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RUNGE TUBES IN STEIN MANIFOLDS WITH THE DENSITY PROPERTY

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ABSTRACT. In this paper we give a simple proof of the existence and plenitude of Runge tubes in $\mathbb{C}^n$ $(n > 1)$ and, more generally, in Stein manifolds with the density property. We show in particular that for any algebraic submanifold X of codimension at least two in a complex Euclidean space $\mathbb{C}^n$, the normal bundle of X admits a holomorphic embedding onto a Runge domain in $\mathbb{C}^n$ which agrees with the inclusion map $X \hookrightarrow \mathbb{C}^n$ on the zero section.

1. INTRODUCTION

It has been an open question for a long time whether it is possible to embed $\mathbb{C}^* \times \mathbb{C}$ as a Runge domain in $\mathbb{C}^2$. (Here, $\mathbb{C}$ is the complex plane and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.) Such hypothetical domains have been called *Runge cylinders* in $\mathbb{C}^2$. The question arose in connection with the classification of Fatou-components for Hénon maps by E. Bedford and J. Smillie in 1991; see [4]. This problem has recently been solved in the affirmative by F. Bracci, J. Raissy, and B. Stensønes [6], who obtained a Runge embedding of $\mathbb{C}^* \times \mathbb{C}$ into $\mathbb{C}^2$ as the basin of attraction of a (nonpolynomial) holomorphic automorphism of $\mathbb{C}^2$ at a parabolic fixed point.

The purpose of this note is to give a very simple proof of the existence of Runge cylinders and, furthermore, of the existence of an abundance of Runge tube domains in all Stein manifolds with the density property. Although our proof is completely different from that given in [6], both proofs depend crucially on the Andersén-Lempert theory (see [14, Chapter 4]).

Recall (see D. Varolin [21, 22] or [14, Definition 4.10.1]) that a complex manifold Y has the *density property* if every holomorphic vector field on Y is a uniform limit on compacts of finite sums of $\mathbb{C}$-complete holomorphic vector fields on Y. In particular, Euclidean spaces $\mathbb{C}^n$ of dimension $n > 1$ have the density property according to E. Andersén and L. Lempert [1].

Our first main result is the following.

Theorem 1.1. *Let X and Y be Stein manifolds with $\dim X < \dim Y$, and assume that Y has the density property. Suppose that $\theta : X \hookrightarrow Y$ is a holomorphic embedding with $\mathcal{O}(Y)$-convex image (this holds in particular if θ is proper), and let*

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$E \to X$ denote the normal bundle associated to θ. Then, θ can be approximated uniformly on compacts in X by holomorphic embeddings of E into Y whose images are Runge domains in Y.

Recall that a locally closed subset Z of a complex manifold Y is said to be $\mathcal{O}(Y)$-*convex* if for every compact set $K \subset Z$, its $\mathcal{O}(Y)$-convex hull

$$\widehat{K}_{\mathcal{O}(Y)} = \{y \in Y : |f(y)| \le \sup_K |f| \ \ \forall f \in \mathcal{O}(Y)\} \tag{1.1}$$

is compact and contained in Z.

To get a Runge embedding of $\mathbb{C}^* \times \mathbb{C}$ into $\mathbb{C}^2$ from Theorem 1.1, one embeds $X = \mathbb{C}^*$ onto the curve $\{zw = 1\} \subset \mathbb{C}^2$ and notes that any holomorphic vector bundle over $\mathbb{C}^*$ (and in fact over any open Riemann surface) is trivial by Oka's theorem [19]. (See also [14, Section 5.2].)

There is a big list of Stein manifolds, and in particular of affine algebraic manifolds, which are known to have the density property; see the list of examples in [2], as well as the recent surveys of Kaliman and Kutzschebauch [18] and of the first named author [14, Section 4.10]. The authors together with R. Andrist and T. Ritter proved in [2, 3] that every Stein manifold X embeds properly holomorphically into any Stein manifold Y with the density property satisfying $\dim Y \ge 2 \dim X + 1$. Every open Riemann surface X embeds properly holomorphically into $\mathbb{C}^3$, and a plenitude of them embed properly into $\mathbb{C}^2$; see [13] and [14, Sections 9.10-9.11] for a discussion of this topic. By Theorem 1.1, every such embedding can be approximated by a Runge embedding of the normal bundle of X into Y. This provides a huge variety of nontrivial Runge tubes in any Stein manifold with the density property. In particular, we have the following corollary to Theorem 1.1.

