



The Calabi–Yau Property of Superminimal Surfaces in Self-Dual Einstein Four-Manifolds

Franz Forstnerič^{1,2} 

Received: 15 April 2020 / Published online: 2 July 2020
© Mathematica Josephina, Inc. 2020

Abstract

In this paper, we show that if (X, g) is an oriented four-dimensional Einstein manifold which is self-dual or anti-self-dual then superminimal surfaces in X of appropriate spin enjoy the Calabi–Yau property, meaning that every immersed surface of this type from a bordered Riemann surface can be uniformly approximated by complete superminimal surfaces with Jordan boundaries. The proof uses the theory of twistor spaces and the Calabi–Yau property of holomorphic Legendrian curves in complex contact manifolds.

Keywords Superminimal surface · Einstein manifold · Twistor space · Complex contact manifold · Holomorphic Legendrian curve

Mathematics Subject Classification Primary 53A10 · 53C25 · 53C26. Secondary 32E30 · 37J55

1 Introduction

It has been known since the 1980s that four-dimensional self-dual Einstein manifolds have a rich theory of *superminimal surfaces*. In the present paper we provide further evidence by showing that such surfaces enjoy the *Calabi–Yau property*; see Theorems 1.2 and 5.3. The latter term was introduced in the recent paper by Alarcón et al. [6, Definition 6.1]. The motivation comes from the classical problem posed by Calabi in 1965 (see [22, p. 170] and [18, p. 212]) and in a more precise form by Yau in 2000 (see [61, p. 360] and [62, p. 241]), asking which open Riemann surfaces

✉ Franz Forstnerič
franc.forstneric@fmf.uni-lj.si

¹ Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

² Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

admit complete conformal minimal immersions with bounded images into Euclidean spaces \mathbb{R}^n , $n \geq 3$, and what is the possible boundary behavior of such surfaces. For the history of this subject and some recent developments, see the survey [2] and the papers [1,3,7].

Superminimal surfaces form an interesting class of minimal surfaces in four-dimensional Riemannian manifolds. Although this term was coined by Bryant in his study [15] of such surfaces in the four-sphere S^4 and their relationship to holomorphic Legendrian curves in \mathbb{CP}^3 , the Penrose twistor space of S^4 , it soon became clear through the work of Friedrich [30,31] that this class of minimal surfaces was described geometrically already by Kommerell in his 1897 dissertation [39] and his 1905 paper [41], and they were subsequently studied by Eisenhart [26], Borōvka [13,14], Calabi [16], and Chern [19,20], among others; see Sect. 2. Unfortunately, at least three different definitions are used in the literature. We adopt the original geometric definition of Kommerell [39] (see also Friedrich [31, Sect. 1]) and explain the role of *spin* in this context.

Assume that (X, g) is a Riemannian four-manifold and $M \subset X$ is a smoothly embedded surface with the induced conformal structure. (Our considerations, being of local nature, will also apply to immersed surfaces.) Then $TX|_M = TM \oplus N$ where $N = N(M)$ is the orthogonal normal bundle to M . A unit normal vector $n \in N_x$ at a point $x \in M$ determines a *second fundamental form* $S_x(n) : T_x M \rightarrow T_x M$, a self-adjoint linear operator on the tangent space of M . For a fixed tangent vector $v \in T_x M$ we consider the closed curve

$$I_x(v) = \{S_x(n)v : n \in N_x, |n|_g = 1\} \subset T_x M. \quad (1.1)$$

Suppose now that M and X are oriented, and coorient the normal bundle N accordingly.

Definition 1.1 A smooth oriented embedded surface M in an oriented Riemannian four-manifold (X, g) is *superminimal of positive (negative) spin* if for every point $x \in M$ and unit tangent vector $v \in T_x M$, the curve $I_x(v) \subset T_x M$ (1.1) is a circle centered at 0 and the map $n \rightarrow S(n)v \in I_x(v)$: $(n \in N_x)$ is orientation preserving (resp. orientation reversing). The last condition is void at points $x \in M$ where the circle $I_x(v)$ reduces to $0 \in T_x M$. The analogous definition applies to a smoothly immersed oriented surface $f : M \rightarrow X$.

Every superminimal surface is a minimal surface; see Friedrich [31, Proposition 3] and the discussion in Sect. 2. The converse is not true except in special cases, see Remark 4.10. The notion of spin, which is only implicitly present in Friedrich’s discussion, is very important in the *Bryant correspondence* described in Theorem 4.6.

The surface M in Definition 1.1 is endowed with the conformal structure which renders the given immersion $M \rightarrow X$ conformal. In the sequel we prefer to work with a fixed conformal structure on M and consider only conformal immersions $M \rightarrow X$. Since M is also oriented, it is a Riemann surface. We denote by $SM^\pm(M, X)$ the spaces of smooth conformal superminimal immersions of positive and negative spin, respectively, and set

$$\mathrm{SM}(M, X) = \mathrm{SM}^+(M, X) \cup \mathrm{SM}^-(M, X). \quad (1.2)$$

The intersection $\mathrm{SM}^+(M, X) \cap \mathrm{SM}^-(M, X)$ of these two spaces consists of immersions for which all circles $I_x(v)$ (1.1) reduce to points; such surfaces are minimal with vanishing normal curvature, hence totally geodesic (see [31]).

Recall that a (finite) bordered Riemann surface is a domain of the form $M = R \setminus \bigcup_i \Delta_i$, where R is a compact Riemann surface and Δ_i are finitely many compact pairwise disjoint discs with smooth boundaries $b\Delta_i$, diffeomorphic images of $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Its closure \bar{M} is a compact bordered Riemann surface. The definition of superminimality clearly applies to smooth conformal immersions $\bar{M} \rightarrow X$ and the notation (1.2) shall be used accordingly.

The following is our first main result; see also Theorem 5.3.

Theorem 1.2 *Let (X, g) be an oriented four-dimensional Einstein manifold whose Weyl tensor $W = W^+ + W^-$ satisfies $W^+ = 0$ or $W^- = 0$. Given any bordered Riemann surface M and a conformal superminimal immersion $f_0 \in \mathrm{SM}^\pm(\bar{M}, X)$ of class \mathcal{C}^3 (with the respective choice of sign \pm), we can approximate f_0 uniformly on \bar{M} by continuous maps $f : \bar{M} \rightarrow X$ such that $f : M \rightarrow X$ is a complete conformal superminimal immersion in $\mathrm{SM}^\pm(M, X)$ and $f : bM \rightarrow X$ is a topological embedding.*

Recall that an immersion $f : M \rightarrow (X, g)$ is said to be *complete* if the Riemannian metric f^*g induced by the immersion is a complete metric on M ; equivalently, for any divergent path $\lambda : [0, 1) \rightarrow M$ (i.e., such that $\lambda(t)$ leaves any compact subset of M as $t \rightarrow 1$) the path $f \circ \lambda : [0, 1) \rightarrow X$ has infinite length: $\int_0^1 \left| \frac{d(f \circ \lambda(t))}{dt} \right|_g dt = +\infty$.

Note that our result is local in the sense that the complete conformal superminimal immersion stays uniformly close to the given superminimal surface. Hence, if Theorem 1.2 holds for a Riemannian manifold X then it also holds for every open domain in X .

Recall (see Atiyah et al. [8, p. 427]) that the Weyl tensor $W = W^+ + W^-$ is the conformally invariant part of the curvature tensor of a Riemannian four-manifold (X, g) , so it only depends on the conformal class of the metric. The manifold is called *self-dual* if $W^- = 0$, and *anti-self-dual* if $W^+ = 0$. Note that $W = 0$ if and only if the metric is conformally flat. A Riemannian manifold (X, g) is called an *Einstein manifold* if the Ricci tensor of g is proportional to the metric, $\mathrm{Ric}_g = kg$ for some constant $k \in \mathbb{R}$. The curvature tensor of g then reduces to the constant scalar curvature (the trace of the Ricci curvature, hence $4k$ when $\dim X = 4$) and the Weyl tensor W (see [8, p. 427]). The Einstein condition is equivalent to the metric being a solution of the vacuum Einstein field equations with a cosmological constant, although the signature of the metric can be arbitrary in this setting, thus not being restricted to the four-dimensional Lorentzian manifolds studied in general relativity. Self-dual Einstein four-manifolds are important as *gravitational instantons* in quantum theories of gravity. A classical reference is the monograph [10] by Besse. The role of these conditions in Theorem 1.2 will be clarified by Theorems 4.11 and 4.12.

The analogue of Theorem 1.2 also holds for bordered Riemann surfaces with countably many boundary curves; see Theorem 5.3. Every such surface is an open domain

$$M = R \setminus \bigcup_{i=0}^{\infty} D_i \quad (1.3)$$

in a compact Riemann surface R , where $D_i \subset R$ are pairwise disjoint smoothly bounded closed discs. By the uniformisation theorem of He and Schramm [36], every open Riemann surface of finite genus and having at most countably many ends is conformally equivalent to a surface of the form (1.3), where D_i lift to round discs or points in the universal covering surface of R . This gives the following corollary to Theorems 1.2 and 5.3.

Corollary 1.3 *Every self-dual or anti-self-dual Einstein four-manifold contains a complete conformally immersed superminimal surface with Jordan boundary parameterized by any given bordered Riemann surface with finitely or countably many boundary curves.*

In particular, every open Riemann surface of finite genus and having at most countably many ends, none of which are point ends, is conformally equivalent to a complete conformal superminimal surface in any self-dual or anti-self-dual Einstein four-manifold.

It is in general impossible to ensure completeness of a minimal surface at a point end unless (X, g) is complete and the immersion $M \rightarrow X$ is proper at such end.

The special case of Theorem 1.2 when X is the four-sphere S^4 is given by [29, Corollary 1.10]; see also [5, Theorem 7.5]. Since the spherical metric is conformally flat, the Weyl tensor vanishes and Theorem 1.2 applies to superminimal surfaces of both positive and negative spin in S^4 . The same holds for the hyperbolic 4-space H^4 ; see Corollary 6.3. While S^4 admits plenty of superminimal surfaces of any given conformal type (see [5, Corollary 7.3]), every minimal surface in H^4 is uniformised by the disc \mathbb{D} (see Corollary 6.3).

A natural question at this point is, how many Riemannian four-manifolds (X, g) are there satisfying the conditions in Theorem 1.2? Among the complete ones with positive scalar curvature, there are not many. The classical Bonnet–Myers theorem (see Myers [45] or do Carmo [23, p. 200]) states that if the Ricci curvature of an n -dimensional complete Riemannian manifold (X, g) is bounded from below by a positive constant, then it has finite diameter and hence X is compact. Further, a theorem of Friedrich and Kurke [32] from 1982 says that a compact self-dual Einstein four-manifold with positive scalar curvature is either isometric to S^4 or diffeomorphic to the complex projective plane \mathbb{CP}^2 . Superminimal surfaces in S^4 and \mathbb{CP}^2 with their natural metrics have been studied extensively; see [12, 15, 33, 34, 44]. Hitchin [37] described in 1974 the topological type all four-dimensional compact self-dual Einstein manifolds with vanishing scalar curvature. He proved that such a space is either flat or a $K3$ -surface, an Enriques surface, or the orbit space of an Enriques surface by an antiholomorphic involution. Conversely, it follows from the solution of the Calabi conjecture by Yau [59, 60] that every $K3$ surface admits a self-dual Einstein metric

($W^- = 0$) with vanishing scalar curvature. On the other hand, there are many self-dual Einstein manifolds with negative scalar curvature including all real and complex space forms. In particular, there is an infinite dimensional family of self-dual Einstein metrics with scalar curvature -1 on the unit ball $\mathbb{B} \subset \mathbb{R}^4$ having prescribed conformal structure of a suitable kind on the boundary sphere $S^3 = b\mathbb{B}$; see Graham and Lee [35], Hitchin [38], and Biquard [11]. Another construction of an infinite dimensional family of self-dual Einstein metrics was given by Donaldson and Fine [24] and Fine [27]. It was shown by Derdzinski [21] that a compact four-dimensional self-dual Kähler manifold is locally symmetric.

In the remainder of this introduction, we outline the proof of Theorem 1.2; the details are provided in Sect. 5. In Sects. 2–4 we provide a sufficiently complete account of the necessary ingredients from the theory of superminimal surfaces and twistor spaces to make the paper accessible to a wide audience. Several different definitions of superminimal surfaces are used in the literature, and hence statement which are formally the same need not be equivalent. We take care to present a coherent picture to an uninitiated reader with basic knowledge of complex analysis and Riemannian geometry.

