

# Every conformal minimal surface in $\mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve

By *Antonio Alarcón* at Granada and *Franc Forstnerič* at Ljubljana

---

**Abstract.** We show that for every conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$  from an open Riemann surface  $M$  to  $\mathbb{R}^3$  there exists a smooth isotopy  $u_t : M \rightarrow \mathbb{R}^3$  ( $t \in [0, 1]$ ) of conformal minimal immersions, with  $u_0 = u$ , such that  $u_1$  is the real part of a holomorphic null curve  $M \rightarrow \mathbb{C}^3$  (i.e.  $u_1$  has vanishing flux). If furthermore  $u$  is nonflat, then  $u_1$  can be chosen to have any prescribed flux and to be complete.

## 1. The main results

Let  $M$  be a smooth oriented surface. A smooth immersion  $u = (u_1, u_2, u_3) : M \rightarrow \mathbb{R}^3$  is *minimal* if its mean curvature vanishes at every point. The requirement that an immersion  $u$  be *conformal* uniquely determines a complex structure on  $M$ . Finally, a conformal immersion is minimal if and only if it is harmonic:  $\Delta u = 0$  (Osserman [20]). A holomorphic immersion  $F = (F_1, F_2, F_3) : M \rightarrow \mathbb{C}^3$  of an open Riemann surface to  $\mathbb{C}^3$  is said to be a *null curve* if its differential  $dF = (dF_1, dF_2, dF_3)$  satisfies the equation

$$(dF_1)^2 + (dF_2)^2 + (dF_3)^2 = 0.$$

The real and the imaginary part of a null curve  $M \rightarrow \mathbb{C}^3$  are conformal minimal immersions  $M \rightarrow \mathbb{R}^3$ . Conversely, the restriction of a conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$  to any simply connected domain  $\Omega \subset M$  is the real part of a holomorphic null curve  $\Omega \rightarrow \mathbb{C}^3$ ;  $u$  is globally the real part of a null curve if and only if its conjugate differential  $d^c u$  satisfies  $\int_C d^c u = 0$  for every closed curve  $C$  in  $M$ . This period vanishing condition means that  $u$  admits a harmonic conjugate  $v$ , and  $F = u + iv : M \rightarrow \mathbb{C}^3$  ( $i = \sqrt{-1}$ ) is then a null curve.

---

Antonio Alarcón is supported by the Ramón y Cajal program of the Spanish Ministry of Economy and Competitiveness, and is partially supported by the MINECO/FEDER grants MTM2011-22547 and MTM2014-52368-P, Spain. Franc Forstnerič is supported in part by the research program P1-0291 and the grant J1-5432 from ARRS, Republic of Slovenia.

In this paper we prove the following result which further illuminates the connection between conformal minimal surfaces in  $\mathbb{R}^3$  and holomorphic null curves in  $\mathbb{C}^3$ . We shall systematically use the term *isotopy* instead of the more standard *regular homotopy* when talking of smooth 1-parameter families of immersions.

**Theorem 1.1.** *Let  $M$  be an open Riemann surface. For every conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$  there exists a smooth isotopy  $u_t : M \rightarrow \mathbb{R}^3$  ( $t \in [0, 1]$ ) of conformal minimal immersions such that  $u_0 = u$  and  $u_1 = \Re F$  is the real part of a holomorphic null curve  $F : M \rightarrow \mathbb{C}^3$ .*

The analogous result holds for minimal surfaces in  $\mathbb{R}^n$  for any  $n \geq 3$ , and the tools used in the proof are available in that setting as well. On a compact bordered Riemann surface we also have an up to the boundary version of the same result (cf. Theorem 4.1).

Given a conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$ , the *flux map* of  $u$  is the group homomorphism  $\text{Flux}_u : H_1(M; \mathbb{Z}) \rightarrow \mathbb{R}^3$  on the first homology group of  $M$  which is given on any closed curve  $C$  in  $M$  by

$$(1.1) \quad \text{Flux}_u(C) = \int_C d^c u.$$

We can view  $\text{Flux}_u$  as the element of the de Rham cohomology group  $H^1(M; \mathbb{R}^3)$  determined by the closed real 1-form  $d^c u = \iota(\bar{\partial}u - \partial u)$  with values in  $\mathbb{R}^3$ . (Note that  $d^c u$  is closed precisely when  $u$  is harmonic:  $dd^c u = 0$ .) A conformal harmonic immersion  $u : M \rightarrow \mathbb{R}^3$  is the real part of a holomorphic null curve  $M \rightarrow \mathbb{C}^3$  if and only if the flux map  $\text{Flux}_u$  is identically zero. Hence Theorem 1.1 can be expressed as follows.

*Every conformal minimal immersion is isotopic to one with vanishing flux.*

Every open Riemann surface  $M$  is homotopy equivalent to a wedge of circles, and its first homology group  $H_1(M; \mathbb{Z})$  is a free abelian group with at most countably many generators. If  $M$  is of finite genus  $g$  with  $m$  topological ends, then  $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^{2g+m-1}$ .

We now describe a more general existence result for isotopies of conformal minimal immersions. Recall that an immersion  $u : M \rightarrow \mathbb{R}^n$  is said to be *complete* if the pullback  $u^*(ds^2)$  of the Euclidean metric  $ds^2$  on  $\mathbb{R}^n$  is a complete metric on  $M$ . It is easily seen that a holomorphic null curve  $M \rightarrow \mathbb{C}^3$  is complete if and only if its real part  $M \rightarrow \mathbb{R}^3$  is complete (cf. Osserman [20]). An immersion  $u : M \rightarrow \mathbb{R}^3$  is *nonflat* if its image  $u(M) \subset \mathbb{R}^3$  is not contained in any affine plane. It is easily seen that every flat conformal minimal immersion from a connected open Riemann surface is the real part of a holomorphic null curve (see Remark 2.1), so Theorem 1.1 trivially holds in this case. For nonflat immersions we have the following second main result of this paper.

**Theorem 1.2.** *Let  $M$  be a connected open Riemann surface and let  $\mathfrak{p} : H_1(M; \mathbb{Z}) \rightarrow \mathbb{R}^3$  be a group homomorphism. For every nonflat conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$  there exists a smooth isotopy  $u_t : M \rightarrow \mathbb{R}^3$  ( $t \in [0, 1]$ ) of conformal minimal immersions such that  $u_0 = u$ ,  $u_1$  is complete, and  $\text{Flux}_{u_1} = \mathfrak{p}$ . Furthermore, if  $u$  is complete, then an isotopy as above can be chosen such that  $u_t$  is complete for every  $t \in [0, 1]$ . In particular, every nonflat conformal minimal immersion  $M \rightarrow \mathbb{R}^3$  is isotopic through conformal minimal immersions to the real part of a complete holomorphic null curve.*

It was shown by Alarcón, Fernández, and López [2, 3] that every group homomorphism  $\mathfrak{p} : H_1(M; \mathbb{Z}) \rightarrow \mathbb{R}^3$  is the flux map  $\mathfrak{p} = \text{Flux}_u$  of a complete conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$ . The novel part of Theorem 1.2 is that one can deform an arbitrary nonflat conformal minimal immersion  $M \rightarrow \mathbb{R}^3$  through a smooth family of such immersions to one that is complete and has the given flux homomorphism.

Our results provide an initial step in the problem of homotopy classification of conformal minimal immersions  $M \rightarrow \mathbb{R}^3$  by their tangent maps; we discuss this in Section 8 below (see in particular Proposition 8.4). The subject of homotopy classification of immersions goes back to Smale [21] and Hirsch [15], and it was later subsumed by Gromov's h-principle in smooth geometry (see Gromov's monograph [12]). The basic result of the Hirsch–Smale theory is that if  $M$  and  $N$  are smooth manifolds and  $1 \leq \dim M < \dim N$ , then regular homotopy classes of immersions  $M \rightarrow N$  are in one-to-one correspondence with the homotopy classes of fiberwise injective vector bundle maps  $TM \rightarrow TN$  of their tangent bundles; the same holds if  $\dim M = \dim N$  and  $M$  is not compact. This has been extended subsequently to several other classes of immersions. In particular, Eliashberg and Gromov obtained the h-principle for holomorphic immersions of Stein manifolds to complex Euclidean spaces; see the discussion and references in [10, Section 8.5].

Here we are interested in regular homotopy classes of conformal minimal immersions of open Riemann surfaces into  $\mathbb{R}^3$ . The  $(1, 0)$ -derivative  $\partial u$  of a conformal (not necessarily harmonic) immersion  $u : M \rightarrow \mathbb{R}^3$  gives a map  $M \rightarrow \mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$  with vanishing real periods into the punctured null quadric (2.2); *harmonic* (minimal) immersions correspond to holomorphic maps  $M \rightarrow \mathfrak{A}^*$ . (See Section 2.) The question is whether regular homotopy classes of conformal minimal immersions are in one-to-one correspondence with the homotopy classes of continuous maps  $M \rightarrow \mathfrak{A}^*$ . (We wish to thank R. Kusner who pointed out (private communication) the connection to the theory of spin structures on Riemann surfaces; we refer to the preprint [18] by R. Kusner and N. Schmitt. Since  $\mathfrak{A}^*$  is an Oka manifold and every open Riemann surface  $M$  is a Stein manifold, the homotopy classes of continuous maps  $M \rightarrow \mathfrak{A}^*$  coincide with the homotopy classes of holomorphic maps by the Oka principle [10, Theorem 5.4.4].) One direction is provided by [4, Theorem 2.6]: *Every continuous map  $M \rightarrow \mathfrak{A}^*$  is homotopic to the derivative of a holomorphic null immersion  $M \rightarrow \mathbb{C}^3$ , hence to the  $(1, 0)$ -derivative of a conformal minimal immersion  $M \rightarrow \mathbb{R}^3$ .* What remains unclear is whether two conformal minimal immersions  $M \rightarrow \mathbb{R}^3$ , whose  $(1, 0)$ -derivatives are homotopic as maps  $M \rightarrow \mathfrak{A}^*$ , are regularly homotopic through conformal minimal immersions. A more precise question is formulated as Problem 8.2 below.

Another main result of the paper is an *h-Runge approximation theorem* for conformal minimal immersions of open Riemann surfaces to  $\mathbb{R}^3$ ; see Theorem 6.3. (Here, h stands for *homotopy*. This terminology is inspired by Gromov's h-Runge approximation theorem which plays a key role in the *Oka principle* for holomorphic maps from Stein manifolds to elliptic and Oka manifolds; cf. [13] and [10, Chapter 6].) Basically our result says that a homotopy of conformal minimal immersions  $u_t : U \rightarrow \mathbb{R}^3$  ( $t \in [0, 1]$ ), defined on a Runge open set  $U$  in an open Riemann surface  $M$  and such that  $u_0$  extends to a nonflat conformal minimal immersion  $M \rightarrow \mathbb{R}^3$ , can be approximated uniformly on compacts in  $U$  by a homotopy of conformal minimal immersions  $\tilde{u}_t : M \rightarrow \mathbb{R}^3$  such that  $\tilde{u}_0 = u_0$ . We also prove a version of this result with a fixed component function. For the usual (nonparametric) version of this result see [5, 6].

We now describe the content and the organization of the paper.

In Section 2 we establish the notation and review the background. In Section 3 we prepare the necessary results concerning the existence of loops with vanishing real or complex periods in the punctured null quadric  $\mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$  (see (2.2)). After suitable approximation, using Mergelyan's theorem, a loop with vanishing real period represents the  $(1, 0)$ -differential  $\partial u$  of a conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$  along a closed embedded Jordan curve in our Riemann surface  $M$ ; similarly, a loop with vanishing complex period represents the differential  $dF = \partial F$  of a holomorphic null curve  $F : M \rightarrow \mathbb{C}^3$  along a curve in  $M$ . A reader familiar with Gromov's *convex integration theory* [8, 12] will notice a certain similarity of ideas in the construction of such loops. In order to use the Mergelyan approximation theorem and at the same time keep the period vanishing condition we work with *period dominating sprays of loops* (cf. Lemma 3.6), using some results from our previous paper [4] on holomorphic null curves.

In Section 4 we prove Theorem 1.1 in the special case when  $M$  has finite topology. The general case is treated in Section 5.

In Section 6 we prove Theorem 6.3 (the h-Runge approximation theorem for conformal minimal immersions of open Riemann surfaces to  $\mathbb{R}^3$ ). By using this h-Runge theorem we obtain in Section 7 several extensions of Theorem 1.1 to isotopies of *complete* conformal minimal immersions; in particular, we prove Theorem 1.2.

In Section 8 we discuss the topology of the space of all conformal minimal immersions  $M \rightarrow \mathbb{R}^3$  and we indicate several open questions related to the results in the paper.

## 2. Notation and preliminaries

A compact set  $K$  in a complex manifold  $X$  is said to be *holomorphically convex* (or  $\mathcal{O}(X)$ -convex) if for every point  $p \in X \setminus K$  there exists a holomorphic function  $f \in \mathcal{O}(X)$  satisfying  $|f(p)| > \max_K |f|$ . This notion is especially important if  $X$  is a Stein manifold (in particular, an open Riemann surface) in view of the Runge approximation theorem (also called the Oka–Weil theorem); see e.g. [16].

Let  $M$  be a Riemann surface. An immersion  $u = (u_1, u_2, u_3) : M \rightarrow \mathbb{R}^3$  is conformal if and only if, in any local holomorphic coordinate  $z = x + iy$  on  $M$ , the partial derivatives  $u_x = (u_{1,x}, u_{2,x}, u_{3,x})$  and  $u_y = (u_{1,y}, u_{2,y}, u_{3,y})$ , considered as vectors in  $\mathbb{R}^3$ , have the same Euclidean length and are orthogonal to each other at every point of  $M$ :

$$(2.1) \quad |u_x| = |u_y| > 0, \quad u_x \cdot u_y = 0.$$

Equivalently,  $u_x \pm iy_y \in \mathbb{C}^3 \setminus \{0\}$  are *null vectors*, i.e. they lie in the *null quadric*

$$(2.2) \quad \mathfrak{A} = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0\}.$$

We shall write  $\mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$ . It is easily seen that  $\mathfrak{A}^*$  is a smooth closed hypersurface in  $\mathbb{C}^3 \setminus \{0\}$  which is the total space of a (nontrivial) holomorphic fiber bundle with fiber  $\mathbb{C}^*$  over  $\mathbb{C}\mathbb{P}^1$  (see [4, p. 741]). In particular,  $\mathfrak{A}^*$  is an *Oka manifold* [4, Proposition 4.5].

The exterior derivative on  $M$  splits into the sum  $d = \partial + \bar{\partial}$  of the  $(1, 0)$ -part  $\partial$  and the  $(0, 1)$ -part  $\bar{\partial}$ . In any local holomorphic coordinate  $z = x + iy$  on  $M$  we have

$$(2.3) \quad 2\partial u = (u_x - iy_y)dz, \quad 2\bar{\partial}u = (u_x + iy_y)d\bar{z}.$$

Hence (2.1) shows that  $u$  is conformal if and only if the differential  $\partial u = (\partial u_1, \partial u_2, \partial u_3)$  satisfies the nullity condition

$$(2.4) \quad (\partial u_1)^2 + (\partial u_2)^2 + (\partial u_3)^2 = 0.$$

Assume now that  $M$  is a connected open Riemann surface and that  $u : M \rightarrow \mathbb{R}^3$  is a conformal immersion. It is classical (cf. Osserman [20]) that

$$\Delta u = 2Hv,$$

where  $H : M \rightarrow \mathbb{R}$  denotes the mean curvature function of  $u$ ,  $v : M \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$  is the Gauss map of  $u$ , and  $\Delta$  is the Laplacian operator with respect to the metric induced on  $M$  by the Euclidean metric of  $\mathbb{R}^3$  via  $u$ . Hence  $u$  is minimal ( $H = 0$ ) if and only if it is harmonic ( $\Delta u = 0$ ). If  $v$  is any local harmonic conjugate of  $u$ , then it follows from the Cauchy–Riemann equations that

$$\partial(u + iv) = 2\partial u = 2i\bar{\partial}v.$$

Thus  $F = u + iv$  is a holomorphic immersion into  $\mathbb{C}^3$  whose differential  $dF = \partial F = 2\partial u$  has values in  $\mathfrak{A}^*$  (2.2); i.e. a *null holomorphic immersion*. In particular, the differential  $\partial u$  of any conformal minimal immersion is a holomorphic 1-form satisfying (2.4).

It is useful to introduce the *conjugate differential*,  $d^c u = i(\bar{\partial}u - \partial u)$ . We have that

$$(2.5) \quad 2\partial u = du + id^c u, \quad dd^c u = 2i\bar{\partial}\bar{\partial}u = \Delta u \cdot dx \wedge dy.$$

If  $u$  is harmonic (hence minimal), then  $d^c u$  is a closed vector valued 1-form on  $M$ , and we have that  $d^c u = dv$  for any local harmonic conjugate  $v$  of  $u$ . The *flux map* of  $u$  is the group homomorphism  $\text{Flux}_u : H_1(M; \mathbb{Z}) \rightarrow \mathbb{R}^3$  given by

$$\text{Flux}_u([C]) = \int_C d^c u, \quad [C] \in H_1(M; \mathbb{Z}).$$

The integral is independent of the choice of a path in a given homology class, and we shall write  $\text{Flux}_u(C)$  for  $\text{Flux}_u([C])$  in the sequel. Furthermore,  $u$  admits a global harmonic conjugate on  $M$  if and only if the 1-form  $d^c u$  is exact on  $M$ , and this holds if and only if

$$(2.6) \quad \text{Flux}_u(C) = \int_C d^c u = 0 \quad \text{for every closed curve } C \subset M.$$

We shall prove Theorem 1.1 by finding an isotopy  $u_t : M \rightarrow \mathbb{R}^3$  ( $t \in [0, 1]$ ) of conformal minimal immersions, with  $u_0 = u$ , such that  $u_1$  satisfies the period condition (2.6).

