

# THE FIRST THIRTY YEARS OF ANDERSÉN–LEMPERT THEORY

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*Dedicated to László Lempert in honour of his 70th birthday*

**Abstract.** In this paper we expose the impact of the fundamental discovery, made by Erik Andersén and László Lempert in 1992, that the group generated by shears is dense in the group of holomorphic automorphisms of a complex Euclidean space of dimension  $n > 1$ . In three decades since its publication, their groundbreaking work led to the discovery of several new phenomena and to major new results in complex analysis and geometry involving Stein manifolds and affine algebraic manifolds with many automorphisms. The aim of this survey is to present the focal points of these developments, with a view towards the future.

## 1. Introduction

Complex Euclidean spaces  $\mathbb{C}^n$  ( $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ ) are the most basic and important objects in analytic and algebraic geometry. It is natural to try understanding holomorphic automorphisms (symmetries) of  $\mathbb{C}^n$  and their role in applications. If  $n = 1$ , these are precisely the affine linear maps  $z \mapsto az + b$  with  $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and  $b \in \mathbb{C}$ . However, for  $n > 1$  the group  $\text{Aut}(\mathbb{C}^n)$  of holomorphic automorphisms of  $\mathbb{C}^n$  is huge. The affine group  $\text{Aff}(\mathbb{C}^n)$ , generated by the general linear group  $\text{GL}_n(\mathbb{C})$  together with translations, acts transitively on  $\mathbb{C}^n$ . Writing complex coordinates on  $\mathbb{C}^n$  as

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$z = (z', z_n)$  with  $z' = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ , we have automorphisms of the form

$$(1.1) \quad \Phi(z) = \Phi(z', z_n) = (z', e^{f(z')} z_n + g(z')), \quad z \in \mathbb{C}^n,$$

where  $f$  and  $g$  are entire functions of  $n - 1$  variables. Such maps and their  $\mathrm{GL}_n(\mathbb{C})$ -conjugates are called *shears*. Maps (1.1) with  $f = 0$  and their  $\mathrm{SL}_n(\mathbb{C})$ -conjugates are *additive shears*; they have complex Jacobian determinant identically equal to 1. Maps (1.1) with  $g = 0$  and their  $\mathrm{GL}_n(\mathbb{C})$ -conjugates are *multiplicative shears*. Shears generate the *shear group*  $\mathcal{S}(n)$ , an infinite dimensional subgroup of  $\mathrm{Aut}(\mathbb{C}^n)$  when  $n > 1$ . Likewise, additive shears generate a subgroup  $\mathcal{S}_1(n)$  of the group  $\mathrm{Aut}_1(\mathbb{C}^n)$  of holomorphic automorphisms with Jacobian one. By composing shears, one obtains many interesting automorphisms. For example, composing a shear in two variables and the switch map  $\delta(z_1, z_2) = (z_2, z_1)$  yields *Hénon maps*  $H(z_1, z_2) = (e^{f(z_1)} z_2 + g(z_1), z_1)$  whose dynamical properties have been studied intensively.

In 1990, Erik Andersén [12] made a fundamental discovery that for every  $n > 1$  the group  $\mathcal{S}_1(n)$  generated by additive shears is dense in  $\mathrm{Aut}_1(\mathbb{C}^n)$  but not equal to it. This was extended by Andersén and László Lempert [14] in 1992 to the pair of groups  $\mathcal{S}(n) \subset \mathrm{Aut}(\mathbb{C}^n)$ . A year later, Forstnerič and Rosay [87] recast the proof of their approximation theorem [14, Theorem 1.3] in terms of complete holomorphic vector fields, showing that the main technical lemma from [12, 14] amounts to the following statement:

(\*) *Every polynomial holomorphic vector field  $V$  on  $\mathbb{C}^n$  for  $n > 1$  is a finite sum of complete polynomial vector fields  $V_1, \dots, V_N$  whose flows consist of shears. If  $V$  has divergence zero then each  $V_j$  can be chosen to have divergence zero, and we only get additive shears.*

The flow of a complete holomorphic vector field is a one-parameter group of holomorphic automorphisms; conversely, the infinitesimal generator of a one-parameter group of automorphisms is a complete holomorphic vector field. Since the flow of the sum  $V = V_1 + V_2 + \dots + V_N$  can be approximated by compositions of flows of vector fields  $V_1, \dots, V_N$ , it follows that the flow of any holomorphic vector field on  $\mathbb{C}^n$  can be approximated by compositions of shears. This implies that the shear group  $\mathcal{S}(n)$  is dense in the automorphism group  $\mathrm{Aut}(\mathbb{C}^n)$ . Forstnerič and Rosay gave the following more general and useful version of this approximation result; see [87, Theorem 1.1 and Erratum], as well as [73, Theorem 1.1].

**THEOREM 1.1.** *Assume that  $\Omega$  is a domain in  $\mathbb{C}^n$ ,  $n > 1$ , and  $\Phi_t: \Omega \rightarrow \mathbb{C}^n$  ( $t \in [0, 1]$ ) is a continuous isotopy of injective holomorphic maps such that  $\Phi_0$  is the identity map on  $\Omega$  and the domain  $\Omega_t = \Phi_t(\Omega)$  is Runge in  $\mathbb{C}^n$  for every  $t \in [0, 1]$ . Then,  $\Phi_1$  can be approximated uniformly on compacts*

in  $\Omega$  by elements of the shear group  $\mathcal{S}(n)$ . If in addition the domain  $\Omega$  is pseudoconvex,  $H^{n-1}(\Omega, \mathbb{C}) = 0$ , and  $\Phi_t$  has Jacobian one for every  $t \in [0, 1]$ , then  $\Phi_1$  can be approximated by elements of the shear group  $\mathcal{S}_1(n)$ .

Recall that a domain  $\Omega$  in  $\mathbb{C}^n$  is called Runge if polynomials are dense in the space  $\mathcal{O}(\Omega)$  of holomorphic functions. The domain  $\Omega$  in the first part of Theorem 1.1 need not be pseudoconvex; however, the Runge property is essential. It is easily seen that if  $\Phi: \Omega \rightarrow \Omega'$  is a biholomorphic map between domains in  $\mathbb{C}^n$  which is a limit of automorphisms of  $\mathbb{C}^n$ , then  $\Omega$  is Runge if and only if  $\Omega'$  is Runge. The condition  $H^{n-1}(\Omega, \mathbb{C}) = 0$  and pseudoconvexity of  $\Omega$  guarantee that holomorphic vector fields with vanishing divergence on  $\Omega$  can be approximated by holomorphic vector fields of the same type defined on  $\mathbb{C}^n$ . (The point is that, on a Stein manifold, the de Rham cohomology can be computed by means of holomorphic differential forms.) A complete proof of Theorem 1.1 is also available in [80, Theorem 4.9.2].

The next important step was made by Dror Varolin in his dissertation (University of Wisconsin-Madison, 1997) and in the papers [172–174]. His vantage point is the observation that the flow of a holomorphic vector field on a complex manifold  $X$ , which is a Lie combination of complete holomorphic vector fields, is a limit of holomorphic automorphisms of  $X$ . Mimicking (\*), Varolin introduced the following notion.

**DEFINITION 1.2.** A complex manifold  $X$  has the *density property* if every holomorphic vector field on  $X$  can be approximated uniformly on compacts by Lie combinations of complete holomorphic vector fields on  $X$ .

Varolin [174] introduced the notion of density property for any Lie algebra of holomorphic vector fields (an *algebraic structure* on  $X$ ), asking that it be densely generated by the complete vector fields which it contains; see Definition 2.1. An important example is the Lie algebra of holomorphic vector fields having zero divergence with respect to a holomorphic volume form  $\omega$  on  $X$ , that is, a nowhere vanishing holomorphic section of the canonical bundle  $K_X = \wedge^n T^*X$  with  $n = \dim X$  (the top exterior power of the cotangent bundle of  $X$ ). The density property for this Lie algebra is called the *volume density property* of  $(X, \omega)$ . The standard volume form on  $\mathbb{C}^n$  with coordinates  $z = (z_1, \dots, z_n)$  is  $dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$ .

The density property of  $X$  is equivalent to the ostensibly weaker condition that every holomorphic vector field on  $X$  can be approximated uniformly on compacts by sums of complete holomorphic vector fields on  $X$ ; however, commutators come handy in calculations. Thus, the Andersén–Lempert observation (\*) says that  $\mathbb{C}^n$  for  $n > 1$  enjoys the density property. By mimicking the proof in the case  $X = \mathbb{C}^n$ , one easily obtains the following analogue of Theorem 1.1; see [80, Theorem 4.10.5].

**THEOREM 1.3.** *Let  $X$  be a Stein manifold with the density property. If  $\Phi_t: \Omega \xrightarrow{\cong} \Omega_t = \Phi_t(\Omega) \subset X$  ( $t \in [0, 1]$ ) is a continuous isotopy of biholomor-*

phic maps between Stein Runge domains in  $X$  with  $\Phi_0 = \text{Id}_\Omega$ , then  $\Phi_1$  can be approximated uniformly on compacts in  $\Omega$  by holomorphic automorphisms of  $X$ .

The assumption that the domains  $\Omega_t$  are Runge and Stein is used to approximate holomorphic vector fields on  $\Omega_t$  by holomorphic vector fields on  $X$ . If  $X = \mathbb{C}^n$  or, more generally, if  $X$  is holomorphically parallelizable, this reduces to approximation of functions, and hence it suffices to assume that  $\Omega_t$  is Runge in  $X$  for each  $t \in [0, 1]$ .

REMARK 1.4. In this paper, a holomorphic vector field  $V$  on a complex manifold  $X$  is said to be complete if it is complete in complex time, i.e., its flow is a complex 1-parameter group of holomorphic automorphisms of  $X$ . One could also consider the density property for holomorphic vector fields which are complete in real time. However, it turns out that every Stein manifold  $X$  with this *real density property* also has the density property as defined above. Indeed, working with  $\mathbb{R}$ -complete holomorphic vector fields, the proof of Theorem 1.3 holds without any changes. Therefore, we can find a Fatou–Bieberbach domain in  $X$  around each point  $x \in X$ , arising by contracting a small coordinate ball around  $x$  and approximating this map by an automorphism of  $X$  with an attracting fixed point at  $x$ . The basin of attraction is biholomorphic to  $\mathbb{C}^n$  with  $n = \dim X$  (see Theorem 4.2). It follows that the manifold  $X$  has the Liouville property, i.e., every negative plurisubharmonic function on  $X$  is constant. On a Stein manifold with the Liouville property, every holomorphic vector field which is complete in real time is also complete in complex time, see [75, Corollary 2.2].

Here is another useful version of Theorems 1.1 and 1.3 for isotopies of compact holomorphically convex sets; see [87, Theorem 2.1] or [80, Theorem 4.12.1] for the case  $X = \mathbb{C}^n$ ,  $n > 1$ . The proof in the general case is the same.

THEOREM 1.5. *Let  $X$  be a Stein manifold with the density property. Assume that  $K$  is a compact holomorphically convex (i.e.,  $\mathcal{O}(X)$ -convex) set in  $X$ ,  $\Omega \subset X$  is an open set containing  $K$ , and  $\Phi_t: \Omega \rightarrow X$  ( $t \in [0, 1]$ ) is a continuous isotopy of injective holomorphic maps such that  $\Phi_0 = \text{Id}_\Omega$  and the set  $K_t = \Phi_t(K)$  is holomorphically convex in  $X$  for every  $t \in [0, 1]$ . Then we can approximate  $\Phi_1$  uniformly on  $K$  by holomorphic automorphisms of  $X$ .*

There is an easy reduction to Theorem 1.3 since a compact holomorphically convex set admits a basis of Stein Runge neighbourhoods in  $X$ . In both version of this result, one can approximate the entire isotopy  $\{\Phi_t\}_{t \in [0, 1]}$  by isotopies of automorphisms of  $X$ .

Several further additions to Theorems 1.1 and 1.3 are possible. The following parametric version of Theorem 1.1 was proved by Kutzschebauch [123, Theorem 2.3].

THEOREM 1.6. *Let  $\Omega$  be an open set in  $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^m$  with  $m > 1$ . For every  $t \in [0, 1]$  let  $\Phi_t: \Omega \rightarrow \mathbb{C}^n$  be a continuous family of injective holomorphic map of the form*

$$(1.2) \quad \Phi_t(z, w) = (z, \varphi_t(z, w)), \quad z \in \mathbb{C}^k, \quad w \in \mathbb{C}^m$$

*with  $\Phi_0 = \text{Id}_\Omega$ . If the domain  $\Phi_t(\Omega)$  is Runge in  $\mathbb{C}^n$  for every  $t \in [0, 1]$ , then  $\Phi_1$  can be approximated uniformly on compacts in  $\Omega$  by holomorphic automorphisms of the form (1.2).*

It is elementary to include interpolation at finitely many points in these approximation theorems. There are considerably more general interpolation results for automorphisms on algebraic subvarieties of  $\mathbb{C}^n$ , which we describe in Sections 2 and 3. In particular, any jet of a local biholomorphism at a point is the jet of an automorphism; see Theorem 3.2 and its parametric version, Theorem 3.3. These results are useful in the construction of holomorphic automorphisms with interesting dynamical properties.

Theorem 1.1 was extended by Kutzschebauch and Wold [131] (2018) to include Carleman approximation on sets of the form  $K \cup \mathbb{R}^s \subset \mathbb{C}^n$ , where  $\mathbb{R}^s$  is an affine totally real subspace of  $\mathbb{C}^n$  for  $s < n$ . Under certain technical assumptions on the isotopy of such sets in  $\mathbb{C}^n$  (they must be polynomially convex, totally real on  $\mathbb{R}^s \setminus K$ , and nearly fixed near infinity on  $\mathbb{R}^s$ ), it is possible to approximate the isotopy in the Carleman sense by isotopies of holomorphic automorphisms of  $\mathbb{C}^n$ .

Theorems 1.1 and 1.3 and their generalisations mentioned in the sequel form the backbone of what is now called *the Andersén–Lempert theory*.

After the initial work of Varolin, the subject of Stein manifolds with the density (or volume density) property was developed by Kaliman and Kutzschebauch with collaborators; see Section 2 for an overview. They also introduced and studied the algebraic (volume) density property on affine complex manifolds, the algebraic analogues of Stein manifolds. A closely related notion is *holomorphically flexibility* in the sense of Arzhantsev et al. [27], asking that complete holomorphic vector fields generate the tangent space of the manifold at every point. This field became known as *elliptic complex geometry*. Most complex Lie groups and homogeneous manifolds have the density property. In the two decades since their introduction, Stein manifolds with the density property came to play a major role in complex geometry. Every such manifold is an Oka manifold (see Section 8); however, the former class of manifolds allows many additional and more precise results concerning holomorphic maps from Stein manifolds. On Stein manifolds with the density property one can find automorphisms with given jets at finitely many points, and also at some discrete set of points; see Section 3.

Sections 4–10 contain a survey of applications of Andersén–Lempert theory.

A fascinating phenomenon in complex analysis is the existence for any  $n > 1$  of proper subdomains  $\Omega \subsetneq \mathbb{C}^n$  which are biholomorphic to  $\mathbb{C}^n$ ; see Section 4. Such a domain is called a *Fatou–Bieberbach domain*. An injective holomorphic map  $\Phi: \mathbb{C}^n \rightarrow \mathbb{C}^n$  whose image  $\Omega = \Phi(\mathbb{C}^n)$  is a proper subdomain of  $\mathbb{C}^n$  (and its inverse  $\Phi^{-1}: \Omega \rightarrow \mathbb{C}^n$ ) is called a *Fatou–Bieberbach map*. The first explicit example was given by Bieberbach [34] in 1933, following earlier examples by Fatou [63] (1922) of non-degenerate (but non-injective) entire maps  $\mathbb{C}^2 \rightarrow \mathbb{C}^2$  with non-dense images. (See also Bochner and Martin [38, Sect. III.1].) These examples arose as limits of holomorphic automorphisms of  $\mathbb{C}^n$ , and hence they are Runge in  $\mathbb{C}^n$ . In particular, if  $p \in \mathbb{C}^n$  is an attracting fixed point of a holomorphic automorphism  $F \in \text{Aut}(\mathbb{C}^n)$  and we denote by  $F^k$  the  $k$ -th iterate of  $F$ , the basin of attraction  $\{z \in \mathbb{C}^n : \lim_{k \rightarrow \infty} F^k(z) = p\}$  is either  $\mathbb{C}^n$  or a Runge Fatou–Bieberbach domain (see Theorem 4.2). Many further examples of Fatou–Bieberbach maps with interesting properties arose as limits of sequences of automorphisms which are not iterates of a fixed automorphism. Theorems 1.1 and 1.3 gave rise to several new constructions of Fatou–Bieberbach domains. An interesting application of the Andersén–Lempert theory was the discovery by Wold [181] (2008) of Fatou–Bieberbach domains in  $\mathbb{C}^n$  for any  $n > 1$  which fail to be Runge, and hence the corresponding Fatou–Bieberbach maps are not limits of automorphisms of  $\mathbb{C}^n$ .

In Section 5 we describe constructions of highly twisted proper holomorphic embeddings  $F: \mathbb{C}^k \hookrightarrow \mathbb{C}^n$  for  $1 \leq k < n$  such that the complements  $\mathbb{C}^n \setminus F(\mathbb{C}^k)$  of their images are  $(n - k)$ -hyperbolic in the sense of Eisenman (see Theorem 5.3). In particular,  $\mathbb{C}^k$  embeds as a closed complex hypersurface  $\Sigma \subset \mathbb{C}^{k+1}$  such that  $\mathbb{C}^{k+1} \setminus \Sigma$  is Kobayashi hyperbolic. This complements the well-known fact that most affine algebraic hypersurfaces in  $\mathbb{C}^n$  of sufficiently big degree are hyperbolic and have hyperbolic complements. This led to the discovery by Derksen and Kutzschebauch [53] of nonlinearizable periodic holomorphic automorphisms of  $\mathbb{C}^n$  for any  $n \geq 4$ ; see Theorem 5.4.

Another classical subject where the Andersén–Lempert theory generated major progress is the *Forster–Bell–Narasimhan Conjecture* [33, 72], asking whether every open Riemann surface embeds properly as a smooth complex curve in the Euclidean plane  $\mathbb{C}^2$ . The most advanced known results on this subject use constructions of Fatou–Bieberbach domains with the aid of Theorem 1.1. We discuss this topic in Section 6; a more comprehensive presentation can be found in [80, Sections 9.10–9.11].

