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A foliation of the ball by complete holomorphic discs

Antonio Alarcón¹ · Franc Forstnerič^{2,3}

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Abstract

We show that the open unit ball \mathbb{B}^n of \mathbb{C}^n ($n > 1$) admits a nonsingular holomorphic foliation by complete properly embedded holomorphic discs.

Keywords Riemann surface · Holomorphic disc · Foliation · Complete Riemannian manifold

Mathematics Subject Classification 32B15 · 32H02 · 32M17 · 53C12

1 Introduction

An open connected submanifold M of a Euclidean space is said to be *complete* if every divergent path in M has infinite Euclidean length; equivalently, if the restriction of the Euclidean metric ds^2 to M is a complete Riemannian metric on M . Recall that a path $\gamma : [0, 1) \rightarrow M$ is called divergent if $\gamma(t)$ leaves any compact subset of M as $t \rightarrow 1$.

For $n > 1$, we denote by \mathbb{B}^n the open unit ball of \mathbb{C}^n . In this paper, we prove the following result.

Theorem 1 *For any integer $n > 1$ there exists a nonsingular holomorphic foliation \mathcal{F} of \mathbb{B}^n all of whose leaves are complete properly embedded holomorphic discs in \mathbb{B}^n .*

Theorem 1 seems to be the first result which provides control of the topology of all leaves in a nonsingular holomorphic foliation of the ball by complete leaves; in our examples, all leaves are the simplest possible ones, namely, discs. Our proof easily adapts to show that we can ensure completeness of leaves in any given Riemannian metric on the ball, and not only

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the Euclidean metric. We do not know whether a comparable result holds for leaves with prescribed but nontrivial topology. See also Remark 1 for a generalization of Theorem 1 to bounded pseudoconvex Runge domains in \mathbb{C}^n for $n > 1$.

Before proceeding, we place our result in the context of what is known.

The question whether there exist bounded (relatively compact) complete complex submanifolds of \mathbb{C}^n for $n > 1$ was raised by Yang [19] in 1977. The first examples were found in 1979 by Jones [17] who showed that the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ admits a complete bounded holomorphic immersion into \mathbb{C}^2 , embedding into \mathbb{C}^3 , and proper embedding into the ball of \mathbb{C}^4 . Interest in this subject has recently been revived due to new construction methods. It was shown that there are complete properly *immersed* holomorphic curves in \mathbb{B}^2 , and embedded ones in \mathbb{B}^3 , with any given topology [8], and also those with the complex structure of any given bordered Riemann surface [2,3]. A related result in higher dimension was obtained by Drinovec Drnovšek [11]. Parallel developments were made in minimal surface theory where the corresponding circle of questions is known as the *Calabi–Yau problem*; see the survey [5] and the paper [4].

It is a considerably more challenging task to construct complete properly *embedded* holomorphic curves in \mathbb{B}^2 and, more generally, complete complex hypersurfaces in \mathbb{B}^n for $n > 1$. The first examples for $n = 2$ were given by Alarcón and López in [9].

In a pair of landmark papers [15,16] in 2015–2016, Globevnik constructed for any pair of integers $1 \leq k < n$ a complete k -dimensional complex submanifold of \mathbb{B}^n and, more generally, of any pseudoconvex Runge domain in \mathbb{C}^n . For $k = n - 1$ his construction provides a possibly singular holomorphic foliation of the ball \mathbb{B}^n by complete complex hypersurfaces, most of which are smooth. Subsequently, Alarcón [1] introduced to this subject Forstnerič's techniques from [12], concerning noncritical holomorphic functions, and showed that every smooth complex hypersurface in the ball \mathbb{B}^n is a leaf of a *nonsingular holomorphic foliation* of \mathbb{B}^n by hypersurfaces such that all except perhaps the initial one are complete. An analogous result was established for foliations of any codimension. This provides both a converse to, and an extension of the aforementioned theorem of Globevnik.

