



The Oka principle for holomorphic Legendrian curves in \mathbb{C}^{2n+1}

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Abstract Let M be a connected open Riemann surface. We prove that the space $\mathcal{L}(M, \mathbb{C}^{2n+1})$ of all holomorphic Legendrian immersions of M to \mathbb{C}^{2n+1} , $n \geq 1$, endowed with the standard holomorphic contact structure, is weakly homotopy equivalent to the space $\mathcal{C}(M, \mathbb{S}^{4n-1})$ of continuous maps from M to the sphere \mathbb{S}^{4n-1} . If M has finite topological type, then these spaces are homotopy equivalent. We determine the homotopy groups of $\mathcal{L}(M, \mathbb{C}^{2n+1})$ in terms of the homotopy groups of \mathbb{S}^{4n-1} . It follows that $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is $(4n - 3)$ -connected.

Keywords Riemann surface · Legendrian curve · Oka principle · Absolute neighborhood retract

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1 Introduction

It is an interesting and important problem to describe the rough shape of mapping spaces that arise in analysis and geometry. Answering such a question typically amounts to proving a *homotopy principle* (h-principle) to the effect that analytic solutions can be classified by topological data; in particular, a solution exists in the absence of topological obstructions.

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For a survey of the h-principle and its applications, see the monographs by Gromov [15], Eliashberg and Mishachev [9], and Spring [20]. In complex analysis, a synonym for h-principle is *Oka principle*. This is a subject with a long and rich history going back to Oka’s paper [19] in 1939; we refer to the monograph [11].

In this paper, we describe the rough shape of the space $\mathcal{L}(M, \mathbb{C}^{2n+1})$ of holomorphic Legendrian immersions of an open Riemann surface M into the complex Euclidean space \mathbb{C}^{2n+1} , $n \geq 1$, with the standard holomorphic contact structure (1.2). Our main result is that $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is weakly homotopy equivalent to the space $\mathcal{C}(M, \mathbb{S}^{4n-1})$ of continuous maps from M to the $(4n - 1)$ -dimensional sphere, and is homotopy equivalent to it if M has finite topological type; see Corollary 1.2. Analogous results for several other mappings spaces were obtained in [12].

We begin by introducing the relevant spaces of maps. All spaces under consideration are endowed with the compact-open topology, unless otherwise specified.

A holomorphic 1-form α on a complex manifold X of odd dimension $2n + 1 \geq 3$ is said to be a *contact form* if it satisfies the nondegeneracy condition $\alpha \wedge (d\alpha)^n \neq 0$ at every point of X . The model is the complex Euclidean space \mathbb{C}^{2n+1} with coordinates

$$x = (x_1, \dots, x_n) \in \mathbb{C}^n, \quad y = (y_1, \dots, y_n) \in \mathbb{C}^n, \quad z \in \mathbb{C}, \tag{1.1}$$

and α the standard contact form

$$\alpha = dz + \sum_{j=1}^n x_j dy_j. \tag{1.2}$$

By Darboux’s theorem, every holomorphic contact form on a $(2n + 1)$ -dimensional complex manifold is given by (1.2) in some local holomorphic coordinates at each point (see [5, Theorem A.2]; for the smooth case, see e.g. [14, Theorem 2.5.1]).

A smooth map $F: M \rightarrow \mathbb{C}^{2n+1}$ from a smooth manifold M is said to be *Legendrian* if $F^*\alpha = 0$ on M . It is an elementary observation that every smooth Legendrian surface in a 3-dimensional complex contact manifold is a complex curve; see Proposition 1.5.

Let M be a connected open Riemann surface. Denote by $\mathcal{S}(M, \mathbb{C}^{2n})$ the space of all holomorphic immersions $M \rightarrow \mathbb{C}^{2n}$, and consider the closed subspace

$$\mathcal{S}_*(M, \mathbb{C}^{2n}) = \left\{ (x, y) \in \mathcal{S}(M, \mathbb{C}^{2n}) : xdy = \sum_{j=1}^n x_j dy_j \text{ is an exact 1-form on } M \right\}.$$

Elements of $\mathcal{S}_*(M, \mathbb{C}^{2n})$ will be called *exact holomorphic immersions*. Let

$$\mathcal{S}_*(M, \mathbb{C}^{2n}) \xleftarrow{\phi} \mathcal{S}(M, \mathbb{C}^{2n}) \tag{1.3}$$

be the inclusion. Note that the map

$$\mathcal{L}(M, \mathbb{C}^{2n+1}) \longrightarrow \mathcal{S}_*(M, \mathbb{C}^{2n}) \times \mathbb{C},$$

given for a fixed choice of a base point $u_0 \in M$ by

$$\mathcal{L}(M, \mathbb{C}^{2n+1}) \ni (x, y, z) \longmapsto (x, y, z(u_0)) \in \mathcal{S}_*(M, \mathbb{C}^{2n}) \times \mathbb{C}, \tag{1.4}$$

is a homeomorphism. This follows immediately from the formula

$$z(u) = z(u_0) - \int_{u_0}^u xdy, \quad u \in M, \tag{1.5}$$

which holds for any Legendrian immersion $(x, y, z) \in \mathcal{L}(M, \mathbb{C}^{2n+1})$, observing also that the integral $\int_{u_0}^u x dy$ is independent of the choice of a path from u_0 to u (and hence defines a Legendrian immersion by the above formula) if and only if $(x, y) \in \mathcal{I}_*(M, \mathbb{C}^{2n})$. It follows that the projection $\pi : \mathcal{L}(M, \mathbb{C}^{2n+1}) \rightarrow \mathcal{I}_*(M, \mathbb{C}^{2n})$ is a homotopy equivalence.

Fix a nowhere vanishing holomorphic 1-form θ on M ; such exists by the Oka-Grauert principle [11, Theorem 5.3.1]. The specific choice of θ will be irrelevant. For every immersion $\sigma \in \mathcal{I}(M, \mathbb{C}^{2n})$, the map $d\sigma/\theta : M \rightarrow \mathbb{C}^{2n}$ is holomorphic and it avoids the origin $0 \in \mathbb{C}^{2n}$. The correspondence $\sigma \mapsto d\sigma/\theta$ defines a continuous map

$$\varphi : \mathcal{I}(M, \mathbb{C}^{2n}) \longrightarrow \mathcal{O}(M, \mathbb{C}_*^{2n}).$$

Here, $\mathbb{C}_*^{2n} = \mathbb{C}^{2n} \setminus \{0\}$. By [12, Theorem 1.4], φ is a weak homotopy equivalence, and a homotopy equivalence if M has finite topological type.

Let $\iota : \mathcal{O}(M, \mathbb{C}_*^{2n}) \hookrightarrow \mathcal{C}(M, \mathbb{C}_*^{2n})$ denote the inclusion of the space of holomorphic maps into the space of continuous maps. Since \mathbb{C}_*^{2n} is a homogeneous space of the complex Lie group $GL_{2n}(\mathbb{C})$, ι is a weak homotopy equivalence by the Oka-Grauert principle [11, Theorem 5.3.2]; if M has finite topological type, then ι is a homotopy equivalence [17].

Finally, the radial projection $\mathbb{C}_*^{2n} \rightarrow \mathbb{S}^{4n-1}$ onto the unit sphere induces a homotopy equivalence $\tau : \mathcal{C}(M, \mathbb{C}_*^{2n}) \rightarrow \mathcal{C}(M, \mathbb{S}^{4n-1})$.

In summary, all the maps in the following sequence except ϕ are known to be weak homotopy equivalences, and to be homotopy equivalences when M has finite topological type:

$$\begin{aligned} \mathcal{L}(M, \mathbb{C}^{2n+1}) &\xrightarrow{\pi} \mathcal{I}_*(M, \mathbb{C}^{2n}) \xleftarrow{\phi} \mathcal{I}(M, \mathbb{C}^{2n}) \xrightarrow{\varphi} \\ &\xrightarrow{\varphi} \mathcal{O}(M, \mathbb{C}_*^{2n}) \xrightarrow{\iota} \mathcal{C}(M, \mathbb{C}_*^{2n}) \xrightarrow{\tau} \mathcal{C}(M, \mathbb{S}^{4n-1}). \end{aligned} \tag{1.6}$$

The following is our main result.

Theorem 1.1 *For every connected open Riemann surface M , the inclusion*

$$\mathcal{I}_*(M, \mathbb{C}^{2n}) \hookrightarrow \mathcal{I}(M, \mathbb{C}^{2n})$$

of the space of exact holomorphic immersions $M \rightarrow \mathbb{C}^{2n}$, $n \geq 1$, into the space of all holomorphic immersions is a weak homotopy equivalence, and a homotopy equivalence if the surface M has finite topological type.

Since a composition of (weak) homotopy equivalences is again a (weak) homotopy equivalence, Theorem 1.1 implies the following.

Corollary 1.2 *All the maps in the sequence (1.6), and compositions thereof, are weak homotopy equivalences, and homotopy equivalences if M has finite topological type. This holds in particular for the map $\mathcal{L}(M, \mathbb{C}^{2n+1}) \rightarrow \mathcal{C}(M, \mathbb{S}^{4n-1})$.*

The first part of Theorem 1.1 follows immediately from Theorem 4.1, which establishes the parametric Oka principle with approximation for the inclusion (1.3). The same proof gives the parametric Oka principle with approximation for holomorphic Legendrian immersions; see Remark 4.2. The basic case of the latter result is [5, Theorem 1.1]. The parametric case considered here is more demanding, but unavoidable when analysing the homotopy type of these mapping spaces. The second part of Theorem 1.1 is proved in Sect. 5. Our proofs bring together tools from complex analysis and geometry, convex integration theory, and the theory of absolute neighborhood retracts.

The examples in [10] show that Theorem 1.1 and Corollary 1.2 have no analogue for more general holomorphic contact structures on Euclidean spaces; see Remark 1.4. In those examples, the contact structure is Kobayashi hyperbolic, and hence it does not admit any nonconstant Legendrian maps from \mathbb{C} or \mathbb{C}_* .

It was shown in [5] that the space $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is very big from the analytic viewpoint. In particular, every holomorphic Legendrian map $K \rightarrow \mathbb{C}^{2n+1}$ from a (neighborhood of) a compact $\mathcal{O}(M)$ -convex subset $K \subset M$ can be approximated on K by proper holomorphic Legendrian embeddings of M into \mathbb{C}^{2n+1} . Furthermore, every bordered Riemann surface carries a *complete* proper holomorphic Legendrian embedding into the ball of \mathbb{C}^3 , and a complete bounded holomorphic Legendrian embedding in \mathbb{C}^3 such that the image surface is bounded by Jordan curves. (An immersion $F: M \rightarrow \mathbb{R}^n$ is said to be complete if the pull-back of the Euclidean metric on \mathbb{R}^n by F is a complete metric on M .) Analogous results for holomorphic immersions $M \rightarrow \mathbb{C}^n$ ($n \geq 2$), null holomorphic curves in \mathbb{C}^n ($n \geq 3$), and conformal minimal immersions in \mathbb{R}^n ($n \geq 3$) were proved in [1, 2].