Corollary 1.2 (Runge tubes over open Riemann surfaces)**.** *If X is an open Riemann surface which admits a proper holomorphic embedding into $\mathbb{C}^2$, then $X \times \mathbb{C}$ admits a Runge embedding into $\mathbb{C}^2$. For every open Riemann surface X and every $k \ge 2$, $X \times \mathbb{C}^k$ admits a Runge embedding into $\mathbb{C}^{k+1}$ and into any Stein manifold Y^{k+1} with the density property.*

Remark 1.3. The Runge embeddings $E \hookrightarrow Y$ of the normal bundle in Theorem 1.1 need not agree with the given embedding $\theta : X \hookrightarrow Y$ on the zero section of E (which we identify with X). Indeed, it is known that for every pair of integers $1 \le k < n$ there exists a proper holomorphic embedding $\theta : X = \mathbb{C}^k \hookrightarrow Y = \mathbb{C}^n$ whose complement is $(n-k)$-hyperbolic in the sense of Eisenman; in particular, there are no nondegenerate holomorphic maps $\mathbb{C}^{n-k} \to \mathbb{C}^n \setminus \theta(\mathbb{C}^k)$ (see Buzzard and Fornæss [7] for the case $k = 1, n = 2$ and Borell and Kutzschebauch [5] and Forstnerič [11, Corollary 5.3] for the general case). Since the normal bundle of the embedding θ is the trivial bundle $E = \mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^k$ and the complement of $\mathbb{C}^k \times \{0\}^{n-k}$ in $\mathbb{C}^n$ is clearly not $(n-k)$-hyperbolic, θ does not extend to a holomorphic embedding $E = \mathbb{C}^n \hookrightarrow \mathbb{C}^n$. □

However, we can ensure this additional interpolation condition for algebraic embeddings of codimension at least 2 into $\mathbb{C}^n$. Here is the precise result.

Theorem 1.4. *Let X be a Stein manifold and let $\theta\colon X \hookrightarrow \mathbb{C}^n$ be a proper holomorphic embedding onto an algebraic submanifold $\theta(X) \subset \mathbb{C}^n$. If $n \ge \dim X + 2$, then θ extends to a holomorphic Runge embedding $\tilde{\theta}\colon E \hookrightarrow Y$ of the normal bundle of θ.*

According to Docquier and Grauert [9] (see also [14, Theorem 3.3.3]), every proper holomorphic embedding $\theta\colon X \hookrightarrow Y$ of a Stein manifold X into a complex manifold Y extends to an embedding of a neighborhood of the zero section in the normal bundle of θ. Theorem 1.4 says that one can in fact embed the entire normal bundle as a Runge domain in $\mathbb{C}^n$.

The proof of Theorem 1.4 is similar to that of Theorem 1.1. It uses the result of Kaliman and Kutzschebauch [17, Theorem 6] that the Lie algebra of all algebraic vector fields on $\mathbb{C}^n$ vanishing on an algebraic submanifold of codimension at least two enjoys the algebraic density property. It follows that flows of such vector fields can be approximated by automorphisms of $\mathbb{C}^n$ fixing the submanifold pointwise.

Corollary 1.5. *If X is an affine algebraic curve, then every proper algebraic embedding $\theta\colon X \hookrightarrow \mathbb{C}^{n+1}$ $(n \geq 2)$ extends to a holomorphic embedding $\tilde{\theta} : X \times \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ onto a Runge domain in $\mathbb{C}^{n+1}$.*

Remark 1.6. Note that holomorphic Runge embeddings of the normal bundle, furnished by Theorem 1.4 and Corollary 1.5, can never be algebraic. Indeed, if $E \to X$ is the algebraic normal bundle of an algebraic submanifold $X \subset \mathbb{C}^n$ and $F\colon E \hookrightarrow \mathbb{C}^n$ is an algebraic embedding, then $\Omega = F(E) \subset \mathbb{C}^n$ is a Zariski open set in $\mathbb{C}^n$ and its complement $A = \mathbb{C}^n \setminus \Omega$ is a Zariski closed set, i.e., an algebraic subvariety of $\mathbb{C}^n$ (see Chevalley [8]). Since Ω is a Stein domain, A must be of pure codimension one, and hence $A = \{f = 0\}$ for some entire function $f \in \mathcal{O}(\mathbb{C}^n)$. Clearly, the function $1/f \in \mathcal{O}(\Omega)$ cannot be approximated uniformly on compacts in Ω by entire functions, and hence the domain $\Omega = F(E)$ is not Runge in $\mathbb{C}^n$.