Foliations in [1,15,16] are given by level sets of suitable holomorphic functions on \mathbb{B}^n (or, more generally, of submersions $\mathbb{B}^n \rightarrow \mathbb{C}^q$ with $1 \leq q < n$), so the topology of their leaves is not controlled. The same can be said about the examples in [9]. By a different technique, using holomorphic automorphisms of \mathbb{C}^n , Alarcón, Globevnik, and López obtained in [7] complete closed complex hypersurfaces in the ball \mathbb{B}^n ($n > 1$) with certain restrictions on the topology of the examples, and with any given topology when $n = 2$ [6]. It follows in particular that the disc \mathbb{D} can be embedded as a complete proper holomorphic curve in \mathbb{B}^2 . However, their results do not apply to foliations.

Our proof of Theorem 1 follows a similar approach as the one in [7], but with an addition which enables us to control the topology and completeness of all leaves in a foliation. By using holomorphic automorphisms we successively twist a holomorphic foliation of \mathbb{C}^n by complex lines in order to make bigger and bigger parts of the foliation avoid pieces of a suitable labyrinth Γ in \mathbb{B}^n . The labyrinth is a closed set in \mathbb{B}^n exhausted by an increasing sequence $K_j = \bigcup_{i=1}^j \Gamma_i$ ($j \in \mathbb{N}$) of compact polynomially convex sets, where Γ_i is contained in a spherical shell $B_{i+1} \setminus B_i$ between two concentric balls in \mathbb{B}^n and consists of finitely many pairwise disjoint closed round balls in suitably chosen affine real hyperplanes. The main property of the labyrinth is that any divergent curve in \mathbb{B}^n which avoids the set $\bigcup_{k=i}^{\infty} \Gamma_k$ for some $i \in \mathbb{N}$ has infinite length. Such labyrinths have been constructed in [7,15]. Note that each connected component of the intersection of \mathbb{B}^n with a properly embedded complex line $L \subset \mathbb{C}^n$ is Runge in L and hence simply connected; since it is also bounded, it is a properly

embedded disc in \mathbb{B}^n . Our construction therefore yields a sequence of foliations $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ of \mathbb{B}^n by discs such that all leaves of \mathcal{F}_i intersecting a compact subset $B_i \subset \mathbb{B}^n$ have intrinsic diameter bigger than a certain number k_i , with $k_i \rightarrow +\infty$ and B_i increasing to \mathbb{B}^n as $i \rightarrow \infty$. In the limit foliation \mathcal{F} , all leaves are discs with infinite intrinsic diameter, hence complete.

2 The construction

Fix an integer $n > 1$. Denote by $\text{Aut}(\mathbb{C}^n)$ the group of holomorphic automorphisms of \mathbb{C}^n . We shall need the following result concerning moving compact convex sets in \mathbb{C}^n by holomorphic automorphisms; see [13, Theorem 2.3] or [14, Corollary 4.12.4, p. 158] for more general statements.

Lemma 1 *Let K_0, K_1, \dots, K_m be pairwise disjoint compact convex sets in \mathbb{C}^n and let $\Psi_j \in \text{Aut}(\mathbb{C}^n)$ ($j = 0, 1, \dots, m$) be such that the images $K'_j = \Psi_j(K_j)$ are pairwise disjoint. If the sets $K = \bigcup_{j=0}^m K_j$ and $K' = \bigcup_{j=0}^m K'_j$ are polynomially convex, then for any $\delta > 0$ there exists $\Psi \in \text{Aut}(\mathbb{C}^n)$ such that*

$$|\Psi(z) - \Psi_j(z)| < \delta \text{ for all } z \in K_j, j = 0, 1, \dots, m. \quad (2.1)$$

The following lemma provides the induction step in the proof of Theorem 1.

