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Hyperbolic Complex Contact Structures on $\mathbb{C}^{2n+1}$

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Abstract We construct complex contact structures on $\mathbb{C}^{2n+1}$ for any $n \geq 1$ with the property that every holomorphic Legendrian map $\mathbb{C} \to \mathbb{C}^{2n+1}$ is constant. In particular, these contact structures are not globally contactomorphic to the standard complex contact structure on $\mathbb{C}^{2n+1}$.

Keywords Complex contact structures · Hyperbolicity · Fatou–Bieberbach domains

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1 Introduction and Main Results

Let M be a complex manifold of odd dimension $2n+1 \geq 3$, where $n \in \mathbb{N} = \{1, 2, \ldots\}$. A holomorphic vector subbundle $\xi \subset TM$ of complex codimension one in the tangent bundle TM is a *holomorphic contact structure* on M if every point $p \in M$ admits an open neighborhood $U \subset M$ such that $\xi|_U = \ker \alpha$ for a holomorphic 1-form α on U satisfying

$$\alpha \wedge (d\alpha)^n \neq 0.$$

A 1-form α satisfying this nondegeneracy condition is called a *holomorphic contact form*, and (M, ξ) is a *complex contact manifold*. We shall also write (M, α)

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when $\xi = \ker \alpha$ holds on all of M. The model is the complex Euclidean space $(\mathbb{C}^{2n+1}_{x_1,y_1,\ldots,x_n,y_n,z}, \xi_0 = \ker \alpha_0)$ where α_0 is the standard complex contact form

$$\alpha_0 = \mathrm{d}z + \sum_{j=1}^{n} x_j \, \mathrm{d}y_j. \tag{1.1}$$

By Darboux's theorem, every holomorphic contact form equals α_0 in suitably chosen local holomorphic coordinates at any given point (see e.g., Geiges [11, Theorem 2.5.1, p. 67] for the smooth case and [1, Theorem A.2] for the holomorphic one). This standard case has recently been considered by Alarcón, López, and the author in [1]. They proved in particular that every open Riemann surface R admits a proper holomorphic embedding $f\colon R \hookrightarrow (\mathbb{C}^{2n+1}, \alpha_0)$ as a *Legendrian curve*, meaning that $f^*\alpha_0 = 0$ holds on R. In the same paper, the authors asked whether there exists a holomorphic contact form α on $\mathbb{C}^3$ which is not globally equivalent to the standard form α_0 (cf. [1, Problem 1.5, p. 4]). In this paper, we provide such examples in every dimension.

Theorem 1.1 *For every $n \in \mathbb{N}$, there exists a holomorphic contact form α on $\mathbb{C}^{2n+1}$ such that any holomorphic map $f\colon \mathbb{C} \to \mathbb{C}^{2n+1}$ satisfying $f^*\alpha = 0$ is constant. In particular, the complex contact manifold $(\mathbb{C}^{2n+1}, \alpha)$ is not contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$.*

Indeed, a contactomorphism sends Legendrian curves to Legendrian curves, and $(\mathbb{C}^{2n+1}, \xi_0)$ admits plenty of embedded Legendrian complex lines $\mathbb{C} \hookrightarrow \mathbb{C}^{2n+1}$. For example, given a point $p = (x_0, y_0, z_0) \in \mathbb{C}^3$ and a vector $v = (v_1, v_2, v_3) \in \ker \alpha_0|_p$, the quadratic map $f\colon \mathbb{C} \to \mathbb{C}^3$ given by

$$f(\zeta) = \left(x_0 + v_1\zeta,\, y_0 + v_2\zeta,\, z_0 + v_3\zeta - v_1 v_2 \zeta^2/2\right)$$

is a holomorphic Legendrian embedding satisfying $f(0) = p$ and $f'(0) = v$.

We expect that our construction actually gives many nonequivalent holomorphic contact structures on $\mathbb{C}^{2n+1}$; however, we do not know how to distinguish them. Eliashberg showed that on $\mathbb{R}^3$ there exist countably many isotopy classes of smooth contact structures [8,9]. His classification is based on the study of *overtwisted disks* in contact 3-manifolds; it is not clear whether a similar invariant could be used in the complex case.