We conclude this introduction by pointing out the following open problem related to Theorem 1.1 and Corollary 1.5.

Problem 1.7. Is there a Runge embedding of the (trivial) normal bundle $E = H \times \mathbb{C} \cong \mathbb{C}^* \times \mathbb{C}$ of the hyperbola $H = \{(z, w) \in \mathbb{C}^2 : zw = 1\}$ extending the inclusion map $H \hookrightarrow \mathbb{C}^2$?

The method of proof breaks down at the point where one would need to know that the Lie algebra of holomorphic vector fields vanishing on H has the density property. To decide about this is a notoriously hard problem well known and open for decades, as is the problem about the density property of $(\mathbb{C}^*)^n$ for $n > 1$.

2. Proof of Theorems 1.1 and 1.4

We begin by recalling some basic facts from the theory of Stein manifolds (see e.g. Gunning and Rossi [15] or Hörmander [16]) and explaining the setup.

A domain D in a complex manifold Y is said to be *Runge* in Y if the set of restrictions $\{f|_D : f \in \mathcal{O}(Y)\}$ is a dense subset of $\mathcal{O}(D)$. If both D and Y are Stein, this holds if and only if for every compact subset $K \subset D$ we have that $\widehat{K}_{\mathcal{O}(D)} = \widehat{K}_{\mathcal{O}(Y)}$. In particular, a domain in a Stein manifold Y which is exhausted by compact $\mathcal{O}(Y)$-convex sets is Runge in Y.

A holomorphic embedding $\theta\colon X \hookrightarrow Y$ of a complex manifold X into a complex manifold Y is said to be Runge if the image $Z = \theta(X) \subset Y$ is an $\mathcal{O}(Y)$-convex subset of Y; i.e., it is exhausted by compact $\mathcal{O}(Y)$-convex subsets. If X and Y are Stein manifolds, then every proper holomorphic embedding $X \hookrightarrow Y$ is Runge.

Assume that $\pi\colon E \to X$ is a holomorphic vector bundle over a Stein manifold X. The total space E is then also a Stein manifold. We shall write elements of E in the

form $e = (x, v)$ where $\pi(e) = x$, identifying X with the zero section $\{(x, 0) : x \in X\}$ of E. For any $t \in \mathbb{C}$ there is a holomorphic fibre-preserving map

$$\psi_t \colon E \to E, \qquad \psi_t(x, v) = (x, tv). \tag{2.1}$$

Clearly, ψ_t is a holomorphic automorphism of E for every $t \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

A subset $Z \subset E$ is called *radial* if $\psi_t(Z) \subset Z$ holds for every $t \in [0, 1]$.

The following lemma provides the induction step in the proof of Theorem 1.1.

Lemma 2.1. *Assume that X is a Stein manifold, $E \to X$ is a holomorphic vector bundle, $K \subset L$ are compact radial $\mathcal{O}(E)$-convex subsets of E, $\Omega \subset E$ is an open set containing $X \cup K$, Y is a Stein manifold with the density property such that $\dim Y = \dim E$, and $\theta : \Omega \hookrightarrow Y$ is a holomorphic embedding such that $\theta|_X \colon X \hookrightarrow Y$ is a Runge embedding and $\theta(K)$ is $\mathcal{O}(Y)$-convex. Then, θ can be approximated as closely as desired uniformly on K by a holomorphic embedding $\tilde\theta \colon \widetilde\Omega \hookrightarrow Y$ of a domain $\widetilde\Omega \subset E$ with $X \cup L \subset \widetilde\Omega$ such that $\tilde\theta|_X \colon X \hookrightarrow Y$ is a Runge embedding and the sets $\tilde\theta(L)$ and $\tilde\theta(K)$ are $\mathcal{O}(Y)$-convex.*

If, in addition to the hypotheses above, $Y = \mathbb{C}^n$ with $n \geq \dim X + 2$ and $\theta(X) \subset \mathbb{C}^n$ is a closed algebraic submanifold of $\mathbb{C}^n$, then the approximating embedding $\tilde\theta : \widetilde\Omega \hookrightarrow \mathbb{C}^n$ can be chosen to agree with θ on X.

The conditions in the lemma imply that $E \to X$ is the normal bundle of the embedding $\theta|_X \colon X \hookrightarrow Y$.

Proof. We identify X with the zero section of E.