Lemma 2 *Let B be a compact convex set contained in the ball $\mathbb{B}^n \subset \mathbb{C}^n$, and let $\Gamma = \bigcup_{j=1}^m \Gamma_j \subset \mathbb{B}^n \setminus B$ be a union of finitely many, pairwise disjoint, compact convex sets Γ_j such that the set $B \cup \Gamma$ is polynomially convex. If $\Phi \in \text{Aut}(\mathbb{C}^n)$, then for any numbers $r > 0$ and $\epsilon > 0$ there exists $\Theta \in \text{Aut}(\mathbb{C}^n)$ such that*

- (a) $\Theta(\Phi(r\overline{\mathbb{D}} \times \mathbb{C}^{n-1})) \cap \Gamma = \emptyset$, and
- (b) $|\Theta(z) - z| < \epsilon$ for all $z \in B$.

Proof Let K_0 be a compact convex neighbourhood of B , and for each $j = 1, \dots, m$ let K_j be a compact convex neighbourhood of Γ_j such that the sets K_0, \dots, K_m are pairwise disjoint and their union $K = \bigcup_{j=0}^m K_j$ is polynomially convex. (We refer to Stout [18] for general results on polynomial convexity.)

Let $\Psi_0 = \text{Id} \in \text{Aut}(\mathbb{C}^n)$ be the identity automorphism. For $j = 1, \dots, m$ we choose automorphisms $\Psi_j \in \text{Aut}(\mathbb{C}^n)$ such that the compact sets $K'_j := \Psi_j(K_j)$ ($j = 0, 1, \dots, m$) are pairwise disjoint, we have that

$$K'_j \cap \Phi(r\overline{\mathbb{D}} \times \mathbb{C}^{n-1}) = \emptyset \text{ for } j = 1, \dots, m, \quad (2.2)$$

and the union $\bigcup_{j=0}^m K'_j$ is polynomially convex. Clearly, such Ψ_j exist: noting that $K'_0 = K_0$, we may squeeze each convex set K_j ($j = 1, \dots, m$) by a dilation into a very small neighbourhood of an interior point of itself and then translate their images into sufficiently small pairwise disjoint balls around some points in the complement of $K_0 \cup \Phi(r\overline{\mathbb{D}} \times \mathbb{C}^{n-1})$. (We refer to [14, proof of Corollary 4.12.4] for the details.)

Now, Lemma 1 furnishes for every $\delta > 0$ an automorphism $\Psi \in \text{Aut}(\mathbb{C}^n)$ satisfying (2.1). Let $\Theta = \Psi^{-1}$. If $\delta > 0$ is small enough then condition (b) holds, and we have $\Psi(\Gamma_j) \subset K'_j$ and hence $\Gamma_j \subset \Theta(K'_j)$ for every $j = 1, \dots, m$, which yields (a). Indeed, if $\Theta(\Phi(z)) \in \Gamma_j$ for some $z \in r\overline{\mathbb{D}} \times \mathbb{C}^{n-1}$ then $\Phi(z) \in K'_j$ which contradicts (2.2). \square

Proof of Theorem 1 Fix an integer $n > 1$. We exhaust the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ by an increasing sequence of closed balls

$$B_1 \subset B_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} B_i = \mathbb{B}^n \quad (2.3)$$

centered at the origin such that each B_i is contained in the interior of the next ball B_{i+1} . Denote by ρ_i the radius of B_i , so we have $0 < \rho_1 < \rho_2 < \cdots < 1$ with $\lim_{i \rightarrow \infty} \rho_i = 1$.

In each spherical shell $\bar{B}_{i+1} \setminus B_i$ ($i \in \mathbb{N}$) we choose a compact set $\Gamma_i = \bigcup_{j=1}^{m_i} \Gamma_{i,j}$ consisting of finitely many, pairwise disjoint, compact convex sets $\Gamma_{i,j}$ and satisfying the following conditions.

- (A) The set $B_i \cup \Gamma_i$ is polynomially convex for every $i \in \mathbb{N}$.
- (B) Every divergent path in \mathbb{B}^n avoiding $\Gamma^i = \bigcup_{k=i}^{\infty} \Gamma_k$ for some $i \in \mathbb{N}$ has infinite length.

Labyrinths with these properties have been constructed in [7, 15]. In [7] the connected components $\Gamma_{i,j}$ of Γ are balls in suitably chosen affine real hyperplanes in \mathbb{C}^n .