Fix a nowhere vanishing holomorphic 1-form  $\theta$  on  $M$ . (Such exists by the Oka–Grauert principle, cf. [10, Theorem 5.3.1, p. 190].) It follows from (2.3) that

$$2\partial u = f\theta,$$

where  $f = (f_1, f_2, f_3) : M \rightarrow \mathfrak{A}^*$  is a holomorphic map satisfying

$$(2.7) \quad \int_C \Re(f\theta) = \int_C du = 0 \quad \text{for any closed curve } C \text{ in } M.$$

Furthermore, we have  $u = \Re F$  for some null holomorphic immersion  $F : M \rightarrow \mathbb{C}^3$  if and only if  $\int_C f\theta = 0$  for all closed curves  $C$  in  $M$ . The meromorphic function on  $M$  given by

$$(2.8) \quad g := \frac{f_3}{f_1 - if_2}$$

is the stereographic projection of the Gauss map of  $u$ , and the map  $f = 2\partial u/\theta$  can be recovered from the pair  $(g, f_3)$  by the expression

$$(2.9) \quad f = \left( \frac{1}{2} \left( \frac{1}{g} - g \right), \frac{i}{2} \left( \frac{1}{g} + g \right), 1 \right) f_3.$$

The pair  $(g, f_3\theta)$  is called the *Weierstrass data* of  $u$ . The Riemannian metric  $ds_u^2$  induced on  $M$  by the Euclidean metric of  $\mathbb{R}^3$  via the immersion  $u$  equals

$$(2.10) \quad ds_u^2 = \frac{1}{2} |f\theta|^2 = \frac{1}{4} \left( \frac{1}{|g|} + |g| \right)^2 |f_3|^2 |\theta|^2.$$

We denote by  $\text{dist}_u$  the distance function induced on  $M$  by  $ds_u^2$ . Conversely, given a meromorphic function  $g$  and a holomorphic function  $f_3$  on  $M$  such that the map  $f$  in (2.9) has no poles, then  $f$  assumes values in  $\mathfrak{A}$ . If in addition  $f$  does not vanish anywhere on  $M$  and satisfies (2.7), then  $ds_u^2 > 0$  everywhere on  $M$  (see (2.10)), and  $f\theta$  integrates to a conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$  given by  $u(x) = \int^x \Re(f\theta)$  for  $x \in M$ .

**Remark 2.1.** If  $u$  is a *flat* (planar) immersion, in the sense that the image  $u(M) \subset \mathbb{R}^3$  lies in an affine 2-plane in  $\mathbb{R}^3$ , we may assume after an orthogonal change of coordinates that  $u_3 = \text{const}$ . In this case (2.3) implies  $\partial u_1 = \pm i\partial u_2$  which gives  $d^c u_1 = \pm du_2$  and  $d^c u_2 = \pm du_1$ , so  $d^c u$  is exact. This shows that *every flat conformal minimal immersion is the real part of a holomorphic null curve*  $F : M \rightarrow \mathbb{C}^3$ . In this case the image of  $dF$  lies in a ray  $\mathbb{C}v \subset \mathbb{C}^3$  spanned by a null vector  $0 \neq v \in \mathfrak{A}$ , and  $F(M)$  is contained in an affine complex line  $a + \mathbb{C}v \subset \mathbb{C}^3$ . Such *flat null curves* are precisely those that are *degenerate* in the sense of [4, Definition 2.2]. Note that every open Riemann surface  $M$  admits flat null holomorphic immersions  $M \rightarrow \mathbb{C}^3$  of the form  $F(x) = e^{g(x)}v$  ( $x \in M$ ), where  $0 \neq v \in \mathfrak{A}$  is a null vector and  $g \in \mathcal{O}(M)$  is a holomorphic function without critical points [14].

We denote by  $\text{CMI}(M)$  the set of all conformal minimal immersions  $M \rightarrow \mathbb{R}^3$  and by  $\text{CMI}_*(M) \subset \text{CMI}(M)$  the subset consisting of all nonflat immersions. By  $\text{NC}(M)$  we denote the space of all null holomorphic immersions  $F : M \rightarrow \mathbb{C}^3$ , and  $\text{NC}_*(M)$  is the subset of  $\text{NC}(M)$  consisting of nonflat immersions. These spaces are endowed with the compact-open topology. We have natural inclusions

$$\Re \text{NC}(M) \hookrightarrow \text{CMI}(M), \quad \Re \text{NC}_*(M) \hookrightarrow \text{CMI}_*(M),$$

where  $\Re \text{NC}(M) = \{\Re F : F \in \text{NC}(M)\}$  is the kernel of the flux map (2.6) on  $\text{CMI}(M)$ .

If  $K \subset M$  is a compact subset, we denote by  $\text{CMI}(K)$  the set of all conformal minimal immersions of unspecified open neighborhoods of  $K$  into  $\mathbb{R}^3$ , and by  $\text{CMI}_*(K) \subset \text{CMI}(K)$  the subset consisting of all immersions which are not flat on any connected component. We define  $\text{NC}(K)$  and  $\text{NC}_*(K)$  in the analogous way.

### 3. Loops with prescribed periods in the null quadric

Assume that  $C$  is a closed, embedded, oriented, real analytic curve in  $M$ . There are an open set  $W \subset M$  containing  $C$  and a biholomorphic map  $z : W \rightarrow \Omega$  onto an annulus

$\Omega = \{z \in \mathbb{C} : r^{-1} < |z| < r\}$  taking  $C$  onto the positively oriented unit circle

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

The exponential map  $\mathbb{C} \ni \zeta = x + iy \mapsto \exp(2\pi i \zeta) \in \mathbb{C}^*$  provides a universal covering of the annulus  $\Omega$  by the strip  $\Sigma = \{x + iy : x \in \mathbb{R}, |y| < (2\pi)^{-1} \log r\} \subset \mathbb{C}$ , mapping the real axis  $\mathbb{R} = \{y = 0\}$  onto the circle  $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ . We shall view  $\zeta = x + iy$  as a *uniformizing coordinate* on  $W$ , with  $C = \{y = 0\}$ . The restriction of a conformal harmonic immersion  $u : M \rightarrow \mathbb{R}^3$  to  $W$  is given in this coordinate by a 1-periodic conformal harmonic immersion  $U : \Sigma \rightarrow \mathbb{R}^3$ . Along  $y = 0$  we have

$$(3.1) \quad U(x + iy) = h(x) - g(x)y + O(y^2),$$

where  $h(x) = U(x + i0)$  and  $g(x) = -U_y(x + i0)$  are smooth 1-periodic maps  $\mathbb{R} \rightarrow \mathbb{R}^3$ ,  $h$  is an immersion, and the remainder  $O(y^2)$  is bounded by  $cy^2$  for some  $c > 0$  independent of  $x$ . We have that

$$2 \frac{\partial}{\partial \zeta} U(x + iy)|_{y=0} = (U_x - iU_y)|_{y=0} = h'(x) + ig(x).$$

The conformality condition (2.1) implies that

$$(3.2) \quad g(x) \cdot h'(x) = 0 \quad \text{and} \quad |g(x)| = |h'(x)| > 0 \quad \text{hold for all } x \in \mathbb{R}.$$

We also have that  $d^c U = -U_y dx + U_x dy$  and hence

$$\int_C d^c u = \int_0^1 d^c U = - \int_0^1 U_y(x + i0) dx = \int_0^1 g(x) dx.$$

Condition (3.2) is equivalent to saying that the map  $\sigma = h' + ig : \mathbb{R} \rightarrow \mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$  is a loop in the null quadric (2.2) whose real part has vanishing period:

$$\int_0^1 \Re \sigma(x) dx = \int_0^1 h'(x) dx = 0.$$

In this section we prove that for every such  $\sigma$  and for any vector  $v \in \mathbb{R}^3$  there is an isotopy of 1-periodic maps  $\sigma_t : \mathbb{R} \rightarrow \mathfrak{A}^*$  ( $t \in [0, 1]$ ) such that  $\sigma_0 = \sigma$  and

$$(3.3) \quad \int_0^1 \Re \sigma_t(x) dx = 0 \quad \text{for all } t \in [0, 1],$$

$$\int_0^1 \Im \sigma_1(x) dx = v.$$

(See Lemmas 3.2 and 3.4.) This will be one of the main steps in the proof of Theorems 1.1 and 1.2. (The special case  $v = 0$  will be of importance for the proof of Theorem 1.1.)

We denote by  $\mathfrak{S}$  the space of all smooth 1-periodic immersions  $\mathbb{R} \rightarrow \mathbb{R}^3$  endowed with the  $\mathcal{C}^\infty$  topology. We identify an immersion  $h \in \mathfrak{S}$  with a smooth immersion  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^3$  of the circle into  $\mathbb{R}^3$ . We shall say that  $h \in \mathfrak{S}$  is *nonflat* on a segment  $L \subset [0, 1]$  if the image  $h(L)$  is not contained in any affine plane of  $\mathbb{R}^3$ .

**Lemma 3.1.** *Any pair of immersions  $h_0, h_1 \in \mathfrak{S}$  can be connected by a smooth path of immersions  $h_t \in \mathfrak{S}$  ( $t \in [0, 1]$ ). If furthermore  $h_0, h_1 \in \mathfrak{S}$  agree on a proper closed subinterval  $I \subset [0, 1]$ , then the isotopy  $h_t$  can be chosen fixed on  $I$ . If  $h_0|_L$  is nonflat on a segment  $L \subset [0, 1]$ , then we can arrange that  $h_t|_L$  is nonflat for every  $t \in [0, 1]$ .*

*Proof.* Connect  $h_0$  and  $h_1$  by  $\tilde{h}_t = (1-t)h_0 + h_1$  ( $t \in [0, 1]$ ). The jet transversality theorem shows that a generic perturbation  $\{h_t\}$  of  $\{\tilde{h}_t\}$  with fixed ends at  $t \in \{0, 1\}$  (or one that is in addition fixed on a subinterval  $I \subset [0, 1]$  on which  $h_0$  and  $h_1$  agree) yields a smooth isotopy of immersions connecting  $h_0$  and  $h_1$ . The nonflatness condition can be satisfied by a generic deformation; observe also that being flat is a closed condition.  $\square$

**Lemma 3.2.** *Every smooth 1-periodic immersion  $h_0 : \mathbb{R} \rightarrow \mathbb{R}^3$  (i.e.  $h_0 \in \mathfrak{S}$ ) can be approximated in  $\mathfrak{S}$  by a smooth 1-periodic immersion  $h : \mathbb{R} \rightarrow \mathbb{R}^3$  for which there exists a smooth 1-periodic map  $g : \mathbb{R} \rightarrow \mathbb{R}^3$  satisfying the following properties:*

- (i)  $g(x) \cdot h'(x) = 0$  and  $|g(x)| = |h'(x)| > 0$  for all  $x \in \mathbb{R}$ , and
- (ii)  $\int_0^1 g(x) dx = 0$ .

A map  $g$  with these properties can be chosen in any given homotopy class of sections of the circle bundle over  $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$  determined by condition (i).

**Remark 3.3.** The last sentence in Lemma 3.2 requires a comment. Denote the coordinates on  $\mathbb{C}^3$  by  $z = \xi + i\eta$ , with  $\xi, \eta \in \mathbb{R}^3$ . Let  $\pi : \mathbb{C}^3 = \mathbb{R}^3 \oplus i\mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the projection  $\pi(\xi + i\eta) = \xi$ . Then  $\pi^{-1}(0) \cap \mathfrak{A} = \{0\}$  and  $\pi : \mathfrak{A}^* \rightarrow \mathbb{R}^3 \setminus \{0\}$  is a real analytic fiber bundle with circular fibers given by

$$(3.4) \quad \mathfrak{A} \cap \pi^{-1}(\xi) = \{\xi + i\eta \in \mathbb{C}^3 : \xi \cdot \eta = 0, |\xi| = |\eta|\} \cong \mathbb{S}^1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

An immersion  $h \in \mathfrak{S}$  determines the circle bundle  $E_h = (h')^*(\mathfrak{A}^*) \mapsto \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$  (the pull-back of  $\mathfrak{A}^* \rightarrow \mathbb{R}^3 \setminus \{0\}$  by the derivative  $h' : \mathbb{R} \rightarrow \mathbb{R}^3 \setminus \{0\}$ ), and sections of  $E_h$  are 1-periodic maps  $g : \mathbb{R} \rightarrow \mathbb{R}^3$  satisfying condition (i) in Lemma 3.2. Every oriented circle bundle  $E_h \rightarrow \mathbb{S}^1$  is trivial, and the set of homotopy classes of its sections  $\mathbb{S}^1 \rightarrow E_h$  can be identified with the fundamental group  $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ .

*Proof of Lemma 3.2.* Pick a  $\delta > 0$  with  $3\delta < 1$ . We approximate  $h_0$  by an immersion  $\tilde{h}_0 \in \mathfrak{S}$  whose derivative is constant on  $J = [0, 3\delta]$  and such that  $\tilde{h}_0$  agrees with  $h_0$  outside a slightly bigger interval  $J' \supset J$ . (We think of intervals in  $[0, 1]$  as arcs in the circle  $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ . The rate of approximation of  $h_0$  by  $\tilde{h}_0$  will of course depend on  $\delta$  which we are free to choose as small as we wish.) Replacing  $h_0$  by  $\tilde{h}_0$  we assume from now on that  $h_0$  satisfies these properties.

Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  denote the standard basis of  $\mathbb{R}^3$ . After an orthogonal rotation and a dilation on  $\mathbb{R}^3$  we may assume that  $h'_0(x) = \mathbf{e}_1$  for  $x \in [0, 3\delta]$ . Let  $\mathbb{B}$  denote the closed unit ball in  $\mathbb{R}^3$ . Pick a number  $0 < \epsilon < 1$  and a family of immersions  $h_p \in \mathfrak{S}$ , depending smoothly on  $p = (p_1, p_2, p_3) = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3 \in \mathbb{B}$  and with  $h_0$  the given immersion, such that

$$h'_p(x) = \begin{cases} \mathbf{e}_1 + \epsilon p, & \text{if } x \in [0, \delta], \\ \mathbf{e}_1, & \text{if } x \in [2\delta, 3\delta]. \end{cases}$$

We choose  $h_p$  to agree with  $h_0$  outside a small neighborhood of the indicated intervals, so every  $h_p$  is close to  $h_0$  (depending on the choice of  $\epsilon$ ). Let  $A \in O(3)$  be the orthogonal linear transformation on  $\mathbb{R}^3$  given by

$$A\mathbf{e}_1 = \mathbf{e}_2, \quad A\mathbf{e}_2 = -\mathbf{e}_1, \quad A\mathbf{e}_3 = \mathbf{e}_3.$$



Define a locally constant map  $\tilde{g}_p : [0, \delta] \cup [2\delta, 3\delta] \rightarrow \mathbb{R}^3$  by setting

$$\tilde{g}_p(x) = \begin{cases} \mathbf{e}_2 + \epsilon Ap - \epsilon^2 \frac{p_3^2}{1+\epsilon p_1} \mathbf{e}_1, & \text{if } x \in [0, \delta], \\ -\mathbf{e}_2, & \text{if } x \in [2\delta, 3\delta]. \end{cases}$$

Since  $0 < \epsilon < 1$  and  $p \in \mathbb{B}$ , we have  $1 + \epsilon p_1 > 0$  and hence  $\tilde{g}_p$  is well defined; furthermore, we have that  $\tilde{g}_p(x) \neq 0$  for all  $x \in [0, \delta] \cup [2\delta, 3\delta]$  and  $p \in \mathbb{B}$ . A calculation shows that  $\tilde{g}_p(x) \cdot h'_p(x) = 0$  for  $x \in [0, \delta] \cup [2\delta, 3\delta]$  and

$$(3.5) \quad \int_0^\delta \tilde{g}_p(x) dx + \int_{2\delta}^{3\delta} \tilde{g}_p(x) dx = \epsilon \delta \left( Ap - \epsilon \frac{p_3^2}{1 + \epsilon p_1} \mathbf{e}_1 \right).$$

Set

$$(3.6) \quad g_p(x) = \begin{cases} \frac{|\mathbf{e}_1 + \epsilon p|}{|\tilde{g}_p(x)|} \tilde{g}_p(x), & \text{if } x \in [0, \delta], \\ \tilde{g}_p(x) = -\mathbf{e}_2, & \text{if } x \in [2\delta, 3\delta]. \end{cases}$$

Clearly,  $|g_p(x)| = |h'_p(x)|$  and  $g_p(x) \cdot h'_p(x) = 0$  for all  $x \in [0, \delta] \cup [2\delta, 3\delta]$  and  $p \in \mathbb{B}$ . For  $x \in [0, \delta]$  we also have  $\tilde{g}_p(x) = \mathbf{e}_2 + \epsilon Ap + O(\epsilon^2) = A(\mathbf{e}_1 + \epsilon p) + O(\epsilon^2)$ . Since  $A$  is orthogonal, we get  $|\tilde{g}_p(x)| = |A(\mathbf{e}_1 + \epsilon p)| + O(\epsilon^2) = |\mathbf{e}_1 + \epsilon p| + O(\epsilon^2)$  and hence

$$\frac{|\mathbf{e}_1 + \epsilon p|}{|\tilde{g}_p(x)|} = \frac{|\mathbf{e}_1 + \epsilon p|}{|\mathbf{e}_1 + \epsilon p| + O(\epsilon^2)} = 1 + O(\epsilon^2).$$

This shows that the rescaling in the definition (3.6) of  $g_p$  changes the integrals in (3.5) only by a term of size  $O(\epsilon^2\delta)$ , so we have

$$(3.7) \quad \int_0^\delta g_p(x) dx + \int_{2\delta}^{3\delta} g_p(x) dx = \epsilon \delta (Ap + O(\epsilon)).$$

We now extend each  $g_p$  to a smooth 1-periodic map  $\mathbb{R} \rightarrow \mathbb{R}^3$ , depending smoothly on  $p \in \mathbb{B}$ , such that the following conditions hold:

- (a)  $g_p(x) \cdot h'_p(x) = 0$  and  $|g_p(x)| = |h'_p(x)| > 0$  for all  $x \in \mathbb{R}$  and  $p \in \mathbb{B}$ ,
- (b)  $|\int_\delta^{2\delta} g_p(x) dx + \int_{3\delta}^1 g_p(x) dx| < \frac{1}{3}\epsilon\delta$  for all  $|p| = 1$ .

Condition (a) is compatible with the definition of  $g_p$  on  $[0, \delta] \cup [2\delta, 3\delta]$ . Condition (b) can be achieved by choosing  $g_p(x)$  to spin sufficiently fast (with nearly constant angular velocity) along the circle bundles defined by condition (a) as  $x$  traces the intervals in the two integrals. Since spinning in both directions is allowed, and we can change the direction on short intervals with an arbitrarily small contribution to the integral, we can arrange that  $g_p$  belongs to any given homotopy class of sections (independent of  $p \in \mathbb{B}$ ).