Choose a compact $\mathcal{O}(X)$-convex subset $X_0 \subset X$ such that $\pi(L) \subset X_0$. Since the embedding $\theta|_X \colon X \hookrightarrow Y$ is Runge, the image $Y_0 = \theta(X_0) \subset \theta(X)$ is $\mathcal{O}(Y)$-convex. Pick a compact $\mathcal{O}(Y)$-convex neighborhood $N \subset \theta(\Omega)$ of Y_0 (such exists since an $\mathcal{O}(Y)$-convex set has a basis of compact $\mathcal{O}(Y)$-convex neighborhoods). Thus, $N = \theta(N_0)$ for a compact set $N_0 \subset \Omega$ with $X_0 \subset \mathring{N}_0$.

Let ψ_t be defined by (2.1). Since $\pi(L) \subset X_0$ and N_0 is a neighborhood of X_0 in E, we can choose $\epsilon > 0$ small enough such that $\psi_\epsilon(L) \subset N_0$. Since L is $\mathcal{O}(E)$-convex and $\psi_\epsilon \in \mathrm{Aut}(E)$, the set $\psi_\epsilon(L)$ is $\mathcal{O}(E)$-convex, and hence a fortiori $\mathcal{O}(N_0)$-convex. Since $\theta \colon \Omega \to \theta(\Omega)$ is a biholomorphism, it follows that the set $\theta(\psi_\epsilon(L))$ is $\mathcal{O}(N)$-convex and hence also $\mathcal{O}(Y)$-convex (since N is $\mathcal{O}(Y)$-convex).

Consider the isotopy of injective holomorphic maps σ_t for $t \in [\epsilon, 1]$, defined on an open neighborhood of $\theta(K)$ in Y by the condition

$$\theta \circ \psi_t = \sigma_t \circ \theta, \qquad t \in [\epsilon, 1]. \tag{2.2}$$

Note that the following hold:

(a) σ_1 is the identity map, and

(b) for every $t \in [\epsilon, 1]$ the compact set $\sigma_t(\theta(K)) \subset Y$ is $\mathcal{O}(Y)$-convex.

Condition (b) holds because $\psi_t(K) \subset K$ is clearly $\mathcal{O}(E)$-convex, so the set $\sigma_t(\theta(K)) = \theta(\psi_t(K))$ is $\mathcal{O}(\theta(K))$-convex and hence $\mathcal{O}(Y)$-convex (since $\theta(K)$ is $\mathcal{O}(Y)$-convex).

By the Andersén-Lempert-Forstnerič-Rosay-Varolin theorem (see [12, Theorem 2.1] for $Y = \mathbb{C}^n$ and [14, Theorem 4.10.5] for the general case), the map σ_ϵ can be approximated uniformly on a neighborhood of $\theta(K)$ by holomorphic automorphisms $\phi \in \mathrm{Aut}(Y)$.

Since $\psi_\epsilon(L \cup X) = \psi_\epsilon(L) \cup X \subset \Omega$ by the choice of ϵ, there is an open neighborhood $\widetilde\Omega \subset E$ of $L \cup X$ such that $\psi_\epsilon(\widetilde\Omega) \subset \Omega$. We claim that the holomorphic

embedding

$$\tilde{\theta} := \phi^{-1} \circ \theta \circ \psi_\epsilon : \widetilde{\Omega} \hookrightarrow Y \tag{2.3}$$

satisfies the lemma provided that ϕ is chosen close enough to σ_ϵ on a fixed neighborhood of $\theta(K)$. Indeed, since the sets $\theta(\psi_\epsilon(L))$ and $\theta(\psi_\epsilon(K))$ are $\mathcal{O}(Y)$-convex and ϕ is an automorphism of Y, the sets $\tilde{\theta}(L)$ and $\tilde{\theta}(K)$ are also $\mathcal{O}(Y)$-convex. Furthermore, $\tilde{\theta}|_X = \phi^{-1} \circ \theta|_X : X \hookrightarrow Y$ is a Runge embedding since $\theta|_X$ is. Finally, on the set K we have in view of (2.2) that

$$\tilde{\theta} = \phi^{-1} \circ \theta \circ \psi_\epsilon = \phi^{-1} \circ \sigma_\epsilon \circ \theta.$$

Since the map $\phi^{-1} \circ \sigma_\epsilon$ is close to the identity on $\theta(K)$ by the choice of ϕ, it follows that $\tilde{\theta}$ is close to θ on K. This proves the first part of the lemma.