We now describe the induction leading to the proof of Theorem 1.

Recall that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathbb{P} = \mathbb{D}^{n-1} \subset \mathbb{C}^{n-1}$ denote the unit $(n-1)$ -dimensional polydisc. By $r\mathbb{P}$ for $r > 0$ we denote the polydisc with polyradius r . Choose a number $\epsilon_0 > 0$ and set $r_0 = 0$, $B_0 = \Gamma_0 = \emptyset$. Let $\Phi_0 = \phi_0 = \text{Id} \in \text{Aut}(\mathbb{C}^n)$ be the identity map.

We shall inductively find sequences $r_i > 0$, $\epsilon_i > 0$, and $\phi_i \in \text{Aut}(\mathbb{C}^n)$ such that, setting $\Phi_i = \phi_i \circ \cdots \circ \phi_1$, the following conditions hold for every $i \in \mathbb{N}$.

- (a_i) $r_i > r_{i-1} + 1$ and $B_i \subset \Phi_{i-1}(r_i \mathbb{P} \times \mathbb{C})$.
- (b_i) $|\phi_i(z) - z| < \epsilon_i$ for all $z \in B_i$.
- (c_i) $\Phi_i(r_j \mathbb{P} \times \mathbb{C}) \cap \Gamma_j = \emptyset$ for $j = 1, \dots, i$.
- (d_i) $0 < \epsilon_i < \min\{\epsilon_{i-1}/2, \rho_{i+1} - \rho_i\}$.
- (e_i) For every holomorphic map $\theta : B_i \rightarrow \mathbb{C}^n$ satisfying $|\theta(z) - z| < \epsilon_i$ for all $z \in B_i$ we have that $\theta(\Phi_{i-1}(r_j \mathbb{P} \times \mathbb{C})) \cap \Gamma_j = \emptyset$ for $j = 1, \dots, i-1$.

Assume inductively that for some $i \in \mathbb{N}$ we have already found these objects for all indices up to $i-1$; this trivially holds for $i=1$.

Choose a number r_i satisfying (a_i). Next, choose $\epsilon_i > 0$ so small that (d_i) and (e_i) are satisfied. When $i=1$, condition (e₁) is vacuous, while for $i > 1$ it can be satisfied by (c_{i-1}); note that for $j < i$ the set Γ_j is contained in the interior of B_i , and the set $\Phi_{i-1}(r_j \mathbb{P} \times \mathbb{C}) \cap B_i$ is compact and disjoint from Γ_j .

By property (A) of the labyrinth, we may apply Lemma 2 with $\Phi = \Phi_{i-1}$ and obtain an automorphism $\phi_i \in \text{Aut}(\mathbb{C}^n)$ (called Θ in the lemma) satisfying the approximation condition (b_i) and such that the automorphism $\Phi_i := \phi_i \circ \Phi_{i-1} \in \text{Aut}(\mathbb{C}^n)$ satisfies (c_i). (The lemma directly ensures that Φ_i satisfies condition (c_i) for $j=i$; it then also satisfies the same condition for indices $j=1, \dots, i-1$ in view of the condition (e_i) on the number ϵ_i .) Thus, the induction may proceed.

In view of (2.3) and conditions (b_i) and (d_i), we see from [14, Proposition 4.4.1] that the sequence $\Phi_i \in \text{Aut}(\mathbb{C}^n)$ converges uniformly on compacts in the domain

$$\Omega = \bigcup_{i=1}^{\infty} \Phi_i^{-1}(B_i) \subset \mathbb{C}^n$$

to a biholomorphic map $\Phi = \lim_{i \rightarrow \infty} \Phi_i : \Omega \rightarrow \mathbb{B}^n$. Moreover, $\Phi_i^{-1}(B_i)$ for $i=1, 2, \dots$ is an increasing sequence of compact polynomially convex sets exhausting Ω . It follows that