We shall use three key ingredients. The first two are provided by the *twistor theory* initiated by Penrose [47] in 1967. One of its main features from mathematical viewpoint is that it provides harmonic maps from a given Riemann surface M into a Riemannian four-manifold (X, g) as projections of suitable holomorphic maps $M \rightarrow Z$ into the total space of the twistor bundle $\pi : Z \rightarrow X$. Although this idea is reminiscent of the Enneper-Weierstrass formula for minimal surfaces in flat Euclidean spaces (see Osserman [46]), it differs from it in certain key aspects. There are two twistor spaces $\pi^\pm : Z^\pm \rightarrow X$, reflecting the spin (see Sect. 4). Their total spaces Z^\pm carry natural almost complex structures J^\pm (nonintegrable in general), and the fibers of π^\pm are holomorphic rational curves in Z^\pm . The Levi-Civita connection of (X, g) determines a complex horizontal subbundle $\xi^\pm \subset TZ^\pm$ projecting by $d\pi^\pm$ isomorphically onto the tangent bundle of X . The key point of twistor theory pertaining to our paper is the *Bryant correspondence*; see Theorem 4.6. This correspondence, discovered by Bryant [15] in the case when X is the four-sphere S^4 (whose twistor spaces Z^\pm are the three-dimensional complex projective space \mathbb{CP}^3 , see Sect. 6 for an elementary explanation), shows that superminimal surfaces in X of \pm spin are precisely the projections of holomorphic horizontal curves in Z^\pm , i.e., curves tangent to the horizontal distribution ξ^\pm .

The second ingredient is provided by a couple of classical integrability results. According to Atiyah et al. [8, Theorem 4.1], the twistor space (Z^\pm, J^\pm) of a smooth oriented Riemannian four-manifold (X, g) is an integrable complex manifold if and only if the conformally invariant Weil tensor $W = W^+ + W^-$ of g satisfies $W^+ = 0$ or $W^- = 0$, respectively. Assuming that this holds, a result of Salamon [51, Theorem 10.1] (see also Eells and Salamon [25, Theorem 4.2]) says that the horizontal bundle ξ^\pm is a holomorphic hyperplane subbundle of TZ^\pm if and only if g is an Einstein metric, and in such case ξ^\pm is a holomorphic contact bundle if and only if the scalar curvature of g is nonzero.

The third main ingredient is a recent result of Alarcón and the author [1, Theorem 1.3] saying that holomorphic Legendrian immersions from bordered Riemann sur-

faces into any holomorphic contact manifold enjoy the Calabi–Yau property, i.e., the analogue of Theorem 1.2 holds for such immersions. (See also [5, Theorem 1.2] for the standard complex contact structure on Euclidean spaces \mathbb{C}^{2n+1} , $n \geq 1$.) Analogous results hold for holomorphic immersions into any complex manifold of dimension > 1 , and for conformal minimal immersions into the flat Euclidean space \mathbb{R}^n for any $n \geq 3$. We refer to the recent survey [2] for an account of these developments. The proof of Theorem 1.2 is then completed and generalized to surfaces M with countably many boundary curves in Sect. 5. In Sect. 6 we take a closer look at the case when X is the sphere S^4 or the hyperbolic space H^4 .

2 Superminimal Surfaces in Riemannian 4-Manifolds

In this section we recall the notion of the indicatrix of a smooth surface in a smooth Riemannian four-manifold (X, g) and the geometric definition of a superminimal surface. We follow the paper by Friedrich [31] from 1997.

Let $M \subset X$ be a smoothly embedded surface endowed with the induced metric. (Since our considerations in this section are local, they also apply to immersions $M \rightarrow X$.) The tangent bundle of X splits along M into the orthogonal direct sum $TX|_M = TM \oplus N$ where N is the normal bundle of M in X . Given a point $p \in M$ we let

$$\text{Sym}(T_p M) = \{A : T_p M \rightarrow T_p M : g(Au, v) = g(u, Av) \text{ for all } u, v \in T_p M\}$$

denote the three-dimensional real vector space of linear symmetric self-maps of $T_p M$. Fixing an orthonormal basis of $T_p M$, we identify $\text{Sym}(T_p M) \cong \text{Sym}(\mathbb{R}^2)$ with the space of real symmetric 2×2 matrices and introduce the isometry $\text{Sym}(T_p M) \xrightarrow{\cong} \mathbb{R}^3$ by

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto \left(\frac{a+c}{\sqrt{2}}, \sqrt{2}b, \frac{a-c}{\sqrt{2}} \right).$$

Each unit normal vector $n \in N_p$, $|n|^2 := g(n, n) = 1$, determines a second fundamental form $S_p(n) : T_p M \rightarrow T_p M$ which belongs to $\text{Sym}(T_p M)$. The unit normal vectors form a circle in the normal plane N_p to M at p , and the curve

$$I_p = \{S_p(n) : n \in N_p, |n| = 1\} \subset \text{Sym}(T_p M) \cong \mathbb{R}^3 \quad (2.1)$$

is called the *indicatrix* of M at p . It was shown by Kommerell [41] that $I_p \subset \mathbb{R}^3$ is either a straight line segment which is symmetric around the origin $0 \in \mathbb{R}^3$ (possibly reducing to 0) or the intersection of a cylinder over an ellipse and a two plane. If M is a minimal surface in X then I_p is a symmetric segment, an ellipse, or a circle; see Kommerell [41] and Eisenhart [26]. For a fixed tangent vector $v \in T_p M$ we also consider the curve

$$I_p(v) = \{S_p(n)v : n \in N_p, |n| = 1\} \subset T_p M. \quad (2.2)$$

Definition 2.1 A smooth surface $M \subset X$ is *superminimal* if every curve $I_p(v) \subset T_p M$ ($p \in M$, $0 \neq v \in T_p M$) is a circle with center 0 (which may reduce to the origin). The same definition applies to a conformally immersed surface $f : M \rightarrow X$.

Remark 2.2 (A) A calculation in [31, pp. 2–3] shows that the indicatrix I_p (2.1) of a superminimal surface $M \subset X$ at any point $p \in M$ is a circle in $\text{Sym}(T_p M) \cong \mathbb{R}^3$ with center 0, and every superminimal surface is a minimal surface (see [31, Proposition 3]). The converse fails in general, but see Remark 4.10 for some special cases.

(B) The above definition does not require orientability. If M and X are oriented, then we can introduce *superminimal surfaces of positive or negative spin* by looking at the direction of the rotation of the point $S_p(n)v \in I_p(v) \subset T_p M$ as the unit normal vector $n \in N_p$ traces the unit circle in a given direction. This gives the two spaces $\text{SM}^\pm(M, X)$ in Definition 1.1 which get interchanged under the reversal of the orientation on X .

(C) The class of superminimal surfaces is invariant under isometries of (X, g) . \square

Superminimal surfaces have been studied by many authors; see in particular Kommerell [41], Eisenhart [26], Borōvka [13, 14], Calabi [16], Chern [19, 20], Bryant [15], Friedrich [30, 31], Eells and Salamon [25], Gauduchon [33, 34], Wood [58], Montiel and Urbano [44], Bolton and Woodward [12], Shen [54, 55], and Baird and Wood [9]. A recent contribution to the theory of superminimal surfaces in S^4 was made in [6, Sect. 7].

3 Almost Hermitian Structures on \mathbb{R}^4 and Quaternions

In this section we recall some basic facts about linear almost Hermitian structures on \mathbb{R}^4 and their representation by quaternionic multiplication. This material is standard (see e.g. [8, 25]), except for Lemma 3.1 which will be used in Sect. 6.

Let $\langle \cdot, \cdot \rangle$ stand for the Euclidean inner product on \mathbb{R}^4 . We denote by $\mathcal{J}^\pm(\mathbb{R}^4)$ the space of *almost Hermitian structures* on \mathbb{R}^4 , i.e., linear operators $J : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ satisfying the following three conditions:

- (a) $J^2 = -\text{Id}$,
- (b) $\langle Jx, Jy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^4$, and
- (c) letting $\omega(x, y) = \langle Jx, y \rangle$ denote the fundamental form of J , we have that $\omega \wedge \omega = \pm \Omega$ where Ω is the standard volume form on \mathbb{R}^4 with its canonical orientation.

Condition (a) lets us identify \mathbb{R}^4 with \mathbb{C}^2 such that J corresponds to the multiplication by i on \mathbb{C}^2 ; any such linear operator is called a (linear) *almost complex structure* on \mathbb{R}^4 . The second condition means that J is compatible with the inner product on \mathbb{R}^4 , hence the word *almost Hermitian*. The third condition specifies the orientation of J . Note that

$$\mathcal{J}^+(\mathbb{R}^4) \cup \mathcal{J}^-(\mathbb{R}^4) \subset SO(4).$$

Any choice of positively oriented orthonormal basis $e = (e_1, e_2, e_3, e_4)$ of \mathbb{R}^4 determines a pair of almost Hermitian structures $J_e^\pm \in \mathcal{J}^\pm(\mathbb{R}^4)$ by

$$J_e^\pm(e_1) = e_2, \quad J_e^\pm(e_3) = \pm e_4. \quad (3.1)$$

If $e' = (e'_1, e'_2, e'_3, e'_4)$ is another orthonormal basis in the same orientation class, there is a unique $A \in SO(4)$ mapping e_i to e'_i for $i = 1, \dots, 4$, and hence

$$J_e^\pm = A^{-1} \circ J_{e'}^\pm \circ A.$$

This shows that for any fixed $J \in \mathcal{J}^+(\mathbb{R}^4)$, conjugation $A \mapsto A^{-1} \circ J \circ A$ by orthogonal rotations $A \in SO(4)$ acts transitively on $\mathcal{J}^+(\mathbb{R}^4)$; the corresponding property also holds for $\mathcal{J}^-(\mathbb{R}^4)$. The stabiliser of this action is the unitary group $U(2)$, the group of orthogonal rotations preserving the given structure J , and $\mathcal{J}^\pm(\mathbb{R}^4)$ can be identified with the quotient $SO(4)/U(2) \cong S^2$. Conjugation by an element $A \in O(4)$ of the orthogonal group with $\det A = -1$ interchanges $\mathcal{J}^+(\mathbb{R}^4)$ and $\mathcal{J}^-(\mathbb{R}^4)$, and $O(4)/U(2) \cong \mathcal{J}^+(\mathbb{R}^4) \cup \mathcal{J}^-(\mathbb{R}^4)$. For instance, the two structures in (3.1) are interchanged by the orientation reversing map $A \in O(4)$ given by $Ae_1 = e_1$, $Ae_2 = e_2$, $Ae_3 = e_4$, $Ae_4 = e_3$. Note however that the structures $\pm J$ belong to the same space $\mathcal{J}^\pm(\mathbb{R}^4)$.

It is classical that every $A \in SO(4)$ is represented by a pair of rotations for angles $\alpha, \beta \in (-\pi, +\pi]$ in orthogonal cooriented 2-planes $\Sigma \oplus \Sigma^\perp = \mathbb{R}^4$. (Such pair of planes is uniquely determined by A if and only if $|\alpha| \neq |\beta|$.) The rotation A is said to be *left isoclinic* if $\alpha = \beta$ (it rotates for the same angle in the same direction on both planes), and *right isoclinic* if $\alpha = -\beta$ (it rotates for the same angle but in the opposite directions). Thus, elements of $\mathcal{J}^+(\mathbb{R}^4)$ are precisely the left isoclinic rotations for the angle $\pi/2$, while those in $\mathcal{J}^-(\mathbb{R}^4)$ are the right isoclinic rotations for the angle $\pi/2$.