Assume now that $Y = \mathbb{C}^n$ and that $\theta(X) = A \subset \mathbb{C}^n$ is a closed algebraic submanifold, where $n \geq \dim A + 2$. By Kaliman and Kutzschebauch [17, Theorem 6], the Lie algebra of algebraic vector fields on $\mathbb{C}^n$ vanishing on A enjoys the algebraic density property. (This means that every algebraic vector field on $\mathbb{C}^n$ vanishing on A can be expressed by sums and commutators of complete algebraic vector fields on $\mathbb{C}^n$ vanishing on A. Indeed, one may use shear vector fields vanishing on A.) This implies (see [14, Proposition 4.10.4]) that the flow of any algebraic vector field vanishing on A can be approximated on each compact polynomially convex subset by holomorphic automorphisms of $\mathbb{C}^n$ fixing A pointwise.

Note that, up to a change of the t-parameter, the isotopy ψ_t (2.1) is the flow of a holomorphic vector field $\mathcal{V}$ on E, tangent to the fibres of the projection $E \to X$ and vanishing on the zero section $X \subset E$. Hence, the isotopy σ_t defined by (2.2) is also the flow of a holomorphic vector field $\mathcal{W}$ on a neighborhood of $\theta(K)$ in $\mathbb{C}^n$ that vanishes on the algebraic submanifold $A = \theta(X) \subset \mathbb{C}^n$. (Indeed, $\mathcal{W} = \theta_*(\mathcal{V})$ is the push-forward of $\mathcal{V}$ by the embedding θ.) By Serre's Theorems A and B [20] we can approximate $\mathcal{W}$ as closely as desired on a neighborhood of the compact polynomially convex set $\theta(K)$ by an algebraic vector field vanishing on A. By what has been said above, this shows that the map σ_ϵ can be approximated uniformly on a neighborhood of $\theta(K)$ by holomorphic automorphisms $\phi \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying $\phi(z) = z$ for all $z \in \theta(X)$. The proof is now completed just as before. In particular, we see from (2.3) that the embedding $\tilde{\theta}$ agrees with θ on X. □

Proof of Theorem 1.1. Pick an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = E$ by compact radial $\mathcal{O}(E)$-convex sets. In fact, we may choose each K_j of the form

$$K_j = \{(x, v) \in E : \varphi_1(x) \leq j \quad \text{and} \quad |v|_{\varphi_2} \leq j\},$$

where φ_1 is a strongly plurisubharmonic exhaustion function on X and φ_2 is a suitably chosen hermitian metric on E. Let $\theta : X \hookrightarrow Y$ be a holomorphic Runge embedding. By a theorem of Docquier and Grauert (see [14, Theorem 3.3.3]) there is a neighborhood $\Omega_0 \subset E$ of the zero section $X \subset E$ such that θ extends to a holomorphic embedding $\theta_0 : \Omega_0 \hookrightarrow Y$. Set $K_0 = \varnothing$. By applying Lemma 2.1 inductively, we find a sequence of open neighborhoods $\Omega_j \subset E$ of $K_j \cup X$ and holomorphic embeddings $\theta_j : \Omega_j \hookrightarrow Y$ satisfying the following conditions for every $j \in \mathbb{N}$:

(a) the compact sets $\theta_j(K_j)$ and $\theta_j(K_{j-1})$ are $\mathcal{O}(Y)$-convex,
(b) the embedding $\theta_j|_X : X \hookrightarrow Y$ is Runge, and
(c) θ_j approximates θ_{j-1} as closely as desired on K_{j-1}.

If the approximations are close enough, the sequence θ_j converges uniformly on compacts in E to a holomorphic embedding $\tilde{\theta}\colon E \hookrightarrow Y$. Since $\mathcal{O}(Y)$-convexity of a compact set in a Stein manifold Y is a stable property for compact strongly pseudoconvex domains (see [10]) and every compact $\mathcal{O}(Y)$-convex set can be approximated from the outside by such domains, it follows that the image of each set K_j remains $\mathcal{O}(Y)$-convex in the limit provided that all approximations were close enough. Hence, $\tilde{\theta}(E)$ is a Runge domain in Y. □

Proof of Theorem 1.4. We follow the proof of Theorem 1.1. By the second part of Lemma 2.1, the sequence of embeddings $\theta_j\colon \Omega_j \hookrightarrow \mathbb{C}^n$ can now be chosen such that, in addition to the above, we have that $\theta_j|_X = \theta_{j-1}|_X$ holds for all $j \in \mathbb{N}$. This ensures that the limit embedding $\tilde{\theta} = \lim_{j\to\infty} \theta_j\colon E \hookrightarrow \mathbb{C}^n$ also satisfies $\tilde{\theta}|_X = \theta|_X$. □

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