Choosing  $\epsilon > 0$  small enough, condition (b) together with (3.7) implies

$$\left| \int_0^1 g_p(x) dx - \epsilon \delta Ap \right| < \frac{\epsilon \delta}{2}, \quad p \in \mathbb{B}.$$

Since  $|Ap| = |p|$  for all  $p \in \mathbb{R}^3$ , it follows that the map  $\mathbb{B} \ni p \mapsto \int_0^1 g_p(x) dx \in \mathbb{R}^3$  is nowhere vanishing on the sphere  $\mathbb{S}^2 = b\mathbb{B} = \{|p| = 1\}$  and the restricted map  $\mathbb{S}^2 \mapsto \mathbb{R}^3 \setminus \{0\} \simeq \mathbb{S}^2$  has the same degree as the map  $p \mapsto Ap$ , which is one. Hence there exists a point  $p \in \mathbb{B}$  for which  $\int_0^1 g_p(x) dx = 0$ . The pair  $(g, h) = (g_p, h_p)$  then clearly satisfies Lemma 3.2.  $\square$

We shall also need the following version of Lemma 3.2 in which all maps remain fixed on a proper subinterval of  $[0, 1]$  and the period of  $g_1$  equals any given vector in  $\mathbb{R}^3$ .

**Lemma 3.4.** *Let  $g_0, h_0 : \mathbb{R} \rightarrow \mathbb{R}^3$  be smooth 1-periodic maps satisfying condition (3.2), and let  $v \in \mathbb{R}^3$ . Given a proper closed subinterval  $I \subset [0, 1]$ , there exist smooth isotopies of 1-periodic maps  $g_t, h_t : \mathbb{R} \rightarrow \mathbb{R}^3$  ( $t \in [0, 1]$ ) such that*

- $g_t(x) = g_0(x)$  and  $h_t(x) = h_0(x)$  for  $x \in I$  and  $t \in [0, 1]$ ,
- condition (3.2) holds for  $(g_t, h_t)$  for every  $t \in [0, 1]$ , and
- $\int_0^1 g_1(x) dx = v$ .

Furthermore, given a segment  $I' \subset [0, 1]$  such that  $h_0|_{I'}$  is nonflat, we can choose  $h_t, g_t$  as above such that  $h_t|_{I'}$  is nonflat for every  $t \in [0, 1]$ .

*Proof.* Choose a pair of nontrivial closed intervals  $J, L \subset [0, 1]$  such that the intervals  $I, J, L \subset [0, 1]$  are pairwise disjoint. We may assume that  $J = [0, 3\delta]$  for a small  $\delta > 0$ . (As before, we consider these intervals as arcs in the circle  $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ .) We explain the individual moves without changing the notation at every step.

We begin by deforming the pair  $(g_0, h_0)$ , keeping it fixed on the segment  $I$ , such that for  $x \in J$  we have  $h'_0(x) = \mathbf{e}_1$  and  $g_0(x) = \mathbf{e}_2$ . Consider the 1-periodic map

$$\sigma_0 := h'_0 + \iota g_0 : \mathbb{R} \rightarrow \mathfrak{A}^* = \mathfrak{A} \setminus \{0\}.$$

Set

$$w = \int_{[0,1] \setminus L} \sigma_0(x) dx \in \mathbb{C}^3.$$

By [4, Lemma 7.3] there is a smooth 1-periodic map  $\sigma_1 : \mathbb{R} \rightarrow \mathfrak{A}^*$  which agrees with  $\sigma_0$  on  $[0, 1] \setminus L$  such that  $\int_L \sigma_1(x) dx \in \mathbb{C}^3$  is arbitrarily close to  $\iota v - w \in \mathbb{C}^3$ , and hence  $\int_0^1 \sigma_1(x) dx$  is close to  $\iota v$ . (The main point of the proof is that  $\mathfrak{A}$  is connected and its convex hull equals  $\mathbb{C}^3$ .) By general position we may assume that  $\Re \sigma_1$  does not assume the value  $0 \in \mathbb{R}^3$ . By another small correction of  $\sigma_1$  on  $L$  we may also arrange that  $\int_0^1 \Re \sigma_1(x) dx = 0$ , while  $\sigma_1$  still assumes values in  $\mathfrak{A}^*$  and  $\int_0^1 \Im \sigma_1(x) dx$  remains close to  $v$ . Fix a point  $x_0 \in I$  and set

$$h_1(x) = h_0(x_0) + \int_{x_0}^x \Re \sigma_1(s) ds, \quad x \in \mathbb{R}.$$

Then  $h_1 \in \mathfrak{S}$  is a smooth 1-periodic immersion which agrees with  $h_0$  on  $[0, 1] \setminus L$  and satisfies  $h'_1 = \Re \sigma_1$ . By Lemma 3.1 we can connect  $h_0$  to  $h_1$  by a smooth isotopy of immersions  $h_t \in \mathfrak{S}$  such that  $h_t(x) = h_0(x)$  for every  $t \in [0, 1]$  and  $x \in [0, 1] \setminus L$ .

By the argument given in Remark 3.3 we can cover the homotopy  $h'_t : \mathbb{R} \rightarrow \mathbb{R}^3 \setminus \{0\}$  by a smooth homotopy of 1-periodic maps  $g_t : \mathbb{R} \rightarrow \mathbb{R}^3$  such that  $g_0$  is the given initial map, the homotopy is fixed on  $[0, 1] \setminus L$ , and  $(g_t, h_t)$  satisfies condition (3.2) for every  $t \in [0, 1]$ . The maps  $g_1$  and  $\Im \sigma_1$  are sections of the circle bundle  $E_1 = (h'_1)^* E \rightarrow \mathbb{S}^1$ , but they need not be homotopic. To correct this, we replace  $\Im \sigma_1$  by another section of the same bundle whose period is still close to  $v$  and which is homotopic to  $g_1$  (see the last sentence in Lemma 3.2). It is then possible to choose the homotopy  $g_t$  as above connecting  $g_0$  to  $g_1 = \Im \sigma_1$ . The homotopy of smooth 1-periodic maps  $\sigma_t = h'_t + \iota g_t : \mathbb{R} \rightarrow \mathfrak{A}^*$  ( $t \in [0, 1]$ ) satisfies all the required properties, except that the period  $\int_0^1 g_1(x) dx \in \mathbb{R}^3$  is only close to  $v \in \mathbb{R}^3$ . It remains to make this period exactly equal to  $v$  by another small deformation of the pair  $(g_1, h_1)$  that is fixed on the segment  $I$ . This can be achieved by the perturbation device on the segment  $J = [0, 3\delta]$  described in the proof of Lemma 3.2.  $\square$

We shall also need a period perturbation lemma for (finite unions of) loops in the null quadric  $\mathfrak{A}$  (see (2.2)). Let  $\mathcal{L} = \mathcal{C}^\infty(\mathbb{S}^1, \mathfrak{A}^*)$  denote the space of smooth loops in  $\mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$ . We may think of  $\sigma \in \mathcal{L}$  as a smooth 1-periodic map  $\mathbb{R} \rightarrow \mathfrak{A}^*$ ; in particular, we write

$$\int_{\mathbb{S}^1} \sigma = \int_0^1 \sigma(x) dx \in \mathbb{C}^3.$$

We identify the tangent space  $T_z \mathfrak{A} \subset T_z \mathbb{C}^3$  at a point  $z \in \mathfrak{A}^*$  with a complex 2-dimensional subspace of  $\mathbb{C}^3$ .

**Definition 3.5.** A loop  $\sigma \in \mathcal{L}$  is said to be *nondegenerate* on a segment  $I \subset \mathbb{S}^1$  if the family of tangent spaces  $\{T_{\sigma(x)} \mathfrak{A} : x \in I\}$  spans  $\mathbb{C}^3$ .

Since  $\mathfrak{A}$  is a complex cone, the tangent space  $T_z \mathfrak{A}$  at any point  $0 \neq z \in \mathfrak{A}$  is spanned by  $z$  together with one more vector, and  $T_z \mathfrak{A} = T_w \mathfrak{A}$  for any point  $w = \lambda z$  with  $\lambda \neq 0$ . It follows that a loop  $\sigma \in \mathcal{L}$  is nondegenerate on the segment  $I \subset \mathbb{S}^1$  if and only if the image  $\sigma(I)$  is not contained in any ray  $\mathbb{C} \cdot v \subset \mathfrak{A}$  of the null quadric.

A continuous map  $\sigma : W \rightarrow \mathcal{L}$  from a complex manifold  $W$  to the loop space  $\mathcal{L}$  is naturally identified with a continuous map  $\sigma : W \times \mathbb{S}^1 \rightarrow \mathbb{C}^3$ . Such a map is said to be *holomorphic* if for every  $x \in \mathbb{S}^1$  the map  $\sigma(\cdot, x) : W \rightarrow \mathbb{C}^3$  is holomorphic; in such case we shall also say that the family  $\sigma_w = \sigma(w, \cdot) \in \mathcal{L}$  is holomorphic in  $w \in W$ .

Consider the period map  $\mathcal{P} : \mathcal{L} \rightarrow \mathbb{C}^3$  defined by

$$\mathcal{P}(\sigma) = \int_0^1 \sigma(x) dx \in \mathbb{C}^3, \quad \sigma \in \mathcal{L}.$$

**Lemma 3.6** (Period dominating sprays of loops). *Let  $I \subset \mathbb{S}^1$  be a nontrivial segment. If a loop  $\sigma \in \mathcal{L} = \mathcal{C}^\infty(\mathbb{S}^1, \mathfrak{A}^*)$  is nondegenerate on  $I$  (Definition 3.5), then there is a holomorphic family of loops  $\{\sigma_w\}_{w \in W} \in \mathcal{L}$ , where  $W \subset \mathbb{C}^3$  is a ball centered at  $0 \in \mathbb{C}^3$ , such that  $\sigma_0 = \sigma$ ,  $\sigma_w(x) = \sigma(x)$  for all  $x \in \mathbb{S}^1 \setminus I$  and  $w \in W$ , and*

$$\left. \frac{\partial \mathcal{P}(\sigma_w)}{\partial w} \right|_{w=0} : T_0 \mathbb{C}^3 \cong \mathbb{C}^3 \rightarrow \mathbb{C}^3 \quad \text{is an isomorphism.}$$

More generally, given a family of loops  $\sigma_q \in \mathcal{L}$  depending continuously on a parameter  $q$  in a compact Hausdorff space  $Q$  such that  $\sigma_q$  is nondegenerate on the segment  $I$  for every  $q \in Q$ , there is a ball  $0 \in W \subset \mathbb{C}^N$  for some  $N \in \mathbb{N}$  and a continuous family of loops  $\sigma_{q,w} \in \mathcal{L}$  ( $q \in Q$ ,  $w \in W$ ), depending holomorphically on  $w \in W$ , such that  $\sigma_{q,0} = \sigma_q$  for all  $q \in Q$ ,  $\sigma_{q,w} = \sigma_q$  on  $\mathbb{S}^1 \setminus I$  for all  $q \in Q$  and  $w \in W$ , and

$$(3.8) \quad \left. \frac{\partial \mathcal{P}(\sigma_{q,w})}{\partial w} \right|_{w=0} : T_0 \mathbb{C}^N \cong \mathbb{C}^N \rightarrow \mathbb{C}^3 \quad \text{is surjective for every } q \in Q.$$

**Definition 3.7.** A family of loops  $\{\sigma_w\}_{w \in W}$  satisfying the domination condition (3.8) is called a *holomorphic period dominating spray of loops*.

For a general notion of a (local) dominating holomorphic spray, see for instance [7, Definition 4.1] or [10, Definition 5.9.1]. Period dominating sprays were first constructed in [4, Lemma 5.1].

*Proof.* The main idea is contained in the proof of [4, Lemma 5.1]; we outline the main idea for the sake of readability. Since  $\sigma$  is nondegenerate on  $I$ , there are points  $x_1, x_2, x_3 \in I$  and holomorphic vector fields  $V_1, V_2, V_3$  on  $\mathbb{C}^3$  which are tangential to the quadric  $\mathfrak{A}$  given by (2.2) such that the vectors  $V_j(\sigma(x_j)) \in \mathbb{C}^3$  for  $j = 1, 2, 3$  are a complex basis of  $\mathbb{C}^3$ . Choose a smooth function  $h_j : \mathbb{S}^1 \rightarrow \mathbb{C}$  supported on a short segment  $I_j \subset I$  around the point  $x_j$  for  $j = 1, 2, 3$ . Let  $\phi_t^j$  denote the flow of the vector field  $V_j$  for time  $t \in \mathbb{C}$  near 0. It is easily seen that for suitable choices of the functions  $h_j$  the holomorphic spray of loops  $\sigma_w \in \mathcal{L}$ , given for any  $w = (w_1, w_2, w_3) \in \mathbb{C}^3$  sufficiently close to the origin by

$$\sigma_w(x) = \phi_{w_1 h_1(x)}^1 \circ \phi_{w_2 h_2(x)}^2 \circ \phi_{w_3 h_3(x)}^3 \sigma(x) \in \mathbb{C}^3, \quad x \in \mathbb{S}^1,$$

enjoys the stated properties; in particular, it is period dominating. The same proof applies to a continuous family of loops  $\{\sigma_q\}_{q \in Q} \subset \mathcal{L}$  by using compositions of flows of finitely many holomorphic vector fields tangential to  $\mathfrak{A}$ .  $\square$

**Remark 3.8.** Lemma 3.6 also applies to a finite union  $C = \bigcup_{j=1}^l C_j$ , where each  $C_j \cong \mathbb{S}^1$  is an embedded oriented analytic Jordan curve in a Riemann surface. Let  $\sigma : C \rightarrow \mathfrak{A}^*$  be a smooth map. Given pairwise disjoint segments  $I_j \subset C_j \setminus \bigcup_{i \neq j} C_i$  ( $j = 1, \dots, l$ ) such that  $\sigma$  is nondegenerate on every  $I_j$  in the sense of Definition 3.5, the same proof furnishes a holomorphic family  $\sigma_w \in \mathcal{C}^\infty(C, \mathfrak{A}^*)$  for  $w$  in a ball  $0 \in W \subset \mathbb{C}^{3l}$  such that  $\sigma_0 = \sigma$ ,  $\sigma_w$  agrees with  $\sigma$  on  $C \setminus \bigcup_{j=1}^l I_j$  for every  $w \in W$ , and the differential at  $w = 0$  of the map  $P = (P_1, \dots, P_l) : W \rightarrow (\mathbb{C}^3)^l$  with the components

$$P_j(w) = \int_{C_j} \sigma_w \in \mathbb{C}^3$$

is an isomorphism. The analogous result also holds for homotopies of maps  $\sigma_q \in \mathcal{C}^\infty(C, \mathfrak{A}^*)$  with the parameter  $q \in Q$  in a compact Hausdorff space. In the present paper we shall use it for 1-parameter homotopies, with the parameter  $q = t \in [0, 1]$ .

**Remark 3.9.** Results in this section apply to real analytic Jordan curves in Riemann surfaces via a holomorphic change of coordinates along the curve, mapping the curve onto the circle  $\mathbb{S}^1 \subset \mathbb{C}$ . However, one can also use them for smooth curves by applying an asymptotically holomorphic change of coordinates (i.e. one whose  $\bar{\partial}$ -derivative vanishes along the curve), mapping the curve onto the circle  $\mathbb{S}^1$ . Such changes of coordinates, which always exist along a smooth curve, respect conditions (3.1) and (3.2) which concern first order jets. However, real analytic curves suffice for the applications in this paper.

#### 4. Proof of Theorem 1.1 for surfaces with finite topology

In this section we prove Theorem 1.1 for Riemann surfaces  $M$  with finitely generated first homology group  $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^l$ ,  $l \in \mathbb{N}$ .

Without loss of generality we may assume that  $M$  is connected and the immersion  $u$  in Theorem 1.1 is nonflat,  $u \in \text{CMI}_*(M)$  (cf. Remark 2.1). Set  $u_0 := u$ .

Fix  $p_* \in M$ . There exist embedded, closed, oriented, real analytic curves  $C_1, \dots, C_l$  in  $M$  such that  $C_i \cap C_j = \{p_*\}$  when  $i \neq j$ , the homology classes  $[C_j] \in H_1(M; \mathbb{Z})$  are a basis of the first homology group  $H_1(M; \mathbb{Z})$ ,  $M$  retracts onto  $C = \bigcup_{j=1}^l C_j$ , and  $C$  is  $\mathcal{O}(M)$ -convex.

Along each curve  $C_j$  we choose a uniformizing holomorphic coordinate  $\zeta_j = x + iy$  (see Section 3). In this coordinate we can represent the differential  $2\partial u_0$  on  $C_j$  by a pair of smooth 1-periodic maps  $g_{j,0}, h_{j,0} : \mathbb{R} \rightarrow \mathbb{R}^3$  such that

$$\sigma_{j,0} := h'_{j,0} + \iota g_{j,0} : \mathbb{R} \rightarrow \mathfrak{X}^*, \quad 2\partial u_0 = \sigma_{j,0} d\zeta_j \quad \text{on } C_j.$$

Lemma 3.4 furnishes smooth homotopies of 1-periodic maps  $h_{j,t}, g_{j,t} : \mathbb{R} \rightarrow \mathbb{R}^3$  that are fixed near the intersection point  $C_i \cap C_j = \{p_*\}$  such that

$$(4.1) \quad \sigma_{j,t} = h'_{j,t} + \iota g_{j,t} : \mathbb{R} \rightarrow \mathfrak{X}^*$$

holds for  $j = 1, \dots, l$  and  $t \in [0, 1]$ , and at  $t = 1$  we also have that

$$(4.2) \quad \int_0^1 g_{j,1}(x) dx = 0, \quad j = 1, \dots, l.$$

Furthermore, given nontrivial arcs  $I_j \subset C_j \setminus \{p_*\}$ , we can choose  $\sigma_{j,t}$  to be nondegenerate on  $I_j$  (see Definition 3.5) for all  $t \in [0, 1]$  and all  $j = 1, \dots, l$ .