Wold’s construction in [182] of non-Runge Fatou–Bieberbach domains led to the construction of uncountably many pairwise non-biholomorphic and non-Stein long  $\mathbb{C}^n$ ’s, i.e., complex manifolds which are increasing unions of biholomorphic copies of  $\mathbb{C}^n$ ; see Wold [182] and Boc Thaler and Forstnerič [37]. Many interesting questions concerning these *long*  $\mathbb{C}^n$ ’s remain open, and we refer to Section 7 for a discussion of this topic.

In Section 8 we discuss the impact of the Andersén–Lempert theory on Oka theory. Every Stein manifold  $X$  with the density or volume density property is an Oka manifold (Theorem 8.1), which means that holomorphic maps  $S \rightarrow X$  from any Stein manifold  $S$  satisfy all forms of the h-principle (see [80, Theorem 5.4.4]). Furthermore, if  $K \subset X$  is a compact  $\mathcal{O}(X)$ -convex subset then  $X \setminus K$  is an Oka manifold (Theorem 8.3). The same holds for complements of certain closed unbounded  $\mathcal{O}(X)$ -convex subsets. However, on a Stein manifold  $X$  the density property is a much stronger condition than the Oka property, implying finer results on the existence of holomorphic maps  $S \rightarrow X$  which do not hold for every Oka manifold  $X$ . For example, a Stein manifold  $X$  with the density property contains every Stein manifold of dimension  $k$  with  $2k + 1 \leq \dim X$  as a properly embedded complex submanifold (immersed if  $2k = \dim X$ ), and one can prescribe the homotopy class of the embedding. Furthermore, Stein manifolds with the density property contain big Stein Runge domains which are total spaces of normal bundles of certain embedded complex submanifolds. For example,  $\mathbb{C}^* \times \mathbb{C}$  embeds as a Runge domain in  $\mathbb{C}^2$ , although not in any obvious way.

In Section 9 we survey recent results concerning the problem of Paul Yang from 1977, asking whether there are bounded (metrically) complete complex submanifolds of  $\mathbb{C}^n$ . (This is holomorphic analogue of the Calabi–Yau problem concerning minimal surfaces in  $\mathbb{R}^n$  for  $n \geq 3$ ; see [8, Chapter 7] for the latter.) It has been discovered fairly recently that the ball of  $\mathbb{C}^n$  and, more generally, any pseudoconvex Runge domain in  $\mathbb{C}^n$  can be foliated by complete complex submanifolds of any codimension and with partial control of the topology of the leaves. The methods of Andersén–Lempert theory play a major role in these constructions.

In Section 10 we discuss an application of the Andersén–Lempert method in the smooth world. A long standing problem in 3-dimensional topology asked whether the fundamental group of any homology 3-sphere different from the 3-sphere  $S^3$  admits an irreducible representation into  $\mathrm{SL}_2(\mathbb{C})$ , i.e., a 2-dimensional irreducible representation. The affirmative answer given by Rafael Zentner [187] in 2018 is a case where shears play a role in the real setting.

We conclude by discussing the *recognition problem in complex analysis* in Section 11. The question is how to decide whether a given Stein manifold, which is contractible and simply connected at infinity, is a complex Euclidean space. To be such, it must have many holomorphic automorphisms. Hence, it is natural to ask whether every Stein manifold with the density property which is diffeomorphic to  $\mathbb{R}^{2n}$  is also biholomorphic to  $\mathbb{C}^n$  (see Problem 11.2). This question, asked by Tóth and Varolin [170] in 2000, remains unsolved. We also discuss other related problems such as the cancellation problem.

## 2. Stein manifolds with density properties

We denote by  $\mathrm{VF}_{\mathrm{hol}}(X)$  the Lie algebra of global holomorphic vector fields on a complex manifold  $X$ . In [174] Varolin introduced the following notion.

**DEFINITION 2.1.** A Lie subalgebra  $\mathfrak{g}$  of the Lie algebra  $\mathrm{VF}_{\mathrm{hol}}(X)$  has the density property if the Lie algebra generated by complete fields in  $\mathfrak{g}$  is dense in  $\mathfrak{g}$  in the compact-open topology.

**2.1. Density property.** The case that  $\mathfrak{g} = \mathrm{VF}_{\mathrm{hol}}(X)$  corresponds to  $X$  having the density property (see Definition 1.2). The importance of this property for Stein manifolds lies in Theorem 1.3 from the introduction. On the other hand, for compact complex manifolds this property is of no interest since every holomorphic vector field is complete, and furthermore there need not exist any nonzero holomorphic vector fields.

Establishing the density property can be rather tricky. After the initial work of Andersén and Lempert [14] which established this property for Euclidean spaces  $\mathbb{C}^n$  of any dimension  $n \geq 2$ , Varolin [173, 174] gave a list of further examples, and in his joint work with Tóth [170] established the density property for semi-simple complex Lie groups and their quotients by reductive subgroups. Later, Kaliman and Kutzschebauch [109] found a strong criterion based on the notion of compatible pairs of vector fields; see Definitions 2.3 and 2.5. Their criterion was initially developed for affine algebraic manifolds. For such manifolds, the algebraic density property was already introduced by Varolin as follows.

**DEFINITION 2.2.** An affine algebraic manifold  $X$  has the algebraic density property if the Lie algebra of algebraic vector fields on  $X$  is generated by complete algebraic vector fields.

Varolin remarked that this condition implies the holomorphic density property. Indeed, the Oka–Weil theorem says that polynomial functions are dense in the space of holomorphic functions; by an application of Theorems A and B the same holds for sections of any coherent algebraic sheaf, in particular, for holomorphic vector fields on an affine algebraic manifold.

**DEFINITION 2.3.** Complete algebraic vector fields  $\nu$  and  $\mu$  on an affine algebraic manifold  $X$  form a *compatible pair* if the following two conditions hold:

- (1) the linear span of the product of the kernels  $\ker \nu \cdot \ker \mu$  contains a nontrivial ideal  $I \subset \mathbb{C}[X]$ , and
- (2) there is a function  $h \in \mathbb{C}[X]$  with  $h \in \ker \mu$  and  $\nu(h) \in \ker \nu \setminus \{0\}$ .

If only condition (1) is satisfied, we call  $(\nu, \mu)$  a semi-compatible pair. The biggest ideal  $I$  with this property is called the *ideal of the pair*  $(\nu, \mu)$ .

The following powerful criterion was found by Kaliman and Kutzschebauch in [109].

**THEOREM 2.4.** *Let  $X$  be an affine algebraic manifold on which the group of algebraic automorphisms  $\text{Aut}_{\text{alg}}(X)$  acts transitively. If there are compatible pairs  $(\nu_i, \mu_i)$  and a point  $p \in X$  such that the vectors  $\mu_i(p)$  form a generating set for  $T_p X$ , then  $X$  has the algebraic density property.*

Here we call a subset  $\{v_1, \dots, v_k\} \subset T_p X$  a *generating set* if the union of orbits of these vectors under the isotropy group  $(\text{Aut}_{\text{alg}})_p(X)$  spans the tangent space  $T_p X$ .

The corresponding notion of a compatible pair, and the analogous theorem for general (not necessarily algebraic) Stein manifolds was given by the same authors in [113].

**DEFINITION 2.5.** A pair  $(\nu, \mu)$  of complete holomorphic vector fields on a Stein manifold  $X$  is a *compatible pair* if the following conditions hold:

(1) the closure of the linear span of the product of the kernels  $\ker \nu \cdot \ker \mu$  contains a nontrivial ideal  $I \subset \mathcal{O}(X)$ , and

(2) there is a function  $h \in \mathcal{O}(X)$  with  $h \in \ker \mu$  and  $\nu(h) \in \ker \nu \setminus \{0\}$ .

The biggest ideal  $I$  with this property is called the *ideal of the pair*  $(\nu, \mu)$ . If only condition (1) is satisfied, we call the pair semi-compatible.

The following is an analogue of Theorem 2.4 for Stein manifolds.

**THEOREM 2.6.** *Let  $X$  be a Stein manifold on which the group of holomorphic automorphisms  $\text{Aut}(X)$  act transitively. If there are compatible pairs  $(\nu_i, \mu_i)$  such that there is a point  $p \in X$  where the vectors  $\mu_i(p)$  form a generating set for  $T_p X$ , then  $X$  has the density property.*

This generalisation of the algebraic case is crucial in proving that the Koras–Russell threefold has the density property, see (5) in the list below. Indeed, algebraic automorphisms do not act transitively on that threefold, whereas holomorphic automorphisms do act transitively.

Before we come to the complete list of examples of Stein manifolds known to have the density property, we would like to show the power of this criterion in some examples.

**EXAMPLE 2.7.** On  $\mathbb{C}^n$ ,  $n \geq 2$ , with coordinates  $z = (z_1, z_2, \dots, z_n)$  the pair of vector fields  $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$  is compatible, with the function  $z_1$  and the ideal  $I = \mathcal{O}(\mathbb{C}^n)$  satisfying the conditions in Definition 2.5. Since we can permute coordinates,  $\{\frac{\partial}{\partial z_2}\}$  is a generating set for each tangent space. Thus,  $\mathbb{C}^n$  has the density property.

EXAMPLE 2.8. Denote an element of the special linear group  $X = \mathrm{SL}_2(\mathbb{C})$  by

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

The pair of vector fields on  $X$  given by

$$\delta_1 = b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2}, \quad \delta_2 = a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2}$$

is compatible, with the ideal  $I = \mathcal{O}(X)$  and the function  $h = a_1$ . Observe that the time- $t$  map of the field  $\delta_1$  is adding  $t$ -times the first row to the second row, and vice versa for the field  $\delta_2$ . Hence these fields are tangent to  $\mathrm{SL}_2(\mathbb{C})$ . Since the adjoint representation of  $\mathrm{SL}_2(\mathbb{C})$  is irreducible,  $\delta_2$  is a generating set at the identity. Thus,  $\mathrm{SL}_2(\mathbb{C})$  has density property. The proof of the density property for  $\mathrm{SL}_n(\mathbb{C})$  and  $\mathrm{GL}_n(\mathbb{C})$ ,  $n \geq 3$ , goes the same way.

**A list of examples of Stein manifolds known to have the density property:** (1) A homogeneous space  $X = G/H$ , where  $G$  is a linear complex algebraic group and  $H$  is a closed algebraic subgroup such that  $X$  is affine and the connected component of  $X$  is different from  $\mathbb{C}$  and from  $(\mathbb{C}^*)^n$ ,  $n \geq 1$ , has the (algebraic) density property.

It is known that if the subgroup  $H$  is reductive then the space  $X = G/H$  is always affine, but there is no known group-theoretic characterisation of  $G$  which would say when is  $X$  affine.

The above result has a long history and includes all previously known examples from works of Andersén–Lempert, Varolin, Tóth–Varolin, Kaliman–Kutzschebauch, and Donzelli–Dvorsky–Kaliman. The final result was obtained by Kaliman and Kutzschebauch in [115]. The manifolds  $\mathbb{C}$  and  $\mathbb{C}^*$  clearly do not have density property; however, the following problem is well known and seems notoriously difficult.

PROBLEM 2.9. Does  $(\mathbb{C}^*)^n$  for  $n \geq 2$  have the density property?

It is conjectured that the answer is negative. More precisely, one expects that all holomorphic automorphisms of  $(\mathbb{C}^*)^n$ ,  $n \geq 2$ , respect the volume form  $\bigwedge_{i=1}^n \frac{dz_i}{z_i}$  up to a sign.

(2) The manifolds  $X$  given as a submanifold of  $\mathbb{C}^{n+2}$  with coordinates  $u \in \mathbb{C}$ ,  $v \in \mathbb{C}$ ,  $z \in \mathbb{C}^n$  by the equation  $uv = p(z)$ , where the zero fibre of the polynomial  $p \in \mathbb{C}[\mathbb{C}^n]$  is smooth and reduced (otherwise  $X$  is not smooth), have (algebraic) density property; see [110].

Before formulating the next result, recall that *Gizatullin surfaces* are by definition the normal affine surfaces on which the algebraic automorphism

group acts with an open orbit whose complement is a finite set of points. By the classical result of Gizatullin [94], they are characterized by admitting a completion with a simple normal crossing chain of rational curves at infinity. Every Gizatullin surface admits a  $\mathbb{C}$ -fibration with at most one singular fibre which however is not always reduced. Since the only affine algebraic 2-manifolds admitting semi-compatible pairs are  $\mathbb{C} \times \mathbb{C}$  and  $\mathbb{C} \times \mathbb{C}^*$  [111], the criterion from Theorem 2.4 is not applicable for surfaces, which makes the proof of the following result more cumbersome.

(3) Smooth Gizatullin surfaces which admit a  $\mathbb{C}$ -fibration with at most one singular and reduced fibre have the density property (Andrist [16]). These surfaces are also called *generalised Danielewski surfaces*. Special cases of this result were proved before in [21, 56, 110].

(4) The only known non-algebraic examples of Stein manifolds with the density property are, firstly, the holomorphic analogues of (2), namely, complex submanifolds  $X$  of  $\mathbb{C}^{n+2}$  with coordinates  $u \in \mathbb{C}$ ,  $v \in \mathbb{C}$ ,  $z \in \mathbb{C}^n$  given by an equation  $uv = f(z)$ , where the zero fibre of the holomorphic function  $f \in \mathcal{O}(\mathbb{C}^n)$  is smooth and reduced (otherwise  $X$  is not smooth); see [110]. Secondly, in the special case of (3) when the Gizatullin surface can be completed by four rational curves, the Stein manifolds given by the same algebraic equations (but using holomorphic functions as in the above example  $uv = f(z)$ ) have the density property [21].

(5) Certain hypersurfaces in  $\mathbb{C}^{n+3}$  with coordinates  $z = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}$ ,  $x \in \mathbb{C}$ ,  $y \in \mathbb{C}$ , given by the polynomial equation  $x^2y = a(z) + xb(z)$  where  $\deg_{z_0} a \leq 2$ ,  $\deg_{z_0} b \leq 1$  and not both degrees are zero, have the density property. (The exact conditions on  $a$  and  $b$  ensuring transitivity of the holomorphic automorphism group are rather technical.) This family includes the Koras–Russell threefold given in  $\mathbb{C}^4$  by the equation

$$(2.1) \quad x + x^2y + s^2 + t^3 = 0.$$

This result of Leuenberger [133] is interesting in connection with the recognition problem for affine spaces; see Section 11.

(6) The Calogero–Moser spaces  $\mathcal{C}_n$ ,  $n \in \mathbb{N}$ , have the algebraic density property according to a recent result of Andrist [17]. The space  $\mathcal{C}_n$  describes the phase space of a Calogero–Moser system, an  $n$ -particle system in classical physics with a certain Hamiltonian.

**2.2. Volume density property.** The volume density property was the very first known density property (before the terminology was introduced), which was discovered on Euclidean spaces  $\mathbb{C}^n$  by Andersén [12] in 1990. (This was the first paper on the subject and a precursor to Andersén’s joint work with Lempert [14].) We consider a holomorphic volume form  $\omega$

on  $X$ , i.e., a nowhere vanishing holomorphic section of the canonical bundle  $K_X = \wedge^n T^*X$ ,  $n = \dim X$ . (Note that  $K_X$ , being a line bundle over  $X$ , must be topologically trivial for such a form to exist, and on a Stein manifold this necessary condition is also sufficient by the Oka–Grauert principle [80, Theorem 5.3.1].) The Lie algebra  $\mathfrak{g}$  from Definition 2.1 is now the Lie algebra  $\mathrm{VF}_\omega(X)$  of volume preserving holomorphic vector fields  $\theta$ , i.e., such that the Lie derivative of  $\omega$  along  $\theta$  vanishes,  $L_\theta(\omega) = 0$ . Let  $\iota_\xi\omega$  denote the interior product of  $\omega$  and a vector field  $\xi \in \mathrm{VF}_{\mathrm{hol}}(X)$ . Since  $d\omega = 0$ , Cartan’s formula for the Lie derivative gives

$$(2.2) \quad L_\theta(\omega) = d(\iota_\theta\omega) = \mathrm{div}_\omega(\theta)\omega.$$

The function  $\mathrm{div}_\omega(\theta)$  is called the divergence of  $\theta$  with respect to  $\omega$ . Hence,  $\mathrm{VF}_\omega(X)$  is the algebra of holomorphic vector fields on  $X$  with vanishing divergence,  $\mathrm{div}_\omega(\theta) = 0$ .

Since volume preserving vector fields do not form an  $\mathcal{O}$ -module, the proof that the algebraic volume density property implies the holomorphic volume density property (see Kaliman and Kutzschebauch [111]) is not straightforward. For the same reason, the version of Theorem 1.1 of Forstnerič and Rosay [87] for holomorphic automorphisms of  $\mathbb{C}^n$  with Jacobian one (preserving the standard holomorphic volume form  $dz_1 \wedge \cdots \wedge dz_n$ ) requires the additional cohomological assumption  $H^{n-1}(\Omega, \mathbb{C}) = 0$  on the domain  $\Omega$  of the isotopy.

The list of Stein manifolds known to enjoy the (algebraic) volume density property was rather short before an efficient criterion was established by Kaliman and Kutzschebauch in [114]. We only state the holomorphic version from [113]; the algebraic version is similar as in Theorem 2.4. Let  $X$  be a Stein manifold of dimension  $n$  with a holomorphic volume form  $\omega$ . Denote by  $\mathcal{Z}_{n-1}(X)$  the space of closed holomorphic differential  $(n-1)$ -forms on  $X$ . The formula (2.2) shows that the map

$$(2.3) \quad \Theta : \mathrm{VF}_\omega(X) \xrightarrow{\cong} \mathcal{Z}_{n-1}(X), \quad \xi \rightarrow \iota_\xi\omega$$

is an isomorphism. By  $\mathrm{Lie}_{\mathrm{hol}}^\omega(X)$  we denote the Lie subalgebra of  $\mathrm{VF}_\omega(X)$  generated by complete holomorphic  $\omega$ -volume preserving vector fields on  $X$ .