Here is another interpretation of the spaces $\mathcal{J}^\pm(\mathbb{R}^4)$; see Atiyah et al. [8, Sect. 1] or Eells and Salamon [25, Sect. 2]. Let $\Lambda^2(\mathbb{R}^4)$ denote the second exterior power of \mathbb{R}^4 . For any oriented orthonormal basis e_1, \dots, e_4 of \mathbb{R}^4 the vectors $e_i \wedge e_j$ for $1 \leq i < j \leq 4$ form an orthonormal basis of $\Lambda^2(\mathbb{R}^4)$, so $\dim_{\mathbb{R}} \Lambda^2(\mathbb{R}^4) = 6$. The Hodge star endomorphism $*$: $\Lambda^2(\mathbb{R}^4) \rightarrow \Lambda^2(\mathbb{R}^4)$ is defined by $\alpha \wedge * \beta = \langle \alpha, \beta \rangle \Omega \in \Lambda^4(\mathbb{R}^4)$. We have that $*^2 = 1$, and the ± 1 eigenspace $\Lambda_\pm^2(\mathbb{R}^4)$ of $*$ has an oriented orthonormal basis

$$e_1 \wedge e_2 \pm e_3 \wedge e_4, \quad e_1 \wedge e_3 \pm e_4 \wedge e_2, \quad e_1 \wedge e_4 \pm e_2 \wedge e_3. \quad (3.2)$$

The Euclidean metric lets us identify \mathbb{R}^4 with its dual $(\mathbb{R}^4)^*$, which gives the inclusion

$$\Lambda^2(\mathbb{R}^4) \hookrightarrow \mathbb{R}^4 \otimes \mathbb{R}^4 \cong (\mathbb{R}^4)^* \otimes \mathbb{R}^4 = \text{End}(\mathbb{R}^4) \cong GL_4(\mathbb{R}). \quad (3.3)$$

Under this identification of $\Lambda^2(\mathbb{R}^4)$ with a subset of $\text{End}(\mathbb{R}^4)$, we have that

$$\mathcal{J}^\pm(\mathbb{R}^4) = S(\Lambda_\pm^2(\mathbb{R}^4)) := \text{the unit sphere of } \Lambda_\pm^2(\mathbb{R}^4) \cong \mathbb{R}^3. \quad (3.4)$$

For example, the vector $e = e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda_+^2(\mathbb{R}^4)$ is sent under the first inclusion in (3.3) to $e_1 \otimes e_2 - e_2 \otimes e_1 + e_3 \otimes e_4 - e_4 \otimes e_3 \in \mathbb{R}^4 \otimes \mathbb{R}^4$, and under the second isomorphism in (3.3) to the almost Hermitian structure given by (3.1):

$$J_e = e_1^* \otimes e_2 - e_2^* \otimes e_1 + e_3^* \otimes e_4 - e_4^* \otimes e_3 \in \mathcal{J}^+(\mathbb{R}^4).$$

We adopt the following convention regarding the orientations. (This essential point in the construction of twistor spaces is difficult to find spelled out in the literature.)

Orientation on $\mathcal{J}^\pm(\mathbb{R}^4)$. Let $e = (e_1, e_2, e_3, e_4)$ be a positively oriented orthonormal basis of \mathbb{R}^4 , and let the spaces $\Lambda_\pm^2(\mathbb{R}^4) \cong \mathbb{R}^3$ be oriented by the pair of bases (3.2). We endow $\mathcal{J}^+(\mathbb{R}^4) = S(\Lambda_+^2(\mathbb{R}^4))$ with the *outward orientation* of the unit 2-sphere in $\Lambda_+^2(\mathbb{R}^4) \cong \mathbb{R}^3$, while $\mathcal{J}^-(\mathbb{R}^4) = S(\Lambda_-^2(\mathbb{R}^4))$ is given the *inward orientation*.

Letting $\overline{\mathbb{R}}^4$ denote \mathbb{R}^4 with the opposite orientation, it is easily checked that we have *orientation preserving* isometric isomorphisms

$$\mathcal{J}^\pm(\mathbb{R}^4) = S(\Lambda_\pm^2(\mathbb{R}^4)) \xrightarrow{\cong} S(\Lambda_\mp^2(\overline{\mathbb{R}}^4)) = \mathcal{J}^\mp(\overline{\mathbb{R}}^4).$$

An oriented 2-plane $\Sigma \subset \mathbb{R}^4$ determines a pair of almost Hermitian structures $J_\Sigma^\pm \in \mathcal{J}^\pm(\mathbb{R}^4)$ which rotate for $\pi/2$ in the positive direction on Σ and for $\pm\pi/2$ on its cooriented orthogonal complement Σ^\perp . Denoting by $G_2(\mathbb{R}^4)$ the Grassmann manifold of oriented 2-planes in \mathbb{R}^4 , we have that (cf. [25, p. 595])

$$G_2(\mathbb{R}^4) \cong S(\Lambda_+^2(\mathbb{R}^4)) \times S(\Lambda_-^2(\mathbb{R}^4)) = \mathcal{J}^+(\mathbb{R}^4) \times \mathcal{J}^-(\mathbb{R}^4). \quad (3.5)$$

Almost Hermitian structures on \mathbb{R}^4 can be represented by quaternionic multiplication. Let \mathbb{H} denote the field of quaternions. An element of \mathbb{H} is written uniquely as

$$q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = z_1 + z_2\mathbf{j}, \quad (3.6)$$

where $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $z_1 = x_1 + x_2\mathbf{i} \in \mathbb{C}$, $z_2 = x_3 + x_4\mathbf{i} \in \mathbb{C}$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternionic units satisfying

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

We identify \mathbb{R}^4 with \mathbb{H} using $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ as the standard positively oriented orthonormal basis. (Some authors write complex coefficients on the right in (3.6); due to noncommutativity this makes for certain differences in the constructions and formulas.) Recall that

$$\bar{q} = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}, \quad q\bar{q} = |q|^2 = \sum_{i=1}^4 x_i^2, \quad q^{-1} = \frac{\bar{q}}{|q|^2} \text{ if } q \neq 0, \quad \overline{pq} = \bar{q}\bar{p}.$$

By \mathbb{H}_0 we denote the real 3-dimensional subspace of purely imaginary quaternions:

$$\mathbb{H}_0 = \{q = x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} : x_2, x_3, x_4 \in \mathbb{R}\} \cong \mathbb{R}^3. \quad (3.7)$$

We also introduce the spheres of unit quaternions and imaginary unit quaternions:

$$\mathcal{S}^3 := \{q \in \mathbb{H} : |q| = 1\} \cong S^3, \quad \mathcal{S}^2 = \{q \in \mathbb{H}_0 : |q| = 1\} \cong S^2. \quad (3.8)$$

We take i, j, k as a positive orthonormal basis of \mathbb{H}_0 and orient the spheres $\mathcal{S}^2 \subset \mathbb{H}_0$ and $\mathcal{S}^3 \subset \mathbb{H}$ by the respective outward normal vector field. In particular, the vectors j, k are a positively oriented orthonormal basis of the tangent space $T_i \mathcal{S}^2$.

Elements of $\mathcal{J}^+(\mathbb{R}^4)$ and $\mathcal{J}^-(\mathbb{R}^4)$ then correspond to left and right multiplications, respectively, on $\mathbb{H} \cong \mathbb{R}^4$ by imaginary unit quaternions $q \in \mathcal{S}^2$. To see this, note that every $J \in \mathcal{J}^+(\mathbb{R}^4)$ is uniquely determined by its value $q = J(1)$ on the first basis vector; this value is orthogonal to 1 and of unit length, hence an element of the unit sphere $\mathcal{S}^2 \subset \mathbb{H}_0$ inside the 3-space of imaginary quaternions (3.8). The pair $1, q$ spans a 2-plane $\Sigma \subset \mathbb{H}$ whose orthogonal complement Σ^\perp is contained in the hyperplane \mathbb{H}_0 . The left multiplication by q on \mathbb{H} then amounts to a rotation for $\pi/2$ in the positive direction on Σ^\perp , while the right multiplication by q yields a rotation for $\pi/2$ in the negative direction on Σ^\perp . The left multiplication by i determines the standard structure $J_i(1) = i$, $J_i(j) = k$.

The following lemma will be used in Sect. 6 to provide an elementary explanation of the fact that \mathbb{CP}^3 is the twistor space of S^4 . The analogous result holds for $\mathcal{J}^-(\mathbb{R}^4)$ as seen by using the right multiplication on \mathbb{H} by nonzero quaternions.

Lemma 3.1 *For every $q \in \mathbb{H} \setminus \{0\}$ the left multiplication by \bar{q} on \mathbb{H} uniquely determines an almost Hermitian structure $J_q \in \mathcal{J}^+(\mathbb{R}^4)$ making the following diagram commute:*

$$\begin{array}{ccccccc} \mathbb{R}^4 & \xrightarrow{\cong} & \mathbb{H} & \xrightarrow{\bar{q} \cdot} & \mathbb{H} & \xrightarrow{\cong} & \mathbb{R}^4 \\ J_i \downarrow & & i \cdot \downarrow & & \downarrow \bar{q} i \bar{q}^{-1} & & \downarrow J_q \\ \mathbb{R}^4 & \xrightarrow{\cong} & \mathbb{H} & \xrightarrow{\bar{q} \cdot} & \mathbb{H} & \xrightarrow{\cong} & \mathbb{R}^4 \end{array}$$

The map $\mathbb{H} \setminus \{0\} \rightarrow \mathcal{J}^+(\mathbb{R}^4)$ given by $q \mapsto J_q$ is equivalent to the canonical projection $\mathbb{H} \setminus \{0\} = \mathbb{C}_*^2 \rightarrow \mathbb{CP}^1$ under an orientation preserving diffeomorphism $\mathcal{J}^+(\mathbb{R}^4) \rightarrow \mathbb{CP}^1$.

Proof From $q\bar{q} = |q|^2$ we see that $\bar{q}^{-1} = q/|q|^2$ and hence

$$\bar{q} i \bar{q}^{-1} = \frac{\bar{q}}{|q|^2} i q = q^{-1} i q \in \mathcal{S}^3.$$

For any $q_1, q_2 \in \mathbb{H}$ we have that $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ and hence

$$\overline{\bar{q} i \bar{q}^{-1}} = q^{-1} (-i) q = -q^{-1} i q,$$

so $q^{-1} i q \in \mathcal{S}^2$ is a purely imaginary unit quaternion. It follows that the left product by $q^{-1} i q$ on \mathbb{H} determines an almost Hermitian structure $J_q \in \mathcal{J}^+(\mathbb{R}^4)$.

Let us consider more closely the map

$$\Phi : \mathbb{H} \setminus \{0\} \rightarrow \mathcal{S}^2, \quad \Phi(q) = q^{-1} i q.$$

We have that $\Phi(q_1) = \Phi(q_2)$ if and only if

$$q_1^{-1} i q_1 = q_2^{-1} i q_2 \iff (q_2 q_1^{-1}) i = i (q_2 q_1^{-1}) \iff q_2 q_1^{-1} \in \mathbb{C}^*,$$

so the fibers of Φ are the punctured complex lines $\mathbb{C}^* q$ for $q \in \mathbb{H} \setminus \{0\}$.

We claim that Φ is a submersion. Since Φ is constant on the lines $\mathbb{C}^* q$, it suffices to show that $\Phi : \mathcal{S}^3 \rightarrow \mathcal{S}^2$ is a submersion. Fix $q \in \mathcal{S}^3$. For any $q' \in \mathbb{H}$ we have that

$$d\Phi_q(q') = \left. \frac{d}{dt} \right|_{t=0} \Phi(q + tq') = \left. \frac{d}{dt} \right|_{t=0} \overline{(q + tq')} i (q + tq') = \bar{q}' i q + \bar{q} i q'.$$

In particular,

$$d\Phi_q(jq) = 2\bar{q} \, \mathfrak{k} q, \quad d\Phi_q(\mathfrak{k} q) = -2\bar{q} \, j q.$$

These two vectors are clearly \mathbb{R} -linearly independent, so $d\Phi_q : T_q \mathcal{S}^3 \rightarrow T_{\Phi(q)} \mathcal{S}^2$ has rank 2 at each point. For $q = i$ we get that $\Phi(i) = i$ and $d\Phi_i(j) = 2j$, $d\Phi_i(\mathfrak{k}) = 2\mathfrak{k}$. Note that (j, \mathfrak{k}) is a positively oriented orthonormal basis of both $T_i \mathcal{S}^2$ and $T_{[1:0]} \mathbb{CP}^1$, the tangent space at the point $[1 : 0]$ to the projective line consisting of complex lines in $\mathbb{H} = \mathbb{C}^2$, with $[1 : 0] = \mathbb{C} \times \{0\}$. It follows that $\Phi = h \circ \phi$ where $\phi : \mathbb{C}_*^2 \rightarrow \mathbb{CP}^1$ is the canonical projection and $h : \mathbb{CP}^1 \rightarrow \mathcal{S}^2$ is an injective orientation preserving local diffeomorphism, hence an orientation preserving diffeomorphism onto \mathcal{S}^2 . (Surjectivity is easily seen by an explicit calculation.) Finally, we identify $\mathcal{J}^+(\mathbb{R}^4)$ with \mathcal{S}^2 acting on $\mathbb{R}^4 = \mathbb{H}$ by left multiplication; this identification is orientation preserving as well.