On a compact bordered Riemann surface M , we define for every integer $r \geq 1$ the corresponding mapping spaces $\mathcal{L}^r(M, \mathbb{C}^{2n+1})$ and $\mathcal{L}_*^r(M, \mathbb{C}^{2n}) \subset \mathcal{L}^r(M, \mathbb{C}^{2n})$ by considering maps of class $\mathcal{C}^r(M)$ that are holomorphic in the interior $\dot{M} = M \setminus bM$; see Sect. 2.2. These spaces are complex Banach manifolds (see Theorem 2.3), and hence absolute neighborhood retracts, and the corresponding maps in the sequence (1.6) are homotopy equivalences (see Remark 4.2 and Sect. 5).

We will now explicitly describe the homotopy type of $\mathcal{L}(M, \mathbb{C}^{2n+1})$ and determine its homotopy groups in terms of the homotopy groups of the sphere \mathbb{S}^{4n-1} .

A connected open Riemann surface M is homotopy equivalent to a bouquet of circles $\bigvee_{i=1}^{\ell} \mathbb{S}^1$, where $\ell \in \{0, 1, \dots, \infty\}$ is the rank of the free abelian group $H_1(M; \mathbb{Z}) = \mathbb{Z}^{\ell}$. For $\ell = 0$, we take the bouquet to be a point. The surface M has finite topological type if and only if ℓ is finite; then M is biholomorphic to the complement of a finite set of points and closed disks in a compact Riemann surface (see Stout [21]).

The bouquet $\bigvee_{i=1}^{\ell} \mathbb{S}^1$ embeds in M as a deformation retract of M . Hence we have a homotopy equivalence

$$\mathcal{C}(M, \mathbb{S}^{4n-1}) \rightarrow \mathcal{C}\left(\bigvee_{i=1}^{\ell} \mathbb{S}^1, \mathbb{S}^{4n-1}\right).$$

For a space Y , let us denote the space $\mathcal{C}(\bigvee_{i=1}^{\ell} \mathbb{S}^1, Y)$ by $\mathcal{L}_{\ell}Y$. Then \mathcal{L}_1Y is the free loop space $\mathcal{L}Y$ of Y . It is well known that if we choose a base point $s \in \mathbb{S}^1$, then the evaluation map $\mathcal{L}Y \rightarrow Y, \gamma \mapsto \gamma(s)$, is a fibration whose fibre is the loop space ΩY of Y [22, Theorem 10]. More generally, taking s to be the common point of the circles in the bouquet $\bigvee_{i=1}^{\ell} \mathbb{S}^1, \ell \geq 1$, the evaluation map $\mathcal{L}_{\ell}Y \rightarrow Y$ is a fibration whose fibre is $(\Omega Y)^{\ell}$.

Corollary 1.2 now implies the first part of the following result.

Corollary 1.3 *Let M be a connected open Riemann surface with $H_1(M; \mathbb{Z}) = \mathbb{Z}^{\ell}, \ell \in \{0, 1, \dots, \infty\}$. For each $n \geq 1$, the spaces $\mathcal{L}(M, \mathbb{C}^{2n+1})$ and $\mathcal{L}_{\ell}\mathbb{S}^{4n-1}$ are weakly homotopy equivalent. If M has finite topological type, then they are homotopy equivalent.*

It follows that $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is path connected and simply connected, and for each $k \geq 2$,

$$\pi_k(\mathcal{L}(M, \mathbb{C}^{2n+1})) = \pi_k(\mathbb{S}^{4n-1}) \times \pi_{k+1}(\mathbb{S}^{4n-1})^{\ell}.$$

In particular, $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is $(4n - 3)$ -connected.

Proof Recall that $\pi_i(\mathbb{S}^m) = 0$ for all $i < m$, and $\pi_m(\mathbb{S}^m) = \mathbb{Z}$. We must prove the second part of the corollary. It is clear for $\ell = 0$, so let us assume that $\ell \geq 1$. Since $Y = \mathbb{S}^{4n-1}$ is

simply connected, $\mathcal{L}_\ell Y$ is path connected. Consider the long exact sequence of homotopy groups associated to the fibration $\mathcal{L}_\ell Y \rightarrow Y$ with fibre $(\Omega Y)^\ell$,

$$\dots \rightarrow \pi_{k+1}(Y) \rightarrow \pi_k((\Omega Y)^\ell) \rightarrow \pi_k(\mathcal{L}_\ell Y) \rightarrow \pi_k(Y) \rightarrow \dots, \quad k \geq 1,$$

and recall that $\pi_i(\Omega Y) = \pi_{i+1}(Y)$ for all $i \geq 0$. We see that $\pi_1(\mathcal{L}_\ell Y) = 0$. The fibration $\mathcal{L}_\ell Y \rightarrow Y$ has a section defined by taking a point in Y to the map that takes the whole wedge of circles to that point. Let $k \geq 2$. The induced sections of the morphisms $\pi_j(\mathcal{L}_\ell Y) \rightarrow \pi_j(Y)$ for $j = k$ and $j = k + 1$ yield a split short exact sequence of abelian groups

$$0 \rightarrow \pi_k((\Omega Y)^\ell) \rightarrow \pi_k(\mathcal{L}_\ell Y) \rightarrow \pi_k(Y) \rightarrow 0,$$

demonstrating that $\pi_k(\mathcal{L}_\ell Y) = \pi_k(Y) \times \pi_{k+1}(Y)^\ell$. □

Corollary 1.3 shows that holomorphic Legendrian immersions of an open Riemann surface M into \mathbb{C}^{2n+1} have no homotopy invariants. Any two such immersions are homotopic through holomorphic Legendrian immersions, and every loop of Legendrian immersions in $\mathcal{L}(M, \mathbb{C}^3)$ is contractible. The first nontrivial invariant of the space $\mathcal{L}(M, \mathbb{C}^3)$ is its second homotopy group; see Remark 1.4. This is very different from the case of smooth Legendrian knots in a contact 3-manifold, where the basic topological invariants are the rotation number and the Thurston–Bennequin number; see e.g. [7, 8, 13].

Remark 1.4 (a) Theorem 1.1 and Corollary 1.2 fail for certain other complex contact structures on \mathbb{C}^{2n+1} . Indeed, for any $n \geq 1$, the first author has constructed a Kobayashi hyperbolic complex contact form η on \mathbb{C}^{2n+1} [10]. In particular, every holomorphic η -Legendrian map $M \rightarrow \mathbb{C}^{2n+1}$ from $M = \mathbb{C}$ or $M = \mathbb{C}_*$ is constant. Thus, the space $\mathcal{L}_\eta(\mathbb{C}_*, \mathbb{C}^3) = \mathbb{C}^3$ is contractible. On the other hand, for the α -Legendrian maps (where $\alpha = dz + xdy$),

$$\pi_2(\mathcal{L}_\alpha(\mathbb{C}_*, \mathbb{C}^3)) = \pi_2(\mathcal{L}\mathbb{S}^3) = \pi_3(\mathbb{S}^3) = \mathbb{Z}$$

- by Corollary 1.3. As observed in [10], the hyperbolic contact forms η constructed there are isotopic to α through a 1-parameter family of holomorphic contact forms on \mathbb{C}^{2n+1} .
- (b) It is easily seen that Corollary 1.2 fails if we include ramified Legendrian maps in the statement. On the other hand, it was shown in [5, Lemma 4.4 and Theorem 5.1] that every holomorphic Legendrian map of an open Riemann surface to \mathbb{C}^{2n+1} can be approximated uniformly on compacts by holomorphic Legendrian embeddings.

In conclusion, we observe that holomorphic Legendrian curves in a 3-dimensional complex contact manifold are the only smoothly immersed Legendrian surfaces. Simple examples show that this fails in complex contact manifolds of dimension at least 5.

Proposition 1.5 *Let (X, ξ) be a 3-dimensional complex contact manifold. If M is a smooth real surface and $F : M \rightarrow X$ is a smooth Legendrian immersion, then $F(M)$ is an immersed complex curve in X . Furthermore, M admits the structure of a Riemann surface such that $F : M \rightarrow X$ is holomorphic.*

Proof Fix a point $p_0 \in M$. By Darboux’s theorem, there exist local holomorphic coordinates (x, y, z) on a neighborhood of the point $F(p_0) \in X$ in which the contact structure ξ is given by $\alpha = dz + xdy$. Choose smooth local coordinates (u, v) on a neighborhood of p_0 in M and write $F(u, v) = (x(u, v), y(u, v), z(u, v))$. Then the map $\sigma(u, v) = (x(u, v), y(u, v))$ is an immersion. Differentiation of the equation $dz + xdy = 0$ gives $dx(u, v) \wedge dy(u, v) = 0$ which is equivalent to $x_u y_v - x_v y_u = 0$. This means that the vectors $\sigma_u = (x_u, y_u)$ and

$\sigma_v = (x_v, y_v)$ in \mathbb{C}^2 are \mathbb{C} -linearly dependent, and hence they span a complex line. Clearly, this line is the image of the tangent space $T_{(u,v)}M$ by the differential of σ at the point (u, v) . Finally, since the equation $dz = -xdy$ is \mathbb{C} -linear, it follows that $dF_p(T_pM)$ is a complex line in $T_{F(p)}X$ for every point $p \in M$.

Let $J : TX \rightarrow TX$ denote the almost complex structure operator induced by the given complex structure on X . Since $dF_p(T_pM)$ is a J -complex line in $T_{F(p)}X$ for every $p \in M$, there exists a unique almost complex structure $J_0 : TM \rightarrow TM$ such that $dF_p(J_0\eta) = JdF_p(\eta)$ for every $p \in M$ and $\eta \in T_pM$. The surface (M, J_0) is then a Riemann surface and $F : M \rightarrow X$ is a holomorphic Legendrian immersion. \square

2 Preliminaries

2.1 Riemann surfaces and mapping spaces

For $n \geq 1$, we denote by $|\cdot|$ the Euclidean norm on \mathbb{C}^n . Given a topological space K and a map $f : K \rightarrow \mathbb{C}^n$, we define

$$\|f\|_{0,K} := \sup\{|f(u)| : u \in K\}.$$

Let M be an open Riemann surface. We denote by $\mathcal{O}(M)$ the algebra of all holomorphic functions on M . If K is a compact subset of M , then $\mathcal{O}(K)$ is algebra of all holomorphic functions on open neighborhoods of K in M , where we identify any pair of functions that agree on some neighborhood of K . If K is a smoothly bounded compact domain in M , then for any integer $r \geq 0$, we denote by $\mathcal{C}^r(K)$ the algebra of all r times continuously differentiable complex valued functions on K , and by $\mathcal{A}^r(K)$ the subalgebra of $\mathcal{C}^r(M)$ consisting of all functions that are holomorphic in the interior $\mathring{K} = K \setminus bK$ of K . We denote by $\|f\|_{r,K}$ the \mathcal{C}^r norm of a function $f \in \mathcal{C}^r(K)$, where the derivatives are measured with respect to a Riemannian metric on M ; the choice of the metric will not be important. The corresponding notation $\mathcal{O}(M)^n$ and $\mathcal{A}^r(K)^n$ and norms $\|\cdot\|_{r,K}$ are used for maps $f = (f_1, \dots, f_n)$ with values in \mathbb{C}^n , whose component functions f_j belong to the respective function spaces.

A *compact bordered Riemann surface* is a compact Riemann surface M whose nonempty boundary bM consists of finitely many smooth Jordan curves. The interior $\mathring{M} = M \setminus bM$ of a compact Riemann surface is a *bordered Riemann surface*. It is classical [21] that every compact bordered Riemann surface M is conformally equivalent to a smoothly bounded compact domain in an open Riemann surface \tilde{M} , so the function spaces $\mathcal{A}^r(M)$ are defined as above. Note that $\mathcal{A}^r(M)$ is a complex Banach algebra for every $r \geq 0$.