**THEOREM 2.10.** *Let  $X$  be a Stein manifold with a holomorphic volume form  $\omega$ . Assume that there are pairs of divergence-free semi-compatible vector fields  $(\xi_j, \eta_j)$  with ideals  $I_j$  satisfying the following two conditions.*

(A) *For every  $x \in X$  the set  $\{I_j(x)\xi_j(x) \wedge \eta_j(x)\}_{j=1}^k$  generates  $\wedge^2 T_x X$ .*

(B) *The image of  $\Theta(\overline{\mathrm{Lie}_{\mathrm{hol}}^\omega(X)}) \subset \mathcal{Z}_{n-1}(X)$  under the de Rham homomorphism  $\Phi_{n-1} : \mathcal{Z}_{n-1}(X) \rightarrow H^{n-1}(X, \mathbb{C})$  equals  $H^{n-1}(X, \mathbb{C})$ .*

*Then  $\Theta(\overline{\mathrm{Lie}_{\mathrm{hol}}^\omega(X)}) = \mathcal{Z}_{n-1}(X)$  and therefore  $\overline{\mathrm{Lie}_{\mathrm{hol}}^\omega(X)} = \mathrm{VF}_\omega(X)$ , i.e., the manifold  $(X, \omega)$  has the volume density property.*

A highly nontrivial fact proved by Kaliman and Kutzschebauch [113, Theorem 8] using this criterion is that the product of two Stein manifolds  $(X_1, \omega_1)$  and  $(X_2, \omega_2)$  with the volume density property again has the volume density property for  $\omega = \omega_1 \wedge \omega_2$ .

**The list of examples of Stein manifolds known to have the volume density property:** (1) A homogeneous space  $X = G/H$ , where  $G$  is a linear algebraic group and  $H$  is a closed algebraic subgroup such that  $X$  is affine and admits a  $G$ -invariant volume form (which is necessarily algebraic) has the (algebraic) volume density property (Kaliman and Kutzschebauch [115]). This includes earlier results of Varolin [174] and Andersén [13] for the tori  $(\mathbb{C}^*)^n$ .

(2) The affine submanifolds  $X$  in  $\mathbb{C}^{n+2}$  with coordinates  $u \in \mathbb{C}$ ,  $v \in \mathbb{C}$ ,  $z \in \mathbb{C}^n$ , given by the equation  $uv = p(z)$  where  $p \in \mathbb{C}[\mathbb{C}^n]$  is a polynomial whose zero fibre is smooth and reduced (to ensure that  $X$  is not smooth), have (algebraic) volume density property with respect to the unique algebraic volume form on them (Kaliman and Kutzschebauch [111]). Uniqueness (up to a multiplicative constant) follows from simple connectedness of the manifolds.

(3) The first, and up to now the only known non-algebraic examples are certain holomorphic analogues of (2). Namely, for a nonconstant holomorphic function  $f$  on  $\mathbb{C}^n$ ,  $n \geq 1$ , with  $X_0 = f^{-1}(0)$  reduced and smooth, and such that the reduced cohomology  $\tilde{H}^{n-2}(X_0)$  vanishes if  $n \geq 2$ , the hypersurface  $X \subset \mathbb{C}^{n+2}$  defined by  $uv = f(z_1, \dots, z_n)$  has the volume density property with respect to a certain volume form (Ramos–Peon [150]).

(4) Certain hypersurfaces  $X$  in  $\mathbb{C}^{n+3}$  with coordinates  $z = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}$ ,  $x \in \mathbb{C}$ ,  $y \in \mathbb{C}$  given by the equation  $x^2y = a(z) + xb(z)$ , where  $\deg_{z_0} a \leq 2$ ,  $\deg_{z_0} b \leq 1$  and not both degrees are zero, have volume density property for the volume form  $\omega = \frac{dx}{x^2} \wedge dz_0 \wedge dz_1 \wedge \dots \wedge dz_n$  (Leuenberger [133]). This family includes the Koras–Russell threefold (2.1). Again, the conditions ensuring transitivity of the group  $\text{Aut}_\omega(X)$  are rather technical.

(5) The smooth fibres of the Gromov–Vaserstein fibration given by equations  $p_n(z_1, \dots, z_n) = a$ , where the polynomials  $p_n \in \mathbb{C}[\mathbb{C}^n]$  are defined inductively by

$$p_{n+1} = z_{n+1}p_n + p_{n-1}, \quad p_0 = 1, \quad p_1(z_1) = z_1,$$

have the volume density property for the unique algebraic volume form on them (De Vito [52]). Uniqueness of the algebraic volume form (up to a constant) again follows from simple connectedness of the manifolds.

**2.3. Relative density properties.** The next results concern density properties for Lie algebras of holomorphic vector fields vanishing on subvarieties  $Y \subset X$ . It makes sense to consider such properties on a Stein space  $X$

when the subvariety  $Y$  contains the singular locus  $X_{\text{sing}}$  of  $X$ . By using the same idea as in the proof of Theorem 1.1, this density property naturally leads to theorems on approximation of isotopies of injective holomorphic maps  $\Omega \rightarrow X$  on Runge domains  $\Omega \subset X$ , fixing  $Y \cap \Omega$ , by holomorphic automorphisms of  $X$  fixing  $Y$ .

The following definitions were introduced in [126]. Let  $X$  be an affine algebraic variety,  $Y \subseteq X$  be an algebraic subvariety containing  $X_{\text{sing}}$ , and let  $I = I(Y) \subseteq \mathbb{C}[X]$  denote the ideal of  $Y$ . Let  $\text{VF}_{\text{alg}}(X, Y)$  be the  $\mathbb{C}[X]$ -module of vector fields vanishing on  $Y$ :

$$\text{VF}_{\text{alg}}(X, Y) = \{\partial \in \text{VF}_{\text{alg}}(X) : \partial(\mathbb{C}[X]) \subseteq I\}.$$

Let  $\text{Lie}_{\text{alg}}(X, Y)$  denote the Lie algebra generated by complete vector fields in  $\text{VF}_{\text{alg}}(X, Y)$ .

DEFINITION 2.11. (Assumptions and notation as above.)

(a)  $X$  has the *strong algebraic density property relative to  $Y$*  if  $\text{VF}_{\text{alg}}(X, Y) = \text{Lie}_{\text{alg}}(X, Y)$ .

(b)  $X$  has the *algebraic density property relative to  $Y$*  if there exists an integer  $\ell \geq 0$  such that  $I^\ell \text{VF}_{\text{alg}}(X, Y) \subseteq \text{Lie}_{\text{alg}}(X, Y)$ .

Note that condition (b) with  $\ell = 0$  is equivalent to condition (a). We say that  $X$  has the (strong) algebraic density property if these conditions hold for  $Y = X_{\text{sing}}$ .

Except for the fact that we consider not necessarily smooth varieties, the strong algebraic density property ( $\ell = 0$ ) is a version of Varolin's Definition 2.1 of the density property for the Lie subalgebra  $\mathfrak{g}$  of vector fields vanishing on  $Y$ . On the other hand, for  $\ell > 0$  our property in (b) is slightly weaker than Varolin's definition since we generate the Lie subalgebra of vector fields vanishing on  $Y$  of order at least  $\ell$  using complete vector fields vanishing on  $Y$  of possibly lower order than  $\ell$ . Still, this version of the algebraic density property has the same remarkable consequences as in Varolin's version of the algebraic density property for the group of holomorphic automorphisms of  $X$  fixing the subvariety  $Y$ . In particular, the following analogue of Theorem 1.1 with interpolation on a subvariety holds.

THEOREM 2.12. *Let  $X$  be an affine algebraic variety and  $Y \subset X$  be a closed algebraic subvariety containing  $X_{\text{sing}}$ . Assume that  $\Omega \subset X$  is a Stein Runge domain and  $\Phi_t: \Omega \rightarrow X$  ( $t \in [0, 1]$ ) is an isotopy of injective holomorphic maps as in Theorem 1.1. If condition (b) in Definition 2.11 holds for an integer  $\ell \geq 0$  and the map  $\Phi_t$  agrees with the identity to order  $\ell$  on  $Y \cap \Omega$  for every  $t \in [0, 1]$ , then  $\Phi_1$  can be approximated uniformly on compacts in  $\Omega$  by holomorphic automorphisms of  $X$  fixing  $Y$  pointwise.*

This result is often used to move a compact  $\mathcal{O}(X)$ -convex subset of  $X \setminus Y$  around by automorphisms of  $X$  fixing  $Y$ . When  $X = \mathbb{C}^n$  with  $n \geq 2$ , this

method can be used to find Fatou–Bieberbach domains containing  $Y$  and omitting a certain compact set (see Section 4). Alternatively, approximating a contraction on a closed ball  $B \subset \mathbb{C}^n \setminus Y$  and fixing  $Y$  gives a Fatou–Bieberbach domain in  $\mathbb{C}^n \setminus Y$  containing  $B$ , thereby generalising Corollary 4.5. (See [80, Corollary 4.12.2] and Theorem 4.7 in Section 4). In fact, this holds whenever  $Y$  is an algebraic subvariety of  $\mathbb{C}^n$  of codimension at least two or, more generally, a tame subvariety of  $\mathbb{C}^n$  of codimension at least two; see [80, Definition 4.11.3 and Theorem 4.12.1].

Here are the main examples when the relative density property is known to hold.

(1) The Euclidean space  $\mathbb{C}^n$  for  $n \geq 2$  has the relative algebraic density property with respect to any algebraic subvariety  $Y \subset \mathbb{C}^n$  of codimension at least 2. If moreover the dimension of the Zariski tangent space  $T_y Y$  at every point of  $Y$  is at most  $n - 1$ , then  $\mathbb{C}^n$  has the strong algebraic density property with respect to  $Y$  [109, Theorems 4 and 6]; this holds in particular if  $Y$  is without singularities. Hence, Theorem 2.12 holds for any such subvariety  $Y \subset \mathbb{C}^n$ .

(2) Recall that a *locally nilpotent derivation* on an affine algebraic manifold is a complete algebraic vector field which generates an algebraic subgroup (isomorphic to  $\mathbb{C}$  with  $+$ ) of the algebraic automorphism group  $\text{Aut}_{\text{alg}}(X)$ . An affine algebraic manifold  $X$  is flexible in the sense of Arzhantsev et al. [27] if locally nilpotent derivations span the tangent space at every point of  $X$ . Kaliman proved in [107, Theorem 2.15 and Remark 2.16] that if  $X$  is a flexible affine algebraic manifold with a compatible pair of locally nilpotent derivations, then  $X$  has the density property relative to any algebraic subvariety  $Y$  of codimension at least 2. Again, it follows that Theorem 2.12 holds for any such subvariety  $Y \subset X$  (cf. [107, Theorem 2.17]). This vastly generalizes the first part of (1), while Example 2.8 shows that  $\text{SL}_n(\mathbb{C})$  has the density property relative to any algebraic subvariety of codimension at least 2.

(3) If  $X$  is a normal affine toric variety of dimension  $n \geq 2$  and  $Y$  is a  $T \cong (\mathbb{C}^*)^n$ -invariant closed subvariety of  $X$  containing  $X_{\text{sing}}$ , then  $X$  has the algebraic density property relative to  $Y$  if and only if  $X \setminus Y \neq \emptyset$  [126, Theorem 3.7]. Affine toric surfaces with the strong algebraic density property were classified by Kutzschebauch, Leuenberger, and Liendo [126, Corollary 5.5]. In the same paper, they gave an upper bound for the vanishing order  $\ell$  in Definition 2.11. This includes the results of Varolin [174] for  $Y$  a linear subspace of  $X = \mathbb{C}^n$ .

**2.4. Fibred density properties.** Let  $\pi: X \rightarrow B$  be a holomorphic map between complex manifolds. We say that  $X$  has the *fibred density property with respect to  $\pi$*  if the Lie algebra  $\mathfrak{g}$  of holomorphic vector fields

$\theta$  tangent to the fibres of  $\pi$ , i.e. fulfilling the condition  $d\pi(\theta) = 0$ , has the density property; see Definition 2.1.

The simplest case is a trivial fibration  $W \times X \rightarrow W$ ,  $(w, x) \rightarrow w$ , where  $X$  is a Stein manifold with the density property and the parameter space  $W$  is a Stein manifold. For this case the fibred density property is easy to prove [123, 129]. The implication for the corresponding group of holomorphic automorphisms (in a simple situation of a projection  $\mathbb{C}^k \times \mathbb{C}^m \rightarrow \mathbb{C}^k$ ) is given by Theorem 1.6 from the introduction.

The next case where the fibred density property is known is much more complicated. To formulate it, we need to recall the classical invariant-theoretic quotient of the action of  $\mathrm{SL}_n(\mathbb{C})$  on its Lie algebra  $\mathfrak{sl}_n(\mathbb{C})$  of complex  $n \times n$  matrices by conjugation (the adjoint representation). Denote by  $\sigma_1, \dots, \sigma_n : \mathbb{C}^n \rightarrow \mathbb{C}$  the elementary symmetric polynomials in  $n$  complex variables. Let  $\mathrm{Eig} : \mathrm{Mat}(n \times n; \mathbb{C}) \rightarrow \mathbb{C}^n$  assign to each matrix a vector of its eigenvalues. Denote by  $\pi_1 := \sigma_1 \circ \mathrm{Eig}$ ,  $\dots$ ,  $\pi_n := \sigma_n \circ \mathrm{Eig}$  the elementary symmetric polynomials in the eigenvalues. By symmetrizing we avoid ambiguities caused by the order of eigenvalues in the definition of  $\mathrm{Eig}$  and obtain a polynomial map  $(\pi_1, \dots, \pi_n)$  such that  $\chi_A(\lambda) = \lambda^n + \sum_{j=1}^n (-1)^j \pi_j(A) \lambda^{n-j}$  is the characteristic polynomial of the matrix  $A$ .

Since  $\mathrm{trace} A = 0$  for  $A \in \mathfrak{sl}_n$ , the map  $\pi_1$ , the sum of the eigenvalues, is the zero map.

Consider the fibration  $\pi := (\pi_2, \dots, \pi_n) : \mathfrak{sl}_n \rightarrow \mathbb{C}^{n-1}$ . A generic fibre, i.e. a fibre above a base point with no multiple eigenvalues, consists of exactly one equivalence class of similar matrices, so it is a homogeneous space of  $\mathrm{SL}_n(\mathbb{C})$  and hence smooth. A fibre above a base point with multiple eigenvalues decomposes into several strata of  $\mathrm{SL}_n(\mathbb{C})$  orbits, the largest one being the orbit of a matrix with the largest possible Jordan blocks. The structure of these fibres is well-studied in classical invariant theory, see e.g. Kraft [119]. The fibration  $\pi$  is the invariant theoretic quotient  $\mathfrak{sl}_n(\mathbb{C}) // \mathrm{SL}_n(\mathbb{C})$ , i.e., the algebra of  $\mathrm{SL}_n(\mathbb{C})$ -invariant polynomial/holomorphic functions on  $\mathfrak{sl}_n(\mathbb{C})$  is exactly the pull-back of the algebra  $\pi^*(\mathcal{O}(\mathbb{C}^{n-1}))$  of holomorphic functions on  $\mathbb{C}^{n-1}$ . The proof that  $\pi$  has relative density property (Andrist and Kutzschebauch [20]) uses a special family of complete holomorphic vector fields in this algebra, derived from certain one-parameter subgroups of  $\mathrm{SL}_n(\mathbb{C})$  exactly as the shears (1.1) on  $\mathbb{C}^n$  are derived from the shear vector fields  $\frac{\partial}{\partial z_i}$ . In the same paper, the authors applied this density property to provide a set of generators for a dense subgroup of the automorphism group of the spectral ball.

**2.5. Symplectic density property.** Let  $X$  be a complex manifold of dimension  $2n$  with a *holomorphic symplectic form*  $\omega$ , a closed holomorphic 2-form whose highest exterior power  $\omega^n$  is nowhere vanish-

ing. The standard holomorphic symplectic form on  $\mathbb{C}^{2n}$  with coordinates  $z_1, \dots, z_n, w_1, \dots, w_n$  is

$$(2.4) \quad \omega = \sum_{i=1}^n dz_i \wedge dw_i.$$

A symplectic form on a complex surface is the same thing as a holomorphic volume form.

Let  $\Omega$  be a domain in a complex symplectic manifold  $(X, \omega)$ . A holomorphic map  $f: \Omega \rightarrow X$  is called *symplectic* if  $f^*\omega = \omega$ . Assume that  $\Phi_t: \Omega \rightarrow X$  is a smooth isotopy of injective symplectic holomorphic maps with the infinitesimal generator  $V_t \in \text{VF}(\Omega_t)$ ,  $t \in [0, 1]$ . Differentiating the identity  $\omega = \Phi_t^*(\omega)$  on  $t$  and taking into account the Cartan formula  $L_V\omega = d(\iota_V\omega) + \iota_V d\omega = d(\iota_V\omega)$  for the Lie derivative gives

$$0 = \frac{d}{dt} \Phi_t^*(\omega) = \Phi_t^*(L_{V_t}\omega) = \Phi_t^*(d(\iota_{V_t}\omega)),$$

which holds if and only if  $d(\iota_{V_t}\omega) = 0$  for all  $t \in [0, 1]$ . A holomorphic vector field  $V$  satisfying  $d(\iota_V\omega) = 0$  is called *symplectic*. This shows that flows of symplectic holomorphic maps are generated by symplectic vector fields, and vice versa. A vector field  $V$  is called *Hamiltonian* if  $\iota_V\omega$  is an exact 1-form, and a holomorphic function  $H \in \mathcal{O}(X)$  satisfying  $dH = \iota_V\omega$  is the Hamiltonian of  $V$ . Conversely, every holomorphic function  $H$  on  $X$  determines a Hamiltonian vector field  $V_H$  by the above equation. On  $\mathbb{C}^{2n}$  with coordinates  $(z, w)$  and the symplectic form (2.4), every symplectic vector field is Hamiltonian of the form

$$V_H = \sum_{i=1}^n \frac{\partial H}{\partial w_j} \frac{\partial}{\partial z_j} - \frac{\partial H}{\partial z_j} \frac{\partial}{\partial w_j}, \quad H \in \mathcal{O}(\mathbb{C}^{2n}).$$

The alternating bilinear form on  $\mathbb{C}^{2n}$  defined by

$$(2.5) \quad \tilde{\omega}(u, v) = \sum_{j=1}^n u_j v_{n+j} - u_{n+j} v_j, \quad u, v \in \mathbb{C}^{2n}$$

is the *standard linear symplectic form* on  $\mathbb{C}^{2n}$ . The corresponding differential form is  $\omega$  (2.4).

The following algebraic density property was proved by Forstnerič [75, Proposition 5.2] in 1996. This was the first known density property following the original ones of Andersén and Lempert, and it predates the formal introduction of this notion.