Fix a nowhere vanishing holomorphic 1-form  $\theta$  on  $M$  (such  $\theta$  exists by the Oka–Grauert principle, see Section 2). We then have that  $2\partial u_0 = \sigma_0 \theta$ , where  $\sigma_0 = 2\partial u_0 / \theta : M \rightarrow \mathfrak{X}^*$  is a holomorphic map. Let  $\sigma_t : C = \bigcup_{i=1}^l C_i \rightarrow \mathfrak{X}^*$  be the smooth map determined by the equations

$$(4.3) \quad \sigma_t \theta|_{C_j} = \sigma_{j,t} d\zeta_j = (h'_{j,t} + \iota g_{j,t}) d\zeta_j, \quad j = 1, \dots, l, \quad t \in [0, 1].$$

Here  $\sigma_{j,t}$  is given by (4.1). Note that  $\sigma_t \in \mathcal{C}^\infty(C, \mathfrak{X}^*)$  depends smoothly on  $t \in [0, 1]$ .

Let  $\mathcal{P} : \mathcal{C}^\infty(C, \mathfrak{X}^*) \rightarrow (\mathbb{C}^3)^l$  denote the period map which associates to any map  $\sigma \in \mathcal{C}^\infty(C, \mathfrak{X}^*)$  the vector  $(\mathcal{P}_j(\sigma))_{j=1}^l$  with the components

$$(4.4) \quad \mathcal{P}_j(\sigma) = \int_{C_j} \sigma \theta = \int_0^1 \sigma_j(x) dx \in \mathbb{C}^3, \quad j = 1, \dots, l.$$

The 1-periodic map  $\sigma_j : \mathbb{R} \rightarrow \mathbb{R}^3$  is just the map  $\sigma$  expressed in the uniformizing coordinate  $\zeta_j = x_j + iy_j$  along  $C_j = \{y_j = 0\}$ , that is,  $\sigma \theta = \sigma_j d\zeta_j$  on  $C_j$ . If  $\sigma$  corresponds to the differential  $2\partial u$  of some  $u \in \text{CMI}(M)$ , in the sense that  $2\partial u = \sigma \theta$  holds on  $C$ , then the real periods of  $\sigma$  vanish,  $\Re \mathcal{P}(\sigma) = 0$ , while the imaginary periods  $\Im \mathcal{P}(\sigma)$  are the flux of  $u$ :

$$\Im \mathcal{P}_j(\sigma) = \text{Flux}_u(C_j) = \int_{C_j} d^c u = \int_0^1 \Im \sigma_j(x) dx \in \mathbb{R}^3, \quad j = 1, \dots, l.$$

Since  $\sigma_{j,t}$  is nondegenerate on the segment  $I_j \subset C_j$  for any  $t \in [0, 1]$  and  $j = 1, \dots, l$ , Lemma 3.6 and Remark 3.8 furnish a spray of maps  $\sigma_{t,w} \in \mathcal{C}^\infty(C, \mathfrak{X}^*)$  ( $t \in [0, 1]$ ), depending holomorphically on a complex parameter  $w$  in a ball  $W \subset \mathbb{C}^N$  around the origin  $0 \in \mathbb{C}^N$  for some big  $N$ , such that

- $\sigma_{t,0} = \sigma_t$  for all  $t \in [0, 1]$ ,
- the map  $P = (P_1, \dots, P_l) : [0, 1] \times W \rightarrow (\mathbb{C}^3)^l$  with the components

$$(4.5) \quad [0, 1] \times W \ni (t, w) \mapsto P_j(t, w) = \int_{C_j} \sigma_{t,w} \theta = \int_0^1 \sigma_{j,t,w}(x) dx \in \mathbb{C}^3$$

is submersive with respect to the variable  $w$  at  $w = 0$ , i.e. the partial differential

$$\partial_w P(t, w)|_{w=0} : \mathbb{C}^N \rightarrow (\mathbb{C}^3)^l$$

is surjective for every  $t \in [0, 1]$ .

In the sequel we shall frequently use that  $\mathfrak{X}^*$  is an Oka manifold (see [4, Proposition 4.5]), and hence maps  $M \rightarrow \mathfrak{X}^*$  from any Stein manifold  $M$  (in particular, from an open Riemann surface) to  $\mathfrak{X}^*$  satisfy the Runge and the Mergelyan approximation theorems in the absence of topological obstructions. The Runge approximation theorem in this setting amounts to the (basic or parametric) *Oka property with approximation for holomorphic maps to Oka manifolds*; see [10, Theorem 5.4.4]. (An introductory survey of Oka theory can be found in [11].) The global Mergelyan approximation theorem on suitable subsets of the source Stein manifold follows by combining the local Mergelyan theorem, which holds for an arbitrary target manifold (see [10, Theorem 3.7.2]), and the Oka property.

In the case at hand, the Riemann surface  $M$  retracts onto the union of curves

$$C = \bigcup_{j=1}^l C_j.$$

Since the ball  $W \subset \mathbb{C}^N$  is contractible, we infer that any continuous map  $[0, 1] \times W \times C \rightarrow \mathfrak{X}^*$  extends to a continuous map

$$[0, 1] \times W \times M \rightarrow \mathfrak{X}^*.$$

Applying the parametric version of Mergelyan's theorem we can approximate the family of maps  $\sigma_{t,w} : C \rightarrow \mathfrak{X}^*$  arbitrarily closely in the smooth topology by a family of holomorphic maps  $f_{t,w} : M \rightarrow \mathfrak{X}^*$  depending holomorphically on  $w \in W$  and smoothly on  $t \in [0, 1]$ . (The ball  $W$  is allowed to shrink around  $0 \in \mathbb{C}^N$ .) Furthermore, as the initial map  $\sigma_{0,0} = 2\partial u_0/\theta$  is holomorphic on  $M$ , the family  $\{f_{t,w}\}$  can be chosen such that  $f_{0,0} = \sigma_{0,0}$  on  $M$ . If the approximation of  $\sigma_{t,w}$  by  $f_{t,w}$  is close enough for every  $t \in [0, 1]$  and  $w \in W$ , it follows from submersivity of the map (4.5) and the implicit function theorem that there is a smooth map  $w = w(t) \in W$  ( $t \in [0, 1]$ ) close to 0 such that  $w(0) = 0$  and the homotopy of holomorphic maps  $f_{t,w(t)} : M \rightarrow \mathfrak{X}^*$  satisfies

$$(4.6) \quad \int_{C_j} f_{t,w(t)}\theta = P_j(t, 0), \quad j = 1, \dots, l, \quad t \in [0, 1].$$

(Here  $P_j$  is defined by (4.5).) By (4.1) and (4.5) we have that

$$\Re P_j(t, 0) = \Re \int_{C_j} \sigma_{t,0}\theta = \int_0^1 h'_{j,t}(x) dx = 0, \quad j = 1, \dots, l, \quad t \in [0, 1].$$

Hence it follows from (4.6) that the real part of the holomorphic 1-form  $f_{t,w(t)}\theta$  integrates to a conformal minimal immersion  $u_t \in \text{CMI}(M)$  given by

$$(4.7) \quad u_t(p) = u_0(p_*) + \int_{p_*}^p \Re(f_{t,w(t)}\theta), \quad p \in M, \quad t \in [0, 1].$$

For  $t = 0$  we get the initial immersion  $u = u_0$  since  $\Re(f_{0,0}\theta) = \Re(2\partial u_0) = du_0$ . In view of (4.2) we also have  $P(1, 0) = 0$ , i.e. the holomorphic 1-form  $f_{1,w(1)}\theta$  with values in  $\mathfrak{X}^*$  has vanishing periods along the curves  $C_1, \dots, C_l$ . It follows that  $u_1$  is the real part of the null holomorphic immersion  $F : M \rightarrow \mathbb{C}^3$  defined by

$$F(p) = u_0(p_*) + \int_{p_*}^p f_{1,w(1)}\theta, \quad p \in M.$$

This completes the proof of Theorem 1.1 when  $H_1(M; \mathbb{Z})$  is finitely generated.

If  $M$  is a compact Riemann surface with smooth boundary, then the above proof also gives the following analogue of Theorem 1.1 for conformal minimal immersions  $M \rightarrow \mathbb{R}^3$  that are smooth up to the boundary.

**Theorem 4.1.** *Let  $M$  be a compact bordered Riemann surface with nonempty smooth boundary  $bM$  and let  $r \geq 1$ . For every conformal minimal immersion  $u : M \rightarrow \mathbb{R}^3$  of class  $\mathcal{C}^r(M)$  there exists a smooth isotopy  $u_t : M \rightarrow \mathbb{R}^3$  ( $t \in [0, 1]$ ) of conformal minimal immersions of class  $\mathcal{C}^r(M)$  such that  $u_0 = u$  and  $u_1 = \Re F$  is the real part of a holomorphic null curve  $F : M \rightarrow \mathbb{C}^3$  which is smooth up to the boundary.*

## 5. Proof of Theorem 1.1: The general case

For an open Riemann surface  $M$  of arbitrary topological type we construct an isotopy of conformal minimal immersion  $\{u_t\}_{t \in [0,1]} \subset \text{CMI}(M)$  satisfying Theorem 1.1 by an inductive procedure. As before, we assume that the initial immersion  $u_0 \in \text{CMI}_*(M)$  is nonflat, and all steps of the proof will be carried out through nonflat immersions.

Pick a smooth strongly subharmonic Morse exhaustion function  $\rho : M \rightarrow \mathbb{R}$ . We can exhaust  $M$  by an increasing sequence  $\emptyset = M_0 \subset M_1 \subset \dots \subset \bigcup_{i=0}^{\infty} M_i = M$  of compact smoothly bounded domains of the form  $M_i = \{p \in M : \rho(p) \leq c_i\}$ , where  $c_0 < c_1 < c_2 < \dots$  is an increasing sequence of regular values of  $\rho$  with  $\lim_{i \rightarrow \infty} c_i = +\infty$ . Each domain  $M_i$  is a bordered Riemann surface, possibly disconnected. We may assume that  $\rho$  has at most one critical point  $p_i$  in each difference  $M_{i+1} \setminus M_i$ . It follows that  $M_i$  is  $\mathcal{O}(M)$ -convex and its interior  $\overset{\circ}{M}_i$  is Runge in  $M$  for every  $i \in \mathbb{Z}_+$ .

We proceed by induction. The initial step is trivial since  $M_0 = \emptyset$ . Assume inductively that an isotopy  $u_t^i \in \text{CMI}_*(M_i)$  ( $t \in [0, 1]$ ) satisfying the conclusion of Theorem 1.1 has already been constructed over a neighborhood of  $M_i$  for some  $i \in \mathbb{Z}_+$ . In particular,  $u_0^i$  agrees on  $M_i$  with the initial immersion  $u_0$ , while  $u_1^i = \Re F^i$  is the real part of a null holomorphic immersion  $F^i$  defined on a neighborhood of  $M_i$ . We will show that  $\{u_t^i\}_{t \in [0,1]}$  can be approximated arbitrarily closely in the smooth topology on  $[0, 1] \times M_i$  by an isotopy  $\{u_t^{i+1}\}_{t \in [0,1]}$  satisfying the analogous properties over a neighborhood of  $M_{i+1}$ . The limit

$$u_t = \lim_{i \rightarrow \infty} u_t^i \in \text{CMI}_*(M)$$

will clearly satisfy Theorem 1.1.

Let  $C_1, \dots, C_l \subset \overset{\circ}{M}_i$  be closed, oriented, real analytic curves whose homology classes form a basis of  $H_1(M_i; \mathbb{Z})$  and which satisfy the other properties as in Section 4. Set

$$C = \bigcup_{j=1}^l C_j \subset M_i$$

and let  $\mathcal{P} : \mathcal{C}^\infty(C, \mathfrak{A}^*) \rightarrow (\mathbb{C}^3)^l$  denote the period map (4.4). Fix a nowhere vanishing holomorphic 1-form  $\theta$  on  $M$  (see Section 2) and write

$$(5.1) \quad 2\partial u_t^i = f_t^i \theta \quad \text{on } M_i, \quad t \in [0, 1],$$

where  $f_t^i : M_i \rightarrow \mathfrak{A}^*$  is a holomorphic map depending smoothly on  $t \in [0, 1]$ . (We adopt the convention that a map is holomorphic on a closed set in a complex manifold if it is holomorphic on an unspecified open neighborhood of that set.) Note that the map  $f_0 = 2\partial u_0/\theta : M \rightarrow \mathfrak{A}^*$  is defined and holomorphic on all of  $M$ .

We consider the following two essentially different cases.

(A) The noncritical case:  $\rho$  has no critical value in  $[c_i, c_{i+1}]$ .

(B) The critical case:  $\rho$  has a critical point  $p_i \in \overset{\circ}{M}_{i+1} \setminus M_i$ .

In case (A) there is no change of topology when passing from  $M_i$  to  $M_{i+1}$ . By [4, Lemma 5.1] (see also Lemma 3.6 above) there is a spray of maps  $f_{t,w}^i \in \mathcal{O}(M_i, \mathfrak{X}^*)$  ( $t \in [0, 1]$ ), depending holomorphically on a complex parameter  $w$  in a ball  $W \subset \mathbb{C}^N$  around the origin  $0 \in \mathbb{C}^N$  for some big  $N$ , satisfying the following two properties:

- $f_{t,0}^i = f_t^i$  for all  $t \in [0, 1]$ ,
- the map  $P = (P_1, \dots, P_l) : [0, 1] \times W \rightarrow (\mathbb{C}^3)^l$  with the components

$$(5.2) \quad [0, 1] \times W \ni (t, w) \mapsto P_j(t, w) = \int_{C_j} f_{t,w}^i \theta \in \mathbb{C}^3$$

is submersive with respect to the variable  $w$  at  $w = 0$ , i.e. the partial differential

$$\partial_w P(t, w)|_{w=0} : \mathbb{C}^N \rightarrow (\mathbb{C}^3)^l$$

is surjective for every  $t \in [0, 1]$ .

In view of (5.1) we have

$$\Re P_j(t, 0) = \int_{C_j} \Re(f_{t,0}^i \theta) = \int_{C_j} du_t^i = 0, \quad j = 1, \dots, l, \quad t \in [0, 1],$$

and

$$P_j(1, 0) = \int_{C_j} f_{1,0}^i \theta = \int_{C_j} 2\partial u_1^i = 0, \quad j = 1, \dots, l,$$

since  $u_1^i = \Re F^i$  for a holomorphic null immersion  $F^i : M_i \rightarrow \mathbb{C}^3$ .

Since  $\mathfrak{X}^*$  is an Oka surface and  $M_i$  is a strong deformation retract of  $M_{i+1}$ , the same argument as in Section 4 (using the Oka principle for maps to  $\mathfrak{X}^*$ ) shows that the spray  $f_{t,w}^i$  can be approximated as closely as desired in the smooth topology on  $M_i$  by a spray of holomorphic maps  $f_{t,w}^{i+1} : M_{i+1} \rightarrow \mathfrak{X}^*$ , depending holomorphically on  $w \in W$  (the ball  $W$  shrinks a little) and smoothly on  $t \in [0, 1]$ , such that  $f_{0,0}^{i+1} = f_0|_{M_i}$ . Assuming that the approximation is close enough, the submersivity property of the period map  $P$  (5.2) furnishes a smooth map  $w = w(t) \in W$  ( $t \in [0, 1]$ ) close to 0 such that  $w(0) = 0$  and we have for every  $j = 1, \dots, l$  that

$$\int_{C_j} \Re(f_{t,w(t)}^{i+1} \theta) = 0, \quad t \in [0, 1], \quad \int_{C_j} f_{1,w(1)}^{i+1} \theta = 0.$$

By integrating the family of 1-forms  $\Re(f_{t,w(t)}^{i+1})$  ( $t \in [0, 1]$ ) with the correct choices of initial values at a chosen initial point in each connected component of  $M_i$  (as in (4.7)) we obtain a smooth family of conformal minimal immersions  $u_t^{i+1} \in \text{CMI}_*(M_{i+1})$  which satisfies the induction step. This completes the discussion of the noncritical case (A).

Consider now the critical case (B), i.e.  $\rho$  has a critical point  $p_i \in \overset{\circ}{M}_{i+1} \setminus M_i$ . By the assumption this is the only critical point of  $\rho$  on  $M_{i+1} \setminus M_i$  and is a Morse point. Now  $M_{i+1}$  admits a strong deformation retraction onto  $M_i \cup E$  where  $E$  is an embedded analytic arc in



the complement of  $M_i$ , passing through  $p_i$ , which is attached transversely with both endpoints to  $bM_i$ . There are two possibilities:

- (a)  $E$  is attached with both endpoints to the same connected component of  $M_i$ .
- (b) The endpoints of  $E$  belong to different connected components of  $M_i$ .

Let us begin with case (a). The arc  $E$  completes inside the domain  $M_i$  to a closed smooth embedded curve  $C_{l+1} \subset M_{i+1}$  which is a new generator of the homology group  $H_1(M_{i+1}; \mathbb{Z})$ ; hence the latter group is generated by the curves  $C_1, \dots, C_{l+1}$ . By approximation we may assume that  $C_{l+1}$  is real analytic. Let  $\zeta = x + iy$  be a uniformizing coordinate in an open annular neighborhood  $W_{l+1} \subset M$  of  $C_{l+1}$  (see Section 2). Let  $f_t^i : U_i \rightarrow \mathfrak{X}^*$  be the isotopy of holomorphic maps from the inductive step, defined on an open neighborhood  $U_i$  of  $M_i$  in  $M$  and satisfying (5.1). In analogy with (4.3) we define the isotopy of maps  $\sigma_t^i : C_{l+1} \cap U_i \rightarrow \mathfrak{X}^*$  by the equation

$$(5.3) \quad f_t^i \theta|_{C_{l+1} \cap U_i} = \sigma_t d\zeta, \quad t \in [0, 1].$$

For  $t = 0$  the same equation defines the map

$$\sigma_0 : C_{l+1} \rightarrow \mathfrak{X}^*$$

on all of  $C_{l+1}$  since  $f_0^i = 2\partial u_0/\theta : M \rightarrow \mathfrak{X}^*$  is globally defined on  $M$ . Furthermore, we have  $\sigma_0(x) = h'_0(x) + \iota g_0(x)$ , where  $h_0(x) = u_0(x)$  is the restricted immersion  $u_0|_{C_{l+1}}$  expressed in the uniformizing coordinate along  $C_{l+1}$ .