Every bordered Riemann surface M admits smooth closed curves $C_1, \dots, C_\ell \subset \mathring{M}$ forming a basis of the homology group $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$ such that the union $C = \bigcup_{j=1}^\ell C_j$ is *Runge* in M , meaning that the Mergelyan approximation theorem [18] holds: every continuous function on C can be uniformly approximated by functions that are holomorphic on M . When M is connected, this holds if and only if $M \setminus C$ has no relatively compact connected component.

2.2 Spaces of Legendrian immersions

Let $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. On the space \mathbb{C}^{2n+1} we use the coordinates (x, y, z) introduced by (1.1). To simplify the notation, we often write the standard contact form (1.2) on \mathbb{C}^{2n+1} in the form

$$\alpha = dz + xdy, \quad \text{where } xdy = \sum_{j=1}^n x_j dy_j.$$

We identify $\mathbb{C}^{2n}_{(x,y)}$ with the subspace $\{z = 0\} \subset \mathbb{C}^{2n+1}$. Recall (see (1.3)) that $\mathcal{S}(M, \mathbb{C}^n)$ denotes the space of holomorphic immersions $M \rightarrow \mathbb{C}^n$, and $\mathcal{S}_*(M, \mathbb{C}^{2n})$ is the closed subspace of $\mathcal{S}(M, \mathbb{C}^{2n})$ consisting of holomorphic immersions $(x, y): M \rightarrow \mathbb{C}^{2n}$ for which the holomorphic 1-form $x dy$ is exact on M : the *exact holomorphic immersions*. The space $\mathcal{L}(M, \mathbb{C}^{2n+1})$ of holomorphic Legendrian immersions $M \rightarrow \mathbb{C}^{2n+1}$ is homeomorphic to $\mathcal{S}_*(M, \mathbb{C}^{2n}) \times \mathbb{C}$ provided M is connected; see (1.4).

On a compact bordered Riemann surface M with smooth boundary we introduce the analogous mapping spaces for any integer $r \geq 1$:

- $\mathcal{S}^r(M, \mathbb{C}^n)$ is the space of holomorphic immersions $M \rightarrow \mathbb{C}^n$ of class $\mathcal{A}^r(M)$;
- $\mathcal{S}_*^r(M, \mathbb{C}^{2n})$ is the space of holomorphic immersions $(x, y): M \rightarrow \mathbb{C}^{2n}$ of class $\mathcal{A}^r(M)$ for which the holomorphic 1-form $x dy = \sum_{j=1}^n x_j dy_j$ is exact;
- $\mathcal{L}^r(M, \mathbb{C}^{2n+1})$ is the space of immersions $F: M \rightarrow \mathbb{C}^{2n+1}$ of class $\mathcal{A}^r(M)$ such that $F^*\alpha = 0$, that is, F is Legendrian with respect to the contact form (1.2).

As in Sect. 1, when M is connected, the map (1.4) induces a homeomorphism

$$\mathcal{L}^r(M, \mathbb{C}^{2n+1}) \rightarrow \mathcal{S}_*^r(M, \mathbb{C}^{2n}) \times \mathbb{C}.$$

2.3 The period map, dominating sprays, and a local structure theorem

Let M be an open Riemann surface of finite topological type. Let $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$ with $\ell \geq 0$. Pick closed curves $C_1, \dots, C_\ell \subset M$ forming a Runge homology basis (see Sect. 2.1). Let

$$\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_\ell): \mathcal{O}(M)^{2n} \rightarrow \mathbb{C}^\ell$$

be the *period map* whose j -th component is given by

$$\mathcal{P}_j(x, y) = \int_{C_j} x dy, \quad x, y \in \mathcal{O}(M)^n. \tag{2.1}$$

Note that $\mathcal{P}(x, y) = 0$ if and only if the 1-form $x dy = \sum_{i=1}^n x_i dy_i$ is exact, and hence

$$\mathcal{S}_*(M, \mathbb{C}^{2n}) = \{(x, y) \in \mathcal{S}(M, \mathbb{C}^{2n}) : \mathcal{P}(x, y) = 0\}.$$

If M is a compact smoothly bordered Riemann surface, then (2.1) defines a period map

$$\mathcal{P}: \mathcal{A}^r(M)^{2n} \rightarrow \mathbb{C}^\ell, \quad r \in \mathbb{N}, \tag{2.2}$$

and

$$\mathcal{S}_*^r(M, \mathbb{C}^{2n}) = \{(x, y) \in \mathcal{S}^r(M, \mathbb{C}^{2n}) : \mathcal{P}(x, y) = 0\}.$$

The following lemma provides an important tool used in the proof of Theorem 4.1. Clearly, the lemma is vacuous if (and only if) $\ell = 0$, that is, M is the closed disk \mathbb{D} .

Lemma 2.1 *Let M be a compact bordered Riemann surface, and let \mathcal{P} be the period map (2.2) associated to a Runge homology basis of M . Assume that \mathcal{P} is a compact Hausdorff space (a parameter space) and $r \in \mathbb{N}$. Given a continuous map $(x, y): \mathcal{P} \times M \rightarrow \mathbb{C}^{2n}$ such that for every $p \in \mathcal{P}$, the map $(x(p, \cdot), y(p, \cdot)): M \rightarrow \mathbb{C}^{2n}$ is nonconstant, of class*

$\mathcal{A}^r(M)$, and its differential is continuous as a function of $(p, u) \in P \times M$, there exist an integer $N \in \mathbb{N}$ and a continuous map $(\tilde{x}, \tilde{y}): P \times M \times \mathbb{C}^N \rightarrow \mathbb{C}^{2n}$ such that the map $(\tilde{x}(p, \cdot, \cdot), \tilde{y}(p, \cdot, \cdot)): M \times \mathbb{C}^N \rightarrow \mathbb{C}^{2n}$ is of class $\mathcal{A}^r(M \times \mathbb{C}^N)$ for every $p \in P$, its differential is continuous on $P \times M \times \mathbb{C}^N$, and the partial differential

$$\frac{\partial}{\partial \zeta} \Big|_{\zeta=0} \mathcal{P}(\tilde{x}(p, \cdot, \zeta), \tilde{y}(p, \cdot, \zeta)): \mathbb{C}^N \longrightarrow \mathbb{C}^\ell \tag{2.3}$$

is surjective for every $p \in P$. (Here, $\zeta = (\zeta_1, \dots, \zeta_N)$ are coordinates on \mathbb{C}^N .)

A map (\tilde{x}, \tilde{y}) with surjective differential (2.3) is called a *period dominating holomorphic spray* of maps $P \times M \rightarrow \mathbb{C}^{2n}$ with the core $(\tilde{x}(\cdot, \cdot, 0), \tilde{y}(\cdot, \cdot, 0)) = (x, y)$.

Note that continuity of a map $(x, y): P \times M \rightarrow \mathbb{C}^{2n}$, which is holomorphic on the interior \mathring{M} for each $p \in P$, implies continuity of its M -derivative of any order on $P \times \mathring{M}$. Since the period basis for M is supported in \mathring{M} , the lemma holds under this weaker assumption, which already ensures continuity of the period map (2.3). However, we shall use the lemma in the more general situation when M is an admissible set (see Remark 2.2). Since such sets may include arcs, we need the stronger hypothesis that the differential is continuous in all variables.

Proof Without loss of generality, we assume that the Riemann surface M is connected. When $P = \{p\}$ is a singleton, a spray with these properties was obtained in [5, proof of Theorem 3.3]. (We drop P from the notation.) An inspection of that proof shows that there exists a spray of this type, with $N = \ell = \text{rank } H_1(M; \mathbb{Z})$, such that all but one of its component functions \tilde{x}_j, \tilde{y}_j are independent of $\zeta \in \mathbb{C}^\ell$. For example, if y_k is nonconstant, there is a map (\tilde{x}, \tilde{y}) satisfying (2.3) such that for all $u \in M$ and $\zeta \in \mathbb{C}^\ell$ we have

$$\begin{aligned} \tilde{y}(u, \zeta) &= y(u), \\ \tilde{x}_j(u, \zeta) &= x_j(u) \quad \text{for } j \in \{1, \dots, n\} \setminus \{k\}, \\ \tilde{x}_k(u, \zeta) &= x_k(u) + \sum_{j=1}^{\ell} g_j(u) \zeta_j, \end{aligned} \tag{2.4}$$

where the functions $g_1, \dots, g_\ell \in \mathcal{A}^r(M)$ are chosen such that $\int_{C_i} g_j dy_k$ approximates the Kronecker symbol $\delta_{i,j}$ for $i, j = 1, \dots, \ell$. The approximation can be as close as desired. One first constructs smooth functions g_j on the curves C_i in the homology basis such that $\int_{C_i} g_j dy_k = \delta_{i,j}$ and then applies Mergelyan’s theorem to obtain functions in $\mathcal{A}^r(M)$. Similarly, if x_k is nonconstant but y_k is constant, the goal is accomplished by letting $\tilde{y}_k(u, \zeta) = y_k + \sum_{j=1}^{\ell} g_j(u) \zeta_j$ for suitably chosen functions $g_1, \dots, g_\ell \in \mathcal{A}^r(M)$, while the other components of the map are independent of $\zeta \in \mathbb{C}^\ell$.

To obtain the parametric case, we observe that the nonparametric case for a given parameter value $p_0 \in P$ automatically satisfies the domination condition (2.3) for all points p in an open neighborhood $U \subset P$ of p_0 . Since P is compact, finitely many such neighborhoods U_1, \dots, U_m cover P , and it suffices to combine the associated sprays, each with the parameter space \mathbb{C}^ℓ , into a single spray with the parameter space $\mathbb{C}^{m\ell}$. □

Remark 2.2 (Admissible sets) Lemma 2.1 also holds, with the same proof, if M is a compact admissible set in an open Riemann surface \tilde{M} ; see [6, Definition 5.1]. This means that $M = K \cup \Gamma$, where $K = \bigcup_j K_j$ is a union of finitely many pairwise disjoint, compact, smoothly bounded domains K_j in \tilde{M} and $\Gamma = \bigcup_i \Gamma_i$ is a union of finitely many pairwise

disjoint smooth arcs or closed curves that intersect K only in their endpoints, or not at all, and such that their intersections with the boundary bK are transverse. By Mergelyan’s theorem [18], every function $f \in \mathcal{A}^r(M)$, $r \geq 0$, can be approximated in the $\mathcal{C}^r(M)$ -topology by functions holomorphic on a neighborhood of M . If in addition M is Runge ($\mathcal{O}(\tilde{M})$ -convex) in \tilde{M} , which holds if and only if the inclusion map $M \hookrightarrow \tilde{M}$ induces an injective homomorphism $H_1(M; \mathbb{Z}) \hookrightarrow H_1(\tilde{M}; \mathbb{Z})$, then the approximation is possible by functions holomorphic on \tilde{M} .

An application of Lemma 2.1 and the implicit function theorem give the following structure theorem for the spaces $\mathcal{S}_*^r(M, \mathbb{C}^{2n})$ and $\mathcal{L}^r(M, \mathbb{C}^{2n+1})$.