PROPOSITION 2.13. *Let  $\omega$  be the symplectic form (2.4) on  $\mathbb{C}^{2n}$ , and let  $\tilde{\omega}$  be given by (2.5). The Lie algebra of polynomial Hamiltonian vector fields on  $(\mathbb{C}^{2n}, \omega)$  is generated by the complete Hamiltonian vector fields of the form*

$$(2.6) \quad V(x) = f(\tilde{\omega}(x, v)) \sum_{j=1}^{2n} v_j \frac{\partial}{\partial x_j}, \quad x \in \mathbb{C}^{2n}, \quad v \in \mathbb{C}^{2n}, \quad f \in \mathbb{C}[\mathbb{C}^{2n}].$$

The polynomial vector field (2.6) generates the flow

$$(2.7) \quad \Phi_t(x) = x + tf(\omega(x, v))v, \quad t \in \mathbb{C}, \quad x \in \mathbb{C}^{2n}$$

consisting of symplectic polynomial shear automorphisms of  $(\mathbb{C}^{2n}, \omega)$ . The proof of Theorem 1.1 gives the following result (see [74, Proposition 2.3 and Remark] and [75, Theorem 5.1]).

THEOREM 2.14. *Assume that  $\Omega$  is a pseudoconvex domain in  $\mathbb{C}^{2n}$ ,  $n \in \mathbb{N}$ , with  $H^1(\Omega, \mathbb{C}) = 0$ . If  $\Phi_t: \Omega \rightarrow \mathbb{C}^{2n}$  ( $t \in [0, 1]$ ) is a  $\mathcal{C}^1$  isotopy of injective symplectic holomorphic maps (with respect to the symplectic form (2.4)) such that  $\Phi_0$  is the identity map on  $\Omega$  and the domain  $\Omega_t = \Phi_t(\Omega)$  is Runge in  $\mathbb{C}^{2n}$  for every  $t \in [0, 1]$ , then  $\Phi_1$  can be approximated uniformly on compacts in  $\Omega$  by compositions of symplectic shears (2.7). In particular, the group generated by symplectic shears (2.7) is dense in the group  $\text{Aut}_\omega(\mathbb{C}^{2n})$  of symplectic holomorphic automorphisms of  $(\mathbb{C}^{2n}, \omega)$ .*

The condition  $H^1(\Omega, \mathbb{C}) = 0$  and pseudoconvexity of  $\Omega$  ensure that every symplectic holomorphic vector field on  $\Omega_t$  is Hamiltonian, and hence by the Runge condition on  $\Omega_t$  it can be approximated uniformly on compacts in  $\Omega_t$  by polynomial Hamiltonian vector fields on  $\mathbb{C}^{2n}$ . (Like in the second part of Theorem 1.1, pseudoconvexity is used in order to know that de Rham cohomology can be computed by means of holomorphic differential forms.)

Of all density properties presented in this section, this symplectic density property is the only one that has not been developed yet for more general Stein symplectic manifolds, and no effective criteria are known (except on Stein surfaces where a symplectic form is just a holomorphic volume form). The problem in mimicking the criteria of Kaliman and Kutzschebauch for density or volume density properties lies in finding a module structure to apply the theory of coherent analytic sheaves, in particular, Theorems A and B. A complete symplectic vector field remains complete when multiplied with a function in its kernel; however, it stays symplectic only if it is multiplied by a function  $f(H)$  of the Hamiltonian  $H$  of the vector field. For example, the symplectic density property would be much more interesting for the Calogero–Moser spaces than the density property proved by Andrist in [17].

Since the holomorphic cotangent bundle of a complex manifold carries a natural holomorphic symplectic structure, it seems reasonable to propose the following problem.

PROBLEM 2.15. Does the cotangent bundle of a Stein manifold with the density property enjoy the symplectic density property?

### 3. Automorphisms with given jets

The Andersén–Lempert theory is about Stein manifolds with large automorphism groups. One aspect of this phenomenon is demonstrated by the approximation theorems mentioned in the introduction and the interpolation theorems discussed in Section 2. One can go further and ask which jets of locally biholomorphic maps at a closed discrete set of points are jets of an automorphism. The following result for finitely many points was proved by Andersén and Lempert [14, Proposition 6.3] and Forstnerič (see [76, Proposition 2.1] and [80, Proposition 4.15.3]). The analogous result on Stein manifolds with the density property is due to Varolin [173]. For symplectic holomorphic automorphisms of  $\mathbb{C}^{2n}$ , see Løw et al. [134].

PROPOSITION 3.1. *Let  $m, n, N \in \mathbb{N}$ , with  $n > 1$ . Assume that*

- (a)  *$K$  is a compact polynomially convex set in  $\mathbb{C}^n$ ,*
- (b)  *$\{a_j\}_{j=1}^s$  is a finite set of points in  $K$ ,*
- (c)  *$p$  and  $q$  are points in  $\mathbb{C}^n \setminus K$ , and*
- (d)  *$P : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a polynomial of order  $m$  with nondegenerate linear part and  $P(0) = 0$ .*

*Given  $\varepsilon > 0$ , there exists  $\Phi \in \text{Aut}(\mathbb{C}^n)$  satisfying the following conditions:*

- (i)  *$\Phi(p) = q$  and  $\Phi(z) = q + P(z - p) + O(|z - p|^{m+1})$  as  $z \rightarrow p$ ,*
- (ii)  *$\Phi(z) = z + O(|z - a_j|^N)$  as  $z \rightarrow a_j$  for each  $j = 1, 2, \dots, s$ , and*
- (iii)  *$|\Phi(z) - z| + |\Phi^{-1}(z) - z| < \varepsilon$  for each  $z \in K$ .*

*If in addition we have  $JP(z) = 1 + O(|z|^m)$  as  $z \rightarrow 0$  then there exists a polynomial automorphism  $\Phi$  with  $J\Phi \equiv 1$  satisfying conditions (i)–(iii).*

These results have proved very useful in the construction of holomorphic automorphisms with interesting dynamical properties.

An inductive application of Proposition 3.1 leads to the following Mittag-Leffler interpolation theorem for automorphisms of  $\mathbb{C}^n$  (Buzzard and Forstnerič [45, Theorem 1.1]). Recall that a discrete sequence  $a_j$  without repetition in  $\mathbb{C}^n$  is said to be *tame* if there is an automorphism  $\Phi \in \text{Aut}(\mathbb{C}^n)$  such that  $\Phi(a_j) = (j, 0, \dots, 0) \in \mathbb{C} \times \{0\}^{n-1}$  for all  $j \in \mathbb{N}$ . This notion was introduced and studied by Rosay and Rudin in [155]. (See also [80, Section 4.6].)

THEOREM 3.2. *Assume that  $n > 1$ ,  $a_j$  and  $b_j$  ( $j \in \mathbb{N}$ ) are tame discrete sequences in  $\mathbb{C}^n$  without repetitions, and  $P_j : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a polynomial map with nondegenerate linear part and  $P_j(0) = 0$  for each  $j \in \mathbb{N}$ . Then there*

exists an automorphism  $F$  of  $\mathbb{C}^n$  such that for every  $j = 1, 2, \dots$  we have  $F(a_j) = b_j$  and

$$F(z) = b_j + P_j(z - a_j) + O(|z - a_j|^{m_j+1}), \quad z \rightarrow a_j.$$

If in addition every  $JP_j(z) = 1 + O(|z|^{m_j})$  for every  $j$  and the sequences  $a_j$  and  $b_j$  are very tame, then there exists  $F \in \text{Aut}_1(\mathbb{C}^n)$  with these properties.

Points in a tame sequence can be permuted by automorphisms, and hence we can speak of tame (closed) discrete sets. Rosay and Rudin [155] gave several criteria for tameness, and they constructed nontame and even rigid discrete sets  $A$  in  $\mathbb{C}^n$  for any  $n > 1$ , i.e., such that no nontrivial automorphism of  $\mathbb{C}^n$  fixes  $A$ . There also exist discrete sets whose complements are volume hyperbolic, meaning in particular that every entire map  $\mathbb{C}^n \rightarrow \mathbb{C}^n \setminus A$  has rank  $< n$  at all points. (See also [80, Secs. 4.6–4.7].) An interesting use of such sets is shown in Sect. 5.

The notion of a tame sequence was generalised to Stein manifolds with the density property in a couple of distinct ways by Andrist and Ugolini [22] and Winkelmann [176].

The following parametric version of Proposition 3.1, due to Ramos–Peon and Ugolini [151], generalises earlier results of Kutzschebauch and Ramos–Peon [129] and Ugolini [171]. It concerns interpolation of jets by holomorphic automorphisms at finitely many points of a Stein manifold  $X$  with the density property, where the jets and the interpolating automorphisms of  $X$  depend holomorphically on a parameter in another Stein manifold  $W$ .

**THEOREM 3.3.** *Let  $W$  and  $X$  be Stein manifolds and suppose that  $X$  has the density property. Let  $k \geq 0$  and  $N \geq 1$  be integers, and let  $x_1, \dots, x_N$  be distinct points in  $X$ . Let  $Y$  denote the space of  $N$ -tuples  $\gamma = (\gamma_1, \dots, \gamma_N)$  of  $k$ -jets at  $x_1, \dots, x_N$ , respectively, with nondegenerate linear parts and distinct values at  $x_1, \dots, x_N$ . Given a null-homotopic holomorphic map  $\gamma = (\gamma_1, \dots, \gamma_N): W \rightarrow Y$ , there exists a null-homotopic holomorphic map  $F: W \rightarrow \text{Aut}(X)$  such that the  $k$ -jet of  $F(w) \in \text{Aut}(X)$  at  $x_i$  is  $\gamma(w)_i$  for  $i = 1, \dots, N$  and all  $w \in W$ .*

#### 4. Fatou–Bieberbach domains

A domain  $\Omega \subsetneq \mathbb{C}^n$  which is biholomorphic to  $\mathbb{C}^n$  is called a *Fatou–Bieberbach domain*. A biholomorphic map  $F: \mathbb{C}^n \rightarrow \Omega$  onto such a domain (and its inverse  $F^{-1}$ ) is called a *Fatou–Bieberbach map*. Every injective polynomial map  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  is an automorphism with a polynomial inverse (see Rudin [157]), so there are no algebraic Fatou–Bieberbach maps.

All early constructions of Fatou–Bieberbach domains relied upon the theory of normal forms of local biholomorphisms at an attractive or a repelling fixed point. The following result has a complex genesis as explained in the sequel.

**THEOREM 4.1.** *Let  $F: U \rightarrow F(U)$  be a biholomorphism on an open neighbourhood  $U \subset \mathbb{C}^n$  of the origin such that  $F(0) = 0$  and the eigenvalues  $\lambda_i$  of the differential  $dF_0$  satisfy*

$$(4.1) \quad 1 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| > 0.$$

*After shrinking  $U$  around the origin, there is a biholomorphism  $\psi: U \rightarrow \psi(U) \subset \mathbb{C}^n$  with  $\psi(0) = 0$  such that  $G = \psi \circ F \circ \psi^{-1}$  is a polynomial automorphism of  $\mathbb{C}^n$  of the form*

$$(4.2) \quad G(z) = Az + (0, g_2(z), \dots, g_n(z)), \quad z \in \mathbb{C}^n,$$

*where  $A$  is a lower-triangular matrix with the eigenvalues  $\lambda_i$ , and every component  $g_j(z)$  is a polynomial in the variables  $z_1, \dots, z_{j-1}$  containing no constant or linear terms.*

In fact, choosing  $k \in \mathbb{N}$  such that  $|\lambda_1|^k < |\lambda_n|$ ,  $G$  may be chosen a polynomial map of degree  $k$  such that every monomial  $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$  ( $m_1 + m_2 + \cdots + m_n \geq 2$ ) in a component  $g_j(z)$  of  $G$  (4.2) is *resonant*:  $\lambda_j = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n}$ . Since the eigenvalues satisfy (4.1), it follows that  $m_j = \cdots = m_n = 0$  and hence  $G$  is lower-triangular.

The first complete proof of Theorem 4.1 was given by Rosay and Rudin [155, Appendix]; see also [80, Sect. 4.3]. The result was claimed by Reich [152, 153] in 1969, and it was used by Dixon and Esterle [54] in 1986. However, a gap in the proof of convergence of the normalization maps in [153] (even on the formal level) was pointed out by Rosay and Rudin [155, p. 49]. In the special case when the matrix  $A$  of the differential  $dF_0$  is diagonalisable and there are no resonances between the eigenvalues, we can conjugate  $F$  locally near 0 to the linear map  $z \mapsto Az$  with  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ . In the special case when all eigenvalues agree, this was proved by Poincaré [147] in 1890. Poincaré indicated that this leads to the existence of injective holomorphic maps  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  with non-dense image (since there are automorphisms with several attracting fixed points), although he did not provide a specific example. The formal normal form in the general case was developed by Leau [132] in 1987, who also obtained analytic solution under an extra technical hypothesis. Further work was done by Picard [145, 146] in the period 1900–1905. Much later, Sternberg [165] (1957) solved the normalization problem for smooth local diffeomorphisms of  $\mathbb{R}^n$  at an attracting fixed point. His remark (see bottom of page 816 in his paper), that [165, proof of Theorem 2] also applies in the real analytic case, does not seem supported by any details in his paper.

Suppose now that  $X$  is a complex manifold of dimension  $n$  and  $F: X \rightarrow X$  is an injective holomorphic map with an attracting fixed point at  $p \in X$ . Denote by  $F^k$  the  $k$ -iterate of  $F$  for  $k \in \mathbb{N}$ . The domain

$$(4.3) \quad \Omega_{F,p} = \left\{ x \in X : \lim_{k \rightarrow +\infty} F^k(x) = p \right\}$$

is called the *basin* of  $F$  at  $p$ . Note that  $\Omega_{F,p}$  is the increasing union of preimages  $(F^k)^{-1}(V)$  for  $k \in \mathbb{N}$ , where  $V \subset F(\mathbb{C}^n)$  is any neighbourhood of  $p$ . By taking  $V$  to be connected, we see that  $\Omega_{F,p}$  is connected. By Theorem 4.1 there are a neighbourhood  $U \subset X$  of  $p$ , with  $F(U) \subset U$ , and a local chart  $\psi: U \rightarrow \psi(U) \subset \mathbb{C}^n$  with  $\psi(p) = 0$  such that  $G = \psi \circ F \circ \psi^{-1}$  is a lower-triangular polynomial map of the form (4.2). Clearly, such  $G$  is an automorphism of  $\mathbb{C}^n$  with a globally attracting fixed point at 0, so  $\Omega_{G,0} = \mathbb{C}^n$ . Then,  $G^k = \psi \circ F^k \circ \psi^{-1}$  for each  $k \in \mathbb{N}$ , which is equivalent to  $\psi = G^{-k} \circ \psi \circ F^k$ . As  $k \rightarrow \infty$ , this defines a biholomorphic map  $\Psi: \Omega_{F,p} \rightarrow \Omega_{G,0} = \mathbb{C}^n$ , and hence shows the following.

**THEOREM 4.2.** *If  $X$  is a complex manifold and  $F: X \rightarrow X$  is an injective holomorphic map with an attracting fixed point  $p \in X$ , then the basin (4.3) is biholomorphic to  $\mathbb{C}^n$ ,  $n = \dim X$ .*

When  $n > 1$ , the basin of an automorphism  $F \in \text{Aut}(\mathbb{C}^n)$  need not be all of  $\mathbb{C}^n$ . In fact,  $F$  may have several (even countably many) attracting fixed points, and their basins are pairwise disjoint Fatou–Bieberbach domains in  $\mathbb{C}^n$ .

Before proceeding, we mention that fixed points  $p \in \mathbb{C}^n$  of automorphisms  $F \in \text{Aut}(\mathbb{C}^n)$  which are hyperbolic, in the sense that the eigenvalues  $\lambda_i$  of  $dF_p$  satisfy  $|\lambda_i| \neq 1$ , have also been studied. In favorable cases,  $F$  can again be linearized at  $p$ . In general there may be infinitely many resonances, which makes the study of the normal form much more involved. There is an outstanding conjecture of Bedford that for a hyperbolic fixed point  $p$  of  $F \in \text{Aut}(\mathbb{C}^n)$ , the stable and the unstable manifolds are biholomorphic to Euclidean spaces of appropriate dimensions. For a survey of this topic see Abbondandolo et al. [1].

Using the idea behind Theorem 4.2, Fatou [62] constructed in 1922 birational self-maps of  $\mathbb{C}^2$  whose images are not dense in  $\mathbb{C}^2$ . It was Bieberbach [34] who in 1933 found the first known example of an injective holomorphic map  $\mathbb{C}^2 \rightarrow \mathbb{C}^2$  with Jacobian one and non-dense image  $\Omega$  which is Runge in  $\mathbb{C}^2$ . Bieberbach's example is also described by Stehlé in [162] (1972), who used it to find a properly embedded holomorphic disc in  $\mathbb{C}^2$ . (The point is that if a complex line  $\Lambda \subset \mathbb{C}^2$  intersects a Runge domain  $\Omega$  but is not contained in it, then any connected component of  $\Lambda \cap \Omega$  is Runge in  $\Lambda$ , hence biholomorphic to the disc  $\mathbb{D}$ .) See also the monograph by Bochner and Martin [38, Sect. III.1] (1948). In 1971, Kodaira [118] gave an example of an injective holomorphic map  $\mathbb{C}^2 \rightarrow \mathbb{C}^2$  with constant Jacobian omitting a complex line of  $\mathbb{C}^2$ . In 1983, Nishimura [141] gave such an example  $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  which omits a neighbourhood  $U$  of a complex line. In [142], Nishimura investigated the shape of  $U$  for some specific  $F$ , and he proved that there is no injective holomorphic map from  $\mathbb{C}^2$  into itself with constant Jacobian whose image omits the union  $E$  of two complex lines in  $\mathbb{C}^2$  and a neighbourhood of

a point of  $E$ . This partially answers the (still open) question whether there is an injective holomorphic map from  $\mathbb{C}^2$  into itself omitting two complex lines:

PROBLEM 4.3. Is there a Fatou–Bieberbach domain in  $\mathbb{C}^* \times \mathbb{C}^*$ ?

The following result concerning basins of polynomial automorphisms of  $\mathbb{C}^2$  was proved by Bedford and Smillie [29] in 1991. Note that such an automorphism has constant Jacobian determinant which is smaller than one in absolute value.