Applying Lemma 3.4 to  $(g_0, h_0)$  we find an isotopy of smooth 1-periodic maps

$$(5.4) \quad \sigma_t(x) = h'_t(x) + \iota g_t(x) \in \mathfrak{X}^*, \quad x \in \mathbb{R}, t \in [0, 1],$$

which agrees with  $\sigma_0$  at  $t = 0$  and satisfies the following conditions:

- (i)  $h_t$  is a nonflat immersion for every  $t \in [0, 1]$ ,
- (ii) the extended map agrees with the previously defined map on the segment in  $[0, 1]$  representing the arc  $C_{l+1} \cap M_i = C_{l+1} \setminus E$ ,
- (iii)  $\int_0^1 g_1(x) dx = 0$ .

We extend the isotopy  $f_t^i$  from  $M_i$  to  $M_i \cup E$  by the equation

$$(5.5) \quad f_t^i \theta = \sigma_t d\zeta, \quad t \in [0, 1].$$

The maps  $f_t^i : M_i \cup E \rightarrow \mathfrak{X}^*$  are smooth (also in  $t \in [0, 1]$ ) and holomorphic on a neighborhood of  $M_i$ , and we have

$$(5.6) \quad \int_{C_{l+1}} \Re(f_t^i \theta) = 0, \quad t \in [0, 1], \quad \int_{C_{l+1}} f_1^i \theta = \int_0^1 g_1(x) dx = 0.$$

We now complete the induction step as in the case of surfaces with finite topology treated in Section 4; let us outline the main steps. First we apply [4, Lemma 5.1] to embed the isotopy  $f_t^i$  into a spray  $f_{t,w}^i : M_i \cup E \rightarrow \mathfrak{X}^*$  of smooth maps which are holomorphic on  $M_i$  and depend holomorphically on a parameter  $w \in W \subset \mathbb{C}^N$  in a ball of  $\mathbb{C}^N$  for some big  $N$  such that  $f_{0,0}^i = 2\partial u_0/\theta$  and the period map  $(t, w) \mapsto P(t, w) \in (\mathbb{C}^3)^{l+1}$  with the components

$$P_j(t, w) = \int_{C_j} f_{t,w}^i \theta \in \mathbb{C}^3, \quad j = 1, \dots, l + 1,$$

is submersive with respect to  $w$  at  $w = 0$ . (Compare with (4.5). This also follows from the proof of Lemma 3.6 and Remark 3.8 above.) Since  $\mathfrak{X}^*$  is an Oka manifold, the parametric

Mergelyan approximation theorem [10, Theorem 5.4.4] allows us to approximate the spray  $f_{t,w}^i$  in the smooth topology on  $M_i \cup E$  by a spray of holomorphic maps  $\tilde{f}_{t,w}^i : M_{i+1} \rightarrow \mathfrak{A}^*$ , depending smoothly on  $t \in [0, 1]$  and holomorphically on  $w \in W$  (the ball  $W$  is allowed to shrink a little) such that  $\tilde{f}_{0,0}^i = f_{0,0}^i = 2\partial u_0/\theta$ . If the approximation is sufficiently close, then the implicit function theorem furnishes a smooth map  $w : [0, 1] \rightarrow W \subset \mathbb{C}^N$  close to 0, with  $w(0) = 0$ , such that the isotopy of holomorphic maps

$$f_t^{i+1} = \tilde{f}_{t,w(t)}^i : M_{i+1} \rightarrow \mathfrak{A}^*, \quad t \in [0, 1],$$

satisfies the following properties:

- ( $\alpha$ )  $f_0^{i+1} = f_0^i = 2\partial u_0/\theta$  on  $M_{i+1}$ ,
- ( $\beta$ )  $f_t^{i+1}$  approximates  $f_t^i$  as closely as desired in the smooth topology on  $M_i \cup E$  (uniformly in  $t \in [0, 1]$ ),
- ( $\gamma$ )  $\int_{C_j} \Re(f_t^{i+1}\theta) = 0$  for all  $j = 1, \dots, l+1$  and  $t \in [0, 1]$ ,
- ( $\delta$ )  $\int_{C_j} f_1^{i+1}\theta = 0$  for all  $j = 1, \dots, l+1$ .

Property ( $\gamma$ ) ensures that the real part  $\Re(f_t^{i+1}\theta)$  of the holomorphic 1-form  $f_t^{i+1}\theta$  integrates to a conformal minimal immersion  $u_t^{i+1} : M_{i+1} \rightarrow \mathbb{R}^3$  depending smoothly on  $t \in [0, 1]$ . Property ( $\alpha$ ) shows that  $u_0^{i+1} = u_0^i = u_0$  with correct choices of constants of integration, and ( $\beta$ ) implies that  $u_t^{i+1}$  approximates  $u_t^i$  in the smooth topology on  $M_i$ . Finally, property ( $\delta$ ) ensures that  $u_1^{i+1} = \Re F^{i+1}$ , where  $F^{i+1} : M_{i+1} \rightarrow \mathbb{C}^3$  is a holomorphic null curve obtained by integrating the holomorphic 1-form  $f_1^{i+1}\theta$ . This completes the induction step in case (a).

In case (b), when the endpoints of the arc  $E$  belong to different connected components of the domain  $M_i$ ,  $E$  does not complete to a closed loop inside  $M_i$ , and the inclusion  $M_i \hookrightarrow M_{i+1}$  induces an isomorphism  $H_1(M_i; \mathbb{Z}) \cong H_1(M_{i+1}; \mathbb{Z})$ . Let  $C_{l+1} \subset M_i \cup E$  be a real analytic arc containing  $E$  in its relative interior. Choose a holomorphic coordinate  $\zeta$  on a neighborhood of  $C_{l+1}$  in  $M$  which maps  $C_{l+1}$  into the real axis and maps  $E$  onto the segment  $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$ . Let  $f_t^i$  and  $\sigma_t^i$  be determined by (5.1) and (5.3), respectively. In the local coordinate  $\zeta$ ,  $\sigma_t$  is of the form (5.4), where  $h_t(x)$  and  $g_t(x)$  are defined for  $x$  near the endpoints 0, 1 of  $\zeta(E) = [0, 1]$ . Clearly, we can extend  $h_t$  and  $g_t$  smoothly to  $[0, 1]$  such that conditions (i) and (ii) (stated just below (5.4)) hold. The map  $f_t^i$  defined by (5.5) then satisfies the first condition in (5.6), and the second condition is irrelevant. We now complete the inductive step exactly as in case (a).

## 6. h-Runge approximation theorem for conformal minimal immersions

The proof of Theorem 1.1, given in Sections 4 and 5, depends on the Mergelyan approximation theorem applied to period dominating sprays with values in  $\mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$ , where  $\mathfrak{A}$  is the null quadric (2.2). We now present a more conceptual approach to this problem. Theorem 6.3 below is a homotopy version of the Runge–Mergelyan approximation theorem for isotopies of conformal minimal immersions, with the additional control of one component function which is globally defined on the Riemann surface. This will be used in Section 7 to prove Theorems 1.2 and 7.3.

We begin by introducing the type of sets that we shall consider for the Mergelyan approximation (cf. [6, Definition 2.2] or [4, Definition 7.1]).

**Definition 6.1.** A compact subset  $S$  of an open Riemann surface  $M$  is said to be *admissible* if  $S = K \cup \Gamma$ , where  $K = \bigcup \overline{D}_i$  is a union of finitely many pairwise disjoint, compact, smoothly bounded domains  $\overline{D}_i$  in  $M$  and  $\Gamma = \bigcup \gamma_j$  is a union of finitely many pairwise disjoint analytic arcs or closed curves that intersect  $K$  only in their endpoints (or not at all), and such that their intersections with the boundary  $\partial K$  are transverse.

An admissible subset  $S \subset M$  is  $\mathcal{O}(M)$ -convex (also called *Runge* in  $M$ ) if and only if the inclusion map  $S \hookrightarrow M$  induces an injective homomorphism  $H_1(S; \mathbb{Z}) \hookrightarrow H_1(M; \mathbb{Z})$ .

Given an admissible set  $S = K \cup \Gamma \subset M$ , we denote by  $\mathfrak{D}(S, \mathfrak{A}^*)$  the set of all smooth maps  $S \rightarrow \mathfrak{A}^*$  which are holomorphic on an unspecified open neighborhood of  $K$  (depending on the map). We denote by  $\mathfrak{D}_*(S, \mathfrak{A}^*)$  the subset of  $\mathfrak{D}(S, \mathfrak{A}^*)$  consisting of those maps mapping no component of  $K$  and no component of  $\Gamma$  to a ray on  $\mathfrak{A}^*$ .

Fix a nowhere vanishing holomorphic 1-form  $\theta$  on  $M$ . (Such exists by the Oka–Grauert principle, see Section 2; the precise choice of  $\theta$  will be unimportant in the sequel.)

The following definition of a conformal minimal immersion of an admissible subset emulates the spirit of the concept of *marked immersion* [6] and provides the natural initial objects for the Mergelyan approximation by conformal minimal immersions.

**Definition 6.2.** Let  $M$  be an open Riemann surface and let  $S = K \cup \Gamma \subset M$  be an admissible subset (Definition 6.1). A *generalized conformal minimal immersion* on  $S$  is a pair  $(u, f\theta)$ , where  $f \in \mathfrak{D}(S, \mathfrak{A}^*)$  and  $u : S \rightarrow \mathbb{R}^3$  is a smooth map which is a conformal minimal immersion on an open neighborhood of  $K$ , such that

- $f\theta = 2\partial u$  on an open neighborhood of  $K$ ,
- for any smooth path  $\alpha$  in  $M$  parametrizing a connected component of  $\Gamma$  we have  $\Re(\alpha^*(f\theta)) = \alpha^*(du) = d(u \circ \alpha)$ .

A generalized conformal minimal immersion  $(u, f\theta)$  is *nonflat* if  $u$  is nonflat on every connected component of  $K$  and also on every curve in  $\Gamma$ , equivalently, if  $f \in \mathfrak{D}_*(S, \mathfrak{A}^*)$ .

We denote by  $\text{GCMI}(S)$  the set of all generalized conformal minimal immersions on  $S$ , and by  $\text{GCMI}_*(S)$  the subset consisting of nonflat generalized conformal minimal immersions. We have natural inclusions

$$\text{CMI}(S) \subset \text{GCMI}(S), \quad \text{CMI}_*(S) \subset \text{GCMI}_*(S),$$

where  $\text{CMI}(S)$  is the set of conformal minimal immersions on open neighborhoods of  $S$ . If  $(u, f\theta) \in \text{GCMI}(S)$ , then clearly  $u|_K \in \text{CMI}(K)$ ,  $\int_C \Re(f\theta) = 0$  for every closed curve  $C$  on  $S$ , and  $u(x) = u(x_0) + \int_{x_0}^x \Re(f\theta)$  for every pair of points  $x_0, x$  in the same connected component of  $S$ .

We say that  $(u, f\theta) \in \text{GCMI}(S)$  can be approximated in the  $\mathcal{C}^1(S)$  topology by conformal minimal immersions in  $\text{CMI}(M)$  if there is a sequence  $v_i \in \text{CMI}(M)$  ( $i \in \mathbb{N}$ ) such that  $v_i|_S$  converges to  $u|_S$  in the  $\mathcal{C}^1(S)$  topology and  $\partial v_i|_S$  converges to  $f\theta|_S$  in the  $\mathcal{C}^0(S)$  topology. (The latter condition is a consequence of the first one on  $K$ , but not on  $\Gamma$ .)

**Theorem 6.3** (h-Runge approximation theorem for conformal minimal immersions). *Let  $M$  be an open Riemann surface and let  $u \in \text{CMI}_*(M)$  be a nonflat conformal minimal immersion  $M \rightarrow \mathbb{R}^3$ . Assume that  $S = K \cup \Gamma \subset M$  is an  $\mathcal{O}(M)$ -convex admissible subset*

(Definition 6.1) and  $(u_t, f_t\theta) \in \text{GCMI}_*(S)$  ( $t \in [0, 1]$ ) is a smooth isotopy of nonflat generalized conformal minimal immersions on  $S$  (Definition 6.2) with  $u_0 = u|_S$  and  $f_0\theta = (2\partial u)|_S$ . Then the family  $(u_t, f_t\theta)$  can be approximated arbitrarily closely in the  $\mathcal{C}^1(S)$  topology by a smooth family  $\tilde{u}_t \in \text{CMI}_*(M)$  ( $t \in [0, 1]$ ) satisfying the following conditions:

- (i)  $\tilde{u}_0 = u$ .
- (ii)  $\text{Flux}_{\tilde{u}_t}(C) = \int_C \Im(f_t\theta)$  for every closed curve  $C \subset S$  and  $t \in [0, 1]$ . (See (1.1).)
- (iii) Assume in addition that for every  $t \in [0, 1]$  the third component function  $u_t^3$  of  $u_t$  extends harmonically to  $M$  and the third component  $f_t^3$  of  $f_t$  extends holomorphically to  $M$  (hence  $2\partial u_t^3 = f_t^3\theta$  on  $M$ ). Then the family  $\tilde{u}_t$  can be chosen to satisfy (i), (ii) and also  $\tilde{u}_t^3 = u_t^3$  for all  $t \in [0, 1]$ .

In the proof of Theorem 6.3 we shall need the following version of the h-Runge approximation property for maps into  $\mathfrak{A}^*$  with the control of the periods.

**Lemma 6.4.** *Let  $M$  be an open Riemann surface and  $f = (f^1, f^2, f^3) : M \rightarrow \mathfrak{A}^*$  a holomorphic map whose image is not contained in a ray in  $\mathbb{C}^3$ . Assume that  $S = K \cup \Gamma \subset M$  is an  $\mathcal{O}(M)$ -convex admissible subset (Definition 6.1). Then every smooth isotopy*

$$f_t = (f_t^1, f_t^2, f_t^3) \in \mathfrak{D}_*(S, \mathfrak{A}^*), \quad t \in [0, 1],$$

with  $f_0 = f|_S$  can be approximated arbitrarily closely in  $\mathcal{C}^1(S)$  by a smooth family of holomorphic maps

$$\tilde{f}_t = (\tilde{f}_t^1, \tilde{f}_t^2, \tilde{f}_t^3) : M \rightarrow \mathfrak{A}^*, \quad t \in [0, 1],$$

satisfying the following conditions:

- (i)  $\tilde{f}_0 = f$ .
- (ii)  $\int_C \tilde{f}_t\theta = \int_C f_t\theta$  for every closed curve  $C \subset S$  and every  $t \in [0, 1]$ .
- (iii) If  $f_t^3$  extends holomorphically to  $M$  for all  $t \in [0, 1]$ , then we can choose the family  $\tilde{f}_t$  such that  $\tilde{f}_t^3 = f_t^3$  for all  $t \in [0, 1]$ .

*Proof of Lemma 6.4.* An isotopy satisfying properties (i) and (ii) is obtained by following the proof of (the special case of) Theorem 1.1 in Section 4. Here is a brief sketch.

By Lemma 3.6 and Remark 3.8 we can embed the isotopy  $\{f_t\}_{t \in [0, 1]}$  into a holomorphic period dominating spray of smooth maps  $f_{t,w} = (f_{t,w}^1, f_{t,w}^2, f_{t,w}^3) : S \rightarrow \mathfrak{A}^*$ . Here,  $w$  is a parameter in a ball  $W \subset \mathbb{C}^N$  around the origin in a complex Euclidean space for some big  $N$ ,  $f_{t,w}$  depends holomorphically on  $w$  and smoothly on  $t \in [0, 1]$ , and  $f_{t,0} = f_t$  for all  $t$ . The phrase *period dominating* refers to a fixed finite set of closed loops in  $S$  forming a basis of the first homology group  $H_1(S; \mathbb{Z})$ .

Since  $\mathfrak{A}^*$  is an Oka manifold, we have the Mergelyan approximation property for maps from Stein manifolds (in particular, from open Riemann surfaces) to  $\mathfrak{A}^*$  in the absence of topological obstructions. (See the argument and the references given in Section 4 above.) In the case at hand, the map  $f = f_0 : M \rightarrow \mathfrak{A}^*$  is globally defined and the domain  $[0, 1] \times W$  of the spray  $f_{t,w}$  is contractible, so there are no topological obstructions to extending these maps continuously to all of  $M$ . Applying the Mergelyan approximation theorem on  $S$  we obtain a spray of holomorphic maps  $\tilde{f}_{t,w} : M \rightarrow \mathfrak{A}^*$ , depending holomorphically on  $w$  (whose domain is allowed to shrink a little) and smoothly on  $t \in [0, 1]$ , such that  $\tilde{f}_{t,w}$  approximates  $f_{t,w}$

in the  $\mathcal{C}^1$  topology on  $S$ , and  $\tilde{f}_{0,0} = f_0$  holds on  $M$ . Within this family we can then pick an isotopy  $\tilde{f}_t = \tilde{f}_{t,w(t)}$  ( $t \in [0, 1]$ ) satisfying properties (i) and (ii). The smooth function  $[0, 1] \ni t \mapsto w(t) \in \mathbb{C}^N$  is chosen by the implicit function theorem such that  $w(0) = 0$  (which implies property (i)),  $w(t)$  is close to  $0 \in \mathbb{C}^N$  for all  $t \in [0, 1]$  (to guarantee the approximation on  $S$ ), and  $\tilde{f}_t$  satisfies the period conditions in property (ii).