Note that the map $\Phi : \mathcal{S}^3 \cong \mathcal{S}^3 \rightarrow \mathcal{S}^2 \cong \mathcal{S}^2$ is the Hopf fibration with circle fibers $\{e^{it} q : t \in \mathbb{R}\} \cong S^1$, $q \in \mathcal{S}^3$. \square

4 Twistor Bundles and the Bryant Correspondence

In 1967, Penrose [47] introduced a new *twistor theory* with an immediate goal of studying representation theory of the 15-parameter Lie group of conformal coordinate transformations on four-dimensional Minkowski space leaving the light-cone invariant. (The mathematical ideas in Penrose's paper are in close relation to those developed in the notes [56] of the seminar conducted by Oswald Veblen and John von Neumann during 1935–1936.) One of his aims was to offer a possible path to understand quantum gravity; see Penrose and MacCallum [48]. Penrose also promoted the idea that twistor spaces should be the basic arena for physics from which space-time itself should emerge.

Mathematically, twistor theory connects four-dimensional Riemannian geometry to three-dimensional complex analysis. A basic example is the complex projective

three-space \mathbb{CP}^3 as the twistor space of S^4 with the spherical metric (see Penrose [47, Sect. VI], Bryant [15], and Sect. 6). Physically it is the space of massless particles with spin. Twistor theory evolved into a branch of mathematics and theoretical physics with applications to differential and integral geometry, nonlinear differential equations and representation theory and in physics to relativity and quantum field theory.

For the theory of twistor spaces, see in particular the papers by Atiyah, Hitchin and Singer [8], Friedrich [30], Eells and Salamon [25], Gauduchon [33,34], the monographs by Ward and Wells [57] and Baird and Wood [9], and the recent survey by Sergeev [53]. Twistor theory also exists for certain Riemannian manifolds of real dimension $4n$ for $n > 1$, in particular for quaternion-Kähler manifolds (see Salamon [50], LeBrun and Salamon [43], and LeBrun [42]).

Associated to an oriented Riemannian four-manifold (X, g) is a pair of almost Hermitian fiber bundles with fiber \mathbb{CP}^1 ,

$$Z^\pm(X) := \mathcal{J}^\pm(TX) = S(\Lambda_\pm^2(TX)) \xrightarrow{\pi^\pm} X,$$

the *positive* and the *negative twistor bundle* of X . The fiber over any point $x \in X$ equals

$$(\pi^\pm)^{-1}(x) = \mathcal{J}^\pm(T_x X) = S(\Lambda_\pm^2(T_x X)) \cong \mathbb{CP}^1,$$

the space of positive or negative almost Hermitian structures on $T_x X \cong \mathbb{R}^4$. (The second equality uses the identification (3.4).) The complex structure on $\mathcal{J}^\pm(T_x X) \cong S^2$ is specified by the choice of orientation in Sect. 3. A local trivialisation of $Z^\pm \rightarrow X$ is provided by an oriented orthonormal frame field $e(x) = (e_1(x), \dots, e_4(x))$ for TX on an open set $x \in U \subset X$. If $e'(x)$ is another such frame field on $U' \subset X$ then the transition map between the associated fiber bundle charts is given by conjugation with the field of linear maps $A(x) \in SO(T_x X) \cong SO(\mathbb{R}^4)$ sending $e(x)$ to $e'(x)$ for $x \in U \cap U'$.

The Levi-Civita connection associated to the metric g on X induces at any point $z \in Z^\pm$ a decomposition of the tangent space $T_z Z^\pm$ into the direct sum

$$T_z Z^\pm = T_z^h Z^\pm \oplus T_z^v Z^\pm = \xi_z^\pm \oplus T_z^v Z^\pm,$$

where $T_z^v Z^\pm = T_z \pi^{-1}(\pi(z))$ is the vertical tangent space (the tangent space to the fiber) and $\xi_z^\pm = T_z^h Z^\pm$ is the *horizontal space*. This defines a *horizontal subbundle* $\xi^\pm \subset TZ^\pm$ such that the differential $d\pi_z^\pm : \xi_z^\pm \rightarrow T_{\pi^\pm(z)} X$ is an isomorphism for each $z \in Z^\pm$. Every path $\gamma(t)$ in X with $\gamma(0) = x$ admits a unique horizontal lift $\lambda(t)$ in Z^\pm (tangent to ξ^\pm) with any given initial point $\lambda(0) = z \in (\pi^\pm)^{-1}(x) = \mathcal{J}^\pm(T_x X)$, obtained as the parallel transport of z with respect to the Levi-Civita connection. However, lifting a surface in X to a horizontal surface in Z^\pm is in general impossible due to noninvolutivity of ξ^\pm .

There is a natural almost complex structure J^\pm on Z^\pm determined by the condition that at each point $z \in Z$, J_z^\pm agrees with the standard almost complex structure on the

vertical space $T_z^v Z^\pm \cong T_z \mathbb{CP}^1$, while on the horizontal space ξ_z^\pm we have that

$$d\pi_z^\pm \circ J_z^\pm = z \circ d\pi_z^\pm. \quad (4.1)$$

It follows that ξ^\pm is a J^\pm -complex subbundle of the tangent bundle TZ^\pm . (The structure J^\pm introduced above is denoted J_1 in [8,25]; the second structure J_2^\pm is obtained by reversing the orientations on the fibers of twistor projections. As shown in [52], the structure J_2^\pm is never integrable, but is nevertheless interesting in view of [25, Theorem 5.3].)

Here is a summary of some basic properties of twistor bundles.

Proposition 4.1 (a) *Denoting by \bar{X} the Riemannian manifold X endowed with the same metric and the opposite orientation, we have that*

$$Z^+(\bar{X}) = Z^-(X), \quad Z^-(\bar{X}) = Z^+(X)$$

as Hermitian fiber bundles over X , and also as almost complex manifolds. In particular, their horizontal bundles and the respective almost complex structures on them agree.

- (b) *There are antiholomorphic involutions $\iota^\pm : Z^\pm \rightarrow Z^\pm$ preserving the fibers of $\pi^\pm : Z^\pm \rightarrow X^\pm$ and taking any $J \in \mathcal{J}^\pm(T_x X)$ to $-J \in \mathcal{J}^\pm(T_x X)$. (Identifying the fiber with \mathbb{CP}^1 , this is the map $z \mapsto -1/\bar{z}$ on each fiber.)*
- (c) *An orientation preserving isometry $\phi : X \rightarrow X$ lifts to holomorphic isometries $\Phi^\pm : Z^\pm \rightarrow Z^\pm$ preserving ξ^\pm such that $\pi^\pm \circ \Phi^\pm = \phi \circ \pi^\pm$. Moreover, the almost complex type of (Z^\pm, J^\pm) only depends on the conformal class of the metric on X , but the horizontal spaces ξ^\pm depend on the choice of metric in that class.*
- (d) *An orientation reversing isometry $\theta : X \rightarrow X$ lifts to a holomorphic isometry $\Theta : (Z^+(X), J^+) \rightarrow (Z^-(X), J^-)$ making the following diagram commute:*

$$\begin{array}{ccc} Z^+(X) & \xrightarrow{\Theta} & Z^-(X) \\ \pi^+ \downarrow & & \downarrow \pi^- \\ X & \xrightarrow{\theta} & X \end{array} \quad (4.2)$$

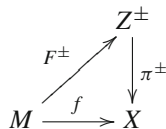
An example of (d) is the antipodal map on $X = S^4$, and in this case $Z^+(S^4) \cong Z^-(S^4) \cong \mathbb{CP}^3$ (see Bryant [15], Gauduchon [33, Sect. III], and Sect. 6).

Example 4.2 (A) The twistor bundle Z^+ of \mathbb{R}^4 with the Euclidean metric is fiberwise diffeomorphic to $\mathbb{R}^4 \times \mathbb{CP}^1$, and its horizontal distribution ξ is involutive with the leaves $\mathbb{R}^4 \times \{z\}$ for $z \in \mathbb{CP}^1$. The almost complex structure J^+ on Z^+ restricted to the leaf $L_z = \mathbb{R}^4 \times \{z\}$ equals $z \in \mathcal{J}^+(\mathbb{R}^4)$, and (L_z, z) is a complex manifold which is biholomorphic to \mathbb{C}^2 under a rotation in $SO(4)$. As a complex manifold, (Z^+, J^+) is biholomorphic to the total space of the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{CP}^1$, and the leaves L_z of ξ are the fibers of this projection. See [30, Remark 2, p. 266] for more details. \square

Recall from (3.5) that an oriented 2-plane $\Sigma \subset T_x X$ determines a pair of almost Hermitian structures $J_\Sigma^\pm \in \mathcal{J}^\pm(T_x X)$. Let M be an oriented surface. To any immersion $f : M \rightarrow X$ we associate the *twistor lifts* $F^\pm : M \rightarrow Z^\pm$ with $\pi^\pm \circ F^\pm = f$ by the condition that for any point $p \in M$ and $x = f(p) \in X$,

$$F^\pm(p) \in \mathcal{J}^\pm(T_x X) \text{ is determined by the oriented 2-plane } df_p(T_p M) \subset T_x X. \quad (4.3)$$

That is, $F^\pm(p)$ rotates for $+\pi/2$ in the oriented plane $\Sigma = df_p(T_p M)$ and for $\pm\pi/2$ in the cooriented orthogonal plane Σ^\perp .



Here is a more explicit description. Assume for simplicity that $M \subset X$ is embedded and let $TX|_M = TM \oplus N$ where N is the orthogonal normal bundle of M in X . Locally near any point $p \in M$ there is an oriented orthonormal frame field (e_1, e_2, e_3, e_4) for TX such that, along M , (e_1, e_2) is an oriented frame for TM while (e_3, e_4) is a frame for N . Then, F^\pm is determined by the conditions $F^\pm e_1 = e_2$, $F^\pm e_3 = \pm e_4$.

Remark 4.3 (A) The twistor lifts F^\pm clearly depend on the first order jet of f . Hence, if the immersion $f : M \rightarrow X$ is of class \mathcal{C}^r ($r \geq 1$) then $F^\pm : M \rightarrow Z^\pm$ are of class \mathcal{C}^{r-1} .

(B) If \tilde{M} is the Riemann surface M with the opposite orientation and $\tilde{F}^\pm : \tilde{M} \rightarrow Z^\pm$ denote the respective twistor lifts of $f : M \rightarrow X$, then $\tilde{F}^\pm = \iota^\pm \circ F^\pm$ where ι^\pm is the antiholomorphic involution on Z^\pm in Proposition 4.1 (B). \square

We have the following additional properties of twistor lifts of a *conformal* immersion. The second statement is the first part of [30, Proposition 3]; note however that in [30] an immersion $f : M \rightarrow X$ is tacitly assumed to be conformal.

Lemma 4.4 *If I is an almost complex structure on M and $f : (M, I) \rightarrow X$ is a conformal immersion, then $F^\pm(p) \in \mathcal{J}^\pm(T_{f(p)} X)$ ($p \in M$) is uniquely determined by the condition*

$$df_p \circ I_p = F^\pm(p) \circ df_p. \quad (4.4)$$

Furthermore, the horizontal part $(dF_p^\pm)^h$ of the differential of F^\pm satisfies

$$(dF_p^\pm)^h \circ I_p = J_{F(p)}^\pm \circ (dF_p^\pm)^h, \quad p \in M. \quad (4.5)$$

In particular, if the twistor lift F^\pm of a conformal immersion $f : M \rightarrow X$ is horizontal, then it is holomorphic as a map from (M, I) into (Z^\pm, J^\pm) .

Proof The formula (4.4) is an immediate consequence of the definition of F^\pm and the conformality of f . Let F denote any of the lifts F^\pm . From $\pi \circ F = f$ we get that

$$d\pi_{F(p)} \circ dF_p^h = d\pi_{F(p)} \circ dF_p = df_p, \quad p \in M, \quad (4.6)$$

and hence

$$\begin{aligned} d\pi_{F(p)} \circ dF_p^h \circ I_p &\stackrel{(4.6)}{=} df_p \circ I_p \stackrel{(4.4)}{=} F(p) \circ df_p \stackrel{(4.6)}{=} F(p) \circ d\pi_{F(p)} \circ dF_p^h \\ &\stackrel{(4.1)}{=} d\pi_{F(p)} \circ J_{F(p)}^\pm \circ dF_p^h. \end{aligned}$$

Since the vectors under $d\pi_{F(p)}$ are horizontal, (4.5) follows. \square

We now consider conformal immersions $f : M \rightarrow X$ which arise as projections to X of holomorphic immersions $F : M \rightarrow Z^\pm$. The following result is [30, Proposition 1].