THEOREM 4.4. *A polynomial basin  $\Omega \subset \mathbb{C}^2$  intersects each algebraic curve  $V \subset \mathbb{C}^2$  in a nonempty set with compact closure  $\overline{\Omega} \cap V$ . On the other hand, the closure  $\overline{\Omega}$  does not contain any closed one dimensional complex subvarieties of  $\mathbb{C}^2$ .*

In 1986, Dixon and Esterle [54] introduced a more general method for constructing Fatou–Bieberbach domains. The underlying idea is that for certain pairs of pairwise disjoint compact sets  $K, L \subset \mathbb{C}^n$  with polynomially convex union  $K \cup L$  one can find holomorphic automorphisms  $\Phi \in \text{Aut}(\mathbb{C}^n)$  which are close to the identity map on one of the sets, say  $K$ , and they push the second set  $L$  far away. If  $K$  and  $L$  are convex, this is easily achieved by a shear, a fact which was also explored by Rosay and Rudin in [155]. In the more general case when  $K$  is polynomially convex and  $L$  is starshaped (or holomorphically contractible), the same can be done by applying Theorem 1.1 (we squeeze  $L$  within itself almost to a point, slide it far away from  $K$ , and approximate the final map by an automorphism). An inductive application of this construction yields a sequence  $\Phi_j \in \text{Aut}(\mathbb{C}^n)$  ( $j \in \mathbb{N}$ ) such that the sequence of compositions  $F_j = \Phi_j \circ \Phi_{j-1} \circ \cdots \circ \Phi_1$  converges on a neighbourhood of  $K$  and diverges to infinity on  $L$  as  $j \rightarrow \infty$ . If in addition the  $\Phi_j$ 's approximate the identity map sufficiently closely on an increasing sequence of compacts exhausting  $\mathbb{C}^n$ , then the domain of convergence  $\Omega$  of the sequence  $F_j$  is a Fatou–Bieberbach domain, and the limit  $F = \lim_{j \rightarrow \infty} F_j: \Omega \rightarrow \mathbb{C}^n$  is a biholomorphic map of  $\Omega$  onto  $\mathbb{C}^n$  such that

$$K \subset \Omega \subset \mathbb{C}^n \setminus L.$$

A similar argument yields a Fatou–Bieberbach domain  $\Omega' \subset \mathbb{C}^n$  with  $L \subset \Omega' \subset \mathbb{C}^n \setminus K$ . See Forstnerič and Ritter [86, Proposition 9] or [80, Proposition 4.4.4] for a precise statement. Taking  $L$  to be a finite set gives the following corollary.

COROLLARY 4.5. *Given a compact polynomially convex set  $K \subset \mathbb{C}^n$ ,  $n > 1$ , and a finite set  $L \subset \mathbb{C}^n \setminus K$  there is a Fatou–Bieberbach domain  $\Omega$  satisfying  $K \subset \Omega \subset \mathbb{C}^n \setminus L$ .*

This result was used by Forstnerič [77] in his construction of holomorphic functions without critical points on any Stein manifold. More generally, Theorem 4.7 below is used in his proof of the basic h-principle for holomorphic submersions  $X \rightarrow \mathbb{C}^q$  from any Stein manifold with  $\dim X > q \geq 1$ , given in the same paper.

This *push-out method* of Dixon and Esterle also became known as *random iteration*, since we are not iterating an automorphism but composing with a new one at every step. It was used even before the Andersén–Lempert theory, or without using it, to provide examples of Fatou–Bieberbach domains with interesting properties and to solve various problems. We refer to the papers by Rosay and Rudin [155], Fornæss and Sibony [68], Globevnik and Stensønes [99], Globevnik [95,96], Stensønes [164], Fornæss and Stensønes [70], among others. Globevnik and Stensønes [99] used random iterations of shears in coordinate directions to show that every planar domain bounded by finitely many Jordan curves admits a proper holomorphic embedding into  $\mathbb{C}^2$ . This was a major advance on the open problem asking which open Riemann surfaces admit a proper holomorphic embedding as a closed complex curve in  $\mathbb{C}^2$  (see Section 6). Random iterations of shears were also used by Stensønes [164] (1997) in her construction of Fatou–Bieberbach domains having  $\mathcal{C}^\infty$  smooth boundaries. Note that a smoothly bounded Fatou–Bieberbach domain in  $\mathbb{C}^2$  has Levi-flat boundary foliated by complex curves. Related results of Globevnik [95,96] give Fatou–Bieberbach domains with  $\mathcal{C}^1$  boundaries and with additional geometric control on their location. Whether there exist Fatou–Bieberbach domains with real analytic boundaries remains an open problem.

A result similar to the one of Globevnik [96] was used by Buzzard and Hubbard [46, Lemma 3.1] to show the following [46, Theorem 4.1].

**THEOREM 4.6.** *For every algebraic subvariety  $A$  of codimension at least two in  $\mathbb{C}^n$ ,  $n \geq 2$ , there exists a Fatou–Bieberbach domain  $\Omega \subset \mathbb{C}^n$  such that  $\overline{\Omega} \cap A = \emptyset$ .*

With the exception of Corollary 4.5, the results mentioned so far were obtained by elementary constructions using shears. The Andersén–Lempert theory provides much more general construction methods. For example, we have the following result.

**THEOREM 4.7.** *Let  $A$  be an algebraic subvariety of  $\mathbb{C}^n$ ,  $n \geq 2$ , of codimension at least 2. Given a compact, polynomially convex and holomorphically contractible set  $K \subset \mathbb{C}^n \setminus A$ , there is a Fatou–Bieberbach domain  $\Omega \subset \mathbb{C}^n$  with  $K \subset \Omega$  and  $\overline{\Omega} \cap A = \emptyset$ .*

Here is a sketch of proof. By the assumption there are a neighbourhood  $U \subset \mathbb{C}^n \setminus A$  of  $K$  and an isotopy of biholomorphic maps  $\phi_t: U \rightarrow \mathbb{C}^n \setminus A$  ( $t \in [0, 1]$ ) such that  $\phi_t(K) \subset K$  for all  $t$  and  $\phi_1(K)$  lies in an arbitrarily

small closed ball  $B$  around a point  $p \in K$ . Clearly, the sets  $K_t = \phi_t(K)$  are polynomially convex. Let  $\Omega' \subset \mathbb{C}^n \setminus A$  be a Fatou–Bieberbach domain such that  $\overline{\Omega'} \cap A = \emptyset$  (see Theorem 4.6). If  $B$  is small enough, we can slide it into  $\Omega'$  by an isotopy of translations which keep the image of  $B$  in  $\mathbb{C}^n \setminus A$ . Applying Theorem 2.12 to the combined isotopy provides an automorphism  $\Phi$  of  $\mathbb{C}^n$  fixing  $A$  such that  $\Phi(K) \subset \Omega'$ . Then,  $\Omega = \Phi^{-1}(\Omega')$  is a Fatou–Bieberbach domain satisfying the conclusion of Theorem 4.7.

Starting from 2005, one of the main contributors of developments on Fatou–Bieberbach domains and their applications has been Wold with collaborators. In his first paper [177], Wold showed the following results.

(1) For any  $m \in \mathbb{N} \cup \{\infty\}$  there exist  $m$  pairwise disjoint Fatou–Bieberbach domains  $\Omega_i$  such that any point  $p \in \mathbb{C}^n \setminus \bigcup_{i=1}^m \Omega_i$  lies in the boundary of every  $\Omega_i$ .

(2) If  $\{L_i\}_{i \in \mathbb{N}}$  is a collection of affine subspaces of  $\mathbb{C}^n$  ( $n > 1$ ), then there exists a Fatou–Bieberbach domain  $\Omega$  such that for every  $i$ ,  $\Omega \cap L_i$  is connected and  $L_i \setminus \Omega \neq \emptyset$ .

(3) If  $\{V_i\}_{i \in \mathbb{N}}$  is a collection of closed proper complex subvarieties of  $\mathbb{C}^n$  ( $n > 1$ ) then there exists a Fatou–Bieberbach domain  $\Omega$  containing  $\bigcup_i V_i$ .

An exciting development was the following result of Wold [181] from 2008.

**THEOREM 4.8.** *For any  $n > 1$  there exists a non-Runge Fatou–Bieberbach domain in  $\mathbb{C}^n$ .*

Note that such domains cannot be limits of automorphisms of  $\mathbb{C}^n$ . The idea behind the construction is the following. Start with a Fatou–Bieberbach domain  $\Omega$  contained in  $\mathbb{C}^* \times \mathbb{C}$ . By Stolzenberg [166] there is a compact set  $K = M_1 \cup M_2 \subset \mathbb{C}^* \times \mathbb{C}$ , consisting of a pair of disjoint closed totally real discs  $M_1$  and  $M_2$ , such that  $K$  is  $\mathcal{O}(\mathbb{C}^* \times \mathbb{C})$ -convex but its polynomial hull  $\widehat{K}$  contains the origin of  $\mathbb{C}^2$ . Since the Stein domain  $\mathbb{C}^* \times \mathbb{C}$  has the density property, Theorem 1.3 can be used to find a holomorphic automorphism  $\Phi$  of  $\mathbb{C}^* \times \mathbb{C}$  such that  $\Phi(K) \subset \Omega$ . (It suffices to construct an isotopy which shrinks each of the discs  $M_1, M_2$  almost to a point and then slide the new small discs into  $\Omega$  within  $\mathbb{C}^* \times \mathbb{C}$  such that the isotopy consists of  $\mathcal{O}(\mathbb{C}^* \times \mathbb{C})$ -convex sets.) Therefore,  $\Omega' = \Phi^{-1}(\Omega) \subset \mathbb{C}^* \times \mathbb{C}$  is a Fatou–Bieberbach domain containing  $K$  but not its polynomial hull  $\widehat{K}$ . It follows that  $\Omega'$  is not Runge in  $\mathbb{C}^2$ , although it is Runge in  $\mathbb{C}^* \times \mathbb{C}$ .

An interesting application of this result was Wold’s construction [182] in 2010 of the first known example of a non-Stein long  $\mathbb{C}^2$ . We describe these developments in Section 7.

In 2010, Baader, Kutzschebauch and Wold [28] used Fatou–Bieberbach domains to construct the first known example of a knotted properly embedded holomorphic disc in  $\mathbb{C}^2$ . Their result was motivated by the problem,

raised by Kirby, whether proper holomorphic embeddings of  $\mathbb{C}$  or the unit disc into  $\mathbb{C}^2$  can be topologically knotted. While the first problem remains open, the second one was solved in the affirmative in [28]. The proof uses well-behaved Fatou–Bieberbach domains in  $\mathbb{C}^2$  constructed by Globevnik in [96], containing small perturbations of the bidisc, and the existence of knotted holomorphic discs in the bidisc. It is unknown whether the disc admits an unknotted proper holomorphic embedding into  $\mathbb{C}^2$ .

In 1998 Globevnik [96] constructed Fatou–Bieberbach domains in  $\mathbb{C}^n$  whose closures intersect the complex line  $\mathbb{C} \times \{0\}^{n-1}$  in closed connected and simply connected domains which are arbitrarily small perturbations of the closed unit disc. In 2012, Wold [183] found a Fatou–Bieberbach domain in  $\mathbb{C}^2$  whose intersection with  $\mathbb{C} \times \{0\}$  contains the unit disc as a connected component, thereby answering a question of Rosay and Rudin [155].

In 2015, Forstnerič and Wold [90] constructed Fatou–Bieberbach domains in  $\mathbb{C}^n$ ,  $n > 1$ , which contain a given compact set  $K$  and avoid a totally real affine subspace  $L \subset \mathbb{C}^n$  with  $\dim_{\mathbb{R}} L < n$  such that  $K \cup L$  is polynomially convex. This was used in [90] to show that  $\mathbb{C}^n \setminus L$  has certain Oka properties. Due to a recent result of Kusakabe it is now clear that such domains are Oka manifolds for  $n \geq 4$ , as follows easily from Theorem 8.4.

A recent development is the following result of Forstnerič and Wold [91, Theorem 1.1] from 2020. An interesting application is given by Theorem 8.3.

**THEOREM 4.9.** *Let  $K$  be a compact polynomially convex set in  $\mathbb{C}^n$  for some  $n > 1$ ,  $L$  be a compact polynomially convex set in  $\mathbb{C}^N$  for some  $N \in \mathbb{N}$ , and  $f: U \rightarrow \mathbb{C}^n$  be a holomorphic map from an open neighbourhood  $U \subset \mathbb{C}^N$  of  $L$  such that  $f(z) \in \mathbb{C}^n \setminus K$  for all  $z \in L$ . Then there are an open neighbourhood  $V \subset U$  of  $L$  and a holomorphic map  $F: V \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that for every  $z \in V$ ,  $F(z, 0) = f(z)$  and the map  $F(z, \cdot): \mathbb{C}^n \rightarrow \mathbb{C}^n \setminus K$  is injective.*

It follows that  $\Omega_z := \{F(z, \zeta) : \zeta \in \mathbb{C}^n\}$  is a Fatou–Bieberbach domain in  $\mathbb{C}^n \setminus K$  with centre  $F(z, 0) = f(z)$  and depending holomorphically on  $z \in V$ .

The proof uses Andersén–Lempert theory with parameters. It also applies to variable fibres  $K_z \subset \mathbb{C}^n$  ( $z \in L$ ) with polynomially convex graph (see [91, Remark 2.2]). For a convex parameter space  $L \subset \mathbb{C}^N$  the analogous result holds if we replace  $\mathbb{C}^n$  by an arbitrary Stein manifold having the density property (see [91, Theorem 3.1]).

So far we have focused on Fatou–Bieberbach domains in Euclidean spaces  $\mathbb{C}^n$ ,  $n \geq 2$ . However, such domains abound in any Stein manifold with the density property as was already observed by Varolin [173, 174]. As an example, we mention the following recent result of Kaliman [107, Corollary 2.18] which generalizes Theorem 4.7. Recall that an affine manifold  $X$  is flexible in the sense of Arzhantsev et al. [27] if locally nilpotent derivations on  $X$  span the tangent space at every point (see Example (2) in Subsection 2.3).

**THEOREM 4.10.** *Let  $X$  be a complex affine flexible manifold and  $Y$  be a closed algebraic subvariety of  $X$  of codimension at least 2. Suppose that*

$X$  admits a pair of compatible vector fields (see Definition 2.5). Then, every point  $x \in X$  has a neighbourhood  $\Omega \subset X \setminus Y$  which is biholomorphic to  $\mathbb{C}^n$  with  $n = \dim X$ .

## 5. Twisted complex lines in $\mathbb{C}^n$ and nonlinearizable automorphisms

It is well known that a generic affine algebraic hypersurface  $A \subset \mathbb{C}^n$  of sufficiently large degree is (Kobayashi) hyperbolic and has hyperbolic complement  $\mathbb{C}^n \setminus A$  (see Brotbek [41]). Such  $A$  is necessarily topologically complicated. On the other hand, proper polynomial embeddings  $F: \mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}$  are believed to have non-hyperbolic complements  $\mathbb{C}^{k+1} \setminus F(\mathbb{C}^k)$ . In particular, it was shown by Suzuki [169] (1974) and Abhyankar and Moh [2] (1975) that for every polynomial embedding  $F: \mathbb{C} \hookrightarrow \mathbb{C}^2$  there is a polynomial automorphism  $\Phi$  of  $\mathbb{C}^2$  such that  $\Phi \circ F(\mathbb{C}) = \mathbb{C} \times \{0\}$ . It is therefore of interest to know that there are proper holomorphic embeddings whose complements are hyperbolic; see Theorem 5.3.

On the way to this result, we begin with the following application of Theorem 1.1 due to Forstnerič, Globevnik, and Rosay [83].

**THEOREM 5.1.** *For every closed discrete set  $B \subset \mathbb{C}^2$  there is a properly embedded complex line  $F: \mathbb{C} \hookrightarrow \mathbb{C}^2$  such that  $B \subset F(\mathbb{C})$ .*

The analogous result holds for embedding  $\mathbb{C} \hookrightarrow \mathbb{C}^n$ ,  $n \geq 3$ . This was proved in a more precise form (with interpolation on a pair of discrete sets) by Rosay and Rudin [156, Theorem I] in 1993. However, their result for  $n \geq 3$  is a very special case of the theorem, due to Acquistapace, Broglia, and Tognoli [3] (1975), that for any Stein manifold  $X$  of dimension  $m \geq 1$ , integer  $n \geq 2m + 1$ , and proper holomorphic embedding  $\phi: X' \hookrightarrow \mathbb{C}^n$  of a closed complex subvariety  $X'$  of  $X$  there is a proper holomorphic embedding  $F: X \hookrightarrow \mathbb{C}^n$  with  $F|_{X'} = \phi$ . (See also [80, Theorem 9.5.5].) The Rosay–Rudin theorem mentioned above amounts to the special case with  $X'$  a closed discrete subset of  $X = \mathbb{C}$ . However, for  $n = 2$  the methods in both mentioned papers only provide immersions  $\mathbb{C} \rightarrow \mathbb{C}^2$  with the desired property (cf. [156, Theorem II]). In this lowest dimensional case, and in the proof of the more general result in Theorem 5.3 for  $n \leq 2k$ , the use of Andersén–Lempert theory is essential.

The main idea behind the proof of Theorem 5.1 is to inductively twist a properly embedded complex line  $\mathbb{C} \hookrightarrow \mathbb{C}^2$  such that it contains more and more points of the discrete set  $B = \{b_1, b_2, \dots\}$ , and the sequence of embeddings converges to a proper holomorphic embedding. In the inductive step we are given a proper holomorphic embedding  $F_k: \mathbb{C} \hookrightarrow \mathbb{C}^2$  such that  $F_k(\mathbb{C})$  contains the first  $k$  points  $b_1, \dots, b_k \in B$  but it does not contain the remaining points of  $B$  (the latter condition is easily arranged by a general position

argument). Choose a disc  $\Delta \subset \mathbb{C}$  with  $\{b_1, \dots, b_k\} \subset F_k(\Delta)$  and a compact ball  $L \subset \mathbb{C}^2$  such that

$$F_k(b\Delta) \cup \{b_{k+1}, b_{k+2}, \dots\} \subset \mathbb{C}^2 \setminus L.$$

The union  $K := L \cup F_k(\overline{\Delta})$  is then a compact polynomially convex set in  $\mathbb{C}^2$ . Theorem 1.1 furnishes an automorphism  $\Phi_k \in \text{Aut}(\mathbb{C}^2)$  which is close to the identity map on  $K$ , it fixes the points  $b_1, \dots, b_k$ , and such that  $b_{k+1} \in \Phi_k \circ F_k(\mathbb{C})$ . This gives the next embedding  $F_{k+1} = \Phi_k \circ F_k: \mathbb{C} \hookrightarrow \mathbb{C}^2$ . An inductive application of this technique gives a sequence of embeddings such that  $F = \lim_{k \rightarrow \infty} F_k: \mathbb{C} \hookrightarrow \mathbb{C}^2$  is an embedding satisfying the conclusion of the theorem. The same proof applies for any  $n \geq 2$ .