It remains to show that we can also fulfill property (iii). Let  $C_1, \dots, C_l$  be closed, oriented, analytic curves in  $S$  whose homology classes form a basis of  $H_1(S; \mathbb{Z})$ . Assume that  $f_t^3$  extends holomorphically to  $M$  for all  $t \in [0, 1]$ . We argue as in [4, Theorem 7.7]. Set  $\mathfrak{X}' = \mathfrak{X} \cap \{z_1 = 1\}$  and observe that  $\mathfrak{X} \setminus \{z_1 = 0\}$  is biholomorphic to  $\mathfrak{X}' \times \mathbb{C}^*$  (in particular,  $\mathfrak{X}' \times \mathbb{C}^*$  is an Oka manifold), and the projection  $\pi_1 : \mathfrak{X} \rightarrow \mathbb{C}$  is a trivial fiber bundle with Oka fiber  $\mathfrak{X}'$  except over  $0 \in \mathbb{C}$  where it is ramified. We may embed the isotopy  $f_t$  into a spray  $f_{t,w} = (f_{t,w}^1, f_{t,w}^2, f_{t,w}^3) : S \rightarrow \mathfrak{X}^*$  of smooth maps which are holomorphic on a neighborhood of  $K$  and depend holomorphically on a parameter  $w \in W \subset \mathbb{C}^N$  in a ball of some  $\mathbb{C}^N$  such that  $f_{0,0} = f_0$ , the third component  $f_{t,w}^3$  of  $f_{t,w}$  equals  $f_t^3$  for all  $(t, w) \in [0, 1] \times W$ , and the period map  $(t, w) \mapsto P(t, w) \in (\mathbb{C}^2)^l$  with the components

$$P_j(t, w) = \int_{C_j} (f_{t,w}^1, f_{t,w}^2) \theta \in \mathbb{C}^2, \quad j = 1, \dots, l,$$

is submersive with respect to  $w$  at  $w = 0$ . Up to slightly shrinking the ball  $W$ , the Oka principle for sections of ramified holomorphic maps with Oka fibers (see [9] or [10, Section 6.13]) enables us to approximate the spray  $f_{t,w}$  in the smooth topology on  $S$  by a spray of holomorphic maps  $\tilde{f}_{t,w} : M \rightarrow \mathfrak{X}^*$ , depending smoothly on  $t \in [0, 1]$  and holomorphically on  $w \in W$ , such that  $\tilde{f}_{0,0} = f$  and the third component  $\tilde{f}_{t,w}^3$  of  $\tilde{f}_{t,w}$  equals  $f_{t,w}^3 = f_t^3$  for all  $(t, w) \in [0, 1] \times W$ . If the approximation is close enough, then the implicit function theorem furnishes a smooth map  $w : [0, 1] \rightarrow W \subset \mathbb{C}^N$  close to 0, with  $w(0) = 0$ , such that the isotopy of holomorphic maps  $\tilde{f}_t := \tilde{f}_{t,w(t)} : M \rightarrow \mathfrak{X}^*$  ( $t \in [0, 1]$ ) satisfies (i), (ii), and (iii). For further details we refer to the proof of [4, Theorem 7.7].  $\square$

Given a compact bordered Riemann surface  $\bar{R} = R \cup bR$  with smooth boundary  $bR$  consisting of finitely many smooth Jordan curves, we denote by  $\mathcal{A}^r(\bar{R})$  the set of all maps  $\bar{R} \rightarrow \mathbb{C}$  of class  $\mathcal{C}^r$  ( $r \in \mathbb{Z}_+$ ) that are holomorphic on the interior  $R$  of  $\bar{R}$ .

*Proof of Theorem 6.3.* Pick a smooth strongly subharmonic Morse exhaustion function  $\rho : M \rightarrow \mathbb{R}$ . We exhaust  $M$  by an increasing sequence

$$M_1 \subset M_2 \subset \dots \subset \bigcup_{j=1}^{\infty} M_j = M$$

of compact smoothly bounded domains of the form

$$M_j = \{p \in M : \rho(p) \leq c_j\},$$

where  $c_1 < c_2 < \dots$  is an increasing sequence of regular values of  $\rho$  with  $\lim_{j \rightarrow \infty} c_j = +\infty$ . Since  $S$  is  $\mathcal{O}(M)$ -convex, we can choose  $\rho$  and  $c_1$  such that  $S \subset M_1$  and  $S$  is a strong deformation retract of  $M_1$ ; in particular, the inclusion  $S \hookrightarrow M_1$  induces an isomorphism

$$H_1(S; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z})$$

of their homology groups. Each domain  $M_j = \overset{\circ}{M}_j \cup b\overset{\circ}{M}_j$  is a compact bordered Riemann

surface, possibly disconnected. We may assume that  $\rho$  has at most one critical point  $p_j$  in each difference  $M_{j+1} \setminus M_j$ . It follows that  $M_j$  is  $\mathcal{O}(M)$ -convex and  $\overset{\circ}{M}_j$  is Runge in  $M$  for every  $j \in \mathbb{N}$ .

We proceed by induction. In the first step we obtain an extension from  $S$  to  $M_1$ .

Assume for simplicity that  $M_1$  and so  $S$  are connected; the same argument works on any connected component. Pick a point  $x_0 \in S$ . By Lemma 6.4, the family  $f_t$  ( $t \in [0, 1]$ ) can be approximated arbitrarily closely in  $\mathcal{C}^1(S)$  by a smooth isotopy of maps

$$f_{t,1} = (f_{t,1}^1, f_{t,1}^2, f_{t,1}^3) : M_1 \rightarrow \mathfrak{X}^*$$

of class  $\mathcal{A}^1(M_1)^3$  such that the family of conformal minimal immersions

$$u_{t,1} = (u_{t,1}^1, u_{t,1}^2, u_{t,1}^3) \in \text{CMI}_*(M_1)$$

given by

$$u_{t,1}(x) = u_t(x_0) + \int_{x_0}^x \Re(f_{t,1}\theta), \quad x \in M_1,$$

is well defined and satisfies

- (i<sub>1</sub>)  $u_{0,1} = u|_{M_1}$ ,
- (ii<sub>1</sub>)  $\text{Flux}_{u_{t,1}}(C) = \int_C \Im(f_t\theta)$  for every closed curve  $C \subset S$  and every  $t \in [0, 1]$ ,
- (iii<sub>1</sub>)  $u_{t,1}^3 = u_t^3|_{M_1}$  for all  $t \in [0, 1]$  provided that the assumptions in Theorem 6.3 (iii) hold.

Assume inductively that for some  $j \in \mathbb{N}$  we have already constructed a smooth isotopy

$$u_{t,j} = (u_{t,j}^1, u_{t,j}^2, u_{t,j}^3) \in \text{CMI}_*(M_j), \quad t \in [0, 1],$$

satisfying

- (i<sub>j</sub>)  $u_{0,j} = u|_{M_j}$ ,
- (ii<sub>j</sub>)  $\text{Flux}_{u_{t,j}}(C) = \int_C \Im(f_t\theta)$  for every closed curve  $C \subset S$  and every  $t \in [0, 1]$ ,
- (iii<sub>j</sub>)  $u_{t,j}^3 = u_t^3|_{M_j}$  for all  $t \in [0, 1]$  provided that the assumptions in Theorem 6.3 (iii) hold.

Let us show that the smooth isotopy  $\{u_{t,j}\}_{t \in [0,1]}$  can be approximated arbitrarily closely in the smooth topology on  $[0, 1] \times M_j$  by a smooth isotopy  $\{u_{t,j+1}\}_{t \in [0,1]} \subset \text{CMI}_*(M_{j+1})$  satisfying the analogous properties. The limit  $\tilde{u}_t = \lim_{j \rightarrow \infty} u_{t,j} \in \text{CMI}_*(M)$  will clearly satisfy Theorem 6.3. Indeed, properties (i<sub>j</sub>), (ii<sub>j</sub>), and (iii<sub>j</sub>) trivially imply (i), (ii), and (iii), respectively.

**The noncritical case.** Assume that  $\rho$  has no critical value in  $[c_j, c_{j+1}]$ . In this case  $M_j$  is a strong deformation retract of  $M_{j+1}$ . As above, we finish by using Lemma 6.4 applied to the family of maps  $f_{t,j} : M_j \rightarrow \mathfrak{X}^*$  ( $t \in [0, 1]$ ) given by  $2\partial u_{t,j} = f_{t,j}\theta$  on  $M_j$ .

**The critical case.** Assume that  $\rho$  has a critical point  $p_{j+1} \in M_{j+1} \setminus M_j$ . By the assumptions on  $\rho$ ,  $p_{j+1}$  is the only critical point of  $\rho$  on  $M_{j+1} \setminus M_j$  and is a Morse point. Since  $\rho$  is strongly, the Morse index of  $p_{j+1}$  is either 0 or 1.

If the Morse index of  $p_{j+1}$  is 0, then a new (simply connected) component of the sublevel set  $\{\rho \leq r\}$  appears at  $p_{j+1}$  when  $r$  passes the value  $\rho(p_{j+1})$ . In this case

$$M_{j+1} = M'_{j+1} \cup M''_{j+1},$$

where  $M'_{j+1} \cap M''_{j+1} = \emptyset$ ,  $M''_{j+1}$  is a simply connected component of  $M_{j+1}$ , and  $M_j$  is

a strong deformation retract of  $M'_{j+1}$ . Let  $\Omega \subset M''_{j+1}$  be a smoothly bounded compact disc that will be specified later. It follows that  $M_j \cup \Omega$  is a strong deformation retract of  $M_{j+1}$ . Extend  $\{u_{t,j} = (u_{t,j}^1, u_{t,j}^2, u_{t,j}^3)\}_{t \in [0,1]}$  to  $\Omega$  as any smooth isotopy of conformal minimal immersions such that  $u_{0,j}|_{\Omega} = u|_{\Omega}$ ; for instance one can simply take  $u_{t,j}|_{\Omega} = u|_{\Omega}$  for all  $t \in [0, 1]$ . If the assumptions in Theorem 6.3 (iii) hold, then take this extension to also satisfy  $u_{t,j}^3|_{\Omega} = u_t^3|_{\Omega}$  for all  $t \in [0, 1]$ . For instance, one can choose  $\Omega$  such that  $f_t^3$  does not vanish anywhere on  $\Omega$  for all  $t \in [0, 1]$ , pick  $x_0 \in \Omega$ , and take

$$u_{t,j}(x) = y_t + \Re \int_{x_0}^x \left( \frac{1}{2} \left( \frac{1}{g} - g \right), \frac{t}{2} \left( \frac{1}{g} + g \right), 1 \right) f_t^3 \theta, \quad x \in \Omega,$$

where  $y_t = (y_t^1, y_t^2, y_t^3) \in \mathbb{R}^3$  depends smoothly on  $t \in [0, 1]$  and satisfies  $y_0 = u(x_0)$  and  $y_t^3 = u_t^3(x_0)$  for all  $t \in [0, 1]$ , and  $g$  is the complex Gauss map of  $u$  (cf. (2.8) and (2.9) and observe that  $g$  is holomorphic and does not vanish anywhere on  $\Omega$ ). This reduces the proof to the noncritical case.

If the Morse index of  $p_{j+1}$  is 1, then the change of topology of the sublevel set  $\{\rho \leq r\}$  at  $p_{j+1}$  is described by attaching to  $M_j$  an analytic arc  $\gamma \subset \overset{\circ}{M}_{j+1} \setminus M_j$ . Observe that  $M_j \cup \gamma$  is an  $\mathcal{O}(M)$ -convex strong deformation retract of  $M_{j+1}$ . Without loss of generality we may assume that  $M_j \cup \gamma$  is admissible in the sense of Definition 6.1. Reasoning as in the critical step in Section 5, we extend the family  $\{u_{t,j}\}_{t \in [0,1]}$  to a smooth isotopy of nonflat generalized conformal minimal immersions  $\{(u_{t,j}, f_{t,j}\theta)\}_{t \in [0,1]} \subset \text{GCMI}_*(M_j \cup \gamma)$  such that

$$(u_{0,j}, f_{0,j}\theta) = (u, 2\partial u)|_{M_j \cup \gamma}.$$

If the assumptions in Theorem 6.3 (iii) hold, then we take this extension such that their third components satisfy  $u_{t,j}^3 = u_t^3|_{M_j \cup \gamma}$  and  $f_{t,j}^3\theta = (2\partial u_t^3)|_{M_j \cup \gamma}$  for all  $t \in [0, 1]$ . Then, to construct the isotopy  $u_{t,j+1} \in \text{CMI}_*(M_{j+1})$  ( $t \in [0, 1]$ ) meeting (ij<sub>+1</sub>), (iij<sub>+1</sub>), and (iiij<sub>+1</sub>), we reason as in the first step of the inductive process. This finishes the inductive step and proves the theorem.  $\square$

## 7. Isotopies of complete conformal minimal immersions

The aim of this section is to prove Theorem 1.2. The core of the proof is given by the following technical result. Recall that given a compact set  $K$  in an open Riemann surface  $M$ , we denote by  $\text{CMI}(K)$  the set of maps  $K \rightarrow \mathbb{R}^3$  extending as conformal minimal immersions to an unspecified open neighborhood of  $K$  in  $M$ , and by  $\text{CMI}_*(K) \subset \text{CMI}(K)$  the subset of those immersions which are nonflat on every connected component of  $K$ .

**Lemma 7.1.** *Let  $\overline{M} = M \cup bM$  be a compact connected bordered Riemann surface. Let  $u = (u^1, u^2, u^3) \in \text{CMI}_*(M)$  be a conformal minimal immersion which is of class  $\mathcal{C}^1(\overline{M})$  up to the boundary. Let  $K \subset M$  be an  $\mathcal{O}(M)$ -convex compact set containing the topology of  $M$ . Let  $u_t = (u_t^1, u_t^2, u_t^3) \in \text{CMI}_*(K)$ ,  $t \in [0, 1]$ , be a smooth isotopy with  $u_0 = u|_K$ . Assume also that  $u_t^3$  extends holomorphically to  $M$  for all  $t \in [0, 1]$ . Let  $x_0 \in K$  and denote by  $\tau$  the positive number given by*

$$(7.1) \quad \begin{aligned} \tau &:= \text{dist}_u(x_0, bM) \\ &= \inf\{\text{length}(u(\gamma)) : \gamma \text{ an arc in } \overline{M} \text{ connecting } x_0 \text{ and } bM\}. \end{aligned}$$

(Here  $\text{length}(\cdot)$  denotes the Euclidean length in  $\mathbb{R}^3$ .) Then, for any  $\delta > 0$ , the family  $u_t$  can

be approximated arbitrarily closely in the smooth topology on  $K$  by a family  $\tilde{u}_t \in \text{CMI}_*(M)$  of class  $\mathcal{C}^1(\overline{M})$ , depending smoothly on  $t \in [0, 1]$  and enjoying the following properties:

- (I)  $\tilde{u}_0 = u$ .
- (II)  $\tilde{u}_t^3 = u_t^3$  for all  $t \in [0, 1]$ .
- (III)  $\int_C d^c(\tilde{u}_t - u_t) = 0$  for every closed curve  $C \subset K$  and every  $t \in [0, 1]$ .
- (IV)  $\text{dist}_{\tilde{u}_t}(x_0, bM) > \tau - \delta$  for all  $t \in [0, 1]$ .
- (V)  $\text{dist}_{\tilde{u}_1}(x_0, bM) > 1/\delta$ .

*Proof.* By Theorem 6.3 we may assume without loss of generality that  $K$  is a compact connected smoothly bounded domain in  $M$ , and the isotopy  $u_t$  extends to a smooth isotopy of conformal minimal immersions in  $\text{CMI}_*(M)$  of class  $\mathcal{C}^1(\overline{M})$ . We emphasize that the latter assumption can be fulfilled while preserving the initial immersion  $u$  (see Theorem 6.3 (i)) and the third component of any immersion  $u_t$  in the family (see Theorem 6.3 (iii)).

Write  $u = u_0$ . Since  $u_t$  depends smoothly on  $t$ , (7.1) ensures that, up to possibly enlarging  $K$ , we may also assume the existence of a number  $t_0 \in ]0, 1[$  such that

$$(7.2) \quad \text{dist}_{u_t}(x_0, bK) > \tau - \frac{\delta}{2} \quad \text{for all } t \in [0, t_0].$$

Let  $\theta$  be a nowhere vanishing holomorphic 1-form of class  $\mathcal{A}^1(\overline{M})$ . Write  $2\partial u_t = f_t \theta$ , where  $f_t = (f_t^1, f_t^2, f_t^3) : \overline{M} \rightarrow \mathfrak{A}^*$  is of class  $\mathcal{A}^0(\overline{M})^3$ .

Denote by  $m \in \mathbb{N}$  the number of boundary components of  $\overline{M}$ . By the assumptions on  $K$ , the open set  $M \setminus K$  consists precisely of  $m$  connected components  $O_1, \dots, O_m$ , each one containing in its boundary a component of  $bM$ . Let  $z_j : O_j \rightarrow \mathbb{C}$  be a conformal parametrization such that  $z_j(O_j)$  is a round open annulus of radii  $0 < r_j < R_j < +\infty$  (observe that  $O_j$  is a bordered annulus),  $j = 1, \dots, m$ .

**Claim 7.2.** *There exist numbers  $t_0 < t_1 < \dots < t_l = 1$ ,  $l \in \mathbb{N}$ , and compact annuli  $A_{j,k} \subset O_j$ ,  $(j, k) \in I := \{1, \dots, m\} \times \{1, \dots, l\}$ , satisfying the following properties:*

- (i)  $A_{j,k}$  contains the topology of  $O_j$  for all  $(j, k) \in I$ . In particular, every arc  $\gamma \subset \overline{M}$  connecting  $x_0$  and  $bM$  contains a sub-arc connecting the two boundary components of  $A_{j,k}$  for all  $k \in \{1, \dots, l\}$ , for some  $j \in \{1, \dots, m\}$ .
- (ii)  $z_j(A_{j,k})$  is a round compact annulus of radii  $r_{j,k}$  and  $R_{j,k}$ , where  $r_j < r_{j,k} < R_{j,k} < R_j$  for all  $(j, k) \in I$ .
- (iii)  $A_{j,k} \cap A_{j',k'} = \emptyset$  for all  $(j, k) \neq (j', k') \in I$ .
- (iv)  $f_t^3$  does not vanish anywhere on  $A_{j,k}$  for all  $t \in [t_{k-1}, t_k]$ , for all  $(j, k) \in I$ .

*Proof.* Let  $t \in [t_0, 1]$ . Since  $u_t$  is nonflat,  $f_t^3$  does not vanish identically and hence its zeros are isolated on  $M$ . Therefore there exist compact annuli  $A_j^t \subset O_j$ ,  $j = 1, \dots, m$ , such that  $A_j^t$  contains the topology of  $O_j$ ,  $z_j(A_j^t)$  is a round compact annulus, and  $f_t^3$  does not vanish anywhere on  $A_j^t$ . Since  $f_t^3$  depends smoothly on  $t$ , there exists an open connected neighborhood  $U_t$  of  $t$  in  $[t_0, 1]$  such that  $f_{t'}^3$  does not vanish anywhere on  $A_j^t$  for all  $t' \in U_t$ . Since  $[t_0, 1] = \bigcup_{t \in [t_0, 1]} U_t$  is compact, there exist numbers  $t_0 < t_1 < \dots < t_l = 1$ ,  $l \in \mathbb{N}$ , such that  $\bigcup_{k=1}^l U_{t_k} = [t_0, 1]$ . Set  $A_{j,k} := A_j^{t_k}$  and observe that properties (i), (ii), and (iv) hold. To finish we simply shrink the annuli  $A_{j,k}$  in order to ensure (iii).  $\square$



Since  $A_{j,k} \times [t_{k-1}, t_k]$  is compact for all  $(j, k)$  in the finite set  $I$ , property (iv) gives a small number  $\epsilon > 0$  such that

$$(7.3) \quad \epsilon < \min \left\{ \left| \frac{f_t^3 \theta}{dz_j} \right| (x) : x \in A_{j,k}, t \in [t_{k-1}, t_k], (j, k) \in I \right\}.$$

The next step in the proof of the lemma consists of constructing on each annulus  $A_{j,k}$  a Jorge–Xavier-type labyrinth of compact sets (see [17] or [1–3]).