In order to prove Theorem 1.1, we consider the *directed Kobayashi metric* associated to a contact complex manifold (M, ξ). Let $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ denote the open unit disk. Given a holomorphic subbundle $\xi \subset TM$, we say that a holomorphic disk $f\colon \mathbb{D} \to M$ is *tangential to* ξ or *horizontal* if

$$f'(\zeta) \in \xi|_{f(\zeta)} \quad \text{holds for all } \zeta \in \mathbb{D}.$$

Consider the function $\xi \to \mathbb{R}_+$ given for any point $p \in M$ and vector $v \in \xi_p$ by

$$|v|_\xi = \inf\left\{\frac{1}{|\lambda|} : \exists f : \mathbb{D} \to M \text{ horizontal},\ f(0) = p,\ f'(0) = \lambda v\right\}.$$

When $\xi = TM$, this is the Kobayashi length of the tangent vector $v \in T_pM$, and its integrated version is the Kobayashi metric on M (cf. Kobayashi [14,15]). The directed version of the Kobayashi metric was studied by Demailly [5] and several other authors, mainly on complex projective manifolds. More general metrics, obtained by integrating a Riemannian metric along horizontal curves in a smooth directed manifold (M, ξ), have been studied by Gromov [13] under the name *Carnot–Carathéodory metrics*. (See also Bellaïche [2].) For this reason, we propose the name *Carnot–Carathéodory–Kobayashi metric* or *CCK metric*, for the pseudodistance function $d_\xi : M \times M \to \mathbb{R}_+$ defined by

$$d_\xi(p, q) = \inf_\gamma \int_0^1 |\gamma'(t)|_\xi \, dt, \quad p, q \in M, \tag{1.2}$$

where the infimum is over all piecewise smooth paths $\gamma : [0, 1] \to M$ satisfying $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(t) \in \xi_{\gamma(t)}$ for all $t \in [0, 1]$. (By Chow's theorem [4], a horizontal path connecting any given pair of points in M exists when the repeated commutators of vector fields tangential to ξ span the tangent space of M at every point. A discussion and proof of Chow's theorem can also be found in Gromov's paper [13, pp. 86, 113]. Another source is Sussmann [17,18].)

The directed complex manifold (M, ξ) is said to be *(Kobayashi) hyperbolic* if d_ξ given by (1.2) is a distance function on M (i.e., if $d_\xi(p, q) > 0$ holds for all pairs of distinct points $p, q \in M$), and is *complete hyperbolic* if d_ξ is a complete metric on M. Clearly, the directed Kobayashi metric on (M, ξ) dominates the standard Kobayashi metric on M.

Now, Theorem 1.1 is an obvious corollary to the following result.

Theorem 1.2 *For every $n \in \mathbb{N}$, there exists a holomorphic contact form α on $\mathbb{C}^{2n+1}$ such that the complex contact manifold* $(\mathbb{C}^{2n+1}, \xi = \ker\alpha)$ *is Kobayashi hyperbolic.*

The contact 1-forms that we shall construct in the proof of Theorem 1.2 are of the form

$$\alpha = \Phi^*\alpha_0$$

where α_0 is the standard contact form (1.1) and $\Phi : \mathbb{C}^{2n+1} \hookrightarrow \mathbb{C}^{2n+1}$ is a *Fatou–Bieberbach map*, i.e., an injective holomorphic map from $\mathbb{C}^{2n+1}$ onto a proper subdomain $\Omega = \Phi(\mathbb{C}^{2n+1}) \subsetneq \mathbb{C}^{2n+1}$ such that $(\Omega, \alpha_0|_\Omega)$ is a hyperbolic contact manifold. Let us describe this construction. Let $C_N > 0$ for $N \in \mathbb{N}$ be a sequence diverging to $+\infty$ and

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}^{2n}_{(x,y)} \times C_N \overline{\mathbb{D}}_z. \tag{1.3}$$

Here, $b\mathbb{D}^{2n}_{(x,y)} \subset \mathbb{C}^{2n}$ denotes the boundary of the unit polydisk in the (x, y)-space and $\overline{\mathbb{D}}_z$ is the closed unit disk in the z direction. Thus, K is the union of a sequence of compact cylinders $K_N = 2^{N-1} b\mathbb{D}^{2n}_{(x,y)} \times C_N \overline{\mathbb{D}}_z$ tending to infinity in all directions. Theorem 1.2 follows immediately from the following two results of possible independent interest. In both results, K is the set given by (1.3).

