomorphic immersion on an open set $U_0 \subset S$ containing K. (We may assume that K is nonempty, since we can deform f to an injective holomorphic immersion on a small open set in S.)

Since K is $\mathscr{O}(S)$-convex, there exists a smooth strongly plurisubharmonic Morse exhaustion function $\rho \colon S \to \mathbb{R}$ such that $\rho < 0$ on K, $\rho > 0$ on $S \setminus U_0$, and 0 is a regular value of ρ. Let $p_1, p_2, \ldots$ be the critical points of ρ in $\{\rho > 0\}$, ordered by $0 < \rho(p_1) < \rho(p_2) < \cdots$. Choose an increasing sequence $0 < c_0 < c_1 < c_2 < \cdots$ with $\lim_{j\to\infty} c_j = +\infty$ such that every c_j is a regular value of ρ and $c_{2j-1} < \rho(p_j) < c_{2j}$ for $j = 1, 2, \ldots$. (If there are only finitely many, say N, critical points p_j, $j = 1, \ldots, N$, we choose c_j, $j > N$, arbitrarily, subject only to the condition $\lim_{j\to\infty} c_j = +\infty$.) Furthermore, we require c_{2j-1} and c_{2j} to be close to $\rho(p_j)$ in a sense to be specified later. For $j = 0, 1, 2, \ldots$, set $K_j = \{z \in S \colon \rho(z) \le c_j\}$ so that $K \subset K_0 \subset K_1 \subset \cdots \subset \bigcup_{j=0}^{\infty} K_j = S$, every K_j is $\mathscr{O}(S)$-convex, and $K_{j-1} \subset \mathring{K}_j$ for all for $j = 1, 2, \ldots$.

Also choose an exhaustion $L_1 \subset L_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} L_j = X$ of X by compact $\mathscr{O}(X)$-convex sets. For convenience, assume that $L_j = \{\sigma \le j\}$ for a smooth strongly plurisubharmonic Morse exhaustion function $\sigma \colon X \to \mathbb{R}$, chosen such that every integer $j \in \mathbb{N}$ is a regular value of σ. For Theorem 1.2, we can assume that $L_1 = L$ is the given $\mathscr{O}(X)$-convex set that satisfies condition (c).

Set $f_0 = f$ and pic a number $\epsilon_0 > 0$. Choose a distance function dist on X induced by a complete Riemannian metric. We now find continuous maps $f_j \colon S \to X$ and positive numbers ϵ_j such that for every $j = 1, 2, \ldots$,

(a) f_j is an injective holomorphic immersion on an open neighborhood W_j of K_j.
(b) $\sup_{z\in K_{j-1}} \operatorname{dist}(f_j(z), f_{j-1}(z)) < \epsilon_{j-1}$.
(c) $f_j(\overline{K_j \setminus K_{j-1}}) \subset X \setminus L_j$.
(d) there exists a homotopy $h_j : S \times [0,1] \to X$ with $h_{j;0} := h_j(\cdot, 0) = f_{j-1}$, $h_{j;1} := h_j(\cdot, 1) = f_j$, and for all $t \in [0,1]$, $h_{j;t} : h_j(\cdot, t)$ is holomorphic on a neighborhood of K_{j-1} and

$$\sup_{z\in K_{j-1}} \operatorname{dist}(h_{j;t}(z), f_{j-1}(z)) < \epsilon_{j-1}. \tag{3.1}$$

(e) $0 < \epsilon_j < \epsilon_{j-1}/2$; and
(f) every holomorphic map $F : S \to X$ satisfying $\sup_{z\in K_j} \operatorname{dist}(F(z), f_j(z)) < 2\epsilon_j$ is an injective immersion on K_{j-1}.

Let us explain the construction. Assuming that the maps $f_0, \ldots, f_{j-1}$ and the corresponding numbers $\epsilon_0, \ldots, \epsilon_{j-1}$ with the required properties have already been found (this is true for $j = 1$), we must explain the contruction of the next map f_j and the choice of the number ϵ_j. There are two distinct cases to consider: the *noncritical case*, in which $K_j \setminus K_{j-1}$ does not contain any critical point of ρ (this happens for odd values of j), and the *critical case*, in which $K_j \setminus K_{j-1}$ contains a critical point of ρ (this happens for even values of j). We explain in detail how to get the maps f_1 and f_2; all subsequent steps are analogous to one of these two cases.