A related result of Buzzard and Forstnerič in [44] (1997) yields Carleman approximation with interpolation on  $\mathbb{R} \subset \mathbb{C}$  of proper holomorphic embeddings  $\mathbb{C} \hookrightarrow \mathbb{C}^n$  for any  $n > 1$ .

Rosay and Rudin proved in [155] that for every  $n > 1$  there exist discrete sets  $B \subset \mathbb{C}^n$  which cannot be mapped into an affine line by any holomorphic automorphism of  $\mathbb{C}^n$ ; such sets are called non-tame. Applying Theorem 5.1 (and its generalisation to any  $n > 1$  mentioned above) with such a set  $B$  gives the following corollary.

**COROLLARY 5.2.** *For every  $n > 1$  there is a properly embedded complex line  $F: \mathbb{C} \hookrightarrow \mathbb{C}^n$  such that no holomorphic automorphism of  $\mathbb{C}^n$  maps  $F(\mathbb{C})$  onto an affine complex line. In other words,  $F(\mathbb{C})$  is not straightenable by automorphisms of  $\mathbb{C}^n$ .*

This is in strong contrast with the result of Suzuki [169] (1974) and Abhyankar and Moh [2] (1975) that every polynomial holomorphic embedding  $\mathbb{C} \hookrightarrow \mathbb{C}^2$  is straightenable by a polynomial automorphism of  $\mathbb{C}^2$ . We refer to [80, Sect. 4.18] for a survey of further results on this subject. This is an example where the answer to a certain problem in the holomorphic category differs from the answer in the algebro-geometric category.

A stronger result in the same spirit was obtained by Buzzard and Fornæss [43], who constructed a proper holomorphic embedding  $F: \mathbb{C} \hookrightarrow \mathbb{C}^2$  such that  $\mathbb{C}^2 \setminus F(\mathbb{C})$  is Kobayashi hyperbolic. Since the complement of an affine line in  $\mathbb{C}^2$  is biholomorphic to  $\mathbb{C}^* \times \mathbb{C}$  and hence is not hyperbolic, their result also implies Corollary 5.2. To prove it, they constructed properly embedded complex lines  $\mathbb{C} \hookrightarrow \mathbb{C}^2$  which contain arbitrarily small deformations of a well-chosen discrete family of closed affine discs in  $\mathbb{C}^2$  with Kobayashi hyperbolic complement. If the approximations are close enough then the complement of the embedded line is also hyperbolic. Their proof uses the same general idea as the proof of Theorem 5.1, but the technical details are more involved. This line of results was developed further by Forstnerič [76] and Borell and Kutzschebauch [39] who proved the following.

**THEOREM 5.3.** *For every pair of integers  $1 \leq k < n$  there is a proper holomorphic embedding  $F: \mathbb{C}^k \hookrightarrow \mathbb{C}^n$  such that  $\mathbb{C}^n \setminus F(\mathbb{C}^k)$  is  $(n - k)$ -hyperbolic in the sense of Eisenman. In particular, any entire map  $\mathbb{C}^p \rightarrow \mathbb{C}^n \setminus F(\mathbb{C}^k)$  ( $p \in \mathbb{N}$ ) has rank less than  $n - k$  at each point.*

The theorem of Buzzard and Fornæss [42] corresponds to the case  $k = 1$ ,  $n = 2$ .

As an application of Theorem 5.1, Derksen and Kutzschebauch [53] showed the following result which answered a long-standing open question.

**THEOREM 5.4.** *For every integer  $n \geq 2$  there exists a nonlinearizable periodic holomorphic automorphism of period  $n$  on  $\mathbb{C}^{2+n}$ . In particular, there is a nonlinearizable holomorphic involution on  $\mathbb{C}^4$ .*

An outline of their proof can also be found in [80, Sect. 4.19]. The problem regarding the existence of nonlinearizable periodic automorphisms remains open on  $\mathbb{C}^2$  and  $\mathbb{C}^3$ . A recent survey of the linearization problem for holomorphic automorphisms is available in [124].

## 6. Embedding open Riemann surfaces in $\mathbb{C}^2$

It has been known since mid-1950s that every Stein manifold  $X$  of dimension  $n$  embeds as a closed complex submanifold in a Euclidean space  $\mathbb{C}^{2n+1}$ . We refer to [80, Sect. 2.4] or [81, Sect. 2] for a discussion and references to the early works on the subject. The smallest possible value of  $N$  for any  $n > 1$  was found by Eliashberg and Gromov [61] and Schürmann [159] who showed that every Stein manifold  $X$  of dimension  $n \geq 1$  immerses properly holomorphically in  $\mathbb{C}^M$  with  $M = \lceil \frac{3n+1}{2} \rceil$ , and if  $n > 1$  then  $X$  embeds properly holomorphically in  $\mathbb{C}^N$  with  $N = \lceil \frac{3n}{2} \rceil + 1$ . Their proof relies on an application of the Oka principle for sections of holomorphic fibre bundles with Oka fibres over a Stein manifold. A complete exposition can also be found in [80, Sections 9.3–9.4].

The proof of this embedding theorem breaks down in the lowest dimensional case  $n = 1$  (i.e.,  $X$  is an open Riemann surface) and  $N = 2$ . The following *Forster–Bell–Narasimhan Conjecture* [33, 72] is one of the oldest open problems in complex analysis.

**PROBLEM 6.1.** Does every open Riemann surface embed properly holomorphically in  $\mathbb{C}^2$ ?

A history of the rather sporadic progress on this problem can be found in [80, Sect. 9.10]. We mention in particular that Globevnik and Stensønes [99] proved in 1995 that every finitely connected domain in  $\mathbb{C}$  without isolated boundary points embeds properly holomorphically into  $\mathbb{C}^2$ . Their proof uses Fatou–Bieberbach domains, constructed as domains of convergence of random sequences of shears in coordinate directions. Further progress using

the same techniques was made by Černe and Globevnik [49] and Černe and Forstnerič [48].

A new method based on Andersén–Lempert theory was introduced into the subject a decade later by Wold [178–180]. Assume that  $M$  is a compact bordered Riemann surface (every such is conformally equivalent to a domain in a compact Riemann surface obtained by removing finitely many pairwise disjoint discs [168, Theorem 8.1]) and  $F: \bar{M} \hookrightarrow \mathbb{C}^2$  is a smooth embedding which is holomorphic on  $M$ . We wish to show that this embedding can be modified so that the boundary curves diverge to infinity while the interior  $M \setminus bM$  becomes embedded in  $\mathbb{C}^2$  as a closed complex curve. The main idea introduced by Wold is the following. Write  $bM = \bigcup_{i=1}^m C_i$  where each  $C_i$  is a smooth closed curve. Assume in addition that each  $C_i$  contains a point  $p_i$  such that the affine complex line in  $\mathbb{C}^2$  through the point  $F(p_i) = (a_i, b_i) \in \mathbb{C}^2$  in the second coordinate direction intersects  $F(M)$  only at  $F(p_i)$ . Such point  $p_i$  is said to be *exposed* by the map  $F$ . We apply to  $F$  a rational shear of the form

$$G(z_1, z_2) = \left( z_1, z_2 + \sum_{i=1}^m \frac{c_i}{z_1 - a_i} \right)$$

for a suitable choice of the numbers  $c_i \in \mathbb{C}^*$ . Each point  $F(p_i)$  is sent to infinity,  $G$  has no other poles on  $F(M)$ , and the surface

$$\Sigma = (G \circ F)(M \setminus \{p_1, \dots, p_m\}) \subset \mathbb{C}^2$$

is holomorphically embedded with smooth properly embedded boundary curves  $\Lambda_i = (G \circ F)(C_i \setminus \{p_i\})$  ( $i = 1, \dots, m$ ) diffeomorphic to  $\mathbb{R}$ . By using Theorem 1.1 and results of Stolzenberg [167] it is then possible to find a sequence of holomorphic automorphisms  $\Phi_j \in \text{Aut}(\mathbb{C}^2)$  converging on the interior  $\mathring{\Sigma} = \Sigma \setminus \bigcup_{i=1}^m \Lambda_i$  of  $\Sigma$  while the boundary curves  $\Lambda_i$  diverge to infinity. If things are done right then the domain of convergence of the sequence  $\Phi_j$  is a Fatou–Bieberbach domain  $\Omega \subset \mathbb{C}^2$  such that  $\mathring{\Sigma} \subset \Omega \cong \mathbb{C}^2$  and  $b\Sigma = \bigcup_{i=1}^m \Lambda_i \subset b\Omega$ . This embeds the interior  $M \setminus bM$  of  $M$  properly into  $\Omega \cong \mathbb{C}^2$ .

In Wold’s joint paper with Forstnerič [88] (2009), this construction was coupled with a newly developed technique of exposing boundary points of bordered Riemann surfaces. This led to the following result (see [88, Theorem 1.1 and Corollary 1.2]).

**THEOREM 6.2.** *Assume that  $M$  is a compact bordered Riemann surface with  $\mathcal{C}^r$  boundary ( $r > 1$ ). Every  $\mathcal{C}^1$  embedding  $M \hookrightarrow \mathbb{C}^2$  that is holomorphic in  $\mathring{M} = M \setminus bM$  can be approximated uniformly on compacts in  $\mathring{M}$  by proper holomorphic embeddings  $\mathring{M} \hookrightarrow \mathbb{C}^2$ .*

This shows that, for bordered Riemann surfaces, the main part of Problem 6.1 is to find a holomorphic embedding of the closed surface (including the boundary) in  $\mathbb{C}^2$ .

In the same year 2009, Kutzschebauch, Løw and Wold [127] provided examples of open Riemann surfaces  $M$  which embed properly holomorphically in  $\mathbb{C}^2$  with interpolation, meaning that for every pair of discrete sequences  $a_j \in M$  and  $b_j \in \mathbb{C}^2$  without repetition there is a proper holomorphic embedding  $F : M \hookrightarrow \mathbb{C}^2$  with  $F(a_j) = b_j$  for all  $j = 1, 2, \dots$ .

It is natural to ask whether an analogue of Theorem 6.2 also holds for Riemann surfaces with infinitely many boundary curves. After some initial developments by Majcen [135, 136], a fairly general result was obtained by Forstnerič and Wold [89, Theorem 5.1] for domains in  $\mathbb{CP}^1$  with at most countably many boundary components. By He and Schramm [103], such a domain is conformally equivalent to a *circled domain*  $\Omega \subset \mathbb{CP}^1$ , i.e. such that every connected component of  $\mathbb{CP}^1 \setminus \Omega$  is a round disc or a point (puncture).

**THEOREM 6.3.** *Let  $\Omega$  be a circled domain in  $\mathbb{CP}^1$ . If all but finitely many punctures in  $\mathbb{CP}^1 \setminus \Omega$  are limit points of discs in  $\mathbb{CP}^1 \setminus \Omega$ , then  $\Omega$  embeds properly holomorphically in  $\mathbb{C}^2$ .*

The proof of this result in [89] uses technical ingredients from the previous papers, but it relies on a considerably more delicate induction scheme. The problem is that the boundary components of  $b\Omega$  may cluster on one another. The main new point is that at every step of the induction process one exposes and opens up a new Jordan curve in  $b\Omega$  as described above, while at the same time pushing a carefully selected finite group of curves close to it towards infinity. The details are considerable. Essentially the same proof gives the analogous result for circled domains in elliptic curves (tori).

The problem of embedding Riemann surfaces with punctures properly into  $\mathbb{C}^2$  is even more delicate, and no general techniques have been developed yet. Recently, Kutzschebauch and Poloni [128] (2020) showed in particular that if  $K$  is a countable closed subset of the Riemann sphere  $\mathbb{CP}^1$  with at most two accumulation points then the complement  $\mathbb{CP}^1 \setminus K$  admits a proper holomorphic embedding into  $\mathbb{C}^2$ . They proved the analogous result for complements of certain closed countable sets in complex tori and in hyperelliptic Riemann surfaces.

## 7. Complex manifolds exhausted by Euclidean spaces

A complex manifold  $X$  of dimension  $n$  is called a *long  $\mathbb{C}^n$*  if it is the union of an increasing sequence of domains  $X_1 \subset X_2 \subset X_3 \subset \dots \subset \bigcup_{j=1}^{\infty} X_j = X$  such that each  $X_j$  is biholomorphic to  $\mathbb{C}^n$ . By the Riemann mapping theorem, every long  $\mathbb{C}$  is biholomorphic to  $\mathbb{C}$ . The first known example of a long  $\mathbb{C}^n$  for any  $n > 1$  which is not holomorphically convex, and hence not Stein,

was found by Wold [182] in 2010. In his example, every pair  $X_k \subset X_{k+1}$  in the exhaustion corresponds to a non-Runge Fatou–Bieberbach domain in  $\mathbb{C}^n$  (see Theorem 4.8) such that, for some compact subset  $K \subset X_1$ , the  $\mathcal{O}(X_{k+1})$ -hull of  $K$  is not contained in  $X_k$  for any  $k \in \mathbb{N}$ . It follows that the  $\mathcal{O}(X)$ -hull of  $K$  is not compact, hence  $X$  is not Stein.

Subsequently, Forstnerič [79] showed that for  $n > 1$  and any pair of disjoint countable sets  $A, B \subset \mathbb{C}$  there is a holomorphic submersion  $F : Z \rightarrow \mathbb{C}$  from an  $(n + 1)$ -dimensional complex manifold  $Z$  such that

- (a) every fibre  $Z_z = F^{-1}(z)$  ( $z \in \mathbb{C}$ ) is a long  $\mathbb{C}^n$ ,
- (b) the fibre  $Z_z$  is non-Stein for every  $z \in A$ , and
- (c)  $Z_z$  is biholomorphic to  $\mathbb{C}^n$  for every  $z \in B$ .

By choosing  $A$  and  $B$  to be everywhere dense in  $\mathbb{C}$ , one gets a submersion onto  $\mathbb{C}$  such that the type of the fibre jumps near every point of  $\mathbb{C}$  from  $\mathbb{C}^n$  to a non-Stein long  $\mathbb{C}^n$ .

The questions whether there exist long  $\mathbb{C}^2$ 's without nonconstant holomorphic functions, or non-biholomorphic non-Stein long  $\mathbb{C}^2$ 's, were answered affirmatively by Boc Thaler and Forstnerič [37] in 2016. One of their results is the following.

**THEOREM 7.1.** *For every  $n > 1$  there exists a long  $\mathbb{C}^n$  without any non-constant holomorphic or plurisubharmonic functions.*

To prove this result, they used Wold's construction of a non-Runge Fatou–Bieberbach domain [182], but applied it inductively in a considerably more intricate manner.

Theorem 7.1 gives an essentially optimal counterexample to the classical *union problem for Stein manifolds*, asking whether an increasing union of Stein manifolds is always Stein. For domains in  $\mathbb{C}^n$  this question was raised by Behnke and Thullen [32] in 1934, and an affirmative answer was given by Behnke and Stein [31] in 1939. Some progress on the general question was made by Stein [163] and Docquier and Grauert [55]. The first counterexample to the union problem in any dimension  $n \geq 3$  was found by Fornæss [65] in 1976. He constructed an increasing union of balls that is not holomorphically convex, hence not Stein. His proof is based on an example of a biholomorphic map  $\Phi : \Omega \xrightarrow{\cong} \Phi(\Omega) \subset \mathbb{C}^3$  from a bounded neighbourhood  $\Omega \subset \mathbb{C}^3$  of any compact set  $K \subset \mathbb{C}^3$  with nonempty interior such that the polynomial hull of  $\Phi(K)$  is not contained in  $\Phi(\Omega)$ . (An example of this phenomenon was discovered by Wermer [175] already in 1959.) In 1977, Fornæss and Stout constructed a three-dimensional increasing union of polydiscs without any nonconstant holomorphic function [71]. Increasing unions of hyperbolic Stein manifolds were studied further by Fornæss and Sibony [67] and Fornæss [66]. For the connection with Bedford's conjecture we refer to the survey by Abbondandolo et al. [1]. In dimension  $n = 2$  the

first counterexample to the union problem was the aforementioned example of Wold [182] of a non-Stein long  $\mathbb{C}^2$ .

To answer the second question concerning the existence of non-biholomorphic non-Stein long  $\mathbb{C}^n$ 's, Boc Thaler and Forstnerič introduced in [37] new biholomorphic invariants of a complex manifold  $X$ , the *stable core* and the *strongly stable core*, which allow one to distinguish some long  $\mathbb{C}^n$ 's from one another.

The stable core of  $X$  is the set of all points  $x \in X$  which admit a compact neighbourhood  $K$  such that, for some (and hence for any) increasing sequence of compact sets  $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$  with  $K \subset K_1$  and  $K_j \subset \mathring{K}_{j+1}$  for all  $j$ , the increasing sequence of hulls  $\widehat{K}_{\mathcal{O}(K_j)}$  stabilizes at some  $j = j_0 \in \mathbb{N}$ . (Such  $K$  is said to have the *stable hull property*). Hence, the stable core is an open subset of  $X$ . A compact set  $K$  in  $X$  is called the *strongly stable core* of  $X$  if  $K$  has the stable hull property but any compact set  $L \subset X$  with  $\mathring{L} \setminus K \neq \emptyset$  fails to have the stable hull property. In a Stein manifold or a compact manifold, the stable core is the entire manifold, and a Stein manifold does not have a strongly stable core. However, these invariants are often nontrivial in complex manifolds obtained as increasing unions of domains which fail to form Runge pairs. Boc Thaler and Forstnerič proved the following result [37, Theorem 1.2].