Proposition 1.3 *If $C_N \geq n2^{3N+1}$ holds for all $N \in \mathbb{N}$, then the domain $\Omega_0 = \mathbb{C}^{2n+1} \backslash K$ is α_0-hyperbolic. (Here, α_0 is the contact form (1.1).)*

Proposition 1.4 *For every choice of constants $C_N > 0$, there exists a Fatou–Bieberbach domain $\Omega \subset \mathbb{C}^{2n+1} \backslash K$.*

Indeed, if a domain $\Omega_0 \subset \mathbb{C}^{2n+1}$ is α_0-hyperbolic then so is any subdomain $\Omega \subset \Omega_0$. Furthermore, a biholomorphic map $\Phi\colon \mathbb{C}^{2n+1} \to \Omega$ is an isometry in the directed Kobayashi metric from the contact manifold $(\mathbb{C}^{2n+1}, \alpha)$ with $\alpha = \Phi^*\alpha_0$ onto the contact manifold (Ω, α_0). Since (Ω, α_0) is hyperbolic by Proposition 1.3, Theorem 1.2 follows.

Remark 1.5 It is worthwhile to observe that the contact structures in Theorem 1.2 are isotopic to the standard contact structure on $\mathbb{C}^{2n+1}$. Indeed, every Fatou–Bieberbach map $\Phi\colon \mathbb{C}^{2n+1} \hookrightarrow \mathbb{C}^{2n+1}$ is isotopic to the identity $\Phi_0 = \mathrm{Id}$ on $\mathbb{C}^{2n+1}$ through a smooth 1-parameter family $\Phi_t\colon \mathbb{C}^{2n+1} \hookrightarrow \mathbb{C}^{2n+1}$ ($t \in [0, 1]$) of injective holomorphic maps, and hence $\alpha_t = \Phi_t^*\alpha_0$ is an isotopy of holomorphic contact forms connecting α_0 to $\alpha = \alpha_1$.

Proposition 1.3 is proved in Sect. 2; the proof uses Cauchy estimates and the explicit expression (1.1) for the standard contact form α_0. The set K given by (1.3) presents obstacles which impose a limitation on the size of holomorphic α_0-Legendrian disks.

Proposition 1.4 is a special case of Theorem 3.1 which provides a more general result concerning the possibility of avoiding certain unions of cylinders in $\mathbb{C}^n$ by Fatou–Bieberbach domains. Its proof is inspired by a result of Globevnik [12, Theorem 1.1] who constructed Fatou–Bieberbach domains in $\mathbb{C}^n$ whose intersection with a ball $R\mathbb{B}^n$ for a given $R > 0$ is approximately equal to the intersection of the cylinder $\mathbb{D}^{n-1} \times \mathbb{C}$ with the same ball. His result implies that one can avoid any cylinder K_N in the set K (1.3) by a Fatou–Bieberbach domain Ω. We shall improve the construction so that Ω avoids all cylinders K_N at the same time. For this purpose, we will use a sequence of holomorphic automorphisms $\theta_k \in \mathrm{Aut}(\mathbb{C}^n)$ such that the sequence of their compositions $\Theta_k = \theta_k \circ \cdots \circ \theta_1$ converges on a certain domain Ω and diverges to infinity on the set K; hence $K \cap \Omega = \emptyset$. We ensure in addition that each θ_k approximates the identity map on the polydisk $k\overline{\mathbb{D}}^n$, and hence the limit $\Theta = \lim_{k\to\infty} \Theta_k\colon \Omega \to \mathbb{C}^{2n+1}$ is a biholomorphic map of Ω onto $\mathbb{C}^{2n+1}$.