Theorem 2.3 *Let M be a compact bordered Riemann surface. For every $r \geq 1$, the spaces $\mathcal{S}_*^r(M, \mathbb{C}^{2n})$ and $\mathcal{L}^r(M, \mathbb{C}^{2n+1})$ are complex Banach manifolds.*

Proof In view of the homeomorphism $\mathcal{L}^r(M, \mathbb{C}^{2n+1}) \rightarrow \mathcal{S}_*^r(M, \mathbb{C}^{2n}) \times \mathbb{C}$ induced by the map (1.4), it suffices to show that $\mathcal{S}_*^r(M, \mathbb{C}^{2n})$ is a closed complex Banach submanifold of $\mathcal{S}^r(M, \mathbb{C}^{2n})$, the latter being an open subset of the complex Banach space $\mathcal{A}^r(M)^{2n}$.

Obviously, $\mathcal{S}_*^r(M, \mathbb{C}^{2n}) = \{\sigma \in \mathcal{S}^r(M, \mathbb{C}^{2n}) : \mathcal{P}(\sigma) = 0\}$ is a closed subset of $\mathcal{S}^r(M, \mathbb{C}^{2n})$. The period map $\mathcal{P}: \mathcal{A}^r(M)^{2n} \rightarrow \mathbb{C}^\ell$ is holomorphic. Lemma 2.1 (with P a singleton) says that \mathcal{P} has maximal rank ℓ at each point $\sigma \in \mathcal{A}^r(M)^{2n}$ that represents a nonconstant map. Hence, the conclusion follows from the implicit function theorem. \square

It is easily seen that the tangent space to the submanifold $\mathcal{S}_*^r(M, \mathbb{C}^{2n})$ of $\mathcal{S}^r(M, \mathbb{C}^{2n})$ at the point $\sigma_0 = (x_0, y_0) \in \mathcal{S}_*^r(M, \mathbb{C}^{2n})$ equals

$$T_{\sigma_0} \mathcal{S}_*^r(M, \mathbb{C}^{2n}) = \left\{ \sigma = (x, y) \in \mathcal{A}^r(M)^{2n} : \int_{C_j} x dy_0 + x_0 dy = 0, \quad j = 1, \dots, l \right\},$$

where the curves C_1, \dots, C_l form a basis of $H_1(M; \mathbb{Z})$.

3 An application of the convex integration lemma

In this section, we establish a key technical result, Lemma 3.2, which will be used in the proof of Theorem 4.1 in order to extend families of Legendrian immersions across a smooth arc attached to a compact smoothly bounded domain in a Riemann surface.

Let P be a compact Hausdorff space; it will serve as the parameter space. Let $\mathcal{C}^{0,1}(P \times [0, 1])$ denote the space of all continuous functions $f: P \times [0, 1] \rightarrow \mathbb{C}$, considered as a family of paths $f_p = f(p, \cdot): [0, 1] \rightarrow \mathbb{C}$ depending continuously on $p \in P$, whose derivative $\dot{f}_p(s) = df_p(s)/ds$ is also continuous in both variables $(p, s) \in P \times [0, 1]$. The analogous notation

$$\mathcal{C}^{0,1}(P \times [0, 1], \mathbb{C}^n) = \mathcal{C}^{0,1}(P \times [0, 1])^n$$

is used for maps $f = (f_1, \dots, f_n): P \times [0, 1] \rightarrow \mathbb{C}^n$.

We shall need the following lemma.

Lemma 3.1 *Let $Q \subset P$ be compact Hausdorff spaces, and let $f \in \mathcal{C}^{0,1}(P \times [0, 1])$ and $h \in \mathcal{C}(P \times [0, 1])$ be complex valued functions, with h nowhere vanishing. Write $f_p = f(p, \cdot)$ and similarly for h . Let $b: P \rightarrow \mathbb{C}$ be a continuous function such that*

$$b(p) = \int_0^1 f_p(s)h_p(s) ds, \quad p \in Q.$$

There is a homotopy $f^t \in \mathcal{C}^{0,1}(P \times [0, 1])$ ($t \in [0, 1]$) satisfying the following conditions:

- (i) $f_p^t = f_p$ for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$;
- (ii) $f_p^t(s) = f_p(s)$ and $\dot{f}_p^t(s) = \dot{f}_p(s)$ for $s = 0, 1$ and for all $(p, t) \in P \times [0, 1]$;
- (iii) $\int_0^1 f_p^1(s)h_p(s)ds = b(p)$ for all $p \in P$.

Proof This is a parametric version of Gromov’s one-dimensional *convex integration lemma* [16, Lemma 2.1.7]. The basic version of Gromov’s lemma says that for any open connected set Ω in a Euclidean space \mathbb{R}^n (or in a Banach space), the set of integrals $\int_0^1 f(s)ds$ over all paths $f: [0, 1] \rightarrow \Omega$, with fixed endpoints $f(0)$ and $f(1)$ in Ω , equals the convex hull of Ω . It is a trivial matter to adapt it to arcs of class \mathcal{C}^1 with the matching conditions for the derivatives at the endpoints of $[0, 1]$. For the parametric version we refer to [20, Theorem 3.4]. The nowhere vanishing function h plays the role of a weight; it would suffice to assume that h is not identically zero and work on the corresponding subinterval. \square

In preparation for the next lemma, we need some additional notation. Given $z = (z_1, \dots, z_n)$, $w = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $zw = \sum_{j=1}^n z_j w_j$. We denote by

$$\mathcal{S}(P \times [0, 1], \mathbb{C}^n) \subset \mathcal{C}^{0,1}(P \times [0, 1], \mathbb{C}^n) \tag{3.1}$$

the set of all $f \in \mathcal{C}^{0,1}(P \times [0, 1], \mathbb{C}^n)$ for which the derivative $\dot{f}_p(s) = df_p(s)/ds \in \mathbb{C}^n$ is nowhere vanishing on $(p, s) \in P \times [0, 1]$. We think of $f \in \mathcal{S}(P \times [0, 1], \mathbb{C}^n)$ as a family of immersed arcs $f_p: [0, 1] \rightarrow \mathbb{C}^n$ depending continuously on the parameter $p \in P$.

The following is the main technical lemma used in the proof of Theorem 4.1.

Lemma 3.2 *Let $Q \subset P$ be compact Hausdorff spaces, let $\xi = (f, g) \in \mathcal{S}(P \times [0, 1], \mathbb{C}^{2n})$ with $f, g \in \mathcal{C}^{0,1}(P \times [0, 1])^n$, and let $\beta: P \rightarrow \mathbb{C}$ be a continuous function such that*

$$\beta(p) = \int_0^1 f_p(s)\dot{g}_p(s)ds, \quad p \in Q. \tag{3.2}$$

Then there exists a homotopy $\xi^t = (f^t, g^t) \in \mathcal{S}(P \times [0, 1], \mathbb{C}^{2n})$ ($t \in [0, 1]$) satisfying the following conditions:

- (a) $\xi_p^t = \xi_p$ for $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$;
- (b) $\xi_p^t(s) = \xi_p(s)$ and $\dot{\xi}_p^t(s) = \dot{\xi}_p(s)$ for $s = 0, 1$ and $(p, t) \in P \times [0, 1]$;
- (c) $\int_0^1 f_p^1(s)\dot{g}_p^1(s)ds = \beta(p)$ for $p \in P$.

In [12, Lemma 3.1] we give more precise analogues of Lemmas 3.1 and 3.2 by controlling the integrals in (iii) and (c) for all $t \in [0, 1]$. This can be proved here as well, but is not needed for the application in the present paper.

Proof Since the derivative $\dot{\xi}_p(s) = (\dot{\xi}_{p,1}(s), \dots, \dot{\xi}_{p,2n}(s)) \in \mathbb{C}^{2n}$ is nowhere vanishing on $(p, s) \in P \times [0, 1]$ and P is compact, an elementary argument gives finitely many pairs of compact sets $U_j \subset V_j$ in P ($j = 1, \dots, m$), with $U_j \subset \mathring{V}_j$ and $\bigcup_{j=1}^m U_j = P$, and pairwise disjoint closed segments I_1, \dots, I_m contained in $[0, 1]$ such that for every $j = 1, \dots, m$, there exists an index $k = k(j) \in \{1, 2, \dots, 2n\}$ such that

$$\dot{\xi}_{p,k}(s) \neq 0 \quad \text{for all } s \in I_j \text{ and } p \in V_j. \tag{3.3}$$

The proof of the lemma proceeds by a finite induction on $j = 1, \dots, m$. The desired homotopy is obtained as a composition of m homotopies, each supported on one of the segments I_1, \dots, I_m . We explain the initial step; the subsequent steps are analogous.

Thus, let $j = 1$ and let $k = k(1) \in \{1, 2, \dots, 2n\}$ be such that (3.3) holds for $j = 1$. Suppose first that $k \in \{n + 1, \dots, 2n\}$. Write $k = n + l$ with $l \in \{1, \dots, n\}$. Recall that $\xi = (f, g)$ where $f, g \in \mathcal{C}^{0,1}(P \times [0, 1])^n$. Then (3.3) means that the function $\dot{g}_{p,l}$ is nowhere vanishing on I_1 for all $p \in V_1$. Let us define the function $b: P \rightarrow \mathbb{C}$ by

$$b(p) = \beta(p) - \int_{[0,1] \setminus I_1} f_{p,l}(s) \dot{g}_{p,l}(s) ds - \int_0^1 \sum_{\substack{i=1 \\ i \neq l}}^n f_{p,i}(s) \dot{g}_{p,i}(s) ds. \tag{3.4}$$

In view of (3.2) we have that

$$b(p) = \int_{I_1} f_{p,l}(s) \dot{g}_{p,l}(s) ds, \quad p \in Q.$$

We now apply Lemma 3.1 with $Q \subset P$ replaced by the pair of parameter sets $V_1 \cap Q \subset V_1$, the interval $[0, 1]$ replaced by the segment I_1 , with the functions on I_1 given by

$$f_p = f_{p,l}, \quad h_p = \dot{g}_{p,l} \quad \text{for } p \in V_1,$$

and with the function b given by (3.4). (When applying Lemma 3.1, we pay attention to the matching condition (ii) at the endpoints of the interval I_1). This gives a homotopy $f_{p,l}^t \in \mathcal{C}^{0,1}(V_1 \times [0, 1])$ ($t \in [0, 1]$) satisfying the following conditions:

- (a') $f_{p,l}^t = f_{p,l}$ for all $(p, t) \in (V_1 \times \{0\}) \cup ((Q \cap V_1) \times [0, 1])$;
- (b') $f_{p,l}^t(s) = f_{p,l}(s)$ for all $s \in [0, 1] \setminus I_1$ and $(p, t) \in V_1 \times [0, 1]$;
- (c') $\int_{I_1} f_{p,l}^t(s) \dot{g}_{p,l}(s) ds = b(p)$ for all $p \in V_1$.

Condition (b') means that the deformation is supported on the segment I_1 .