**THEOREM 7.2.** *To every compact, strongly pseudoconvex and polynomially convex domain  $B \subset \mathbb{C}^n$ ,  $n > 1$ , we can associate a complex manifold  $X(B)$ , which is a long  $\mathbb{C}^n$  containing a biholomorphic copy of  $B$  as its strongly stable core, such that every biholomorphic map  $\Phi: X(B) \rightarrow X(C)$  between two such manifolds takes  $B$  onto  $C$ . In particular, for every  $\Phi \in \text{Aut}(X(B))$  the restriction  $\Phi|_B$  is a holomorphic automorphism of  $B$ .*

It follows that if  $X(B)$  is biholomorphic to  $X(C)$  then  $B$  is biholomorphic to  $C$ . The construction likely gives many non-equivalent long  $\mathbb{C}^n$ 's associated to the same domain  $B$ .

The manifold  $X(B)$  is an increasing union  $X_1 \subset X_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} X_i = X$  of domains  $X_i \cong \mathbb{C}^n$  such that  $B \subset X_1$ ,  $\widehat{B}_{\mathcal{O}(X_i)} = B$  for every  $i \in \mathbb{N}$ , but for any compact neighbourhood  $K$  of a point  $p \in X \setminus B$  the hull  $\widehat{K}_{\mathcal{O}(X_{i+1})}$  is not contained in  $X_i$  for large  $i \in \mathbb{N}$ . Every inclusion  $X_i \hookrightarrow X_{i+1}$  is given by a Fatou–Bieberbach map  $\phi_i: \mathbb{C}^n \hookrightarrow \mathbb{C}^n$  such that  $\phi_i(\mathbb{C}^n)$  is not Runge (like in Wold's example [181]), and the  $\mathcal{O}(X_{i+1})$ -hull of the image of  $K$  in  $X_i$  intersects  $X_{i+1} \setminus X_i$ . Thus, the increasing sequence of hulls  $\widehat{K}_{\mathcal{O}(X_i)}$  does not stabilize.

It was shown by Poincaré in 1907 [148] that most pairs of smooth strongly pseudoconvex hypersurfaces in  $\mathbb{C}^n$  are not biholomorphic to each other. A complete set of countably many local holomorphic invariants of such hyper-

surfaces is provided by the Chern–Moser normal form [51]. Hence, Theorem 7.2 implies the following.

**COROLLARY 7.3.** *For every  $n > 1$  there exist uncountably many non-equivalent long  $\mathbb{C}^n$ 's such that none of them has any holomorphic automorphisms different from the identity map.*

The following challenging problems remain open.

**PROBLEM 7.4.** (a) Which open subsets  $U \subset \mathbb{C}^n$  are the stable core of a long  $\mathbb{C}^n$ ?

(b) Is there a long  $\mathbb{C}^n$  which is Stein but not biholomorphic to  $\mathbb{C}^n$ ?

(c) Is there a non-Stein long  $\mathbb{C}^2$  with a nonconstant holomorphic function?

(d) Is there a long  $\mathbb{C}^2$  without any nonconstant meromorphic functions?

Question (d) is motivated by the observation that some meromorphic functions survive in the construction leading to Theorems 7.1 and 7.2.

## 8. Stein manifolds with the density property and Oka manifolds

In this section we discuss the role that Stein manifolds with the density property play in the theory of Oka manifolds, in holomorphic embedding problems, and in complex dynamics.

**8.1. Stein manifolds with the density property are Oka.** Recall (see [80, Section 5.4]) that a complex manifold  $X$  is said to be an *Oka manifold* if every holomorphic map  $U \rightarrow X$  from a neighbourhood of any given compact convex set  $K \subset \mathbb{C}^n$ ,  $n \in \mathbb{N}$ , can be approximated uniformly on  $K$  by entire maps  $\mathbb{C}^n \rightarrow X$ . This *convex approximation property* (CAP) is one of several equivalent characterisations of Oka manifolds. Recall that holomorphic maps  $S \rightarrow X$  from any Stein manifold  $S$  to an Oka manifold  $X$  satisfy all forms of the h-principle; see [80, Theorem 5.4.4] for a precise statement.

Another more recent characterisation of Oka manifolds is due to Kusakabe [122]. Consider the following condition on a complex manifold  $X$ :

(\*) For any compact convex set  $L \subset \mathbb{C}^m$  ( $m \in \mathbb{N}$ ), open set  $U \subset \mathbb{C}^m$  containing  $L$ , and holomorphic map  $f: U \rightarrow X$  there are an open set  $V$  with  $L \subset V \subset U$  and a holomorphic map  $F: V \times \mathbb{C}^n \rightarrow X$  for some  $n \geq \dim X$  such that  $F(\cdot, 0) = f|_V$  and

$$\left. \frac{\partial}{\partial t} \right|_{t=0} F(z, t): \mathbb{C}^n \rightarrow T_{f(z)}X \text{ is surjective for every } z \in V.$$

Here,  $t = (t_1, \dots, t_n) \in \mathbb{C}^n$ . Such  $F$  is called a *dominating holomorphic spray over  $f|_V$* .

This is a restricted version of condition  $\text{Ell}_1$  introduced by Gromov [101, p. 72]; see also [102]. In [122, Theorem 1.3], Kusakabe used the technique of gluing sprays from [80, Sect. 5.9] to show that this condition implies CAP, so a complex manifold satisfying  $(*)$  is an Oka manifold. Conversely, every Oka manifold satisfies condition  $\text{Ell}_1$  by [80, Corollary 8.8.7].

A holomorphic map  $F: X \times \mathbb{C}^n \rightarrow X$  satisfying condition  $(*)$  with  $V = X$  and  $f = \text{Id}_X$  is a *dominating spray* on  $X$ . A complex manifold  $X$  which admits a dominating spray is called *elliptic*. This terminology was introduced by Gromov [102], who proved that every elliptic manifold is an Oka manifold; the details can be found in [85] and [80, Chapter 5]. Conversely, every Stein Oka manifold is elliptic; see [102, 3.2.A] or [80, Proposition 5.6.15].

Suppose now that  $X$  is a Stein manifold with the density property. It is easily seen that complete holomorphic vector fields on  $X$  span the tangent space  $T_x X$  at every point, so  $X$  is holomorphically flexible in the sense of Arzhantsev et al. [26, 27]. It follows that for any compact subset  $K$  of  $X$  there exist finitely many complete holomorphic vector fields which generate  $TX$  over  $K$ . Denoting their flows by  $\phi^1, \dots, \phi^n$ , we obtain a spray  $F: X \times \mathbb{C}^n \rightarrow X$  of the form

$$(8.1) \quad F(x, t_1, \dots, t_n) = \phi_{t_1}^1 \circ \phi_{t_2}^2 \circ \dots \circ \phi_{t_n}^n(x), \quad x \in X, t_j \in \mathbb{C}, j = 1, \dots, n$$

which is dominating at every point of  $K$ . This means that  $X$  is *weakly elliptic*, and hence an Oka manifold [78, Corollary 5.5.12]. With some more work, one can show that finitely many complete holomorphic vector fields span the tangent bundle of  $X$  at every point, and hence  $X$  admits a globally dominating spray of the form (8.1); see [110, Theorem 4] or [80, Proposition 5.6.22 (b)]. Let us record this and a few related results.

**THEOREM 8.1.** (a) *Every Stein manifold with the density property is an Oka manifold.*

(b) *Every Stein manifold with the volume density property is an Oka manifold.*

(c) *Every complex manifold with the density property whose tangent bundle is pointwise spanned by globally defined holomorphic vector fields is an Oka manifold.*

The argument leading to statement (a) was given above (see also [78, Theorem 5.5.18]). Part (b) follows from [112, Lemma 4.1], which says that on a Stein manifold with the volume density property there are finitely many complete divergence-free holomorphic vector fields which span the tangent space at each point, so the manifold is elliptic. The main point is to use the isomorphism (2.3) between holomorphic vector fields with vanishing divergence and closed holomorphic  $(n-1)$ -forms, where  $n$  is the dimension of the manifold, and the fact that Cartan's Theorem A provides many exact (hence

closed) holomorphic  $(n - 1)$ -forms. Part (c), which does not use Steinness, is obtained by noting that the two conditions ensure that the tangent bundle  $TX$  is spanned over any compact subset of  $X$  by finitely many  $\mathbb{C}$ -complete holomorphic vector fields, so  $X$  is weakly elliptic and hence Oka.

REMARK 8.2. Theorem 8.1(c) corresponds to [80, Proposition 5.6.23]. In the latter source the hypothesis that the tangent bundle  $TX$  is spanned by globally defined holomorphic vector fields is accidentally missing, but is tacitly used in the proof.

The connection between Stein manifolds with the density property and Oka manifolds goes way beyond what has been said so far. As shown in [91], Theorem 4.9 together with the characterisation of Oka manifolds by Condition Ell<sub>1</sub> (see (\*) above) easily implies the following result of Kusakabe [121, Theorem 1.2 and Corollary 1.3] from 2020.

THEOREM 8.3. *Let  $X$  be a Stein manifold with the density property. For every compact  $\mathcal{O}(X)$ -convex subset  $K$  of  $X$  the complement  $X \setminus K$  is an Oka manifold. In particular, the complement of any compact polynomially convex set in  $\mathbb{C}^n$ ,  $n > 1$ , is an Oka manifold.*

For  $X = \mathbb{C}^n$  the proof goes as follows. Given a holomorphic map  $f: V \rightarrow \mathbb{C}^n \setminus K$  as in (\*) on a neighbourhood of a compact convex set  $L \subset \mathbb{C}^m$ , Theorem 4.9 gives a dominating spray  $F: V \times \mathbb{C}^n \rightarrow \mathbb{C}^n \setminus K$  such that for every  $z \in V$ ,  $F(z, \cdot): \mathbb{C}^n \rightarrow \mathbb{C}^n \setminus K$  is a Fatou–Bieberbach map and  $F(z, 0) = f(z)$ . Thus,  $\mathbb{C}^n \setminus K$  satisfies Condition Ell<sub>1</sub>, and hence is Oka by Kusakabe’s theorem [122]. The general case follows from [91, Theorem 3.1].

Kusakabe also proved that complements of certain unbounded closed polynomially convex sets  $E \subset \mathbb{C}^n$  are Oka. (A closed set is polynomially convex if it is exhausted by an increasing sequence of compact polynomially convex sets.) The following is [121, Theorem 1.6].

THEOREM 8.4. *Let  $E$  be a closed polynomially convex set in  $\mathbb{C}^n$ ,  $n \geq 2$ , such that*

$$(8.2) \quad E \subset \{(z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : |w| \leq C(1 + |z|)\}$$

*for some  $C > 0$ . Then  $\mathbb{C}^n \setminus E$  is an Oka manifold.*

To prove this result, Kusakabe first showed that the restricted coordinate projection  $\pi: \mathbb{C}^n \setminus E \rightarrow \mathbb{C}^{n-2}$ ,  $\pi(z, w) = z$ , has the Oka property for liftings; see [80, Corollary 5.5.11] for this notion. In particular, any holomorphic map  $f: V \rightarrow \mathbb{C}^n \setminus E$  from a Stein manifold  $V$  (again, it suffices to consider convex domains in Euclidean spaces) is the core of a fibre-dominating spray  $F: V \times \mathbb{C}^m \rightarrow \mathbb{C}^n \setminus E$  with  $F(\cdot, 0) = f$  and  $\pi \circ F(z, t) = \pi \circ f(z)$  for all  $z \in V$  and  $t \in \mathbb{C}^m$ . By (8.2) the same holds for linear projections  $\pi': \mathbb{C}^n \rightarrow \mathbb{C}^{n-2}$

with kernels close to  $\ker \pi = \{0\}^{n-2} \times \mathbb{C}^2$ . This gives sprays over  $f$  with values in  $\mathbb{C}^n \setminus E$  which are fibre-dominating in directions spanning the tangent space to  $\mathbb{C}^n$  at any point. By [122, Corollary 4.1] it follows that  $\mathbb{C}^n \setminus E$  is Oka. (The argument amounts to composing such sprays to obtain a dominating spray over  $f$  and then applying [122, Theorem 1.3].)

Kusakabe also proved the following result (see [121, Theorem 4.2]), which is an interesting and powerful application of the fibred density property discussed in Subsection 2.4.

**THEOREM 8.5.** *Let  $\pi: Y \rightarrow B$  be a holomorphic submersion between reduced complex spaces. Assume that  $E$  is a closed subset of  $Y$  such that every point  $b \in B$  admits an open neighbourhood  $U \subset B$  satisfying the following conditions:*

- (i)  $Y_U := \pi^{-1}(U)$  is a Stein space,
- (ii)  $E_U := E \cap \pi^{-1}(U)$  is holomorphically convex in  $Y_U$ , and
- (iii) the projection  $\pi: Y_U \setminus E_U \rightarrow U$  enjoys the fibred density property (see Subsection 2.4).

*Then the restriction  $\pi: Y \setminus E \rightarrow B$  enjoys the Oka property for liftings.*

We also mention the result of Forstnerič and Lárusson [84] which says that the holomorphic automorphism group  $\text{Aut}(\mathbb{C}^n)$  for  $n \geq 2$  enjoys most Oka properties for holomorphic maps  $X \rightarrow \text{Aut}(\mathbb{C}^n)$  from Stein manifolds. Although  $\text{Aut}(\mathbb{C}^n)$  does not carry the structure of an (infinite dimensional) complex manifold, it is natural to consider a map  $f: X \rightarrow \text{Aut}(\mathbb{C}^n)$  holomorphic if the associated evaluation map  $F: X \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ , given by  $F(x, z) = f(x)(z)$  ( $x \in X$ ,  $z \in \mathbb{C}^n$ ), is holomorphic. It is an open problem whether the same is true for the automorphism group of every Stein manifold with the density property in place of  $\mathbb{C}^n$ . In fact, it seems that this is not known for any other example besides  $\mathbb{C}^n$ .

**8.2. Embedding Stein manifolds in Stein manifolds with the density property.** We consider the problem of embedding Stein manifolds into model complex manifolds. The ideal class of models would be Oka manifolds. The following result (see [80, Corollary 8.9.3]) is obtained by combining the main result of Oka theory [80, Theorem 5.4.4] with the jet transversality theorem for holomorphic maps from Stein manifolds to Oka manifolds [80, Theorem 8.9.1].

**THEOREM 8.6.** *Let  $X$  be a Stein manifold and  $Y$  be an Oka manifold. If  $\dim Y \geq 2 \dim X$ , then every continuous map  $X \rightarrow Y$  is homotopic to a holomorphic immersion with simple double points. If  $\dim Y \geq 2 \dim X + 1$  then the immersion can be chosen injective.*

On the other hand, there are Oka manifolds which do not admit any proper holomorphic images of even the simplest manifolds such as the disc;

see [58, Example 1.3] which involves certain punctured tori of dimension  $> 1$  (these are Oka). Stein manifolds with the density property are much better in this respect, as demonstrated by the following result.

**THEOREM 8.7.** *Let  $X$  and  $Y$  be Stein manifolds, and assume that  $Y$  has the density property or the volume density property. Then, the following hold.*

(a) *If  $2 \dim X + 1 \leq \dim Y$  then any continuous map  $f: X \rightarrow Y$  is homotopic to a proper holomorphic embedding  $F: X \hookrightarrow Y$ . If in addition  $f$  is holomorphic on a neighborhood of a compact  $\mathcal{O}(X)$ -convex set  $K \subset X$  and  $X'$  is a closed complex subvariety of  $X$  such that the restriction  $f|_{X'}: X' \hookrightarrow Y$  is a proper holomorphic embedding, then  $F$  can be chosen to agree with  $f$  on  $X'$  and to approximate  $f$  uniformly on  $K$ .*

(b) *If  $2 \dim X = \dim Y$  then any continuous map  $X \rightarrow Y$  is homotopic to a proper holomorphic immersion  $X \rightarrow Y$  with simple double points, with additions as in part (a) concerning approximation and interpolation.*

Part (a) was proved by Andrist, Forstnerič, Ritter, and Wold [18] in 2016. (The special case for Riemann surfaces was obtained beforehand by Andrist and Wold [23].) Part (b) is due to Forstnerič [82] (2019). The proofs strongly depend on the Andersén–Lempert theory. Using Theorem 1.3, one inductively constructs a sequence of continuous maps  $X \rightarrow Y$  which are holomorphic embeddings (or immersions) on larger and larger domains in  $X$  such that the sequence converges uniformly on compacts to a proper holomorphic embedding or immersion  $X \rightarrow Y$ . The assumption that  $Y$  has the (volume) density property is crucial in this proof.

Theorem 8.7 is classical when  $Y$  is a Euclidean space  $\mathbb{C}^N$  [35, 140, 154]. In this case, the optimal embedding dimension is  $N = \left\lceil \frac{3 \dim X}{2} \right\rceil + 1$  if  $\dim X > 1$  according to Eliashberg and Gromov [61] and Schürmann [159] (see also Section 6). It is not known whether the embedding or immersion dimension in Theorem 8.7 can be lowered for more general Stein manifolds with the density property as targets. Another problem is the following.

**PROBLEM 8.8.** Does Theorem 8.7 hold for every Oka Stein manifold  $Y$ ?

The following is a corollary to Theorem 8.7 (b) and the fact that the space  $(\mathbb{C}^*)^n$  with coordinates  $z = (z_1, \dots, z_n)$  enjoys the volume density property with respect to the volume form  $\omega = \frac{dz_1 \wedge \dots \wedge dz_n}{z_1 \cdots z_n}$ . (See [174] or [80, Theorem 4.10.9(c)].)

**COROLLARY 8.9.** *Every Stein manifold  $X$  of complex dimension  $n \geq 1$  admits a proper holomorphic immersion to  $(\mathbb{C}^*)^{2n}$  and a proper pluriharmonic map to  $\mathbb{R}^{2n}$ .*

This provides a counterexample in any dimension to the conjecture of Schoen and Yau [158] that the unit disc  $\mathbb{D}$  does not admit any proper harmonic maps to  $\mathbb{R}^2$ . We refer to [81, Subsect. 3.3] and [8, Sect. 3.10] for

the discussion of this topic, also in the context of minimal surfaces, and for references to the earlier counterexamples for the disc  $X = \mathbb{D}$ .