Several interesting questions remain open. One is whether there exists a *complete hyperbolic* complex contact structure on $\mathbb{C}^{2n+1}$. Another is whether there exist *algebraic* contact forms α on $\mathbb{C}^{2n+1}$ (i.e., with polynomial coefficients) such that $(\mathbb{C}^{2n+1}, \alpha)$ is hyperbolic. (Our construction only furnishes transcendental examples.) If so, what is the minimal degree of such examples, and for which degrees is a generic (or very generic) contact form hyperbolic? In the integrable case, for affine algebraic and projective manifolds, this is the famous *Kobayashi Conjecture*; see Demailly [6], Brotbek [3], and Deng [7] for recent results on this subject.

Perhaps the most ambitious question is to classify complex contact structures on Euclidean spaces up to isotopy, in the spirit of Eliashberg's classification [8,9] of smooth contact structures on $\mathbb{R}^3$.

Holomorphic contact structures on compact complex manifolds $M = M^{2n+1}$ seem much better understood than those on open manifolds; see for example the paper by LeBrun [16] and the references therein. In particular, the space of all holomorphic contact subbundles of TM, if nonempty, is a connected complex manifold [16, p. 422]. Furthermore, if M is simply connected then any two holomorphic contact structures on M are equivalent via some holomorphic automorphism of M [16, Proposition 2.3]. In particular, the only complex contact structure on the projective space $\mathbb{CP}^{2n+1}$ (up to projective linear automorphisms) is the standard one, given in homogeneous coordinates by the 1-form $\theta = \sum_{j=0}^{n}(z_j \mathrm{d}z_{n+j+1} - z_{n+j+1}\mathrm{d}z_j)$. This structure is obtained by contracting the holomorphic symplectic form $\omega = \sum_{j=0}^{n} \mathrm{d}z_j \wedge \mathrm{d}z_{n+j+1}$ on $\mathbb{C}^{2n+2}$ with the radial vector field $\sum_{k=0}^{2n+1} z_k \frac{\partial}{\partial z_k}$. Its restriction to any affine chart $\mathbb{C}^{2n+1} \subset \mathbb{CP}^{2n+1}$ is equivalent to the standard contact structure given by (1.1). It follows that the projective space $\mathbb{CP}^{2n+1}$ does not carry any hyperbolic complex contact structures.

2 Hyperbolic Contact Structures on Domains in $\mathbb{C}^{2n+1}$

In this section, we prove Proposition 1.3. For simplicity of notation, we consider the case $n = 1$; the same proof applies in every dimension.

Thus, let (x, y, z) be complex coordinates on $\mathbb{C}^3$ and $\alpha_0 = dz + x dy$ be the standard contact form (1.1) on $\mathbb{C}^3$. Recall that $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $\overline{\mathbb{D}} = \{\zeta \in \mathbb{C} : |\zeta| \le 1\}$. The definition of the directed Kobayashi metric shows that Proposition 1.3 is an immediate corollary to the following lemma.

Lemma 2.1 *Assume that $C_N \ge 2^{3N+1}$ for every $N \in \mathbb{N}$ and let*

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}^2_{(x,y)} \times C_N \overline{\mathbb{D}}_z.$$

For every holomorphic α_0-horizontal disk $f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)) \in \mathbb{C}^3 \setminus K$ ($\zeta \in \mathbb{D}$) with $f(0) \in 2^{N_0}\mathbb{D}^3$ for some $N_0 \in \mathbb{N}$, we have the estimates

$$|x'(0)| < 2^{N_0+1}, \quad |y'(0)| < 2^{N_0+1}, \quad |z'(0)| < 2^{2N_0+1}. \tag{2.1}$$

Proof Replacing f by the disk $\zeta \mapsto f(r\zeta)$ for some $r < 1$ close to 1, we may assume that f is holomorphic on $\overline{\mathbb{D}}$. Pick a number $N \in \mathbb{N}$ with $N > N_0$ such that $|x(\zeta)| < 2^N$ and $|y(\zeta)| < 2^N$ for all $\zeta \in \overline{\mathbb{D}}$. By the Cauchy estimates applied with $\delta = 2^{-N}$, we then have

$$|y'(\zeta)| < 2^{2N} \quad \text{and} \quad |x(\zeta)y'(\zeta)| < 2^{3N} \quad \text{for } |\zeta| \le 1 - 2^{-N}.$$