Let $\xi_p^t = (f_p^t, g_p): [0, 1] \rightarrow \mathbb{C}^{2n}$ ($t \in [0, 1]$) denote the homotopy whose l -th component equals $f_{p,l}^t$ and whose other components agree with the corresponding components of ξ_p . Note that ξ_p^t agrees with ξ_p on $[0, 1] \setminus I_1$ for all $t \in [0, 1]$ and $p \in V_1$, and hence is an immersion (since its component $\dot{g}_{p,l}$ is nowhere vanishing on I_1 and $\xi_p^t = \xi_p$ on $[0, 1] \setminus I_1$). Clearly, ξ_p^t satisfies conditions (a) and (b) in Lemma 3.2 for $(p, t) \in (V_1 \times \{0\}) \cup ((Q \cap V_1) \times [0, 1])$, and it satisfies condition (c) for all $p \in V_1$ in view of the definition (3.4) of the function b .

Pick a continuous function $\chi: P \rightarrow [0, 1]$ such that $\chi = 1$ on U_1 and $\text{supp } \chi \subset \mathring{V}_1$. Replacing f_p^t by $f_p^{\chi(p)t}$ and ξ_p^t by $\xi_p^{\chi(p)t}$ yields a homotopy, defined for all $p \in P$, which satisfies conditions (a) and (b), and it satisfies condition (c) for $p \in U_1$.

This concludes the first step if $k(1) \in \{n + 1, \dots, 2n\}$. If on the other hand $k = k(1) \in \{1, \dots, n\}$, we apply the same argument with the roles of the components reversed, using the integration by parts formula

$$\int_0^1 f_{p,k}(s) \dot{g}_{p,k}(s) ds = f_{p,k}(1)g_{p,k}(1) - f_{p,k}(0)g_{p,k}(0) - \int_0^1 g_{p,k}(s) \dot{f}_{p,k}(s) ds.$$

In this case, the assumption is that $\dot{f}_{p,k}(s) \neq 0$ for all $s \in I_1$ for $p \in V_1$. The same argument as above gives a homotopy $g_{p,k}^t$, supported on I_1 , which achieves condition (c) for all $p \in U_1$. As before, the other components of the map are kept fixed.

This concludes the first step of the induction.

In the second step with $j = 2$, we take as our datum the map $\xi^1 \in \mathcal{S}(P \times [0, 1], \mathbb{C}^{2n})$ (the final map at $t = 1$ in the homotopy obtained in step 1). By following the proof of step 1 with the pair of parameter sets $Q_2 = Q \cup U_1 \subset P$, we find a family of immersions

$$\xi_p^{1,t} = (f_p^{1,t}, g_p^{1,t}): [0, 1] \rightarrow \mathbb{C}^{2n}, \quad (p, t) \in P \times [0, 1],$$

satisfying the following conditions:

- $\xi_p^{1,t} = \xi_p^1$ for $(p, t) \in (P \times \{0\}) \cup (Q_2 \times [0, 1])$;
- $\xi_p^{1,t}(s) = \xi_p^1(s)$ for all $s \in [0, 1] \setminus I_2$ and $(p, t) \in P \times [0, 1]$;
- $\int_0^1 f_p^{1,1}(s) \dot{g}_p^{1,1}(s) ds = \beta(p)$ for all $p \in U_1 \cup U_2$.

Since the deformation $\xi_p^{1,t}$ is supported on I_2 which is disjoint from I_1 , it does not destroy the immersion property of the individual maps $[0, 1] \rightarrow \mathbb{C}^{2n}$ in the family. Also, since the deformation is fixed for $p \in Q_2 = Q \cup U_1$, it does not change the values of the integrals in (c) for $p \in Q_2$, and in addition it achieves the correct values for points $p \in U_2$.

We now take $\xi^2 = \xi^{1,1} \in \mathcal{S}(P \times [0, 1], \mathbb{C}^{2n})$ as the datum in step 3, let $Q_3 = Q_2 \cup U_2$, and proceed as before. After m steps of this kind, the proof is complete. □

4 A parametric Oka principle for Legendrian immersions

Let M be an open Riemann surface. In this section we prove the parametric Oka principle with approximation for the inclusion $\mathcal{S}_*(M, \mathbb{C}^{2n}) \hookrightarrow \mathcal{S}(M, \mathbb{C}^{2n})$ in Theorem 1.1.

Let P be a compact Hausdorff space. We introduce the following mapping spaces:

$$\begin{aligned} \mathcal{S}(P \times M, \mathbb{C}^{2n}) &= \{\sigma \in \mathcal{C}(P \times M, \mathbb{C}^{2n}) : \sigma_p \in \mathcal{S}(M, \mathbb{C}^{2n}) \text{ for every } p \in P\}; \\ \mathcal{S}_*(P \times M, \mathbb{C}^{2n}) &= \{\sigma \in \mathcal{S}(P \times M, \mathbb{C}^{2n}) : \sigma_p \in \mathcal{S}_*(M, \mathbb{C}^{2n}) \text{ for every } p \in P\}. \end{aligned}$$

Here, $\sigma_p = \sigma(p, \cdot) : M \rightarrow \mathbb{C}^{2n}$. Given a compact set $K \subset M$, we write

$$\|\sigma\|_{1, P \times K} = \sup_{x \in K} |\sigma_p(x)| + \sup_{x \in K} |d\sigma_p(x)|$$

where the norm $|d\sigma_p|$ of the differential is measured with respect to a fixed Hermitian metric on TM (whose precise choice will not be important) and the Euclidean norm on \mathbb{C}^{2n} .

Theorem 4.1 *Assume that M is an open Riemann surface, $Q \subset P$ are compact Hausdorff spaces, $D \Subset M$ is a smoothly bounded domain whose closure \bar{D} is $\mathcal{O}(M)$ -convex, and $\sigma = (x, y) \in \mathcal{S}(P \times M, \mathbb{C}^{2n})$ ($n \geq 1$) satisfies the following two conditions:*

- (a) $\sigma|_{Q \times M} \in \mathcal{S}_*(Q \times M, \mathbb{C}^{2n})$;
- (b) *there is an open set $U \subset M$, with $\bar{D} \subset U$, such that $\sigma|_{P \times U} \in \mathcal{S}_*(P \times U, \mathbb{C}^{2n})$.*

Given $\epsilon > 0$, there is a homotopy $\sigma^t \in \mathcal{S}(P \times M, \mathbb{C}^{2n})$ ($t \in [0, 1]$) satisfying the following conditions:

- (1) $\sigma_p^t = \sigma_p$ for every $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$;
- (2) $\sigma^t|_{P \times D} \in \mathcal{S}_*(P \times D, \mathbb{C}^{2n})$ for every $t \in [0, 1]$;
- (3) $\|\sigma^t - \sigma\|_{1, P \times \bar{D}} < \epsilon$ for every $t \in [0, 1]$;
- (4) $\sigma^1 \in \mathcal{S}_*(P \times M, \mathbb{C}^{2n})$.

If a continuous map $\varphi : X \rightarrow Y$ satisfies the parametric h-principle (without approximation), then φ is a weak homotopy equivalence. Hence, the first part of Theorem 1.1 is an immediate corollary of Theorem 4.1.

Remark 4.2 (a) The proof of Theorem 4.1 gives the analogous result for a compact bordered Riemann surface M ; in this case, the proof is completed in finitely many steps.

(b) The proof of Theorem 4.1 also gives the parametric Oka principle with approximation for Legendrian immersions. However, a minor difference in the proof is explained in the paragraph following the proof of Theorem 4.1. It has to do with the fact that the map $\mathcal{L}(M, \mathbb{C}^{2n+1}) \rightarrow \mathcal{I}_*(M, \mathbb{C}^{2n}) \times \mathbb{C}$ (see (1.4)) is a homeomorphism only when M is connected. Hence, when extending an exact holomorphic immersion $\sigma = (x, y)$ (the projection of a Legendrian immersion (x, y, z)) across a smooth arc E connecting a pair of disjoint domains in M , we must ensure that the integral of the 1-form $x dy$ on E equals the difference of the values of the last component z at the respective endpoints of the arc; in view of (1.5), this ensures the correct extension of the z -component.

Proof of Theorem 4.1 Pick a smooth strongly subharmonic Morse exhaustion function $\rho: M \rightarrow \mathbb{R}$ and exhaust M by sublevel sets

$$D_j = \{u \in M : \rho(u) < c_j\}, \quad j \in \mathbb{N},$$

where $c_1 < c_2 < c_3 < \dots$ is an increasing sequence of regular values of ρ chosen such that $\lim_{j \rightarrow \infty} c_j = \infty$. We may assume that each interval $[c_j, c_{j+1}]$ contains at most one critical value of the function ρ , and that D_1 coincides with the given domain D in Theorem 4.1. Let $U_1 = U \supset \bar{D}_1$ be the open neighborhood of \bar{D}_1 as in the theorem.

To begin the induction, set $\epsilon_0 = \epsilon$ and

$$\sigma^{t,1} = \sigma|_{P \times U_1} \in \mathcal{I}_*(P \times U_1, \mathbb{C}^{2n}), \quad t \in [0, 1].$$

We shall inductively find a sequence of open sets $U_j \supset \bar{D}_j$ in M , homotopies

$$\sigma^{t,j} \in \mathcal{I}(P \times U_j, \mathbb{C}^{2n}), \quad t \in [0, 1], \quad j \in \mathbb{N}$$

and numbers $\epsilon_j > 0$ satisfying the following conditions for $j = 1, 2, 3, \dots$:

- (a_j) $\sigma_p^{t,j} = \sigma_p|_{U_j}$ for every $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$;
- (b_j) $\sigma^{t,j}|_{P \times D_1} \in \mathcal{I}_*(P \times D_1, \mathbb{C}^{2n})$ for every $t \in [0, 1]$;
- (c_j) $\|\sigma^{t,j} - \sigma^{t,j-1}\|_{1, P \times \bar{D}_{j-1}} < \epsilon_j$ for every $t \in [0, 1]$;
- (d_j) $\sigma^{1,j}|_{P \times D_j} \in \mathcal{I}_*(P \times D_j, \mathbb{C}^{2n})$;
- (e_j) $\epsilon_j < \epsilon_{j-1}/2$;
- (f_j) If $\tilde{\sigma}^t: P \times \bar{D}_{j-1} \rightarrow \mathbb{C}^{2n}$ satisfies $\|\tilde{\sigma}^t - \sigma^{t,j-1}\|_{1, P \times \bar{D}_{j-1}} < 2\epsilon_j$ for every $t \in [0, 1]$, then $\tilde{\sigma}^t(p, \cdot): \bar{D}_{j-1} \rightarrow \mathbb{C}^{2n}$ is an immersion for every $p \in P$ and $t \in [0, 1]$.

Conditions (a₁), (b₁) and (d₁) hold by the definition of $\sigma^{t,1}$, (e₁) is fulfilled by choosing $0 < \epsilon_1 < \epsilon_0/2$, while (c₁) and (f₁) are vacuous.

Assume for a moment that sequences with these properties exist. Conditions (c_j), (e_j) and (f_j) ensure that the sequence $(\sigma^{t,j})_{j \in \mathbb{N}}$ converges to a limit

$$\sigma^t = \lim_{j \rightarrow \infty} \sigma^{t,j}: P \times M \longrightarrow \mathbb{C}^{2n}, \quad t \in [0, 1]$$

such that $\sigma_p^t: M \rightarrow \mathbb{C}^{2n}$ is a holomorphic immersion for every $p \in P$ and $t \in [0, 1]$ and (3) holds. Condition (a_j) ensures that all homotopies $\sigma^{t,j}$ are fixed on the parameter set $(P \times \{0\}) \cup (Q \times [0, 1])$, which gives (1). Condition (b_j) shows that $\sigma_p^t: D \rightarrow \mathbb{C}^{2n}$ is an exact holomorphic immersion for every $p \in P$ and $t \in [0, 1]$, so (2) holds. Condition (d_j) shows that $\sigma_p^1: M \rightarrow \mathbb{C}^{2n}$ is an exact holomorphic immersion for every $p \in P$, which gives (4). This shows that the theorem holds if we can construct such a sequence of homotopies.