The initial map $f_0 : S \to X$ is holomorphic on the open set $U_0 \supset K_0$. Choose a smoothly bounded strongly pseudoconvex open domain D_0 in S such that $K_0 \subset D_0 \Subset U_0$. (We may simply take $D_0 = \{\rho < c\}$ for sufficiently small $c > 0$.) By [8, Theorem 1.1], there exists a holomorphic map $g : \overline{D}_0 \to X$ such that

$$g(bD_0) \subset X \setminus L_1 \quad \text{and} \quad \sup_{z\in K_0} \operatorname{dist}(g(z), f_0(z)) < \frac{\epsilon_0}{2}.$$

(For Theorem 1.2, we can choose K_0 and D_0 such that $f_0(\overline{D_0 \setminus K_0}) \subset X \setminus L_1$, so the above condition holds with $g = f_0$.) Furthermore, g can be chosen so that there is a homotopy from f_0 to g, consisting of holomorphic maps on D_0 and satisfying the approximation condition in (d) for $j = 1$ (that is, on the set K_0), with ϵ_0 replaced by $\epsilon_0/2$. (Such a homotopy exists whenever the approximation of f_0 by g is sufficiently close on K_0.) The map g, and the homotopy from f_0 to g, can be extended continuously to all of S, without changing them on a small neighborhood of K_0, by using a cut-off function in the parameter of the homotopy.

Since the set $K_1 = \{\rho \le c_1\}$ is a noncritical strongly pseudoconvex extension of the set $K_0 = \{\rho \le 0\}$, Proposition 2.3 furnishes an injective holomorphic immersion $f_1 \colon W_1' \to X$ on an open set $W_1' \supset K_1$ in S such that

$$f_1(\overline{K_1 \setminus K_0}) \subset X \setminus L_1 \quad \text{and} \quad \sup_{z \in K_0} \mathrm{dist}(f_1(z), g(z)) < \frac{\epsilon_0}{2}.$$

Furthermore, we can ensure that there exists a homotopy of holomorphic maps from g to f_1 on a neighborhood of K_0 satisfying the estimate (3.1) for $j = 1$, with ϵ_0 replaced by $\epsilon_0/2$ and f_0 replaced by g. As before, we extend the map f_1 and the homotopy continuously to all of S without changing their values on a smaller neighborhood W_1 of the set K_1. By combining these two homotopies (the first one from f_0 to g, and the second one from g to f_1) we get a homotopy $h_{1,t}$ from f_0 to f_1, consisting of maps that are holomorphic on a neighborhood of K_0 and satisfy the estimate (3.1) for $j = 1$. Clearly the map f_1 satisfies properties (a)-(d) for $j = 1$. Now pick a number ϵ_1 satisfying $0 < \epsilon_1 < \epsilon_0/2$ such that condition (f) is satisfied for $j = 1$ (this holds whenever $\epsilon_1 > 0$ is sufficiently small). This completes the first step of the induction.

In the next step we must find the map $f_2 \colon S \to X$. This *critical case* is accomplished in finitely many substeps which we now describe; for further details we refer to [12, Section 5.11, pp. 222–223].

We may assume that c_1 is so close to the critical value $\rho(p_1)$ that we can work in a coordinate neighborhood of that $p_1 \in S$ in which ρ assumes the quadratic normal form furnished by [12, Lemma 3.9.1, p. 88]. We attach to the strongly pseudoconvex domain $K_1 = \{\rho \le c_1\}$ the local stable manifold E of the critical point p_1. In the local holomorphic coordinate in which ρ assumes the normal form, E is a linear totally real disc of dimension k that equals the Morse index of p (indeed, E is a closed ball in $\mathbb{R}^k \subset \mathbb{R}^d \subset \mathbb{C}^d$), $E \setminus bE \subset \{\rho > c_1\} = X \setminus K_1$, and E is attached to K_1 along a legendrian (complex tangential) sphere $S^{k-1} \cong bE$ contained in the strongly pseudoconvex hypersurface $bK_1 = \{\rho = c_1\}$.

Choose c_1' satisfying $c_1 < c_1' < \rho(p_1)$ and close to $\rho(p_1)$. (How close is to determined in the sequel.) By the noncritical case explained above, we may assume that f_1 is holomorphic on a neighborhood of the set $D_1 := \{\rho \le c_1'\}$ and $f_1(\overline{D_1 \setminus K_0}) \subset X \setminus L_1$. We may assume that conditions (a)-(f) hold for this new f_1. Let $E' = E \cap \{\rho \ge c_1'\}$; this is again a totally real k-disc attached with its boundary sphere $bE' \cong S^{k-1}$ to the domain D_1.

Consider the continuous map $f_1|_{E'} \colon E' \to X$. Note that $f_1(bE') \subset X \setminus L_1$. We claim that $f_1|_{E'}$ can be homotopically deformed to a map with values in $X \setminus L_1$, keeping the homotopy fixed near bE'. To see this, observe that X is obtained from $X \setminus \mathring{L}_1$ by attaching to the latter set handles of index at least n. (This is because the critical points of the function $-\sigma \colon X \to \mathbb{R}$ have Morse indices at least n, and $X \setminus \mathring{L}_1 = \{-\sigma \le -1\}$.) It follows that the relative homotopy groups $\pi_l(X, X \setminus L_1)$

vanish for $l = 0, \dots, n-1$. Since $k \le d < n$, any map $(E', bE') \mapsto (X, X \setminus L_1)$ is homotopic to a map with values in $X \setminus L_1$, so the claim follows. We can extend this homotopy to all of X without changing it on D_1. We denote this new map also by f_1. By construction, f_1 maps the compact set $(D_1 \cup E') \setminus \mathring{K}_0$ to $X \setminus L_1$.