Let  $N$  be a large natural number that will be specified later.

Assume that  $2/N < \min\{R_{j,k} - r_{j,k} : (j, k) \in I\}$ . For any  $n \in \{1, \dots, 2N^2\}$ , we set  $s_{j,k;n} := R_{j,k} - n/N^3$  and observe that  $r_{j,k} < s_{j,k;n} < R_{j,k}$ . We set

$$(7.4) \quad L_{j,k;n} := \left\{ x \in A_{j,k} : s_{j,k;n} + \frac{1}{4N^3} \leq |z_j(x)| \leq s_{j,k;n-1} - \frac{1}{4N^3}, \right. \\ \left. \frac{1}{N^2} \leq \arg((-1)^n z_j(x)) \leq 2\pi - \frac{1}{N^2} \right\} \subset A_{j,k}.$$

By (iii), the compact sets  $L_{j,k;n} \subset M \setminus K$  are pairwise disjoint. We also set

$$L_{j,k} := \bigcup_{n=1}^{2N^2} L_{j,k;n}, \quad L := \bigcup_{(j,k) \in I} L_{j,k},$$

and observe that  $K \cap L = \emptyset$  and  $K \cup L$  is  $\mathcal{O}(M)$ -convex. (See Figure 7.1.)

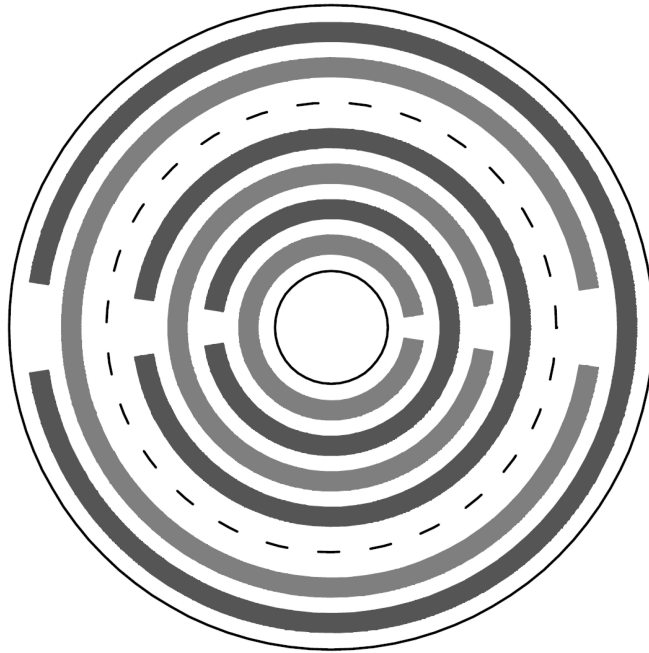


Figure 7.1. The labyrinth  $z_j(L_{j,k})$  inside the round annulus  $z_j(A_{j,k}) \subset \mathbb{C}$ .

Denote by  $g_t$  the complex Gauss map of  $u_t$ , i.e. the meromorphic function on  $M$

$$g_t = \frac{f_t^3}{f_t^1 - t f_t^2}, \quad t \in [0, 1],$$

and recall that

$$(7.5) \quad f_t = \left( \frac{1}{2} \left( \frac{1}{g_t} - g_t \right), \frac{i}{2} \left( \frac{1}{g_t} + g_t \right), 1 \right) f_t^3, \quad t \in [0, 1]$$

(see (2.8) and (2.9)). Since  $f_t$  is holomorphic on  $M$ , (iv) ensures that  $g_t$  has neither zeros nor poles on  $A_{j,k}$  for all  $t \in [t_{k-1}, t_k]$ ,  $(j, k) \in I$ . In particular, since  $I$  is finite and  $[t_{k-1}, t_k]$  is compact, there exists a constant  $c_0 > 0$  such that  $|g_t| > c_0$  on  $A_{j,k}$  for all  $t \in [t_{k-1}, t_k]$  and all  $(j, k) \in I$ . Therefore, we may take a number  $\lambda > 0$  large enough so that

$$(7.6) \quad |1 + \lambda t| \cdot |g_t| > 2N^4 \quad \text{on } A_{j,k} \text{ for all } t \in [t_{k-1}, t_k] \text{ and all } (j, k) \in I,$$

recall that  $t_0 > 0$ .

Consider the family of holomorphic maps  $h_t = (h_t^1, h_t^2, h_t^3) : K \cup L \rightarrow \mathfrak{A}^*$ ,  $t \in [0, 1]$ , given by

$$(7.7) \quad h_t = f_t \quad \text{on } K$$

and

$$(7.8) \quad h_t = \left( \frac{1}{2} \left( \frac{1}{(1 + \lambda t)g_t} - (1 + \lambda t)g_t \right), \frac{i}{2} \left( \frac{1}{(1 + \lambda t)g_t} + (1 + \lambda t)g_t \right), 1 \right) f_t^3 \quad \text{on } L.$$

The map  $h_t$  is said to be obtained from  $f_t$  on  $L$  by a *López–Ros transformation*; see [19].

Since  $f_t$  depends smoothly on  $t \in [0, 1]$ , it is clear from (7.5), (7.7), and (7.8) that the family  $h_t$  depends smoothly on  $t \in [0, 1]$  as well. Notice that, since  $1 + \lambda t \neq 0$ , the holomorphicity of  $f_t$  implies the one of  $h_t$ ,  $t \in [0, 1]$ . Moreover, equations (7.7), (7.8), and (7.5) also give that

$$(7.9) \quad h_0 = f_0|_{K \cup L}, \quad h_t^3 = f_t^3|_{K \cup L} \quad \text{for all } t \in [0, 1].$$

On the other hand, since  $K \cap L = \emptyset$  and  $L$  is the union of finitely many pairwise disjoint closed discs, (7.7) ensures that

$$(7.10) \quad \int_C (h_t - f_t)\theta = 0 \quad \text{for every closed curve } C \subset K \cup L \text{ and every } t \in [0, 1].$$

In view of (7.9) and (7.10), Lemma 6.4 provides a family of holomorphic maps

$$\tilde{f}_t = (\tilde{f}_t^1, \tilde{f}_t^2, \tilde{f}_t^3) : M \rightarrow \mathfrak{A}^*,$$

depending smoothly on  $t \in [0, 1]$ , such that

- (v)  $\tilde{f}_0 = f_0$ ,
- (vi)  $\tilde{f}_t^3 = f_t^3$  for all  $t \in [0, 1]$ ,
- (vii)  $\int_C (\tilde{f}_t - f_t)\theta = 0$  for every closed curve  $C \subset K$  and every  $t \in [0, 1]$ ,
- (viii)  $\tilde{f}_t$  is as close to  $h_t$  in the smooth topology on  $K \cup L$  as desired,  $t \in [0, 1]$ .

For each  $t \in [0, 1]$ , consider the conformal minimal immersion  $\tilde{u}_t \in \text{CMI}_*(M)$  given by

$$\tilde{u}_t(x) := u_t(x_0) + \Re \int_{x_0}^x \tilde{f}_t \theta, \quad x \in M.$$

Observe that  $\tilde{u}_t$  is well defined; see (vii) and recall that the periods of  $f_t \theta = 2\partial u_t$  are purely imaginary.

Let us check that the family  $\{\tilde{u}_t\}_{t \in [0,1]}$  satisfies the conclusion of the lemma. Indeed, since the family  $\tilde{f}_t$  depends smoothly on  $t \in [0, 1]$ , the same is true for the family  $\tilde{u}_t$ . Moreover, (viii) and (7.7) ensure that  $\tilde{u}_t$  can be chosen as close as desired to  $u_t$  in the smooth topology on  $K$  (uniformly with respect to  $t \in [0, 1]$ ); recall that  $K$  is a compact smoothly bounded domain in  $M$ , hence arc-connected. On the other hand, we have that

$$(7.11) \quad \tilde{u}_t(x_0) = u_t(x_0), \quad 2\partial\tilde{u}_t = \tilde{f}_t\theta \quad \text{for all } t \in [0, 1],$$

hence properties (I), (II), and (III) directly follow from (v), (vi), and (vii), respectively.

Let us prove (IV) provided that the number  $N$  is big enough and the approximation in (viii) is close enough. Indeed, fix  $t \in [0, 1]$  and let us distinguish cases.

First assume that  $t \in [0, t_0]$ . In this case (7.2) ensures that

$$\text{dist}_{\tilde{u}_t}(x_0, bM) \geq \text{dist}_{\tilde{u}_t}(x_0, bK) > \tau - \delta$$

provided that  $\tilde{u}_t$  is close enough to  $u_t$  on  $K$ .

Assume now that  $t \in [t_0, 1]$ ; hence  $t \in [t_{k-1}, t_k]$  for some  $k \in \{1, \dots, l\}$ . Recall that the Riemannian metric  $ds_{\tilde{u}_t}^2$  induced on  $M$  by the Euclidean metric of  $\mathbb{R}^3$  via  $\tilde{u}_t$  is given by

$$(7.12) \quad ds_{\tilde{u}_t}^2 = \frac{1}{2} |\tilde{f}_t\theta|^2 \geq |\tilde{f}_t^3\theta|^2$$

(see (2.10) and take into account that  $x + \frac{1}{x} \geq 2$  for all  $x > 0$ ). In particular, (viii) ensures that

(ix)  $ds_{\tilde{u}_t}^2$  is as close to  $\frac{1}{2}|h_t\theta|^2$  as desired in the smooth topology on  $K \cup L$ .

In view of property (i) above, it suffices to show that  $\text{length}_{\tilde{u}_t}(\gamma) > \max\{\tau - \delta, \frac{1}{\delta}\}$  for any arc  $\gamma \subset A_{j,k}$  connecting the two boundary components of the annulus  $A_{j,k}$ ,  $j = 1, \dots, m$ , where  $\text{length}_{\tilde{u}_t}$  denotes the length function in the Riemannian surface  $(M, ds_{\tilde{u}_t}^2)$ . This will also prove (V).

Indeed, let  $j \in \{1, \dots, m\}$  and let  $\gamma \subset A_{j,k}$  be an arc connecting the two boundary components of  $A_{j,k}$ . On the one hand, (7.8) give that

$$\frac{1}{2}|h_t\theta|^2 = \frac{1}{4} \left( \frac{1}{|1 + \lambda t||g_t|} + |1 + \lambda t||g_t| \right)^2 |f_t^3|^2 |\theta|^2 \quad \text{on } L.$$

This, (ix), (7.6), and (7.3), imply that

$$(7.13) \quad ds_{\tilde{u}_t}^2 > N^8 \epsilon^2 |dz_j|^2 \quad \text{on } L_{j,k}.$$

On the other hand, (7.12), (vi), and (7.3) give that

$$(7.14) \quad ds_{\tilde{u}_t}^2 \geq |\tilde{f}_t^3\theta|^2 = |f_t^3\theta|^2 > \epsilon^2 |dz_j|^2 \quad \text{on } A_{j,k}.$$

The above two estimates ensure that

$$(7.15) \quad \text{length}_{\tilde{u}_t}(\gamma) > \min \left\{ \frac{1}{2}, r_{j,k} \right\} \in N,$$

where  $r_{j,k} > 0$  is the inner radius of  $z_j(A_{j,k})$  (see Claim 7.2 (ii)). Indeed, assume first that  $\gamma$  crosses  $L_{j,k;n}$ , for some  $n \in \{1, \dots, 2N^2\}$ , in the sense that  $\gamma$  contains a subarc  $\hat{\gamma} \subset L_{j,k;n}$  such that  $z_j(\hat{\gamma})$  connects the two circumferences defining  $z_j(L_{j,k;n})$ ; see (7.4) and Figure 7.1.

It follows that the Euclidean length of  $z_j(\hat{\gamma})$  is at least  $1/2N^3$  (cf. (7.4)) and hence (7.13) ensures that  $\text{length}_{\tilde{u}_t}(\gamma) > \text{length}_{\tilde{u}_t}(\hat{\gamma}) > \frac{1}{2}\epsilon N$ . Assume now that the arc  $\gamma$  crosses  $L_{j,k;n}$  for no  $n \in \{1, \dots, 2N^2\}$ . In this case, for any  $n \in \{1, \dots, 2N^2 - 1\}$ ,  $z_j(\gamma)$  surrounds the set  $z_j(L_{j,k;n})$  in order to scape by the opening of  $z_j(L_{j,k;n+1})$ ; see (7.4) and Figure 7.1. Since this phenomenon happens at least  $2N^2 - 1$  times, the Euclidean length of  $z_j(\gamma)$  is larger than  $(2N^2 - 1)r_{j,k} > Nr_{j,k}$  and hence (7.14) gives that  $\text{length}_{\tilde{u}_t}(\gamma) > r_{j,k}\epsilon N$ .

In view of (7.15), to conclude the proof it suffices to choose  $N$  large enough so that  $\min\{\frac{1}{2}, r_{j,k}\}\epsilon N > \max\{\tau - \delta, \frac{1}{8}\}$  for all  $(j, k) \in I$ .  $\square$

*Proof of Theorem 1.2.* Let  $\rho : M \rightarrow \mathbb{R}$  be a smooth strongly subharmonic Morse exhaustion function. We can exhaust  $M$  by an increasing sequence

$$M_0 \subset M_1 \subset \dots \subset \bigcup_{i=0}^{\infty} M_i = M$$

of compact smoothly bounded domains of the form

$$M_i = \{p \in M : \rho(p) \leq c_i\},$$

where  $c_0 < c_1 < \dots$  is an increasing sequence of regular values of  $\rho$  with  $\lim_{i \rightarrow \infty} c_i = +\infty$ . Each domain  $M_i = \overset{\circ}{M}_i \cup bM_i$  is a compact bordered Riemann surface, possibly disconnected. We may further assume that  $\rho$  has at most one critical point  $p_i$  in each difference  $M_{i+1} \setminus M_i$ . It follows that  $M_i$  is  $\mathcal{O}(M)$ -convex and its interior  $\overset{\circ}{M}_i$  is Runge in  $M$  for every  $i \in \mathbb{Z}_+$ .

We proceed by induction. Choose a point  $x_0 \in \overset{\circ}{M}_0$  and set

$$(7.16) \quad \tau_i := \text{dist}_u(x_0, bM_i) > 0 \quad \text{for all } i \in \mathbb{Z}_+.$$

The initial step is the smooth isotopy  $\{u_t^0 := u|_{M_0} \in \text{CMI}_*(M_0)\}_{t \in [0,1]}$ . Assume inductively that we have already constructed for some  $i \in \mathbb{Z}_+$  a smooth isotopy  $u_t^i \in \text{CMI}_*(M_i)$  ( $t \in [0, 1]$ ) satisfying the following conditions:

- (a<sub>i</sub>)  $u_0^i = u|_{M_i}$ ,
- (b<sub>i</sub>)  $\text{Flux}_{u_t^i}(C) = \mathfrak{p}(C)$  for every closed curve  $C \subset M_i$ ,
- (c<sub>i</sub>)  $\text{dist}_{u_t^i}(x_0, bM_i) > \tau_i - \frac{1}{t}$  for all  $t \in [0, 1]$  (this condition is omitted for  $i = 0$ ),
- (d<sub>i</sub>)  $\text{dist}_{u_1^i}(x_0, bM_i) > i$ .

We will show that  $\{u_t^i\}_{t \in [0,1]}$  can be approximated arbitrarily closely in the smooth topology on  $[0, 1] \times M_i$  by an isotopy  $\{u_t^{i+1}\}_{t \in [0,1]}$  satisfying the analogous properties over a neighborhood of  $M_{i+1}$ . The limit  $u_t = \lim_{i \rightarrow \infty} u_t^i \in \text{CMI}_*(M)$  will clearly satisfy Theorem 1.2. Indeed, properties (a<sub>i</sub>) imply that  $u_0 = u$ , (b<sub>i</sub>) ensure that  $\text{Flux}_{u_1} = \mathfrak{p}$ , and (d<sub>i</sub>) give that  $u_1$  is complete. Finally, if  $u$  is complete, then

$$\lim_{i \rightarrow \infty} \left( \tau_i - \frac{1}{i} \right) = +\infty$$

(see (7.16)); hence properties (c<sub>i</sub>) guarantee the completeness of  $u_t$  for all  $t \in [0, 1]$ .

Observe that property (c<sub>i</sub>) will not be required in the construction of  $u_t^{i+1}$ . Therefore the construction is consistent with the fact that (c<sub>i</sub>) does not make sense for  $i = 0$ .

**The noncritical case.** Assume that  $\rho$  has no critical value in  $[c_i, c_{i+1}]$ . In this case  $M_i$  is a strong deformation retract of  $M_{i+1}$ . In view of (a<sub>*i*</sub>), (b<sub>*i*</sub>), and (7.16), Lemma 7.1 can be applied to the data

$$(M = M_{i+1}, u = u|_{M_{i+1}}, K = M_i, u_t = u_t^i, x_0, \tau = \tau_{i+1}, \delta = 1/(i + 1)),$$

furnishing a smooth isotopy  $u_t^{i+1} \in \text{CMI}_*(M_{i+1})$  which satisfies conditions (a<sub>*i+1*</sub>)–(d<sub>*i+1*</sub>) and is as close as desired to  $u_t^i$  in the smooth topology on  $M_i$ .

**The critical case.** Assume that  $\rho$  has a critical point  $p_{i+1} \in M_{i+1} \setminus M_i$ . By the assumptions on  $\rho$ ,  $p_{i+1}$  is the only critical point of  $\rho$  on  $M_{i+1} \setminus M_i$  and is a Morse point. Since  $\rho$  is strongly, the Morse index of  $p_{i+1}$  is either 0 or 1.