We now explain the induction. Assume that the quantities satisfying the above conditions have been found up to an index $j \in \mathbb{N}$. Then, conditions (e_{j+1}) and (f_{j+1}) hold provided

that the number $\epsilon_{j+1} > 0$ is chosen small enough; fix such a number. We shall now explain how to obtain $\sigma^{t,j+1}$ and U_{j+1} satisfying conditions $(a_{j+1})-(d_{j+1})$. We distinguish two topologically different cases: (a) the noncritical case, and (b) the critical case.

(a) *The noncritical case:* ρ has no critical values in $[c_j, c_{j+1}]$. In this case, \bar{D}_j is a deformation retract of \bar{D}_{j+1} . (In the critical case considered below, we use the noncritical case also for certain noncritical pairs of sets $K \subset L$ defined by another strongly subharmonic function.)

Pick a Runge homology basis $\mathcal{B} = \{\gamma_i\}_{i=1}^l$ for $H_1(D_j; \mathbb{Z})$, that is, such that the union of supports $\bigcup_{i=1}^l |\gamma_i|$ is $\mathcal{O}(D_j)$ -convex. Let \mathcal{P} denote the associated period map (2.1):

$$\mathcal{P}(\sigma) = \left(\int_{\gamma_i} x dy \right)_{i=1, \dots, l} \in \mathbb{C}^l, \quad \sigma = (x, y) \in \mathcal{S}(D_j, \mathbb{C}^{2n}).$$

Note that the pair $(\mathcal{B}, \mathcal{P})$ also applies to the domain D_{j+1} since \bar{D}_j is a deformation retract of \bar{D}_{j+1} . Let $\zeta = (\zeta_1, \dots, \zeta_N)$ denote the coordinates on \mathbb{C}^N . Shrinking $U_j \supset \bar{D}_j$ if necessary, Lemma 2.1, applied with the parameter space $P' = P \times [0, 1]$, gives an integer $N \in \mathbb{N}$ and a spray

$$\tilde{\sigma}^t = (\tilde{x}^t, \tilde{y}^t): P \times U_j \times \mathbb{C}^N \rightarrow \mathbb{C}^{2n}, \quad t \in [0, 1],$$

such that the map $\tilde{\sigma}_p^t = \tilde{\sigma}^t(p, \cdot, \cdot): U_j \times \mathbb{C}^N \rightarrow \mathbb{C}^{2n}$ satisfies the following conditions:

- (i) $\tilde{\sigma}_p^t$ is holomorphic on $U_j \times \mathbb{C}^N$ for every $(p, t) \in P \times [0, 1]$;
- (ii) $\tilde{\sigma}_p^t(\cdot, 0) = \sigma_p^t(\cdot, 0)$ at $\zeta = 0 \in \mathbb{C}^N$ for every $(p, t) \in P \times [0, 1]$;
- (iii) the partial differential

$$\left. \frac{\partial}{\partial \zeta} \right|_{\zeta=0} \mathcal{P}(\tilde{\sigma}_p^t(\cdot, \zeta)): \mathbb{C}^N \longrightarrow \mathbb{C}^\ell \tag{4.1}$$

is surjective for every $(p, t) \in P \times [0, 1]$.

Furthermore, in view of Mergelyan’s theorem [18], the functions g_j used in the construction of $\tilde{\sigma}^t$ (see (2.4)) can be chosen holomorphic on M . Since the spray $\tilde{\sigma}^t$ is linear in $\zeta \in \mathbb{C}^N$ and the core $\tilde{\sigma}_p^t(\cdot, 0) = \sigma_p^t(\cdot, 0)$ is holomorphic on M for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$, $\tilde{\sigma}_p^t$ is holomorphic on $M \times \mathbb{C}^N$ for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$. Pick an open relatively compact neighborhood $U_{j+1} \Subset M$ of \bar{D}_{j+1} which deformation retracts onto \bar{D}_{j+1} . Since the map $\tilde{\sigma}_p^t(\cdot, 0) = \sigma_p^t(\cdot, 0)$ is an immersion on the respective domain for every $(p, t) \in P \times [0, 1]$, we can shrink U_j slightly around \bar{D}_j and choose a ball $B \subset \mathbb{C}^N$ around the origin such that

- (iv) $\tilde{\sigma}_p^t(\cdot, \zeta): U_j \rightarrow \mathbb{C}^{2n}$ is an immersion for every $(p, t) \in P \times [0, 1]$ and $\zeta \in \bar{B}$, and
- (v) $\tilde{\sigma}_p^t(\cdot, \zeta): \bar{U}_{j+1} \rightarrow \mathbb{C}^{2n}$ is an immersion for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$ and $\zeta \in \bar{B}$.

Claim: $\tilde{\sigma}^t$ can be approximated as closely as desired in the \mathcal{C}^1 norm on $\bar{D}_j \times \bar{B}$, and uniformly in the parameters $(p, t) \in P \times [0, 1]$, by a homotopy

$$\tau^t: P \times U_{j+1} \times B \rightarrow \mathbb{C}^{2n}, \quad t \in [0, 1],$$

satisfying conditions (i)–(v) above and also the following two conditions:

- $\tau^t(p, \cdot, \zeta): U_{j+1} \rightarrow \mathbb{C}^{2n}$ is a holomorphic immersion for every $(p, t) \in P \times [0, 1]$ and $\zeta \in B$, and
- $\tau^t(p, \cdot, \cdot) = \tilde{\sigma}^t(p, \cdot, \cdot)$ for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$.

Proof of the claim Such τ^t can be found by following the noncritical case in [12, proof of Theorem 5.3] when the cone A equals \mathbb{C}^{2n} . The only difference is that, in the present situation, the maps $\tilde{\sigma}_p^t$ depend holomorphically on the additional complex parameter $\zeta \in B \subset \mathbb{C}^N$. We outline the main steps and refer to the cited source for the details.

Fix a nowhere vanishing holomorphic 1-form θ on M . Let d denote the exterior differential on M . Consider the family of holomorphic maps

$$\tilde{\phi}_p^t(\cdot, \zeta) = d\tilde{\sigma}_p^t(\cdot, \zeta)/\theta : U_j \rightarrow \mathbb{C}_*^{2n} \tag{4.2}$$

for $(p, t) \in P \times [0, 1]$ and $\zeta \in \bar{B}$. Their ranges avoid the origin since the maps $\tilde{\sigma}_p^t(\cdot, \zeta)$ are immersions by condition (iv). Furthermore, for each $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$ and $\zeta \in \bar{B}$, the map $\tilde{\phi}_p^t(\cdot, \zeta) : \bar{U}_{j+1} \rightarrow \mathbb{C}_*^{2n}$ is holomorphic on \bar{U}_{j+1} in view of condition (v).

Let \mathcal{Q} denote the period map defined for any map $\phi : D_j \rightarrow \mathbb{C}^{2n}$ by

$$\mathcal{Q}(\phi) = \left(\int_{C_i} \phi \theta \right)_{i=1, \dots, l} \in (\mathbb{C}^{2n})^l.$$

Here, $\{C_i\}_{i=1}^l$ is a Runge homology basis of $H_1(D_j; \mathbb{Z})$. We embed the family of maps (4.2) as the core of a spray $\phi_p^t(\cdot, \zeta, w)$ (that is, $\phi_p^t(\cdot, \zeta, 0) = \tilde{\phi}_p^t(\cdot, \zeta)$), depending holomorphically on another set of parameters $w \in \mathbb{C}^{N'}$ for some integer $N' \in \mathbb{N}$, such that the partial differential

$$\frac{\partial}{\partial w} \Big|_{w=0} \mathcal{Q}(\phi_p^t(\cdot, \zeta, w)) : \mathbb{C}^{N'} \rightarrow (\mathbb{C}^{2n})^l$$

is surjective for every $(p, t) \in P \times [0, 1]$ and $\zeta \in B$. Such \mathcal{Q} -period dominating sprays were constructed in [4, Lemma 5.1]; see also [3, Lemma 3.6] for the parametric case.

Fix a ball $B' \subset \mathbb{C}^{N'}$ centered at the origin. Since \mathbb{C}_*^{2n} is an Oka manifold, the parametric Oka principle with approximation [11, Theorem 5.4.4] shows that we can approximate the family of holomorphic maps $\phi_p^t : U_j \times \bar{B} \times \bar{B}' \rightarrow \mathbb{C}_*^{2n}$ in the \mathcal{C}^r topology on $\bar{D}_j \times B \times B'$ by a continuous family of holomorphic maps

$$\psi_p^t : U_{j+1} \times B \times B' \rightarrow \mathbb{C}_*^{2n}, \quad (p, t) \in P \times [0, 1],$$

such that $\psi_p^t(\cdot, \zeta, w) = \phi_p^t(\cdot, \zeta, w)$ for all $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$ and $(\zeta, w) \in B \times B'$. Assuming that the approximation is close enough, the implicit function theorem gives a continuous function $w = w(p, t, \zeta)$ on $P \times [0, 1] \times \bar{B}$ with values in $\mathbb{C}^{N'}$ and close to 0, such that w is holomorphic in $\zeta \in B$, vanishes for $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$ and $\zeta \in B$, and we have the period vanishing conditions

$$\mathcal{Q}(\psi_p^t(\cdot, \zeta, w(p, t, \zeta))) = 0 \quad \text{for all } (p, t, \zeta) \in P \times [0, 1] \times B. \tag{4.3}$$

Pick an initial point $u_0 \in D_j$. It is straightforward to verify that the family of maps

$$\tau^t(p, u, \zeta) = \tilde{\sigma}^t(p, u_0, \zeta) + \int_{u_0}^u \psi_p^t(\cdot, \zeta, w(p, t, \zeta)) \theta, \quad u \in U_{j+1},$$

then satisfies the claim. (Since \bar{D}_j is a deformation retract of U_{j+1} , the integral is independent of the choice of the path in U_{j+1} due to the period vanishing condition (4.3).) If D_j is disconnected, the same argument applies on each connected component. \square

We continue with the proof of the theorem. Assuming as we may that the approximation of $\tilde{\sigma}^t$ by τ^t is close enough, the period domination property (4.1) of the spray $\tilde{\sigma}^t$ and the implicit function theorem give a continuous map

$$\zeta : P \times [0, 1] \rightarrow B \subset \mathbb{C}^N,$$

with values close to 0 (depending on how close τ^t is to $\tilde{\sigma}^t$), such that

$$\zeta \text{ vanishes on the set } (p, t) \in (P \times \{0\}) \cup (Q \times [0, 1]), \tag{4.4}$$

and the family of holomorphic immersions

$$\sigma_p^{t,j+1} = \tau^t(p, \cdot, \zeta(p, t)) : U_{j+1} \rightarrow \mathbb{C}^{2n}$$

satisfies the period conditions

$$\mathcal{P}(\sigma_p^{t,j+1}) = \mathcal{P}(\sigma_p^{t,j}), \quad (p, t) \in P \times [0, 1]. \tag{4.5}$$

In view of (4.4), $\sigma_p^{t,j+1}$ satisfies condition (a_{j+1}) . Writing $\sigma_p^{t,j+1} = (x_p^{t,j+1}, y_p^{t,j+1})$, it follows from (4.5) that for every loop $C \subset D_1$ and for all $(p, t) \in P \times [0, 1]$, we have

$$\int_C x_p^{t,j+1} dy_p^{t,j+1} = \int_C x_p^{t,j} dy_p^{t,j} = 0.$$

This shows that $\sigma^{t,j+1}$ satisfies condition (b_{j+1}) . The same argument for loops $C \subset D_{j+1}$ and $t = 1$ shows that (d_{j+1}) holds. (Note that it suffices to verify the period vanishing condition for loops in \bar{D}_j , which is a deformation retract of \bar{D}_{j+1} .) Finally, condition (c_{j+1}) holds if the approximations are close enough. This completes the inductive step in the noncritical case.