Lemma 4.5 *Let (Z, J) denote any of the two twistor manifolds $(Z^\pm(X), J^\pm)$. If $F : (M, I) \rightarrow (Z, J)$ is a holomorphic immersion such that $dF_p(T_p M)$ intersects the vertical tangent space $T_{F(p)}^v Z$ only at 0 for every $p \in M$, then F agrees with the twistor lift F^\pm (4.3) of its projection $f = \pi \circ F : M \rightarrow X$.*

Proof The conditions on F implies that f is an immersion. Fix a point $p \in M$. Since F is holomorphic and the horizontal space $T_{F(p)}^h Z$ in J -invariant, (4.5) holds and hence

$$\begin{aligned} df_p \circ I_p &\stackrel{(4.6)}{=} d\pi_{F(p)} \circ dF_p^h \circ I_p \stackrel{(4.5)}{=} d\pi_{F(p)} \circ J_{F(p)} \circ dF_p^h \stackrel{(4.1)}{=} \\ &F(p) \circ d\pi_{F(p)} \circ dF_p^h \stackrel{(4.6)}{=} F(p) \circ df_p. \end{aligned}$$

This shows that f is conformal and F is its twistor lift (cf. (4.4)). \square

The following key statement combines the above observations with [30, Proposition 4]. When $X = S^4$ with the spherical metric, this is due to Bryant [15, Theorems B, B']; the general case was proved by Friedrich [30, Proposition 4].

Theorem 4.6 (The Bryant correspondence) *Let M be a Riemann surface, and let (X, g) be an oriented Riemannian four-manifold. The following conditions are pairwise equivalent for a smooth conformal immersion $f : M \rightarrow X$ (with the same choice of \pm in every item).*

- (a) f is superminimal of \pm spin (see Definition 1.1).
- (b) f admits a holomorphic horizontal lift $M \rightarrow Z^\pm(X)$.
- (c) The respective twistor lift $F^\pm : M \rightarrow Z^\pm(X)$ of f (see (4.3)) is horizontal.
- (d) We have that $\nabla F^\pm = 0$, where ∇ is the covariant derivative on the vector bundle $f^*(TX) \rightarrow M$ induced by the Levi-Civita connection on X .

Sketch of proof The equivalence of (b) and (c) follows from Lemma 4.5.

Consider now (a) \Leftrightarrow (c). In [30, Proposition 4], horizontality of the twistor lift $F^- : M \rightarrow Z^-$ (condition (c)) is characterized by a certain geometric property of the second fundamental forms $S_p(n) : T_p M \rightarrow T_p M$ of f at $p \in M$ in unit normal directions $n \in N_p$. An inspection of the proof shows that this property is equivalent to f being a superminimal surface of negative spin in the sense of Definition 1.1, hence to condition (a). Although not stated in [30], the same proof gives the analogous conclusion for conformal superminimal immersions $f : M \rightarrow X$ of positive spin with respect to the twistor lift $F^+ : M \rightarrow Z^+$. The crux of the matter can be seen from the display on the middle of [30, p. 266] which shows that the rotation of the unit normal vector $n \in N_p M$ in a given direction corresponds to the rotation of the point $S_p(n)v \in I_p(v) \subset T_p M$ (1.1) in the opposite direction (assuming that the spaces $T_p M$ and $N_p M$ are cooriented). Reversing the orientation on X , F^- is replaced by F^+ and the respective curves now rotate in the same direction, so F^+ is horizontal if and only if f is superminimal of positive spin. The direction of rotation is irrelevant (only) at points $p \in M$ where the normal curvature of the immersion f vanishes and hence the circle $I_p(v)$ reduces to the origin.

Concerning (c) \Leftrightarrow (d), Friedrich showed in [30, Proposition 5, p. 270] that the twistor lift F^- is horizontal if and only if the immersion f is *negatively oriented-isoclinic*. It is immediate from his description that the latter property simply says that the almost complex structure on the vector bundle $f^*TX = TM \oplus N$ adapted to f (which is precisely the structure F^-) is invariant under parallel transport along curves in M ; equivalently, F^- is parallel with respect to the covariant derivative ∇ on f^*TX induced by the Levi-Civita connection on X : $\nabla F^- = 0$. Reversing the orientation on X , the analogous conclusion shows that F^+ is horizontal if and only if f is *positively oriented-isoclinic* if and only if $\nabla F^+ = 0$. (See also [33, Proposition 17] and [44, Proposition 1].) \square

In light of the Bryant correspondence, it is a natural question whether not necessarily horizontal holomorphic curves in twistor spaces $Z^\pm(X)$ might yield a larger class or minimal surfaces in the given Riemannian four-manifold X . In fact, this is not so as shown by the following result of Friedrich [30, Proposition 3]. (Note that in [30] an immersion $f : M \rightarrow X$ is called superminimal if and only if its twistor lift is horizontal.)

Lemma 4.7 *The following are equivalent for a smooth conformal immersion $f : M \rightarrow X$.*

- (i) *The twistor lift $F^\pm : M \rightarrow Z^\pm$ of f is horizontal. (By Theorem 4.6, this is equivalent to saying that f is superminimal of \pm spin.)*
- (ii) *f is a minimal surface in X and it admits a holomorphic lift $\tilde{f} : M \rightarrow Z^\pm$.*

Sketch of proof If (i) holds then F^\pm is holomorphic by Lemma 4.4. Conversely, if f admits a holomorphic lift \tilde{f} then $\tilde{f} = F^\pm$ by Lemma 4.5. Friderich showed in [30, Proposition 3] that if F^- is holomorphic then the vertical derivative $(dF_p^-)^v$ equals the mean curvature vector of f at $p \in M$. Since a surface is minimal if and only if its mean curvature vector vanishes, the equivalence (i) \Leftrightarrow (ii) follows for the $-$ sign.

It also holds for the $+$ sign since $Z^+(X) = Z^-(\bar{X})$ (cf. Proposition 4.1 (a)) and the space of minimal surfaces in X does not depend on the choice of orientation of X . \square

Remark 4.8 A conformal immersion $M \rightarrow X$ whose twistor lifts $M \rightarrow Z^\pm(X)$ are both holomorphic parameterizes a totally umbilic surface in X (cf. [25, Proposition 6.1]). Note also that both twistor lifts F^\pm are horizontal precisely when all circles $I_p(v) \subset T_p M$ (2.2) are points, so the normal curvature vanishes and the surface is totally geodesic. \square

Remark 4.9 A smooth conformal immersion $f : M \rightarrow X$ may admit several horizontal lifts $M \rightarrow Z^\pm$, or no such lift. For example, if $X = \mathbb{R}^4$ with the flat metric then the horizontal distribution on $Z^\pm \cong \mathbb{R}^4 \times \mathbb{CP}^1$ is involutive and each leaf projects diffeomorphically onto X (see Example 4.2), so f admits a horizontal lift to every leaf; however, only the twistor lift can be holomorphic in view of Lemma 4.5. The situation is quite different if the horizontal distribution $\xi^\pm = T^h Z^\pm$ is a holomorphic contact bundle on Z^\pm (which holds if the metric g on X has nonzero scalar curvature). In such case, any horizontal lift $M \rightarrow Z^\pm$ is a conformal Legendrian surface (tangential to the contact bundle ξ^\pm), hence holomorphic or antiholomorphic by [1, Lemma 5.1]. By Lemma 4.5, this lift equals the twistor lift or its antiholomorphic reflection (see Proposition 4.1 (b)). \square

Remark 4.10 Another characterization of superminimal surfaces is given by the vanishing of a certain quartic form which was first studied by Calabi [16] and Chern [19, 20]; see also Bryant [15] and Gauduchon [33, Proposition 7]. This shows that every minimal immersion $S^2 \rightarrow S^4$ is superminimal; see [15, Theorem C] or [33, Proposition 25]. The same holds for minimal immersions $S^2 \rightarrow \mathbb{CP}^2$ (see [33, Proposition 28]). \square

We now recall two classical integrability theorems pertaining to twistor spaces. The first one is due to Atiyah, Hitchin, and Singer [8, Theorem 4.1].

Theorem 4.11 *The twistor space (Z^\pm, J^\pm) of a smooth oriented Riemannian four-manifold (X, g) is an integrable complex manifold if and only if the conformally invariant Weil tensor $W = W^+ + W^-$ of (X, g) satisfies $W^+ = 0$ or $W^- = 0$, respectively.*

Let us say that (X, g) is \pm self-dual if $W^\pm = 0$. The next result is due to Salamon [51, Theorem 10.1]; see also Eells and Salamon [25, Theorem 4.2].

Theorem 4.12 *Assume that (X, g) is a \pm self-dual Riemannian four-manifold, so (Z^\pm, J^\pm) is a complex manifold. Then, the horizontal bundle ξ^\pm is a holomorphic hyperplane subbundle of TZ^\pm if and only if X is an Einstein manifold. Assuming that this holds, ξ^\pm is a holomorphic contact bundle if and only if the scalar curvature of X (the trace of the Ricci curvature) is nonzero.*

In short, the complex structures J^\pm on twistor spaces Z^\pm depend only on the conformal class of the metric on X , but the horizontal distribution is defined by a choice of metric in that conformal class, and it is holomorphic precisely when the metric is Einstein.

Example 4.13 (*Twistor spaces of a Kähler manifold*) A smooth section $\sigma : X \rightarrow Z^\pm(X)$ of the twistor bundle determines an almost Hermitian structure J_σ on TX given at a point $x \in X$ by $\sigma(x) \in \mathcal{J}^\pm(T_x X)$. Conversely, an almost Hermitian structure J on TX determines a section $\sigma_J : X \rightarrow Z^\pm(X)$, where the sign depends on whether J agrees or disagrees with the orientation of X . These structures are not integrable in general.

Suppose now that (X, g, J) is an integrable Hermitian manifold endowed with the natural orientation determined by J . Then, the associated holomorphic section $\sigma_J : X \rightarrow Z^+(X)$ is horizontal if and only if (X, g, J) is a Kähler manifold. Indeed, the Kähler condition is equivalent to J being invariant under the parallel transport along curves in X , which means that $\nabla J = 0$. This shows that the horizontal bundle $\xi^+ \subset TZ^+(X)$ associated to a Kähler manifold X is never a holomorphic contact bundle. (Note also that Z^+ is in general not an integrable complex manifold.) Any holomorphic or antiholomorphic curve in X is a superminimal surface of positive spin since σ_J provides a horizontal lift to Z^+ . Another type of superminimal surfaces of positive spin are the Lagrangian ones, i.e., those for which the image of the tangent space at any point by the complex structure J is orthogonal to itself. If the holomorphic sectional curvature of X is nonvanishing then any superminimal surface of positive spin in X is of one of these three types (see [25]).

On the Kähler manifold $\mathbb{R}^4 = \mathbb{C}^2$ with the flat metric, ξ^+ is involutive (cf. Example 4.2). The twistor space $Z^+(\mathbb{CP}^2)$ of the projective plane is not integrable, and the superminimal surfaces in \mathbb{CP}^2 of positive spin are described above. On the other hand, $Z^-(\mathbb{CP}^2)$ is integrable and can be identified with the projectivised tangent bundle of \mathbb{CP}^2 . There is a natural correspondence between superminimal surfaces of negative spin in \mathbb{CP}^2 and holomorphic curves in \mathbb{CP}^2 (see Gauduchon [33, p. 178]). Superminimal surfaces in \mathbb{CP}^2 (and in S^4) were also studied by Montiel and Urbano [44] and others. \square

5 Proofs of Theorems 1.2 and 5.3

Proof of Theorem 1.2 Let (X, g) be a Riemannian manifold satisfying the hypotheses of Theorem 1.2. Let $W = W^+ + W^-$ denote the Weyl tensor of X (see [8, p. 427]). Assume without loss of generality that X is self-dual, meaning that $W^- = 0$; the analogous argument applies if $W^+ = 0$ by reversing the orientation on X (see Proposition 4.1 (a)). Denote by $\pi : Z = Z^-(X) \rightarrow X$ the negative twistor space of X and by $\xi \subset TZ$ its horizontal bundle (see Sect. 4). Also, let \tilde{g} denote a metric on Z for which the differential $d\pi : TZ \rightarrow TX$ maps ξ isometrically onto TX . Such \tilde{g} is obtained by adding to the horizontal component π^*g a positive multiple $\lambda > 0$ of the spherical metric on \mathbb{CP}^1 . By Theorems 4.11 and 4.12, Z is an integrable complex manifold and the horizontal bundle ξ is a holomorphic subbundle of the tangent bundle TZ .