Since f is a horizontal disk, we have $z'(\zeta) = -x(\zeta)y'(\zeta)$ for $\zeta \in \mathbb{D}$ and hence

$$|z(\zeta)| \leq |z(0)| + \left| \int_0^\zeta x dy \right| < 2^{N_0} + 2^{3N} < 2^{3N+1} \leq C_N \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$

From this estimate, the definition of the set K and the fact that $f(\mathbb{D}) \cap K = \emptyset$, it follows that

$$(x(\zeta), y(\zeta)) \notin 2^{N-1} b\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$

Since $2^{N-1} b\mathbb{D}^2$ disconnects the disk $2^N \mathbb{D}^2$ and we have $(x(0), y(0)) \in 2^{N_0}\mathbb{D}^2 \subset 2^{N-1}\mathbb{D}^2$, we conclude that

$$(x(\zeta), y(\zeta)) \in 2^{N-1}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$

If $N - 1 > N_0$, we can repeat the same argument with the restricted horizontal disk $f : (1 - 2^{-N})\overline{\mathbb{D}} \to \mathbb{C}^3$ to obtain

$$(x(\zeta), y(\zeta)) \in 2^{N-2}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N} - 2^{-(N-1)}.$$

After finitely steps of the same kind, we get that

$$(x(\zeta), y(\zeta)) \in 2^{N_0}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N} - \cdots - 2^{-(N_0+1)}.$$

Since $2^{-N} + \cdots + 2^{-(N_0+1)} < 1/2$, we see that $(x(\zeta), y(\zeta)) \in 2^{N_0}\mathbb{D}^2$ for $|\zeta| \leq 1/2$. Applying once again the Cauchy estimates gives $|x'(0)|, |y'(0)| \leq 2^{N_0+1}$ and hence $|z'(0)| = |x(0)y'(0)| \leq 2^{2N_0+1}$; these are precisely the conditions in (2.1). □

3 Fatou–Bieberbach Domains Avoiding a Union of Cylinders

In this section, we prove the following result on avoiding certain closed cylindrical sets in $\mathbb{C}^n$ by Fatou–Bieberbach domains. This includes Proposition 1.4 as a special case.

Theorem 3.1 *Let* $0 < a_1 < b_1 < a_2 < b_2 < \cdots$ *and* $c_i > 0$ *be sequences of real numbers such that* $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = +\infty$. *Let* $n > 1$ *be an integer and*

$$K = \bigcup_{i=1}^{\infty} \left(b_i \overline{\mathbb{D}}^{n-1} \backslash a_i \mathbb{D}^{n-1} \right) \times c_i \overline{\mathbb{D}} \subset \mathbb{C}^n. \tag{3.1}$$

Then there exists a Fatou–Bieberbach domain $\Omega \subset \mathbb{C}^n \backslash K$.

As said in Sect. 1, the proof is inspired by [12, Proof of Theorem 1.2] to a certain point and is based on the so-called push-out method. Since the set K (3.1) is noncompact, the construction of automorphisms used in the proof is somewhat more involved in our case. On the other hand, since our goal is merely to avoid K by a

Fatou–Bieberbach domain, and not to approximate a given cylinder as Globevnik did in [12], the construction is less precise in certain other aspects.

Proof We denote by $\mathrm{Aut}(\mathbb{C}^n)$ the group of all holomorphic automorphisms of $\mathbb{C}^n$. We first give the proof for $n = 2$ and explain in the end how to treat the general case.

Let (z_1, z_2) be complex coordinates on $\mathbb{C}^2$, and let $K = K_1$ be the set (3.1). Up to a dilation of coordinates, we may assume without loss of generality that $a_1 > 1$.

Pick sequence $\epsilon_k \in (0, 1)$ satisfying $\sum_{k=1}^{\infty} \epsilon_k < +\infty$. We shall construct sequences of automorphisms $\phi_k, \psi_k \in \mathrm{Aut}(\mathbb{C}^2)$ $(k \in \mathbb{N})$ of the following form:

$$\phi_k(z_1, z_2) = (z_1, z_2 + f_k(z_1)), \quad \psi_k(z_1, z_2) = (z_1 + g_k(z_2), z_2), \tag{3.2}$$

where f_k and g_k are suitably chosen entire functions on $\mathbb{C}$ to be specified. Set

$$\theta_k = \psi_k \circ \phi_k, \quad \Theta_k = \theta_k \circ \cdots \circ \theta_1, \quad k \in \mathbb{N}. \tag{3.3}$$