(b) The critical case: ρ has a (unique, Morse) critical point in $D_{j+1} \setminus \bar{D}_j$. In this case, \bar{D}_{j+1} deformation retracts onto a compact set of the form $S = \bar{D}_j \cup E$, where E is a smooth embedded arc contained in $D_{j+1} \setminus \bar{D}_j$, except for its endpoints which lie in bD_j . We may assume that E intersects bD_j transversely at both endpoints. Hence, S is an *admissible Runge set* in D_{j+1} (see Remark 2.2 and [6, Definition 5.1]).

There are two topologically different cases to consider.

Case 1: the arc E closes inside the domain D_j to a Jordan curve C such that $E = C \setminus D_j$. This happens when the endpoints of E belong to the same connected component of \bar{D}_j . In this case, $H_1(D_{j+1}; \mathbb{Z}) = H_1(D_j; \mathbb{Z}) \oplus \mathbb{Z}$ where C represents the additional generator.

Case 2: the endpoints of the arc E belong to different connected components of \bar{D}_j . In this case, no new element of the homology basis appears.

We begin with case 1. Let C be a smooth Jordan curve in M such that $E = C \setminus D_j$. Recall that $\sigma = (x, y) \in \mathcal{S}(P \times M, \mathbb{C}^{2n})$ is the given map in the theorem, and $\sigma^{t,j} = (x^{t,j}, y^{t,j}) \in \mathcal{S}(P \times U_j, \mathbb{C}^{2n})$ is a homotopy from the j -th step. After shrinking the neighborhood U_j around \bar{D}_j if necessary, we can extend $\sigma^{t,j}$ from $P \times U_j$ to a homotopy

$$\sigma^{t,j} = (x^{t,j}, y^{t,j}) : P \times (U_j \cup E) \rightarrow \mathbb{C}^{2n}, \quad t \in [0, 1]$$

such that $\sigma_p^{t,j}|_E : E \rightarrow \mathbb{C}^{2n}$ is a \mathcal{C}^1 immersion for every $(p, t) \in P \times [0, 1]$ and

$$\sigma_p^{t,j}|_E = \sigma_p|_E \text{ for all } (p, t) \in (P \times \{0\}) \cup (Q \times [0, 1]).$$

In particular, condition (a) on σ (in the theorem) implies

$$\int_C x_p^{t,j} dy_p^{t,j} = 0 \text{ for all } (p, t) \in Q \times [0, 1]. \tag{4.6}$$

Our goal is to deform the homotopy $\sigma^{t,j}$ (only) on the relative interior of E , keeping it fixed for the parameter values $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$, to a new homotopy (still denoted $\sigma^{t,j} = (x^{t,j}, y^{t,j})$) such that at $t = 1$ we have

$$\int_C x_p^{1,j} dy_p^{1,j} = 0 \quad \text{for all } p \in P. \tag{4.7}$$

This can be done by using Lemma 3.2 as follows. Choose a smooth regular parametrization $\lambda: [0, 1] \rightarrow E$ with $\lambda(0), \lambda(1) \in bD_j$. Consider the family of immersed arcs $\xi_p^t = (f_p^t, g_p^t): [0, 1] \rightarrow \mathbb{C}^{2n}$ for $(p, t) \in P \times [0, 1]$ defined by

$$\xi_p^t(s) = \sigma_p^{t,j}(\lambda(s)) = (f_p^t(s), g_p^t(s)), \quad s \in [0, 1]. \tag{4.8}$$

It follows that

$$\int_E x_p^{t,j} dy_p^{t,j} = \int_0^1 f_p^t(s) g_p^t(s) ds.$$

Define the function $\beta: P \rightarrow \mathbb{C}$ by

$$\beta(p) = - \int_{C \setminus E} x_p^{1,j} dy_p^{1,j}, \quad p \in P. \tag{4.9}$$

We now apply Lemma 3.2 to the family $(\xi_p^t)_{p,t}$, the pair of parameter spaces

$$(p, t) \in P' = P \times [0, 1], \quad Q' = (P \times \{0\}) \cup (Q \times [0, 1]),$$

the function β given by (4.9), taking into account condition (4.6). This provides a deformation of $(\xi_p^t)_{(p,t) \in P'}$ through a family of immersions $[0, 1] \rightarrow \mathbb{C}^{2n}$ of class \mathcal{C}^1 (the parameter of the homotopy $\tau \in [0, 1]$ shall be omitted) such that the homotopy is fixed for $(p, t) \in Q'$, it is fixed near the endpoints of $[0, 1]$ for all $(p, t) \in P'$, and the new family obtained at $\tau = 1$ satisfies the condition

$$\int_0^1 f_p^1(s) \dot{g}_p^1(s) ds = \beta(p), \quad p \in P.$$

By using the parametrization $\lambda: [0, 1] \rightarrow E$ as in (4.8), this provides a homotopy of the family of immersions $\sigma_p^{t,j} = (x_p^{t,j}, y_p^{t,j}): U_j \cup E \rightarrow \mathbb{C}^{2n}$ which is fixed on U_j such that the new family satisfies the condition

$$\int_E x_p^{1,j} dy_p^{1,j} = \int_0^1 f_p^1(s) \dot{g}_p^1(s) ds = \beta(p), \quad p \in P. \tag{4.10}$$

Now, (4.7) follows immediately from (4.9) and (4.10).

Denote by \mathcal{P}' the period map (2.1) with respect to the homology basis \mathcal{B} of D_j and the additional loop C . It follows from the above that $\mathcal{P}'(\sigma_p^{1,j}) = 0$ for all $p \in P$.

The inductive step can now be completed as in the noncritical case; here is an outline. By Lemma 2.1 we can embed the family of immersions $\sigma_p^{t,j}: U_j \cup E \rightarrow \mathbb{C}^{2n}$ ($(p, t) \in P \times [0, 1]$) as the core of a period dominating spray depending on an additional set of variables $\zeta \in \mathbb{C}^N$. (The set U_j may shrink around \bar{D}_j .) Since $\bar{D}_j \cup E$ is an admissible set in D_{j+1} and a deformation retract of \bar{D}_{j+1} , we can apply the Mergelyan theorem for holomorphic immersions to \mathbb{C}^{2n} to approximate this spray, as closely as desired in the \mathcal{C}^1 -topology on $\bar{D}_j \cup E$, by a spray consisting of holomorphic immersions from a neighborhood $U_{j+1} \subset M$ of \bar{D}_{j+1} into \mathbb{C}^{2n} . As in the proof of the noncritical case, replacing the parameter

ζ by a suitably chosen function $\zeta(p, t)$ with values in \mathbb{C}^N and close to 0 gives a homotopy $\sigma^{t, j+1} \in \mathcal{S}(P \times U_{j+1}, \mathbb{C}^{2n})$ satisfying conditions $(a_{j+1})-(d_{j+1})$.

This completes the induction step in case 1 of the critical case (b).

In case 2, the arc E connects two distinct connected components of \bar{D}_j . We follow the construction in case 1 to obtain an extension of the family $\sigma_p^t: U_j \rightarrow \mathbb{C}^{2n}$ across E to a family of immersions $U_j \cup E \rightarrow \mathbb{C}^{2n}$; however, there is no need to adjust the value of the integral (4.10). On the other hand, when approximating this family of maps on $\bar{D}_j \cup E$ by maps on $U_{j+1} \supset \bar{D}_{j+1}$, we still need to use a dominating spray as in case 1 in order to keep the period vanishing condition on curves in the homology basis \mathcal{B} for D_j . \square

Returning to Remark 4.2, we note that a nontrivial difference appears in the final paragraph of the above proof when proving the parametric Oka property for the space of Legendrian immersions. Recall that the map $\mathcal{L}(M, \mathbb{C}^{2n+1}) \rightarrow \mathcal{S}_*(M, \mathbb{C}^{2n}) \times \mathbb{C}$, given by (1.4), is a homeomorphism only if M is connected. When the arc E connects two distinct connected components of the set \bar{D}_j , we must ensure the correct value of the integral (4.10) in order to match the z -component of the Legendrian map (which is already defined on a neighborhood of \bar{D}_j) near the endpoints of E . This can be achieved just like in case 1.

5 Strong homotopy equivalence for surfaces of finite topological type

In this section, we complete the proof of Theorem 1.1 by showing that if M is a connected open Riemann surface of finite topological type, then the inclusion $\mathcal{S}_*(M, \mathbb{C}^{2n}) \hookrightarrow \mathcal{S}(M, \mathbb{C}^{2n})$, already known to be a weak homotopy equivalence, is in fact a homotopy equivalence. It is even the inclusion of a strong deformation retract. We closely follow the proof of a similar result in [12, Section 6], which in turn is based on [17].

Our approach to showing that the weak homotopy equivalence $\mathcal{S}_*(M, \mathbb{C}^{2n}) \hookrightarrow \mathcal{S}(M, \mathbb{C}^{2n})$ is the inclusion of a strong deformation retract is to prove that the metrizable spaces $\mathcal{S}_*(M, \mathbb{C}^{2n})$ and $\mathcal{S}(M, \mathbb{C}^{2n})$ are absolute neighborhood retracts (ANR). Namely, an ANR has the homotopy type of a CW complex, and a weak homotopy equivalence between CW complexes is a homotopy equivalence. Hence, if $j : A \hookrightarrow B$ is the inclusion of a closed subspace in a metrizable space B , both spaces are ANRs, and j is a weak homotopy equivalence, then j is a homotopy equivalence. Moreover, j is a cofibration (in the sense of Hurewicz), so j is the inclusion of a strong deformation retract. For more information on what is involved, we refer to [12, Section 6].

The space $\mathcal{S}(M, \mathbb{C}^{2n})$ is an open subset of the Fréchet space of all holomorphic maps $M \rightarrow \mathbb{C}^{2n}$, so it is an ANR.

To show that the space $\mathcal{S}_*(M, \mathbb{C}^{2n})$ is an ANR, we verify that it satisfies the so-called Dugundji-Lefschetz property. Once we have prepared two ingredients for the proof, it proceeds exactly as the proof of [12, Theorem 6.1].