Let M be a relatively compact domain with smooth boundary in an ambient Riemann surface R , and let $f_0 : \overline{M} \rightarrow X$ be a conformal superminimal immersion of negative spin, $f_0 \in \text{SM}^-(\overline{M}, X)$, and of class \mathcal{C}^r for some $r \geq 3$ (see Definition 1.1). Let $F_0 : \overline{M} \rightarrow Z$ denote the twistor lift of f_0 (see (4.3)). By Remark 4.3 the map F_0 is of

class $\mathcal{C}^{r-1}(\overline{M})$, and by Theorem 4.6 its restriction to M is a horizontal holomorphic immersion $M \rightarrow Z$.

Assume first that the (constant) scalar curvature of X is nonzero, so ξ is a holomorphic contact subbundle of TZ . According to [29, Theorem 1.2], the Legendrian immersion F_0 can be approximated in the $\mathcal{C}^{r-1}(\overline{M})$ topology by holomorphic Legendrian immersions $F_1 : U \rightarrow Z$ from open neighbourhoods U of \overline{M} in R , and we may choose F_1 to agree with F_0 at any given finite set of points $A \subset M$. (We assumed that $r \geq 3$ since [29, Theorem 1.2] applies to Legendrian immersions of class $\mathcal{C}^2(\overline{M})$.) Projecting down to X yields a conformal superminimal immersion $f_1 = \pi \circ F_1 : U \rightarrow X$ of negative spin satisfying the conclusion of the following proposition which seems worthwhile recording.

Proposition 5.1 (Mergelyan approximation theorem for superminimal surfaces)

Assume that (X, g) is an Einstein self-dual ($W^+ = 0$ or $W^- = 0$) four-manifold. If \overline{M} is a compact domain with smooth boundary in a Riemann surface R and $f_0 : \overline{M} \rightarrow X$ is a conformal superminimal immersion in $\text{SM}^\pm(\overline{M}, X)$ (see (1.2)) of class \mathcal{C}^r for some $r \geq 3$, then f_0 can be approximated in the $\mathcal{C}^{r-1}(\overline{M})$ topology by conformal superminimal immersions $f \in \text{SM}^\pm(U, X)$ from open neighbourhoods U of \overline{M} in R . Furthermore, f may be chosen to agree with f_0 to any given finite order at any given finite set of points $A \subset M$.

We continue with the proof of Theorem 1.2. By [1, Theorem 1.3] we can approximate the holomorphic Legendrian immersion $F_1 : U \rightarrow Z$ found above, uniformly on \overline{M} , by topological embeddings $F : \overline{M} \rightarrow Z$ whose restrictions to M are complete holomorphic Legendrian embeddings. Again, we can choose F to match F_1 (and hence F_0) at any given finite set of points in M . The proof of the cited theorem uses Darboux neighbourhoods furnished by [1, Theorem 1.1], thereby reducing the problem to the standard contact structure on \mathbb{C}^3 for which the mentioned result is given by [5, Theorem 1.2].

Since the differential of the twistor projection $\pi : Z \rightarrow X$ maps the horizontal bundle $\xi \subset TZ$ isometrically onto TX , the projection $f := \pi \circ F : \overline{M} \rightarrow X$ is a continuous map whose restriction to M is a complete superminimal immersion $M \rightarrow X$. By the construction, f approximates f_0 as closely as desired uniformly on \overline{M} , and it can be chosen to agree with f_0 to any given finite order at the given finite set of points in M .

By using also the general position theorem for holomorphic Legendrian immersions (see [1, Theorem 1.2]) and the transversality argument given (for the special case of the twistor map $\mathbb{CP}^3 \rightarrow S^4$) in [6, proof of Theorem 7.5], we can arrange that the boundary $f|_{bM} : bM \rightarrow X$ is a topological embedding whose image consists of finitely many Jordan curves. As shown in [3, proof of Theorem 1.1], we can also arrange that the Jordan curves in $f(bM)$ have Hausdorff dimension one.

It remains to consider the case when the manifold (X, g) has vanishing scalar curvature. By Theorem 4.12 the horizontal distribution ξ on the twistor space Z is then an involutive holomorphic subbundle of codimension one in TZ , hence defining a holomorphic foliation of Z by smooth complex surfaces. The (horizontal, holomorphic) twistor lift F_0 of f_0 lies in a leaf of this foliation. It is known (see [2,3]) that complex curves parameterized by bordered Riemann surfaces in any complex manifold of

dimension > 1 enjoy the Calabi–Yau property. Projecting such a surface (contained in the same leaf of ξ as $F_0(M)$) to X gives an immersed complete superminimal surface, and we can arrange by a general position argument (see the proof of Theorem 1.2) that its boundary is topologically embedded. \square

The argument in the above proof gives the following lemma.

Lemma 5.2 (Increasing the intrinsic diameter of a superminimal surface) *Let M and (X, g) be as in Theorem 1.2. Every conformal superminimal immersion $f_0 \in \text{SM}^\pm(\overline{M}, X)$ of class \mathcal{C}^3 can be approximated as closely as desired uniformly on \overline{M} by a smooth conformal superminimal immersion $f \in \text{SM}^\pm(\overline{M}, X)$ with embedded boundary $f(bM) \subset X$ such that the intrinsic diameter of the Riemannian surface (M, f^*g) is arbitrarily big.*

By an inductive application of this lemma, we obtain the following generalisation of Theorem 1.2. Let R be a compact Riemann surface and $M = R \setminus \bigcup_{i=0}^\infty D_i$ be an open domain of the form (1.3) in R whose complement is a countable union of pairwise disjoint, smoothly bounded closed discs D_i . For every $j \in \mathbb{Z}_+$ we consider the compact domain in R given by $M_j = R \setminus \bigcup_{k=0}^j \mathring{D}_k$. This is a compact bordered Riemann surface with boundary $bM_j = \bigcup_{k=0}^j bD_k$, and $M_0 \supset M_1 \supset M_2 \supset \cdots \supset \bigcap_{j=1}^\infty M_j = \overline{M}$.

Theorem 5.3 (Assumptions as above) *Assume that (X, g) is an Einstein four-manifold with the Weyl tensor $W = W^+ + W^-$. If $W^\pm = 0$ then every $f_j \in \text{SM}^\pm(M_j, X)$ ($j \in \mathbb{Z}_+$) of class \mathcal{C}^3 can be approximated as closely as desired uniformly on \overline{M} by continuous maps $f : \overline{M} \rightarrow X$ such that $f : M \rightarrow X$ is a complete conformal superminimal immersion in $\text{SM}^\pm(M, X)$ and $f(bM) = \bigcup_i f(bD_i)$ is a union of pairwise disjoint Jordan curves of Hausdorff dimension one.*

Proof We outline the main idea and refer for the details to [3, proof of Theorem 5.1] where the analogous result is proved for conformal minimal surfaces in Euclidean spaces.

Let $f_j \in \text{SM}^\pm(M_j, X)$ be a smooth conformal superminimal immersion. Using Lemma 5.2 we inductively construct a sequence $f_i \in \text{SM}^\pm(M_i, X)$ ($i = j + 1, j + 2, \dots$) such that at every step the map $f_i : M_i \rightarrow X$ approximates the previous map $f_{i-1} : M_{i-1} \rightarrow X$ uniformly on $M_i \subset M_{i-1}$ as closely as desired, the intrinsic diameter of (M_i, f_i^*g) is as big as desired, and the boundary $f_i(bM_i) \subset X$ is embedded. (Note that at each step a new disc is taken out and hence an additional boundary curve appears.) By choosing the approximations to be close enough at every step and the intrinsic diameters of the Riemannian surfaces (M_i, f_i^*g) growing fast enough, the sequence f_i converges uniformly on \overline{M} to a limit $f = \lim_{i \rightarrow \infty} f_i : \overline{M} \rightarrow X$ satisfying the conclusion of the theorem. For the details of this argument in an analogous situation we refer to [3, Sect. 3]. \square

6 Twistor Spaces of the 4-Sphere and of the Hyperbolic 4-Space

It was shown by Penrose [47, Sect. VI], and more explicitly by Bryant [15, Sect. 1] that the twistor space of the four-sphere S^4 with the spherical metric can be identified

with the complex projective space \mathbb{CP}^3 with the Fubini-Study metric (defined by the homogeneous $(1, 1)$ -form $\omega = dd^c \log |z|^2$ on \mathbb{C}_*^4) such that the horizontal distribution $\xi \subset T\mathbb{CP}^3$ of the twistor projection $\pi : \mathbb{CP}^3 \rightarrow S^4$ is a holomorphic contact bundle given in homogeneous coordinates $[z_1 : z_2 : z_3 : z_4]$ by the homogeneous 1-form

$$\alpha = z_1 dz_2 - z_2 dz_1 + z_3 dz_4 - z_4 dz_3. \quad (6.1)$$

(This complex contact structure \mathbb{CP}^3 is unique up to holomorphic contactomorphisms; see LeBrun and Salamon [43, Corollary 2.3].) Proofs can also be found in many other sources, see Eells and Salamon [25, Sect. 9], Gauduchon [33, pp. 170–175], Bolton and Woodward [12], Baird and Wood [9, Example 7.1.4], among others.

Due to the overall importance of this example we offer here a totally elementary explanation using only basic facts along with Lemma 3.1. We consider $Z^+(S^4)$; the same holds for $Z^-(S^4)$ by applying (4.2) to the antipodal orientation reversing isometry on S^4 . In Example 6.2 we also take a look at the twistor space of the hyperbolic four-space H^4 .

Example 6.1 [The twistor space of S^4] The geometric scheme follows Bryant [15] and Gauduchon [33, pp. 171–175] and [34]. We identify the quaternionic plane \mathbb{H}^2 with \mathbb{C}^4 by

$$\mathbb{H}^2 \ni q = (q_1, q_2) = (z_1 + z_2j, z_3 + z_4j) = (z_1, z_2, z_3, z_4) = z \in \mathbb{C}^4, \quad (6.2)$$

and we identify S^4 with the unit sphere in $\mathbb{R}^5 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$ oriented by the outward vector field. Write $\mathbb{H}_*^2 = \mathbb{H}^2 \setminus \{0\}$ and consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{C}_*^4 & \xrightarrow{\cong} & \mathbb{H}_*^2 & \xrightarrow{\phi_1} & \mathbb{CP}^3 \\ & & \downarrow \phi & \swarrow \phi_2 & \downarrow \pi \\ \mathbb{C}^2 \cup \{\infty\} & \xrightarrow{\cong} & \mathbb{HP}^1 & \xrightarrow{\psi} & S^4 \end{array}$$

where

- $\phi_1 : \mathbb{H}_*^2 = \mathbb{C}_*^4 \rightarrow \mathbb{CP}^3$ is the canonical projection with fiber \mathbb{C}^* sending $q = (q_1, q_2) \in \mathbb{H}_*^2$ to the complex line $\mathbb{C}q \in \mathbb{CP}^3$;
- $\phi_2 : \mathbb{CP}^3 \rightarrow \mathbb{HP}^1$ is the fiber bundle sending a complex line $\mathbb{C}q$, $q \in \mathbb{H}_*^2$, to the quaternionic line $\mathbb{H}q = \mathbb{C}q \oplus \mathbb{C}jq$. Thus, \mathbb{HP}^1 is the quaternionic one-dimensional projective space which we identify with $\mathbb{H} \cup \{\infty\} = \mathbb{R}^4 \cup \{\infty\}$ such that $\mathbb{H}_2 := \{0\} \times \mathbb{H} = \mathbb{H} \cdot (0, 1)$ corresponds to ∞ . The fiber $\phi_2^{-1}(\phi_2(q))$ is the linear rational curve $\mathbb{CP}^1 \subset \mathbb{CP}^3$ of complex lines in the quaternionic line $\mathbb{H}q$;
- $\phi = \phi_2 \circ \phi_1 : \mathbb{H}_*^2 \rightarrow \mathbb{HP}^1$ sends $q \in \mathbb{H}_*^2$ to $\mathbb{H}q \in \mathbb{HP}^1$. Restricting ϕ to the unit sphere $S^7 \subset \mathbb{H}_*^2$ gives a Hopf map $S^7 \rightarrow S^4$ with fiber S^3 ;
- $\psi : \mathbb{HP}^1 \cong \mathbb{R}^4 \cup \{\infty\} \rightarrow S^4 \subset \mathbb{R}^5$ is the orientation preserving stereographic projection mapping ∞ to the south pole $\mathfrak{s} = (0, 0, 0, 0, -1) \in S^4$;
- $\rho := \psi \circ \phi : \mathbb{H}_*^2 \rightarrow S^4$.