We will also ensure that for every $k \in \mathbb{N}$ we have

$$|\theta_k(z) - z| < \epsilon_k \quad \text{for } z \in k\overline{\mathbb{D}}^2.$$

Granted the last condition, it follows (cf. [10, Proposition 4.4.1 and Corollary 4.4.2]) that the sequence $\Theta_k \in \mathrm{Aut}(\mathbb{C}^2)$ converges uniformly on compacts in the open set

$$\Omega = \bigcup_{k=1}^{\infty} \Theta_k^{-1}(k\mathbb{D}^2) = \{z \in \mathbb{C}^2 : (\Theta_k(z))_{k\in\mathbb{N}} \text{ is a bounded sequence}\}$$

to a biholomorphic map $\Theta = \lim_{k\to\infty} \Theta_k \colon \Omega \to \mathbb{C}^2$ of Ω onto $\mathbb{C}^2$. We will also ensure that

$$|\Theta_k(z)| \to +\infty \quad \text{for all points } z \in K, \tag{3.4}$$

and hence $K \cap \Omega = \emptyset$. This will prove the theorem when $n = 2$.

We begin by explaining how to choose the first two maps ϕ_1 and ψ_1; all subsequent steps will be analogous. Set $b_0 = 1$. Pick a sequence r_j satisfying $b_{j-1} < r_j < a_j$ for all $j = 1, 2, \ldots$. Let $N_j \in \mathbb{N}$ be a sequence of integers to be specified later. Set

$$f(\zeta) = \sum_{j=1}^{\infty} \left(\frac{\zeta}{r_j}\right)^{N_j}.$$

This function will define the first automorphism ϕ_1 (cf. (3.2)). Let $f_i(\zeta) = \sum_{j=1}^{i} \left(\frac{\zeta}{r_j}\right)^{N_j}$ denote the ith partial sum of the series defining $f(\zeta)$, where we set $f_0 = 0$. By choosing the exponent N_i big enough, we can ensure that the summand $(\zeta/r_i)^{N_i}$ is arbitrarily small on the disk $b_{i-1}\overline{\mathbb{D}}$ and is arbitrarily big on the annulus

$$A_i := b_i\overline{\mathbb{D}} \backslash a_i\mathbb{D} = \{\zeta : a_i \le |\zeta| \le b_i\}. \tag{3.5}$$

In particular, we may ensure that for every $i \in \mathbb{N}$, we have

$$\sup_{|\zeta| \le b_{i-1}} \left| \frac{\zeta}{r_i} \right|^{Ni} < 2^{-i-1} \epsilon_1. \tag{3.6}$$

It follows that the power series defining $f(\zeta)$ converges on all of $\mathbb{C}$ and satisfies

$$\sup_{|\zeta| \le b_{i-1}} |f(\zeta) - f_{i-1}(\zeta)| < 2^{-i} \epsilon_1, \quad i \in \mathbb{N}. \tag{3.7}$$

Note that the inequalities (3.6) and (3.7) persist if we increase the exponents N_i. We can inductively choose the sequence $N_i \in \mathbb{N}$ to grow fast enough such that the following inequalities hold for every $i \in \mathbb{N}$ with an increasing sequence of numbers $M_i \ge i+1$:

$$\sup_{|\zeta| \le b_{i-1}} |f_{i-1}(\zeta)| + c_{i-1} + \epsilon_1 < M_i < \inf_{\zeta \in A_i} \left(\left| \frac{\zeta}{r_i} \right|^{N_i} - |f_{i-1}(\zeta)| \right) - c_i - \epsilon_1. \tag{3.8}$$

(Recall that A_i is the annulus (3.5). Here, $c_0 \ge 0$ is arbitrary, while $c_i > 0$ for $i \in \mathbb{N}$ are the constants in the definition (3.1) of the set K.) In view of the inequalities (3.6), (3.7), and (3.8), there exist numbers $\beta_{i-1} < \alpha_i$ such that for all $i \in \mathbb{N}$ we have

$$\sup_{|\zeta| \le b_{i-1}} |f(\zeta)| + c_{i-1} < \beta_{i-1} < M_i < \alpha_i < \inf_{\zeta \in A_i} |f(\zeta)| - c_i. \tag{3.9}$$