If the Morse index of  $p_{i+1}$  is 0, then a new (simply connected) component of the sublevel set  $\{\rho \leq r\}$  appears at  $p_{i+1}$  when  $r$  passes the value  $\rho(p_{i+1})$ . In this case

$$M_{i+1} = M'_{i+1} \cup M''_{i+1},$$

where  $M'_{i+1} \cap M''_{i+1} = \emptyset$ ,  $M''_{i+1}$  is a simply connected component of  $M_{i+1}$ , and  $M_i$  is a strong deformation retract of  $M'_{i+1}$ . Extend  $u_t^i$  by setting  $u_t^i = u$  on  $M''_{i+1}$  for all  $t \in [0, 1]$ . This reduces the proof to the noncritical case.

If the Morse index of  $p_{i+1}$  is 1, then the change of topology of the sublevel set  $\{\rho \leq r\}$  at  $p_{i+1}$  is described by attaching to  $M_i$  an analytic arc  $\gamma \subset \overset{\circ}{M}_{i+1} \setminus M_i$ . In this case we take  $r' \in ]\rho(p_{i+1}), c_{i+1}[$  and set  $W = \{\rho \leq r'\}$ . By the assumptions, we have that  $W = \overset{\circ}{W} \cup bW$  is an  $\mathcal{O}(M)$ -convex compact bordered Riemann surface which is a strong deformation retract of  $M_{i+1}$ . Arguing as in the critical case in Section 5 we may approximate  $\{u_t^i\}_{t \in [0,1]}$  arbitrarily closely in the smooth topology on  $[0, 1] \times M_i$  by an isotopy  $\{\hat{u}_t^i\}_{t \in [0,1]} \subset \text{CMI}_*(W)$  satisfying  $\hat{u}_0^i = u|_W$  and  $\text{Flux}_{\hat{u}_1^i}(C) = \mathfrak{p}(C)$  for every closed curve  $C \subset W$  (take into account (a<sub>*i*</sub>) and (b<sub>*i*</sub>)). Further, (7.16) ensures that  $\text{dist}_{\hat{u}_0^i}(x_0, bW) > \tau_i$ . Again this reduces the construction to the noncritical case and concludes the proof of the theorem. □

In a different direction, we can construct an isotopy of conformal minimal immersions from a given immersion to a complete one without changing the flux map.

**Theorem 7.3.** *Let  $M$  be a connected open Riemann surface of finite topology. For every smooth isotopy  $u_t \in \text{CMI}_*(M)$  ( $t \in [0, 1]$ ) there exists a smooth isotopy  $\tilde{u}_t \in \text{CMI}_*(M)$  ( $t \in [0, 1]$ ) of conformal minimal immersions such that  $\tilde{u}_0 = u_0$ ,  $\tilde{u}_1$  is complete, the third component of  $\tilde{u}_t$  equals the one of  $u_t$  for all  $t \in [0, 1]$ , and the flux map of  $\tilde{u}_t$  equals the one of  $u_t$  for all  $t \in [0, 1]$ . Furthermore, if  $u_0$  is complete, then there exists such an isotopy where  $\tilde{u}_t$  is complete for all  $t \in [0, 1]$ .*

*Proof.* Let  $\rho : M \rightarrow \mathbb{R}$  be a smooth strongly subharmonic Morse exhaustion function. Since  $M$  is of finite topology, we can exhaust it by a sequence

$$M_0 \subset M_1 \subset \dots \subset \bigcup_{i=0}^{\infty} M_i = M$$

of compact smoothly bounded domains of the form

$$M_i = \{p \in M : \rho(p) \leq c_i\},$$

where  $c_0 < c_1 < \dots$  is an increasing sequence of regular values of the exhaustion function  $\rho$

such that  $\lim_{i \rightarrow \infty} c_i = +\infty$  and there is no critical point of  $\rho$  in  $M \setminus M_0$ . Then each domain  $M_i = \overset{\circ}{M}_i \cup bM_i$  is a connected compact bordered Riemann surface which is an  $\mathcal{O}(M)$ -convex strong deformation retract of  $M_{i+1}$  and of  $M$ . Pick  $x_0 \in \overset{\circ}{M}_0$  and set

$$\tau_i = \text{dist}_{u_0}(x_0, bM_i) > 0 \quad \text{for all } i \in \mathbb{Z}_+.$$

We proceed by induction. The initial step is the isotopy

$$\{u_t^0 := u_t|_{M_0}\}_{t \in [0,1]}.$$

Assume inductively that we have already constructed for some  $i \in \mathbb{Z}_+$  a smooth isotopy

$$u_t^i \in \text{CMI}_*(M_i), \quad t \in [0, 1],$$

satisfying the following conditions:

- $u_0^i = u_0|_{M_i}$ .
- the third component of  $u_t^i$  equals the third component of  $u_t$  restricted to  $M_i$  for all  $t \in [0, 1]$ ,
- $\text{Flux}_{u_t^i}(C) = \text{Flux}_{u_t}(C)$  for every closed curve  $C \subset M_i$  and all  $t \in [0, 1]$ ,
- $\text{dist}_{u_t^i}(x_0, bM_i) > \tau_i - \frac{1}{t}$  for all  $t \in [0, 1]$ ,  $i \in \mathbb{N}$ ,
- $\text{dist}_{u_1^i}(x_0, bM_i) > i$ .

Reasoning as in the proof of Theorem 1.2, Lemma 7.1 ensures that  $\{u_t^i\}_{t \in [0,1]}$  can be approximated arbitrarily closely in the smooth topology on  $[0, 1] \times M_i$  by an isotopy  $\{u_t^{i+1}\}_{t \in [0,1]}$  satisfying the analogous properties over  $M_{i+1}$ . The limit  $\tilde{u}_t = \lim_{i \rightarrow \infty} u_t^i \in \text{CMI}_*(M)$  clearly satisfies Theorem 7.3.  $\square$

## 8. On the topology of the space of conformal minimal immersions

Theorem 1.1 amounts to saying that every path connected component of  $\text{CMI}(M)$  contains a path connected component of  $\mathfrak{RNC}(M)$ . (See Section 2 for the notation.) The proof (see Sections 4 and 5) shows that a nonflat  $u_0 \in \text{CMI}_*(M)$  can be connected by a path in  $\text{CMI}_*(M)$  to some  $u_1 \in \mathfrak{RNC}_*(M)$ . The following natural question appears:

**Problem 8.1.** Are the natural inclusions

$$\mathfrak{RNC}(M) \hookrightarrow \text{CMI}(M), \quad \mathfrak{RNC}_*(M) \hookrightarrow \text{CMI}_*(M),$$

weak (or even strong) homotopy equivalences?

In order to show that the inclusion  $\iota : \mathfrak{RNC}(M) \hookrightarrow \text{CMI}(M)$  is a weak homotopy equivalence (i.e.  $\pi_k(\iota) : \pi_k(\mathfrak{RNC}(M)) \xrightarrow{\cong} \pi_k(\text{CMI}(M))$  is an isomorphism of the homotopy groups for each  $k = 0, 1, \dots$ ), it suffices to prove that  $\iota$  satisfies the following:

**Parametric h-principle.** Given a pair of compact Hausdorff spaces  $Q', Q$  with  $Q' \subset Q$  (it suffices to consider Euclidean compacts, or even just finite polyhedra) and a continuous

map  $F : Q \rightarrow \text{CMI}(M)$  such that  $F(Q') \subset \Re\text{NC}(M)$ , we can deform  $F$  through a homotopy  $F_t : Q \rightarrow \text{CMI}(M)$  ( $t \in [0, 1]$ ) that is fixed on  $Q'$  to a map  $F_1 : Q \rightarrow \Re\text{NC}(M)$ , as illustrated by the following diagram:

$$\begin{array}{ccc} Q' & \longrightarrow & \Re\text{NC}(M) \\ \text{incl} \downarrow & \nearrow F_1 & \downarrow \iota \\ Q & \xrightarrow{F} & \text{CMI}(M). \end{array}$$

See [8, 12] and [10, Chapter 5] for more details.

We now describe a connection to the underlying topological questions. Fix a nowhere vanishing holomorphic 1-form  $\theta$  on  $M$ . (Such a 1-form exists by the Oka–Grauert principle, cf. [10, Theorem 5.3.1, p. 190].) It follows from (2.3) that for every  $u \in \text{CMI}(M)$  we have

$$2\partial u = f\theta,$$

where  $f = (f_1, f_2, f_3) : M \rightarrow \mathfrak{A}^*$  is a holomorphic map satisfying

$$\int_C \Re(f\theta) = \int_C du = 0$$

for any closed curve  $C$  in  $M$ . Furthermore, we have that  $u = \Re F$  for some  $F \in \text{NC}(M)$  if and only if  $\int_C f\theta = 0$  for all closed curves  $C$  in  $M$ .

**Problem 8.2.** Is the map

$$(8.1) \quad \Theta : \text{CMI}(M) \rightarrow \mathcal{O}(M, \mathfrak{A}^*), \quad \Theta(u) = 2\partial u/\theta$$

a weak homotopy equivalence? Does it satisfy the parametric h-principle?

Let  $\iota : \mathcal{O}(M, \mathfrak{A}^*) \hookrightarrow \mathcal{C}(M, \mathfrak{A}^*)$  denote the inclusion of the space of holomorphic maps  $M \rightarrow \mathfrak{A}^*$  into the space of continuous maps. Since  $\mathfrak{A}^*$  is an *Oka manifold* [4, Section 4],  $\iota$  is a weak homotopy equivalence [10, Corollary 5.4.8]. Hence the map

$$\tilde{\Theta} = \iota \circ \Theta : \text{CMI}(M) \rightarrow \mathcal{C}(M, \mathfrak{A}^*)$$

is a weak homotopy equivalence if and only if  $\Theta$  is.

By [4, Theorem 2.6] every  $f_0 \in \mathcal{C}(M, \mathfrak{A}^*)$  can be connected by a path in  $\mathcal{C}(M, \mathfrak{A}^*)$  to a holomorphic map  $f_1 \in \mathcal{O}(M, \mathfrak{A}^*)$  such that  $f_1\theta$  is an exact holomorphic 1-form in  $M$ ; thus  $f_1 = \Theta(\Re F)$  for some  $F \in \text{NC}(M)$ . In particular, we have the following consequence.

**Corollary 8.3.** *Let  $M$  be an open Riemann surface and let  $\theta$  be a nonvanishing holomorphic 1-form on  $M$ . Every connected component of  $\mathcal{C}(M, \mathfrak{A}^*)$  contains a map of the form  $2\partial u/\theta$ , where  $u \in \text{CMI}(M)$ .*

It is natural to ask how many connected components does  $\mathcal{C}(M, \mathfrak{A}^*)$  have. The answer comes from the theory of spin structures on Riemann surfaces; we refer to the preprint [18] by Kusner and Schmitt. Here we give a short self-contained explanation; we wish to thank Jaka Smrekar for his help at this point.

Denote the coordinates on  $\mathbb{C}^3$  by  $z = \xi + \iota\eta$ , with  $\xi, \eta \in \mathbb{R}^3$ , and let

$$\pi : \mathbb{C}^3 = \mathbb{R}^3 \oplus \iota\mathbb{R}^3 \rightarrow \mathbb{R}^3$$

be the projection  $\pi(\xi + \iota\eta) = \xi$ . Then  $\pi : \mathfrak{A}^* \rightarrow \mathbb{R}^3 \setminus \{0\}$  is a real analytic fiber bundle with

circular fibers

$$(8.2) \quad \mathfrak{A} \cap \pi^{-1}(\xi) = \{\xi + \iota\eta \in \mathbb{C}^3 : \xi \cdot \eta = 0, |\xi| = |\eta| \cong \mathbb{S}^1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Let  $\mathbb{S}^2 = \{\xi \in \mathbb{R}^3 : |\xi| = 1\}$  be the unit sphere of  $\mathbb{R}^3$ . Then  $\mathfrak{A}^*$  is homotopy equivalent to  $\mathfrak{A}^* \cap \pi^{-1}(\mathbb{S}^2)$ , and by (8.2) this is the unit circle bundle of the tangent bundle of  $\mathbb{S}^2$ :

$$\mathfrak{A}^* \cap \pi^{-1}(\mathbb{S}^2) = S(T\mathbb{S}^2) \cong \text{SO}(3).$$

It follows that

$$(8.3) \quad \pi_1(\mathfrak{A}^*) \cong \pi_1(\text{SO}(3)) \cong \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}.$$

An open Riemann surface  $M$  has the homotopy type of a finite or countable wedge of circles, one for each generator of  $H_1(M; \mathbb{Z})$ . Fix a pair of points  $p \in M$  and  $q \in \mathfrak{A}^*$ , and let  $\mathcal{C}_*(M, \mathfrak{A}^*)$  denote the space of all continuous maps sending  $p$  to  $q$ . It is easily seen that  $\mathcal{C}(M, \mathfrak{A}^*) \cong \mathcal{C}_*(M, \mathfrak{A}^*) \times \mathfrak{A}^*$ . The space  $\mathcal{C}_*(M, \mathfrak{A}^*)$  is homotopy equivalent to the cartesian product of loop spaces  $\Omega(\mathfrak{A}^*) = \mathcal{C}_*(\mathbb{S}^1, \mathfrak{A}^*)$ , one for each generator of  $H_1(M; \mathbb{Z})$ . Since the connected components of  $\Omega(\mathfrak{A}^*)$  coincide with the elements of the fundamental group  $\pi_1(\mathfrak{A}^*) \cong \mathbb{Z}_2$  (see (8.3)) and  $\mathfrak{A}^*$  is connected, we infer the following.

**Proposition 8.4.** *If  $M$  is an open Riemann surface and  $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^l$  ( $l \in \mathbb{Z}_+ \cup \{\infty\}$ ), then the connected components of  $\mathcal{C}(M, \mathfrak{A}^*)$  are in one-to-one correspondence with the elements of the free abelian group  $(\mathbb{Z}_2)^l$ . Hence each of the spaces  $\text{NC}(M)$  and  $\text{CMI}(M)$  has at least  $2^l$  connected components.*

The last statement follows from Theorem 1.1 and Corollary 8.3.

**Acknowledgement.** The authors wish to thank Jaka Smrekar for his help with the topological matters in Section 8. We also thank the referee for useful suggestions which lead to improved presentation.

## References

- [1] A. Alarcón and I. Fernández, Complete minimal surfaces in  $\mathbb{R}^3$  with a prescribed coordinate function, *Differential Geom. Appl.* **29** (2011), no. 1, S9–S15.
- [2] A. Alarcón, I. Fernández and F. J. López, Complete minimal surfaces and harmonic functions, *Comment. Math. Helv.* **87** (2012), 891–904.
- [3] A. Alarcón, I. Fernández and F. J. López, Harmonic mappings and conformal minimal immersions of Riemann surfaces into  $\mathbb{R}^N$ , *Calc. Var. Partial Differential Equations* **47** (2013), 227–242.
- [4] A. Alarcón and F. Forstnerič, Null curves and directed immersions of open Riemann surfaces, *Invent. Math.* **196** (2014), 733–771.
- [5] A. Alarcón, F. Forstnerič and F. J. López, Embedded conformal minimal surfaces in  $\mathbb{R}^n$ , preprint 2014, <http://arxiv.org/abs/1409.6901>; *Math. Z.* (2015), DOI 10.1007/s00209-015-1586-5.
- [6] A. Alarcón and F. J. López, Minimal surfaces in  $\mathbb{R}^3$  properly projecting into  $\mathbb{R}^2$ , *J. Differential Geom.* **90** (2012), 351–382.
- [7] B. Drinovec Drnovšek and F. Forstnerič, Holomorphic curves in complex spaces, *Duke Math. J.* **139** (2007), 203–254.
- [8] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, *Grad. Stud. Math.* **48**, American Mathematical Society, Providence 2002.
- [9] F. Forstnerič, The Oka principle for multivalued sections of ramified mappings, *Forum Math.* **15** (2003), 309–328.



- [10] *F. Forstnerič*, Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis, *Ergeb. Math. Grenzgeb. (3)* **56**, Springer, Berlin 2011.
- [11] *F. Forstnerič*, Oka manifolds: From Oka to Stein and back. With an appendix by F. Lárusson, *Ann. Fac. Sci. Toulouse Math. (6)* **22** (2013), no. 4, 747–809.
- [12] *M. Gromov*, Partial differential relations, *Ergeb. Math. Grenzgeb. (3)* **9**, Springer, Berlin 1986.
- [13] *M. Gromov*, Oka's principle for holomorphic sections of elliptic bundles, *J. Amer. Math. Soc.* **2** (1989), 851–897.
- [14] *R. C. Gunning* and *R. Narasimhan*, Immersion of open Riemann surfaces, *Math. Ann.* **174** (1967), 103–108.
- [15] *M. Hirsch*, Immersions of manifolds, *Trans. Amer. Math. Soc.* **93** (1959), 242–276.
- [16] *L. Hörmander*, An Introduction to complex analysis in several variables, 3rd ed., North-Holland Math. Library **7**, North Holland, Amsterdam 1990.
- [17] *L. P. Jorge* and *F. Xavier*, A complete minimal surface in  $\mathbb{R}^3$  between two parallel planes, *Ann. of Math. (2)* **112** (1980), 203–206.
- [18] *R. Kusner* and *N. Schmitt*, The spinor representation of surfaces in space, preprint 1996, <http://arxiv.org/abs/dg-ga/9610005>.
- [19] *F. J. López* and *A. Ros*, On embedded complete minimal surfaces of genus zero, *J. Differential Geom.* **33** (1991), 293–300.
- [20] *R. Osserman*, A survey of minimal surfaces, 2nd ed., Dover Publications, New York 1986.
- [21] *S. Smale*, The classification of immersions of spheres in Euclidean spaces, *Ann. of Math. (2)* **69** (1959), 327–344.

---

Antonio Alarcón, Departamento de Geometría y Topología and Instituto de Matemáticas IEMath-GR,  
Universidad de Granada, E-18071 Granada, Spain  
e-mail: alarcon@ugr.es

Franc Forstnerič, Faculty of Mathematics and Physics, University of Ljubljana,  
and Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia  
e-mail: franc.forstneric@fmf.uni-lj.si

Eingegangen 22. August 2014, in revidierter Fassung 28. Juni 2015