The stereographic projection $\psi : \mathbb{R}^4 \cup \{\infty\} \rightarrow S^4 \subset \mathbb{R}^5$ with $\psi(\infty) = \mathfrak{s}$ is given by

$$\psi(x) = \left(\frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right). \quad (6.3)$$

Using coordinates (6.2) it is elementary to find the following explicit formulas:

$$\begin{aligned} \phi(q_1, q_2) &= q_1^{-1} q_2 = \frac{1}{|q_1|^2} \bar{q}_1 q_2 \\ &= \frac{1}{|z_1|^2 + |z_2|^2} (\bar{z}_1 z_3 + z_2 \bar{z}_4, \bar{z}_1 z_4 - z_2 \bar{z}_3), \end{aligned} \quad (6.4)$$

$$\rho(q_1, q_2) = \frac{1}{|q_1|^2 + |q_2|^2} (2\bar{q}_1 q_2, |q_1|^2 - |q_2|^2) \in S^4 \subset \mathbb{R}^5, \quad (6.5)$$

$$\pi([z_1 : z_2 : z_3 : z_4]) = \frac{1}{|z|^2} (2(\bar{z}_1 z_3 + z_2 \bar{z}_4), 2(\bar{z}_1 z_4 - z_2 \bar{z}_3), |q_1|^2 - |q_2|^2). \quad (6.6)$$

We begin by considering the fiber $\pi^{-1}(\mathfrak{n}) \subset \mathbb{CP}^3$ over the point $\mathfrak{n} := (0, 0, 0, 0, 1) \in S^4 \subset \mathbb{R}^5$. This fiber is the space of complex lines in $\mathbb{H}_1 := \mathbb{H} \times \{0\}$ (hence isomorphic to \mathbb{CP}^1), and its normal space at every point in the Fubini-Study metric is $\mathbb{H}_2 = \{0\} \times \mathbb{H}$. Using (6.2) we have that $\mathbb{H}_1 = \{z_3 = z_4 = 0\}$, and the form α (6.1) along \mathbb{H}_1 equals $z_1 dz_2 - z_2 dz_1$. Its kernel is the complex 3-plane $\mathbb{C} \cdot (z_1, z_2) \oplus \mathbb{H}_2$, so $\xi = \ker \alpha \subset T\mathbb{CP}^3$ coincides with \mathbb{H}_2 at every point of $\pi^{-1}(\mathfrak{n})$. This shows that ξ is orthogonal to the fiber $\pi^{-1}(\mathfrak{n})$ in the Fubini-Study metric. We identify the tangent space $T_{\mathfrak{n}} S^4 = \mathbb{R}^4 \times \{0\}$ with \mathbb{H} and let $J_i \in \mathcal{J}^+(T_{\mathfrak{n}} S^4)$ denote the almost Hermitian structure $J_i(1) = i$, $J_i(j) = k$. Fix a point $q \in \mathbb{H}_1$ with $|q| = 1$. Consider the differential

$$d\rho_{(q,0)} : T_{(q,0)} \mathbb{H}^2 = \mathbb{H}_1 \oplus \mathbb{H}_2 \rightarrow T_{\mathfrak{n}} S^4 \cong \mathbb{H}.$$

We see from (6.5) that the restriction of $d\rho_{(q,0)}$ to the horizontal subspace $\mathbb{H}_2 = \xi$ equals

$$\mathbb{H}_2 \ni q_2 \mapsto 2\bar{q}q_2,$$

so it is an isometry with an appropriate choice of the constant for the metrics. If J_q is the almost Hermitian structure on $T_{\mathfrak{n}}(S^4) \cong \mathbb{H}$ furnished by Lemma 3.1, then

$$d\rho_{(q,0)} \circ J_i = J_q \circ d\rho_{(q,0)} \quad \text{on } \mathbb{H}_2.$$

This means the restriction of $d\pi_{(q,0)}$ to the horizontal subspace $\mathbb{H}_2 = \xi$ intertwines J_i with J_q as in the definition of the twistor space (see (4.1)). Hence, $\pi : \mathbb{CP}^3 \rightarrow S^4$ satisfies all properties of the twistor bundle $Z^+(S^4) \rightarrow S^4$ along the fiber $\pi^{-1}(\mathfrak{n})$.

To complete the proof, it suffices to show that the situation is the same on every fiber of the projection $\pi : \mathbb{CP}^3 \rightarrow S^4$. To this end, we must find a group of \mathbb{C} -linear

isometries of $\mathbb{C}^4 \cong \mathbb{H}^2$, hence a subgroup of $U(4)$, which commutes with the left multiplication of \mathbb{H} on \mathbb{H}^2 and passes down to a transitive group of isometries of S^4 . This requirement is fulfilled by the subgroup of $U(4)$ preserving the quaternionic inner product on \mathbb{H}^2 given by

$$\mathbb{H}^2 \times \mathbb{H}^2 \ni (p, q) \longmapsto p\bar{q}^t = p_1\bar{q}_1 + p_2\bar{q}_2 \in \mathbb{H}.$$

(We consider elements of \mathbb{H}^2 as row vectors acted upon by right multiplication.) Writing

$$p = (z_1 + z_2j, z_3 + z_4j) = z, \quad q = (w_1 + w_2j, w_3 + w_4j) = w,$$

a calculation gives

$$p\bar{q}^t = z\bar{w}^t + \alpha_0(z, w)j, \quad \alpha_0(z, w) = z_2w_1 - z_1w_2 + z_4w_3 - z_3w_4. \quad (6.7)$$

Note that $\alpha_0(z, dz) = \alpha$ is the contact form (6.1). If $J_0 \in SU(4)$ denotes the matrix with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as the diagonal blocks and zero off-diagonal blocks, then $\alpha_0(z, w) = zJ_0w^t$. It follows that the group we are looking for is

$$G = \{A \in U(4) : AJ_0A^t = J_0\} = U(4) \cap Sp_2(\mathbb{C}),$$

where $Sp_2(\mathbb{C})$ is the complexified symplectic group. Its projectivization $\mathbb{P}G$ acts on \mathbb{CP}^3 by holomorphic contact isometries. This shows that \mathbb{CP}^3 is indeed the twistor space of S^4 .

Explicit formulas for the twistor lift of an immersion $M \rightarrow S^4$ into \mathbb{CP}^3 can be found in [15, Sect. 2], [25, Sect. 9], [12, Proposition 2.1], among others. The antiholomorphic fiber preserving involution $\iota : \mathbb{CP}^3 \rightarrow \mathbb{CP}^3$ (cf. Proposition 4.1 (b)) is given by

$$\iota([z_1 : z_2 : z_3 : z_4]) = [-\bar{z}_2 : \bar{z}_1 : -\bar{z}_4 : \bar{z}_3].$$

The formula (6.6) immediately shows that $\pi \circ \iota = \text{Id}_{S^4}$. Identifying S^4 with $\mathbb{R}^4 \cup \{\infty\} = \mathbb{C}^2 \cup \{\infty\}$ via the stereographic projection ψ (6.3) and using complex coordinates $w = (w_1, w_2) \in \mathbb{C}^2$, the spherical metric of constant sectional curvature $+1$ is given by

$$g_s = \frac{4|dw|^2}{(1 + |w|^2)^2}, \quad w \in \mathbb{C}^2,$$

and (6.4) shows that the twistor projection $\phi_2 = \psi^{-1} \circ \pi : \mathbb{CP}^3 \rightarrow \mathbb{C}^2 \cup \{\infty\}$ is given in homogeneous coordinates $[z_1 : z_2 : z_3 : z_4]$ on \mathbb{CP}^3 by

$$w_1 = \frac{\bar{z}_1z_3 + z_2\bar{z}_4}{|z_1|^2 + |z_2|^2}, \quad w_2 = \frac{\bar{z}_1z_4 - z_2\bar{z}_3}{|z_1|^2 + |z_2|^2}, \quad |w|^2 = \frac{|z_3|^2 + |z_4|^2}{|z_1|^2 + |z_2|^2}. \quad (6.8)$$

Example 6.2 (The twistor space of H^4) The geometric model of the *hyperbolic space* H^4 of constant sectional curvature -1 is the hyperquadric

$$H^4 = \{x = (x_1, \dots, x_5) \in \mathbb{R}^5 : x_1^2 + x_2^2 + x_3^2 + x_4^2 + 1 = x_5^2, \ x_5 > 0\} \quad (6.9)$$

in the Lorentzian space $\mathbb{R}^{4,1}$, that is, \mathbb{R}^5 endowed with the Lorentzian inner product

$$x \circ y = x_1 y_1 + \dots + x_4 y_4 - x_5 y_5.$$

(See Ratcliffe [49, Sect. 4.5].) Note that H^4 is one of the two connected components of the the unit ball $\{x \in \mathbb{R}^{4,1} : x \circ x = -1\}$ of imaginary radius $i = \sqrt{-1}$, the other component being given by the same equation (6.9) with $x_5 < 0$.

Consider the stereographic projection $\tilde{\psi} : \mathbb{B} = \{x \in \mathbb{R}^4 : |x|^2 < 1\} \xrightarrow{\cong} H^4$ given by

$$\tilde{\psi}(x) = \left(\frac{2x_1}{1 - |x|^2}, \dots, \frac{2x_4}{1 - |x|^2}, \frac{1 + |x|^2}{1 - |x|^2} \right), \quad x \in \mathbb{B}. \quad (6.10)$$

The pullback by $\tilde{\psi}$ of the Lorentzian pseudometric $\|x\|^2 = x \circ x$ on $\mathbb{R}^{4,1}$ is the hyperbolic metric of constant curvature -1 on the ball \mathbb{B} :

$$g_h = \frac{4|dx|^2}{(1 - |x|^2)^2}, \quad x \in \mathbb{B}.$$

The Riemannian manifold (\mathbb{B}, g_h) is the *Poincaré ball model* for H^4 . We see from (6.8) that the preimage of \mathbb{B} by the projection $\phi_2 : \mathbb{CP}^3 \rightarrow \mathbb{C}^2 \cup \{\infty\}$ is the domain

$$\Omega = \phi_2^{-1}(\mathbb{B}) = \left\{ [z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^3 : |z_1|^2 + |z_2|^2 > |z_3|^2 + |z_4|^2 \right\}. \quad (6.11)$$

Since the hyperbolic metric is conformally flat, Ω is the twistor space $Z^+(H^4)$ as a complex manifold (cf. Theorem 4.11). The twistor metric \tilde{g} on Ω is obtained from the hyperbolic metric g_h on the base \mathbb{B} and the Fubini-Study metric on the fibers \mathbb{CP}^1 . Explicit formulas for the metric \tilde{g} and the horizontal bundle $\tilde{\xi} \subset T\Omega$ can be found in [31, Sect. 4]. (In the cited paper, the opposite inequality is used in (6.11) which amounts to interchanging the variables q_1, q_2 in (6.4), i.e., passing to another affine coordinate chart of \mathbb{HP}^1 .) The metric \tilde{g} on Ω is a complete Kähler metric, and $\tilde{\xi}$ is a holomorphic contact bundle.

Corollary 6.3 *Superminimal surfaces of both positive and negative spin in the hyperbolic 4-space H^4 satisfy the Calabi–Yau property. Furthermore, the twistor contact manifold $(\Omega, \tilde{\xi})$ of H^4 is Kobayashi hyperbolic. The same holds for domains in any complete Riemannian four-manifold of constant negative sectional curvature (a space-form).*

For the notion of Kobayashi hyperbolicity of complex contact manifolds, see [28].