This gives increasing sequences $0 < \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$ diverging to ∞. Set

$$\phi_1(z_1, z_2) = (z_1, z_2 + f(z_1)).$$

The right-hand side of (3.9) shows that for every point $z = (z_1, z_2) \in A_i \times c_i \overline{\mathbb{D}}$ we have

$$|z_2 + f(z_1)| \ge |f(z_1)| - c_i > \alpha_i,$$

while the left-hand side of (3.9) gives

$$|z_2 + f(z_1)| \le c_i + |f(z_1)| < \beta_i.$$

Since these inequalities hold for every $i \in \mathbb{N}$, it follows that

$$\phi_1(K) \subset L := \bigcup_{i=1}^{\infty} b_i \overline{\mathbb{D}} \times \left(\beta_i \overline{\mathbb{D}} \backslash \alpha_i \mathbb{D} \right) \subset \mathbb{C}^2.$$

Note that the set L is of the same kind as K (3.1) with the reversed roles of the variables, i.e., the cylinders in L are horizontal instead of vertical. Furthermore, since the sequence α_i is increasing and $\alpha_1 > M_1 \geq 2$ by (3.9), we also see that

$$L \cap (\mathbb{C} \times 2\overline{\mathbb{D}}) = \emptyset.$$

The same argument as above with the set L furnishes a shear automorphism

$$\psi_1(z_1, z_2) = (z_1 + g(z_2), z_2)$$

for some $g \in \mathscr{O}(\mathbb{C})$ (cf. (3.2)) and a set K_2 of the same kind as $K = K_1$ (3.1) (this time again with vertical cylinders) such that, setting $\theta_1 := \psi_1 \circ \phi_1 \in \mathrm{Aut}(\mathbb{C}^2)$, we have

$$\theta_1(K_1) \subset K_2, \quad K_2 \cap 2\overline{\mathbb{D}}^2 = \emptyset, \quad \sup_{z \in \overline{\mathbb{D}}^2} |\theta_1(z) - z| < \epsilon_1. \tag{3.10}$$

Continuing inductively, we find a sequence of automorphisms $\theta_k \in \mathrm{Aut}(\mathbb{C}^2)$ and of closed sets $K_k \subset \mathbb{C}^2$ of the form (3.1) such that for every $k \in \mathbb{N}$, we have

$$\theta_k(K_k) \subset K_{k+1}, \quad K_k \cap k\overline{\mathbb{D}}^2 = \emptyset, \quad \sup_{z \in k\overline{\mathbb{D}}^2} |\theta_k(z) - z| < \epsilon_k. \tag{3.11}$$

Each step of the recursion is of exactly the same kind as the initial one. This implies that

$$\Theta_k(K) \subset K_{k+1} \subset \mathbb{C}^2 \backslash (k+1)\overline{\mathbb{D}}^2, \quad k \in \mathbb{N}$$

and hence (3.4) also holds. This completes the proof when $n = 2$.

Suppose now that $n > 2$. In this case, each automorphism $\theta_k = \psi_k \circ \phi_k \in \mathrm{Aut}(\mathbb{C}^n)$ in the sequence (3.3) is a composition of two shear-like maps of the form

$$\begin{aligned}
\phi_k(z_1, z_2, \ldots, z_n) &= \big(z_1, z_2 + f_k(z_1), z_3 + f_k(z_2), \ldots, z_n + f_k(z_{n-1})\big), \\
\psi_k(z_1, z_2, \ldots, z_n) &= \big(z_1 + g_k(z_2), z_2 + g_k(z_3), \ldots, z_{n-1} + g_k(z_n), z_n\big).
\end{aligned}$$

A suitable choice of entire functions $f_k, g_k \in \mathscr{O}(\mathbb{C})$ ensures as before that condition (3.11) holds for each k (with $\overline{\mathbb{D}}^2$ replaced by $\overline{\mathbb{D}}^n$). We leave the details to an interested reader. Further details in the case $n > 2$ are also available in [12, proof of Theorem 1.2]. □

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