First, we note the homeomorphism

$$\mathcal{S}(M, \mathbb{C}^{2n}) \rightarrow \mathcal{O}_0(M, \mathbb{C}_*^{2n}) \times \mathbb{C}, \quad \sigma \mapsto (d\sigma/\theta, \sigma(p)),$$

where $\mathcal{O}_0(M, \mathbb{C}_*^{2n})$ is the space of holomorphic maps $M \rightarrow \mathbb{C}_*^{2n}$ with vanishing periods, θ is a nowhere vanishing holomorphic 1-form on M , and $p \in M$ is a chosen base point. We put together the parametric Oka principles with approximation for the inclusion $\mathcal{S}_*(M, \mathbb{C}^{2n}) \hookrightarrow \mathcal{S}(M, \mathbb{C}^{2n})$ (Theorem 4.1), for the inclusion $\mathcal{O}_0(M, \mathbb{C}_*^{2n}) \hookrightarrow \mathcal{O}(M, \mathbb{C}_*^{2n})$ [12, Theorem 5.3], and for the inclusion $\mathcal{O}(M, \mathbb{C}_*^{2n}) \hookrightarrow \mathcal{C}(M, \mathbb{C}_*^{2n})$, which comes from \mathbb{C}_*^{2n} being an Oka

manifold. This yields the first ingredient: the parametric Oka principle with approximation for the inclusion $\mathcal{S}_*(M, \mathbb{C}^{2n}) \hookrightarrow \mathcal{C}(M, \mathbb{C}^{2n}) \times \mathbb{C}$.

The second ingredient is the following lemma, which is analogous to [12, Lemma 6.4]. The proof that $\mathcal{S}_*(M, \mathbb{C}^{2n})$ is an ANR is then so similar to the proof of [12, Theorem 6.1] that we omit further details.

Lemma 5.1 *Let M be an open Riemann surface, let $r \geq 1$ be an integer, and let $\rho : M \rightarrow [0, \infty)$ be a smooth exhaustion function. Let $L_0 \supset L_1 \supset \dots \supset K$ be compact smoothly bounded domains in M of the form $\rho^{-1}([0, c])$, such that K contains all the critical points of ρ . Let $\sigma_0 \in \mathcal{S}_*(M, \mathbb{C}^{2n})$ and let W be a neighborhood of $\sigma_0|_K$ in $\mathcal{S}_*(M, \mathbb{C}^{2n})$. Then there are contractible neighborhoods C_m of $\sigma_0|_{L_m}$ in $\mathcal{S}_*(L_m, \mathbb{C}^{2n})$ such that $C_m|_{L_{m+1}} \subset C_{m+1}$ and $C_m|_K \subset W$ for all $m \geq 0$.*

Proof Since K contains all the critical points of ρ , there is a homology basis $\mathcal{B} = \{\gamma_i\}_{i=1, \dots, l}$ of $H_1(M; \mathbb{Z})$ whose support $|\mathcal{B}| = \bigcup_{j=1}^l |\gamma_j|$ is contained in K and is Runge in M . Let $\mathcal{P} : \mathcal{O}(M, \mathbb{C}^{2n}) \rightarrow \mathbb{C}^l$ denote the associated period map (2.1):

$$\mathcal{P}(\sigma) = \left(\int_{C_j} \sigma \, dy \right)_{j=1, \dots, l}, \quad \sigma = (x, y) \in \mathcal{O}(M, \mathbb{C}^{2n}).$$

Fix a map $\sigma_0 \in \mathcal{S}_*(M, \mathbb{C}^{2n})$. Let M_0 be a compact smoothly bounded domain in M (say a sublevel set of ρ) with the same topology as M and containing L_0 . Note that $\mathcal{S}^r(M_0, \mathbb{C}^{2n})$ is an open subset of the complex Banach space $\mathcal{A}^r(M_0, \mathbb{C}^{2n})$. Pick $\epsilon_0 > 0$ such that the ϵ_0 -ball around σ_0 in $\mathcal{A}^r(M_0, \mathbb{C}^{2n})$ is contained in $\mathcal{S}^r(M_0, \mathbb{C}^{2n})$.

By Lemma 2.1, the differential of the period map $\mathcal{P} : \mathcal{A}^r(M_0, \mathbb{C}^{2n}) \rightarrow \mathbb{C}^l$ at σ_0 is surjective. Let us denote it by

$$D = d_{\sigma_0} \mathcal{P} : \mathcal{A}^r(M_0, \mathbb{C}^{2n}) \longrightarrow \mathbb{C}^l.$$

Its kernel

$$\Lambda_0 = \ker D = \{\sigma \in \mathcal{A}^r(M_0, \mathbb{C}^{2n}) : D(\sigma) = 0\} \tag{5.1}$$

is a closed complex subspace of codimension l in $\mathcal{A}^r(M_0, \mathbb{C}^{2n})$; it is precisely the tangent space to the submanifold $\mathcal{S}_*(M, \mathbb{C}^{2n})$ at the point σ_0 . Pick $h_1, \dots, h_l \in \mathcal{A}^r(M_0, \mathbb{C}^{2n})$ such that the vectors $D(h_1), \dots, D(h_l) \in \mathbb{C}^l$ span \mathbb{C}^l ; then

$$\mathcal{A}^r(M_0, \mathbb{C}^{2n}) = \Lambda_0 \oplus \text{span}_{\mathbb{C}}\{h_1, \dots, h_l\}.$$

Note that the period map $\mathcal{P}(\sigma)$ is defined whenever the domain L of σ contains the support $|\mathcal{B}|$ of the homology basis. Hence, the map $D = d_{\sigma_0} \mathcal{P}$ is well defined on $\mathcal{C}^r(L, \mathbb{C}^{2n})$ whenever $|\mathcal{B}| \subset L \subset M_0$. Taking $L = |\mathcal{B}|$, it follows that the complex Banach space $\mathcal{C}^r(|\mathcal{B}|, \mathbb{C}^{2n})$ decomposes as a direct sum of closed complex Banach subspaces

$$\mathcal{C}^r(|\mathcal{B}|, \mathbb{C}^{2n}) = \ker D|_{\mathcal{C}^r(|\mathcal{B}|, \mathbb{C}^{2n})} \oplus \text{span}_{\mathbb{C}}\{h_1|_{|\mathcal{B}|}, \dots, h_l|_{|\mathcal{B}|}\} = \Lambda \oplus H. \tag{5.2}$$

By the implicit function theorem for Banach spaces, there are a number $\epsilon_1 \in (0, \epsilon_0)$ and smooth bounded complex functions c_1, \dots, c_l on the set $\Lambda_{\epsilon_1} = \{\sigma \in \Lambda : \|\sigma\|_{r, |\mathcal{B}|} < \epsilon_1\}$, vanishing at the origin $0 \in \Lambda$, such that for every $\sigma \in \Lambda_{\epsilon_1}$ the map

$$\tilde{\sigma} = \sigma_0|_{|\mathcal{B}|} + \sigma + \sum_{j=1}^l c_j(\sigma) h_j|_{|\mathcal{B}|} \in \mathcal{C}^r(|\mathcal{B}|, \mathbb{C}^{2n}) \tag{5.3}$$

satisfies the period vanishing equation $\mathcal{P}(\tilde{\sigma}) = 0$. Moreover, (5.3) gives a local representation of the set $\{\tilde{\sigma} \in \mathcal{C}^r(|\mathcal{B}|, \mathbb{C}^{2n}) : \mathcal{P}(\tilde{\sigma}) = 0\}$ in a neighborhood of $\sigma_0|_{|\mathcal{B}|}$ as a graph over the affine linear subspace $\sigma_0|_{|\mathcal{B}|} + \Lambda \subset \mathcal{C}^r(|\mathcal{B}|, \mathbb{C}^{2n})$.

If L is any smoothly bounded compact set with $|\mathcal{B}| \subset L \subset M_0$ and $\sigma \in \mathcal{A}^r(L, \mathbb{C}^{2n})$ satisfies $D\sigma = 0$ and $\|\sigma\|_{r,L} < \epsilon_1$, then (5.3) yields a map

$$\psi_L(\sigma) = \sigma_0|_L + \sigma + \sum_{j=1}^l c_j(\sigma|_{|\mathcal{B}|})h_j|_L \in \mathcal{A}^r(L, \mathbb{C}^{2n})$$

such that $\mathcal{P}(\psi_L(\sigma)) = 0$. Note that $\psi_L(0) = \sigma_0$. Hence, $\psi_L(\sigma) \in \mathcal{A}^r_*(L, \mathbb{C}^{2n})$ provided that $\|\psi_L(\sigma) - \sigma_0\|_{r,L} < \epsilon_0$; the latter condition is satisfied if $\epsilon_1 > 0$ is small enough. As before, this gives a local representation of the set $\{\tilde{\sigma} \in \mathcal{A}^r(L, \mathbb{C}^{2n}) : \mathcal{P}(\tilde{\sigma}) = 0\}$ in a neighborhood of $\sigma_0|_L$ as a graph over the affine linear subspace $\sigma_0|_L + \Lambda_0|_L \subset \mathcal{A}^r(L, \mathbb{C}^{2n})$. Here, $\Lambda_0 = \ker D \subset \mathcal{A}^r(M_0, \mathbb{C}^{2n})$ (see (5.1)).

Note that for any compacts L and L' with $|\mathcal{B}| \subset L \subset L' \subset M_0$, we have

$$\psi_L(\sigma|_L) = \psi_{L'}(\sigma)|_L \tag{5.4}$$

for every $\sigma \in \mathcal{A}^r(L', \mathbb{C}^{2n})$ such that $D(\sigma) = 0$ and $\|\sigma\|_{r,|\mathcal{B}|} < \epsilon_1$.

Since the functions c_j are bounded on a neighborhood of the origin in Λ (see (5.2)), there is a number $\epsilon \in (0, \epsilon_1)$ such that the set

$$C_0 = \{\psi_{M_0}(\sigma) : \sigma \in \Lambda_0, \|\sigma\|_{r,M_0} < \epsilon\} \subset \mathcal{A}^r_*(M_0, \mathbb{C}^{2n})$$

is a neighborhood of $\sigma_0|_{M_0}$ in $\mathcal{A}^r_*(M_0, \mathbb{C}^{2n})$. Furthermore, being a smooth graph over the ball $\{\sigma \in \Lambda_0 : \|\sigma\|_{r,M_0} < \epsilon\}$ in the Banach space Λ_0 , C_0 is contractible. Similarly, for every $m \in \mathbb{N}$, the set

$$C_m = \{\psi_{L_m}(\sigma) : \sigma \in \mathcal{A}^r(L_m, \mathbb{C}^{2n}), D(\sigma) = 0, \|\sigma\|_{r,L_m} < \epsilon\} \subset \mathcal{A}^r_*(L_m, \mathbb{C}^{2n})$$

is a contractible neighborhood of $\sigma_0|_{L_m}$ in $\mathcal{A}^r(L_m, \mathbb{C}^{2n})$.

Taking into account that for any $\sigma \in \mathcal{A}^r(L_m, \mathbb{C}^{2n})$, we have $\|\sigma\|_{r,L_{m+1}} \leq \|\sigma\|_{r,L_m}$ by the maximum principle, the formula (5.4) shows that the restriction map associated to the inclusion $L_m \supset L_{m+1}$ maps C_m into C_{m+1} for every $m \geq 0$. By choosing $\epsilon > 0$ small enough, we can also ensure that the restriction map associated to $L_m \supset K$ maps C_m into a given neighborhood W of $\sigma_0|_K$ in $\mathcal{A}^r_*(K, \mathbb{C}^{2n})$. □

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