Proof The first statement follows directly from Theorems 1.2 and 5.3. Let M be a Riemann surface and $f : M \rightarrow (H^4, g_h)$ be a conformal minimal immersion. The induced metric f^*g_h on M is then a Kähler metric with curvature bounded above by -1 , the curvature of H^4 (see [17, Corollary 2.2]). By the Ahlfors lemma (see [40, Theorem 2.1, p. 3]) it follows that any holomorphic map $h : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \rightarrow M$ from the disc satisfies an upper bound on the derivative at any point $p \in \mathbb{D}$ depending only on $h(p) \in M$. Hence, M is Kobayashi hyperbolic and its universal covering is the disc. Since superminimal surfaces in H^4 lift isometrically to holomorphic Legendrian curves in $(\Omega, \tilde{\xi})$, the contact structure $\tilde{\xi}$ is hyperbolic. (Note that Ω itself is not Kobayashi hyperbolic since the fibers of $\phi_2 : \Omega \rightarrow \mathbb{B}$ are rational curves.) The same argument applies to domains in any space-form X since its universal metric covering space is H^4 ; see [23, Theorem 4.1]. \square

Acknowledgements This research is supported by research program P1-0291 and grant J1-9104 from ARRS, Republic of Slovenia, and by the Stefan Bergman Prize 2019 from the American Mathematical Society. I wish to thank Antonio Alarcón, Joel Fine, Finnur Lárusson, and Francisco Urbano for helpful discussions on this subject.

References

1. Alarcón, A., Drinovec Drnovšek, B., Forstnerič, F., López, F.J.: Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves. *Proc. Lond. Math. Soc.* (3) **111**(4), 851–886 (2015)
2. Alarcón, A., Forstnerič, F.: New complex analytic methods in the theory of minimal surfaces: a survey. *J. Aust. Math. Soc.* **106**(3), 287–341 (2019)
3. Alarcón, A., Forstnerič, F.: The Calabi-Yau problem for Riemann surfaces with finite genus and countably many ends. *arXiv e-prints*. [arXiv:1904.08015](https://arxiv.org/abs/1904.08015) (2019). *Rev. Mat. Iberoam.*, to appear
4. Alarcón, A., Forstnerič, F.: Darboux charts around holomorphic Legendrian curves and applications. *Int. Math. Res. Not. IMRN* **3**, 893–922 (2019)
5. Alarcón, A., Forstnerič, F., Lárusson, F.: Holomorphic Legendrian curves in \mathbb{CP}^3 and superminimal surfaces in \mathbb{S}^4 . *arXiv e-prints*. [arXiv:1910.12996](https://arxiv.org/abs/1910.12996) (2019)
6. Alarcón, A., Forstnerič, F., López, F.J.: Holomorphic Legendrian curves. *Compos. Math.* **153**(9), 1945–1986 (2017)
7. Alarcón, A., Forstnerič, F., López, F.: New complex analytic methods in the study of non-orientable minimal surfaces in \mathbb{R}^n . *Mem. Am. Math. Soc.* **264**(1283) (2020)
8. Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self-duality in four-dimensional Riemannian geometry. *Proc. R. Soc. Lond. Ser. A* **362**(1711), 425–461 (1978)
9. Baird, P., Wood, J.C.: *Harmonic Morphisms Between Riemannian Manifolds*. London Mathematical Society Monographs, vol. 29. New Series. The Clarendon Press, Oxford University Press, Oxford (2003)
10. Besse, A.L.: *Einstein Manifolds*. Classics in Mathematics. Springer, Berlin (2008). Reprint of the 1987 edition
11. Biquard, O.: Métriques autoduales sur la boule. *Invent. Math.* **148**(3), 545–607 (2002)
12. Bolton, J., Woodward, L.M.: Higher singularities and the twistor fibration $\pi : \mathbb{CP}^3 \rightarrow S^4$. *Geom. Dedicata* **80**(1–3), 231–245 (2000)
13. Borōvka, O.: Sur une classe de surfaces minima plongées dans un espace à cinq dimensions à courbure constante. *C. R. Acad. Sci. Paris* **187**, 1271–1273 (1928)
14. Borōvka, O.: Sur une classe de surfaces minima plongées dans un espace à quatre dimensions à courbure constante. *C. R. Acad. Sci. Paris* **187**, 334–336 (1928)
15. Bryant, R.L.: Conformal and minimal immersions of compact surfaces into the 4-sphere. *J. Differ. Geom.* **17**(3), 455–473 (1982)

16. Calabi, E.: Minimal immersions of surfaces in Euclidean spheres. *J. Differ. Geom.* **1**, 111–125 (1967)
17. Chen, B.-Y.: *Differential Geometry of Warped Product Manifolds and Submanifolds*. World Scientific Publishing, Hackensack, NJ (2017). With a foreword by Leopold Verstraeten
18. Chern, S.S.: The geometry of G -structures. *Bull. Am. Math. Soc.* **72**, 167–219 (1966)
19. Chern, S.: On minimal spheres in the four-sphere. In: *Studies and Essays (Presented to Yu-why Chen on his 60th Birthday, April 1, 1970)*, pp. 137–150. Mathematics Research Center, National Taiwan University, Taipei (1970)
20. Chern, S.S.: On the minimal immersions of the two-sphere in a space of constant curvature. In: *Problems in Analysis (Lectures at the Symposium in honor of Salomon Bochner, Princeton University, Princeton, NJ, 1969)*, pp. 27–40 (1970)
21. Derdziński, A.: Self-dual Kähler manifolds and Einstein manifolds of dimension four. *Compos. Math.* **49**(3), 405–433 (1983)
22. Diagaku, K., Kenkyūjo, S.K.: *Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965*. Nippon Hyoronsha Co., Ltd., Tokyo (1966)
23. do Carmo, M.P.: *Riemannian Geometry. Mathematics: Theory & Applications*. Birkhäuser Boston, Inc., Boston, MA (1992). Translated from the second Portuguese edition by F. Flaherty
24. Donaldson, S., Fine, J.: Toric anti-self-dual 4-manifolds via complex geometry. *Math. Ann.* **336**(2), 281–309 (2006)
25. Eells, J., Salamon, S.: Twistorial construction of harmonic maps of surfaces into four-manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **12**(4), 589–640 (1985)
26. Eisenhart, L.P.: Minimal surfaces in Euclidean four-space. *Am. J. Math.* **34**(3), 215–236 (1912)
27. Fine, J.: Toric anti-self-dual Einstein metrics via complex geometry. *Math. Ann.* **340**(1), 143–157 (2008)
28. Forstnerič, F.: Hyperbolic complex contact structures on \mathbb{C}^{2n+1} . *J. Geom. Anal.* **27**(4), 3166–3175 (2017)
29. Forstnerič, F.: Mergelyan approximation theorem for holomorphic Legendrian curves. arXiv e-prints (2020). [arXiv:2001.04379](https://arxiv.org/abs/2001.04379)
30. Friedrich, T.: On surfaces in four-spaces. *Ann. Glob. Anal. Geom.* **2**(3), 257–287 (1984)
31. Friedrich, T.: On superminimal surfaces. *Arch. Math. (Brno)* **33**(1–2), 41–56 (1997)
32. Friedrich, T., Kurke, H.: Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature. *Math. Nachr.* **106**, 271–299 (1982)
33. Gauduchon, P.: La correspondance de Bryant. Number 154–155, pp. 10, 181–208, 351 (1988). 1987. *Théorie des variétés minimales et applications* (Palaiseau, 1983–1984)
34. Gauduchon, P.: Les immersions super-minimales d’une surface compacte dans une variété riemannienne orientée de dimension 4. Number 154–155, pp. 9, 151–180, 351 (1988). 1987. *Théorie des variétés minimales et applications* (Palaiseau, 1983–1984)
35. Graham, C.R., Lee, J.M.: Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.* **87**(2), 186–225 (1991)
36. He, Z.-X., Schramm, O.: Fixed points, Koebe uniformization and circle packings. *Ann. Math. (2)* **137**(2), 369–406 (1993)
37. Hitchin, N.: Compact four-dimensional Einstein manifolds. *J. Differ. Geom.* **9**, 435–441 (1974)
38. Hitchin, N.J.: Twistor spaces, Einstein metrics and isomonodromic deformations. *J. Differ. Geom.* **42**(1), 30–112 (1995)
39. Kommerell, K.: *Die Krümmung der zweidimensionalen Gebilde im ebenen Raum von vier Dimensionen*. Tübingen. 53 S. 8° (1897)
40. Kobayashi, S.: *Hyperbolic Manifolds and Holomorphic Mappings. An Introduction*, 2nd edn. World Scientific Publishing, Hackensack, NJ (2005)
41. Kommerell, K.: Riemannsche Flächen im ebenen Raum von vier Dimensionen. *Math. Ann.* **60**(4), 548–596 (1905)
42. LeBrun, C.: Fano manifolds, contact structures, and quaternionic geometry. *Int. J. Math.* **6**(3), 419–437 (1995)
43. LeBrun, C., Salamon, S.: Strong rigidity of positive quaternion-Kähler manifolds. *Invent. Math.* **118**(1), 109–132 (1994)
44. Montiel, S., Urbano, F.: Second variation of superminimal surfaces into self-dual Einstein four-manifolds. *Trans. Am. Math. Soc.* **349**(6), 2253–2269 (1997)
45. Myers, S.B.: *Riemannian manifolds with positive mean curvature*. *Duke Math. J.* **8**, 401–404 (1941)
46. Osserman, R.: *A Survey of Minimal Surfaces*, 2nd edn. Dover, New York (1986)

47. Penrose, R.: Twistor algebra. *J. Math. Phys.* **8**, 345–366 (1967)
48. Penrose, R., MacCallum, M.A.H.: Twistor theory: an approach to the quantisation of fields and space-time. *Phys. Rep.* **6C**(4), 241–315 (1973)
49. Ratcliffe, J.G.: *Foundations of Hyperbolic Manifolds*. Graduate Texts in Mathematics, vol. 149, 2nd edn. Springer, New York (2006)
50. Salamon, S.: Quaternionic Kähler manifolds. *Invent. Math.* **67**(1), 143–171 (1982)
51. Salamon, S.: Topics in four-dimensional Riemannian geometry. In: *Geometry Seminar “Luigi Bianchi”* (Pisa, 1982). *Lecture Notes in Mathematics*, vol. 1022, pp. 33–124. Springer, Berlin (1983)
52. Salamon, S.: Harmonic and holomorphic maps. In: *Geometry Seminar “Luigi Bianchi” II—1984*. *Lecture Notes in Mathematics*, vol. 1164, pp. 161–224. Springer, Berlin (1985)
53. Sergeev, A.G.: Twistor geometry and gauge fields. *Trans. Moscow Math. Soc.* **79**, 135–175 (2018)
54. Shen, Y.B.: On conformal superminimal immersions of surfaces into \mathbb{CP}^n . In: *Geometry and Global Analysis* (Sendai, 1993), pp. 457–463. Tohoku University, Sendai (1993)
55. Shen, Y.: The Riemannian geometry of superminimal surfaces in complex space forms. *Acta Math. Sin. (N.S.)*, **12**(3), 298–313 (1996). A Chinese summary appears in *Acta Math. Sin.* **40**(1), 158 (1997)
56. Veblen, O.: *Geometry of Complex Domains*, A Seminar Conducted by Professors Oswald Veblen and John von Neumann, 1935–1936. Lectures by Veblen, O. and Givens, J.W., Notes by Taub, A.H. and Givens, J.W., Rev. ed. The Institute for Advanced Study, Princeton, NJ (1955)
57. Ward, R.S., Wells Jr., R.O.: *Twistor Geometry and Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (1990)
58. Wood, J.C.: Twistor constructions for harmonic maps. In: *Differential Geometry and Differential Equations* (Shanghai, 1985). *Lecture Notes in Mathematics*, vol. 1255, pp. 130–159. Springer, Berlin (1987)
59. Yau, S.T.: Calabi’s conjecture and some new results in algebraic geometry. *Proc. Natl Acad. Sci. U.S.A.* **74**(5), 1798–1799 (1977)
60. Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Commun. Pure Appl. Math.* **31**(3), 339–411 (1978)
61. Yau, S.-T.: Review of geometry and analysis. In: *Mathematics: Frontiers and Perspectives*, pp. 353–401. American Mathematical Society, Providence, RI (2000)
62. Yau, S.-T.: Review of geometry and analysis. *Asian J. Math.* **4**(1), 235–278 (2000). Kodaira’s issue

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.