We now apply Mergelyan's Theorem [12, Theorem 3.7.2, p. 81] to make f_1 holomorphic on a neighborhood $\Omega \subset S$ of the set $D_1' \cup E'$. To simplify notation, we denote this new map by f_1 and assume that conditions (a)-(f) hold for $j = 1$.

Assuming that c_1' is close enough to $\rho(p_1)$, Lemma 3.10.1 in [12, p. 92] gives a smooth strongly plurisubharmonic function τ on S and constants $0 < c < c'$ such that

(1) $K_1 \cup E \subset \{\tau < c\} \subset \Omega$,
(2) $K_2 = \{\rho \le c_2\} \subset \{\tau < c'\}$, and
(3) τ has no critical values in the interval $[c, c']$.

Applying the noncritical case explained above with τ and f_1, we find a map f_2 and a homotopy from f_1 to f_2 satisfying properties (a)-(d) for $j = 2$. Choose $\epsilon_2 > 0$ such that conditions (e) and (f) hold for $j = 2$. This completes the construction of f_2.

The subsequent steps in the induction are exactly the same as in one of these two cases, depending on the parity of the index j, so the induction proceeds.

Conclusion of the proof of Theorem 1.1 (assuming $S' = \varnothing$). Conditions (a) and (e) ensure that the sequence $\{f_j\}$ converges uniformly on compacta in S to a holomorphic map $F = \lim_{j\to\infty} f_j : S \to X$. Condition (d) shows that the sequence of homotopies $\{h_{j,t}\}$ also converges uniformly on compacta to a homotopy $h_t : S \to X$ ($t \in [0,1]$) from $h_0 = f_0 = f$ to $h_1 = F$. Properties (b) and (e) imply

$$\sup_{z \in K_j} \operatorname{dist}(F(z), f_j(z)) < 2\epsilon_j \quad \text{for } j = 0, 1, 2, \dots. \tag{3.2}$$

In particular, $\sup_{z\in K} \operatorname{dist}(F(z), f(z)) < 2\epsilon_0$. The estimate (3.2), together with property (f) of the sequence $\{\epsilon_j\}$, shows that $F : S \to X$ is an injective holomorphic immersion on K_{j-1}. Since this holds for every j, F is an injective immersion on all of S. Finally, condition (c) together with (3.2) shows that F is proper.

This completes the proof of Theorems 1.1 and 1.2 in the case $S' = \varnothing$.

Proof of Theorem 1.1 (the general case). By assumption, the restricted map $f_{S'} : S' \to X$ is a proper holomorphic embedding. The construction of a proper holomorphic embedding $F : S \hookrightarrow X$ satisfying $F|_{S'} = f|_{S'}$ requires a minor modification of the induction scheme. One can proceed as in [13, Section 4], except that the individual steps in the induction are accomplished as described in this paper. An important point is that none of the bumps that are used in this construction intersects the subvariety S'; hence, we can glue at every step, so as to satisfy the required interpolation condition on S'.

Remark 3.1. Assuming that X has the density property, one can also prove Theorem 1.2 by following the approach in [13]. The main difference with respect to this paper lies in the proof of Lemma 2.2. When attaching a bump B to a set A, with an embedding $f : A \to X$, we may assume by [8, Theorem 1.1] that f is proper holomorphic on some neighborhood $W \subset S$ of A, so $f(C)$ is an embedded holomorphically contractible set such that $L \cup f(C)$ is holomorphically convex in X. (Here $C = A \cap B$.) Thus we can pick $x \in f(C)$ and construct a sequence $\{\phi_j\}$ of holomorphic automorphisms of X such that $f(C)$ is contained in the basin of attraction $\Omega \subset X$ of $(\phi_j)_{j \in \mathbb{N}}$, but the set L does not intersect Ω. Assuming, as we may, that $\{\phi_j\}$ satisfies the uniform attraction condition $a|z| \le |\phi_j(z)| \le b|z|$ with $0 < b^2 < a < b < 1$ on a ball centered at the point x (in some local holomorphic coordinates z around x with $z(x) = 0$), we conclude that the basin Ω is biholomorphic to $\mathbb{C}^n$; see [22, Theorem 9.1] for the special case of iteration of an automorphism, and [26] for the general case of a random iteration. Hence one can approximate and glue maps just as was done in [13]. One can find a sequence $\{\phi_j\}$ with these properties using the Andersén-Lempert-Forstnerič-Rosay theorem. This approach does not seem to work by using the volume density property of X.

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