Ω is a pseudoconvex Runge domain in \mathbb{C}^n . Conditions (b_i) and (d_i) also show that for any $z \in B_i$ and $k > i$ we have

$$|\Phi_k \circ \Phi_i^{-1}(z) - z| = |\phi_k \circ \dots \circ \phi_{i+1}(z) - z| < \sum_{j=i+1}^k \epsilon_j < \epsilon_i.$$

Passing to the limit as $k \rightarrow \infty$ gives

$$|\Phi \circ \Phi_i^{-1}(z) - z| < \epsilon_i, \quad z \in B_i. \quad (2.4)$$

Writing $\Phi = (\Phi \circ \Phi_i^{-1}) \circ \Phi_i = \theta \circ \Phi_i$, we see from (2.4) and (e_{i+1}) that

$$\Phi((r_i \bar{\mathbb{P}} \times \mathbb{C}) \cap \Omega) \cap \Gamma_i = \emptyset, \quad i = 1, 2, \dots \quad (2.5)$$

Write $z = (z', z_n) \in \mathbb{C}^n$ where $z' = (z_1, \dots, z_{n-1})$. Let \mathcal{G} be the foliation of Ω by the connected components of the sets $(\{z' = c\} \times \mathbb{C}) \cap \Omega$ for $c \in \mathbb{C}^{n-1}$, and let $\mathcal{F} = \Phi(\mathcal{G})$ be the image foliation of \mathbb{B}^n . Since Ω is pseudoconvex and Runge in \mathbb{C}^n , the leaves of \mathcal{G} are discs or complex lines which are proper in Ω ; hence the analogous condition holds for the leaves of \mathcal{F} in \mathbb{B}^n . Since \mathbb{B}^n is bounded, all leaves of \mathcal{F} (and hence of \mathcal{G}) are discs.

It remains to show that all leaves of \mathcal{F} are complete. Let $F \in \mathcal{F}$. Fix a point $w = (w', w_n) \in G := \Phi^{-1}(F)$. Note that G is a disc in the line $L = \{(w', \zeta) : \zeta \in \mathbb{C}\}$. Choose $i_0 \in \mathbb{N}$ so large that $w \in r_{i_0} \mathbb{P} \times \mathbb{C}$ and $\Phi(w) \in B_{i_0}$. Clearly these conditions persist if we increase i_0 . Since $\Phi_i \rightarrow \Phi$ uniformly on compacts in Ω as $i \rightarrow \infty$, we can increase i_0 if necessary so that we also have $\Phi_{i_0-1}(w) \in B_{i_0}$. Since $L = \{w'\} \times \mathbb{C} \subset r_{i_0} \mathbb{P} \times \mathbb{C}$ by the choice of i_0 , we see from (2.5) that $F \cap \Gamma_i \subset \Phi(L \cap \Omega) \cap \Gamma_i = \emptyset$ for all $i \geq i_0$, and hence $F \cap \Gamma^{i_0} = \emptyset$. Since F is proper in \mathbb{B}^n and in view of the property (B) of the labyrinth, it follows that F is complete. \square

Remark 1 Combining the proof of Theorem 1 with results of Ł. Kosiński and S. Charpentier [10], one can easily see that Theorem 1 holds in an arbitrary bounded (or hyperbolic) pseudoconvex Runge domain D in \mathbb{C}^n for $n > 1$. Indeed, it is shown in [10] that every pseudoconvex Runge domain $D \subset \mathbb{C}^n$, endowed with a Riemannian metric g , contains a closed polynomially convex set Γ , every connected component of which is a holomorphically contractible (for example, convex) compact set, such that any divergent path in D avoiding all but finitely many connected components of Γ has infinite length in the metric g ; this answers a question that the authors posed in a preliminar version of the present paper. Our proof of Theorem 1 applies to any labyrinth with these properties. Note also that each connected component of the intersection of a closed embedded complex line in \mathbb{C}^n with a pseudoconvex Runge domain D is simply connected, hence a disc or \mathbb{C} . If D is hyperbolic, it does not contain any complex lines, and hence all leaves of the foliation on D , obtained by the proof of Theorem 1, are complete properly embedded discs.

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