**8.3. Open Runge embeddings.** It was a long-standing problem whether  $\mathbb{C}^* \times \mathbb{C}$  embeds as a Runge domain in  $\mathbb{C}^2$ ; such hypothetical domains were called *Runge cylinders* in  $\mathbb{C}^2$ . This question arose in connection with the classification of Fatou components for Hénon maps by Bedford and Smillie [30] in 1991. In 2021, Bracci, Raissy and Stensønes [40] obtained a Runge embedding  $\mathbb{C}^* \times \mathbb{C} \hookrightarrow \mathbb{C}^2$  as the basin of a non-polynomial holomorphic automorphism of  $\mathbb{C}^2$  at a parabolic fixed point. In 2020, Forstnerič and Wold [92] showed that Runge tubes are abundant in Stein manifolds with the density property. Although their proof is completely different from the one in [40], both use Andersén–Lempert theory. Their first result [92, Theorem 1.1] is the following.

**THEOREM 8.10.** *Let  $X$  and  $Y$  be Stein manifolds with  $\dim X < \dim Y$ , and assume that  $Y$  has the density property. Suppose that  $\theta: X \hookrightarrow Y$  is a holomorphic embedding with  $\mathcal{O}(Y)$ -convex image (this holds in particular if  $\theta$  is proper), and let  $E \rightarrow X$  denote the holomorphic normal bundle associated to  $\theta$ . Then,  $\theta$  can be approximated uniformly on compacts in  $X$  by holomorphic embeddings of  $E$  into  $Y$  whose images are Runge domains in  $Y$ .*

To get a Runge embedding of  $\mathbb{C}^* \times \mathbb{C}$  into  $\mathbb{C}^2$ , it suffices to embed  $X = \mathbb{C}^*$  onto the curve  $\{zw = 1\} \subset \mathbb{C}^2$  and note that any holomorphic vector bundle over  $\mathbb{C}^*$  (indeed, over any open Riemann surface) is trivial by Oka’s theorem [143]. (See also [80, Sect. 5.2].) This argument gives the following corollary to Theorem 8.10.

**COROLLARY 8.11** (Runge tubes over open Riemann surfaces). *If  $X$  is an open Riemann surface which admits a proper holomorphic embedding into  $\mathbb{C}^2$ , then  $X \times \mathbb{C}$  is biholomorphic to a Runge domain in  $\mathbb{C}^2$ . For every open Riemann surface  $X$  and  $k \geq 2$ ,  $X \times \mathbb{C}^k$  admits a Runge embedding into  $\mathbb{C}^{k+1}$ , and into any Stein manifold  $Y^{k+1}$  with the density property.*

The Runge embedding  $E \hookrightarrow Y$  of the normal bundle in Theorem 8.10 need not agree with the given embedding  $\theta: X \hookrightarrow Y$  on the zero section  $X$  of  $E$ ; this is impossible in general due to Theorem 5.3. However, we can ensure this condition for algebraic embeddings of codimension at least 2 into  $\mathbb{C}^n$ ; see [92, Theorem 1.4 and Corollary 1.5].

**THEOREM 8.12.** *Let  $X$  be a Stein manifold and  $\theta: X \hookrightarrow \mathbb{C}^n$  be a proper holomorphic embedding onto an algebraic submanifold of  $\mathbb{C}^n$ . If  $n \geq \dim X + 2$  then  $\theta$  extends to a holomorphic Runge embedding  $E \hookrightarrow \mathbb{C}^n$  of the total space of the normal bundle of  $\theta$ . In particular, every proper algebraic embedding  $X \hookrightarrow \mathbb{C}^n$  ( $n \geq 3$ ) of an affine algebraic curve extends to a holomorphic Runge embedding  $X \times \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ .*

The proof of Theorem 8.12 uses the result of Kaliman and Kutzschebauch [109, Theorems 4 and 6] that the Lie algebra  $\text{Lie}_{\text{alg}}(\mathbb{C}^n, Y)$  of algebraic vector fields on  $\mathbb{C}^n$  vanishing on an algebraic submanifold  $Y \subset \mathbb{C}^n$  of dimension  $\dim Y \leq n - 2$  enjoys the strong algebraic density property (see Example (1) in Subsection 2.3).

**8.4. Holomorphic dynamics on Stein manifolds with the density property.** Holomorphic dynamics is a lively field of complex analysis. So far, dynamical phenomena have mainly been studied on complex Euclidean spaces, which have an abundance of holomorphic endomorphisms and automorphisms. It is natural to ask to what extent can these results be extended to Stein manifolds with the density property, and how does the possibly nontrivial topology of the manifold impact the global behaviour of maps under consideration.

The first steps in this emerging field were made recently by Arosio and Lárusson. In [24] they proved that automorphisms with chaotic behaviour are generic among volume preserving automorphisms of a Stein manifold  $X$  having the density property for an exact volume form. For  $X = \mathbb{C}^n$ ,  $n \geq 2$ , with the standard volume form this was proved by Fornæss and Sibony [69], and the authors follow their approach. They also showed that a generic volume preserving automorphism has a hyperbolic fixed point whose stable manifold is dense in  $X$ , generalizing a result of Peters, Vivas, and Wold on  $\mathbb{C}^n$  [144]. In their second paper [25], they proved closing lemmas for automorphisms of a Stein manifold with the density property and for endomorphisms of an Oka Stein manifold. In the former case they needed to impose a new tameness condition. It follows that hyperbolic periodic points are dense in the tame non-wandering set of a generic automorphism of a Stein manifold with the density property and in the non-wandering set of a generic endomorphism of an Oka Stein manifold.

## 9. Complete complex submanifolds

In 1977, Paul Yang asked [184, 185] whether there exist bounded complete immersed or embedded complex submanifolds in complex Euclidean spaces. Here, an immersion  $X \rightarrow \mathbb{C}^n$  is said to be complete if the image of any divergent path  $\gamma: [0, 1) \rightarrow X$  (i.e., one that leaves every compact subset of  $X$  as  $t \rightarrow 1$ ) has infinite Euclidean length. Equivalently, the Riemannian metric on  $X$  induced by the immersion is a complete metric. Yang's problem is an analogue of the *Calabi–Yau problem* for minimal surfaces in  $\mathbb{R}^n$ . We refer to [8, Chapter 7] for background and a discussion of recent results on this subject.

The first such examples were constructed by Jones in 1979 [106] (a complete immersed holomorphic disc in the ball  $\mathbb{B}^2$  of  $\mathbb{C}^2$ , and an embedded

one in the ball of  $\mathbb{C}^3$ ). Much later, Alarcón and Forstnerič proved in [5] (2013) that every finite bordered Riemann surface admits a complete proper holomorphic immersion in  $\mathbb{B}^2$  and embedding in  $\mathbb{B}^3$ . For further and more recent developments on this subject we refer to the papers [4, 7, 9–11, 57, 97]. In particular, the following is a compilation of results by Globevnik [97, 98], Alarcón, Globevnik and López [10], Alarcón [4, Corollary 1.2], and Alarcón and Forstnerič [7].

**THEOREM 9.1.** *For every pair of integers  $1 \leq q < n$  there exists a holomorphic submersion  $f: \mathbb{B}^n \rightarrow \mathbb{C}^q$  whose fibres are complete complex submanifolds of  $\mathbb{B}^n$ . In particular, the ball  $\mathbb{B}^n$  can be foliated by complete properly embedded holomorphic discs.*

The techniques used in the last three mentioned papers rely on Andersén–Lempert theory, and they provide some additional information on topology of the leaves. The main idea is to place in  $\mathbb{B}^n$  a polynomially convex labyrinth consisting of countably many pairwise disjoint compact sets (the authors used closed balls in affine hyperplanes) such that any divergent curve in  $\mathbb{B}^n$  avoiding all but finitely many pieces of the labyrinth has infinite length. One then uses holomorphic automorphisms of  $\mathbb{C}^n$  to successively twist a given holomorphic foliation on  $\mathbb{C}^n$  such that the resulting sequence of foliations converges in  $\mathbb{B}^n$  to a foliation each of whose leaves avoids all but finitely many pieces of the labyrinth, so it is complete.

On the other hand, Globevnik’s technique in [97, 98] relies on a different idea, based on the construction of holomorphic functions on the ball which grow sufficiently fast on pieces of a suitable labyrinth. This technique does not allow any control of the topology of the leaves.

The methods in the cited papers, together with the construction of suitable labyrinths by Charpentier and Kosiński [50] (2020), show that the same result holds in all pseudoconvex Runge domains in  $\mathbb{C}^n$ ,  $n > 1$ . Globevnik extended his original construction of complete complex submanifolds of  $\mathbb{B}^n$  [97] to this more general setting in [98] (2016), but in his case (for constructing fast growing holomorphic functions) less precise labyrinths suffice.

In a related direction, Alarcón and Forstnerič constructed in [6] a complete injective holomorphic immersion  $\mathbb{C} \rightarrow \mathbb{C}^2$  whose image is dense in  $\mathbb{C}^2$ . The analogous result was obtained for any closed complex submanifold  $X \subset \mathbb{C}^n$  ( $n > 1$ ) in place of  $\mathbb{C}$ .

## 10. An application in 3-dimensional topology

It is well-known that diffeomorphism groups of smooth or real analytic manifolds of positive dimension are huge, in particular infinite dimensional. Complete vector fields on such manifolds can be constructed using cutoff functions and approximation in the strong Whitney topology (to obtain real

analytic ones). It seems therefore not very interesting to study real shears on  $\mathbb{R}^n$  or, say, multiplicative shears on a torus.

However, there are situations where real shears gain importance for some specific reason. In this section we describe a recent example of this type. It concerns the affirmative answer, given by Rafael Zentner [187] in 2018, to the long standing problem in 3-dimensional topology asking whether the fundamental group of any homology 3-sphere different from the 3-sphere  $S^3$  admits an irreducible representation into  $\mathrm{SL}_2(\mathbb{C})$ , i.e. a 2-dimensional irreducible representation. Here, a homology 3-sphere is a compact 3-manifold  $X$  whose homology groups are those of  $S^3$ . Its fundamental group is nontrivial unless  $X$  is  $S^3$  itself.

Let us explain the main points in Zentner's analysis. There are three types of homology 3-spheres, and the one type where the answer was not known are those homology 3-spheres which admit a degree 1-map to a splicing of two nontrivial knots in  $S^3$ . Since a degree 1-map is  $\pi_1$ -surjective, it remained to find the answer for those homology 3-spheres which arise by the following construction, called splicing. Take a pair of knots  $K_1, K_2$  in  $S^3$ . Remove the tubular neighborhood  $N(K_i)$  of the knot from  $S^3$ , i.e. set  $X_i := S^3 \setminus N(K_i)$ , and glue these two manifolds along the boundaries of  $N(K_i)$  (each isomorphic to a 2-dimensional torus) so that the longitude of one torus is glued onto the meridian of the other torus. The resulting manifold is a homology 3-sphere, called the *splicing*  $Y_{K_1, K_2}$  of the knots  $K_1$  and  $K_2$ .

In order to produce an irreducible representation of the fundamental group  $\pi_1(Y_{K_1, K_2})$  into  $\mathrm{SL}_2(\mathbb{C})$ , it was important to prove that for any two nontrivial knots  $K_1$  and  $K_2$  the images of their representation varieties in the representation variety  $R(T^2)$  of the boundary torus  $T^2$  (the place where we glued) intersect and thus yields a representation of  $\pi_1(Y_{K_1, K_2})$ . This is the place where the real version of the Andersén–Lempert theory comes into play.

The representation variety, i.e. the space of  $SU(2)$ -representations of the fundamental group of a two-dimensional torus  $T^2$  modulo conjugation,

$$R(T^2) = \mathrm{Hom}(\mathbb{Z}^2, SU(2))/SU(2)$$

is homeomorphic to the *pillowcase*, a 2-dimensional sphere. In fact, if we denote generators of  $\pi_1(T^2) \cong \mathbb{Z}^2$  by  $m$  and  $l$ , then for a representation  $\rho$  we may suppose that

$$\rho(m) = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \quad \text{and} \quad \rho(l) = \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix},$$

and hence we can associate to  $\rho$  a pair  $(\alpha, \beta) \in [0, 2\pi] \times [0, 2\pi]$ , which we also can think of as being a point on the two-dimensional torus  $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ .

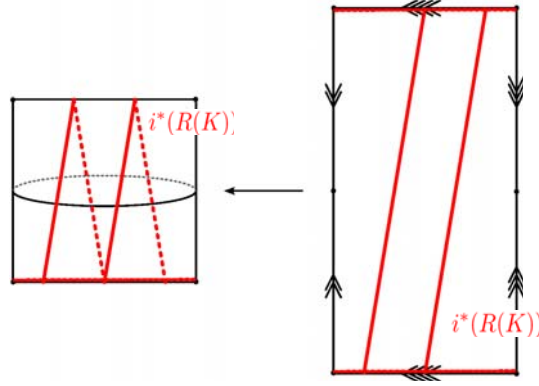


Fig. 10.1: The gluing pattern for obtaining the pillowcase from a rectangle, and the image of the representation variety  $R(K)$  of the trefoil knot in the pillowcase

However, it is easily seen that a representation to which we associate  $(2\pi - \alpha, 2\pi - \beta)$  is conjugate to  $\rho$ . This is the only ambiguity, as the trace of an element in  $SU(2)$  determines its conjugacy class. Therefore  $R(T^2)$  is isomorphic to the quotient of the torus  $T^2$  by the hyperelliptic involution  $\tau : (\alpha, \beta) \mapsto (-\alpha, -\beta)$ . This has four fixed points, and its quotient

$$R(T^2) = T/\tau$$

is homeomorphic to the 2-sphere. It can also be seen as the quotient of the fundamental domain  $[0, \pi] \times [0, 2\pi]$  for  $\tau$  by identifications on the boundary as indicated in Figure 10.1.

All abelian representations map to the red line ‘at the bottom’.

As explained above, one has to understand the image (under the map induced by restriction) of the representation variety  $R(K)$  of the knot complement  $S^3 \setminus K$  in the representation variety of the boundary torus  $R(T^2)$  (the pillowcase). As an example, in Figure 10.1 the image of  $R(K)$  for  $K$  being the trefoil knot is drawn in red.

Using results of Kronheimer and Mrowka, for any nontrivial knot  $K$  the existence of points in the image of  $R(K)$  lying on the straight line ‘on the top’ of the pillowcase can be deduced, as well as on any holonomy perturbation of this top line.

The simplest class of holonomy perturbations are shearing maps on the torus,

$$\Phi_f : \mathbb{R}^2/2\pi\mathbb{Z}^2 \rightarrow \mathbb{R}^2/2\pi\mathbb{Z}^2, \quad (x, y) \mapsto (x, y + f(x))$$

for some odd  $2\pi$ -periodic function  $f$ . Of course one can also change the roles of  $x$  and  $y$ . The following version of Theorem 1.1 for a real torus is due to Zentner [187, Theorem 3.3].

**THEOREM 10.1.** *Let  $\psi: T \rightarrow T$  be an area-preserving map of the  $n$ -dimensional torus  $T$  for  $n \geq 2$ , which is volume preservingly isotopic to the identity, and let  $\varepsilon > 0$  be given. Then there is a finite composition of shearing maps  $\phi: T \rightarrow T$  which is  $\varepsilon$ -close to  $\psi$ . Moreover, the whole isotopy can be realized  $\varepsilon$ -close to an isotopy through finitely many shearing isotopies.*

This result, or rather an equivariant version of it with respect to the involution, was the crucial ingredient to prove the existence of points in the image of  $R(K)$  in the pillowcase  $R(T^2)$  on any path connecting the left and right upper corners. (Namely, using Theorem 10.1 and the Moser trick, any path can be approximated arbitrarily well by the straight line on the top of the pillowcase moved by holonomy perturbations; here, shears. A closedness argument gives the desired conclusion.) This latter fact can in turn be used to show that the images of the representation varieties of the two nontrivial knots in the pillowcase always intersect. Indeed, the image of the representation variety of  $K_1$  has to wrap around the pillowcase in one direction (its image under the lift to the torus contains a path in the homology class of the longitude), while the image of the representation variety of  $K_2$  has to wrap around the pillowcase in the other direction (its image under the lift to the torus contains a path in the homology class of the meridian). Thus, these images meet, thereby yielding a nontrivial representation of  $Y_{K_1, K_2}$ .

## 11. The recognition problem for complex Euclidean spaces

The complex affine spaces  $\mathbb{C}^n$  are the most natural and basic objects in algebraic and complex analytic geometries, comparable to the role played by the real Euclidean spaces for topological or differentiable manifolds. One of the most famous achievements of geometric topology for open real manifolds is the complete solution of the Open Poincaré Conjecture, namely, the characterisation of Euclidean spaces among open topological manifolds as the unique ones which are simultaneously contractible and simply connected at infinity. This conjecture was established by Stallings for PL-manifolds of dimension  $n \geq 5$ , Freedman in the case  $n = 4$ , and finally Perelman in dimension  $n = 3$ .

Nothing even remotely close to this exists in the complex algebraic or holomorphic case, with the exception of the work of Ramanujam [149] on affine algebraic surfaces (see below). Indeed, even extremely simple affine algebraic manifolds, such as the Koras–Russell cubic threefold  $KR$  in  $\mathbb{C}^4$  defined by the polynomial equation  $x + x^2y + s^2 + t^3 = 0$  (see (2.1)), are not fully understood in this context.

In his landmark paper [149] (1971), Ramanujam proved that a smooth contractible affine algebraic surface is isomorphic to  $\mathbb{C}^2$  if it is simply connected at infinity. At the same time, he constructed many examples of

smooth contractible affine algebraic surface with non-trivial fundamental group at infinity, thereby opening the way for the construction of higher dimensional algebraic or analytic varieties which were later on verified, thanks to the cancellation theorems due to Iitaka–Fujita in the algebraic case and Zaidenberg in the holomorphic one, to be diffeomorphic to Euclidean spaces, while neither algebraically nor holomorphically isomorphic to  $\mathbb{C}^n$ . In the holomorphic case, even dimension 2 remains a mystery. These types of varieties, nowadays called *exotic affine spaces*, are challenging objects.

These examples show that additional properties must be imposed on a variety which is diffeomorphic to  $\mathbb{R}^{2n}$  in order to be biholomorphic or algebraically isomorphic to  $\mathbb{C}^n$ . A natural attempt is to use symmetries. Affine spaces are homogeneous under the action of algebraic or holomorphic one-parameter flows. Recall that vector fields generating algebraic flows are called *locally nilpotent derivations*, LND’s for short. A seminal breakthrough of Makar–Limanov [137] in 1996 was to realize that the Koras–Russell cubic KR has some rigidity with respect to such algebraic flows. He introduced an invariant, now called the *Makar–Limanov invariant*, which measures the richness of algebraic flows and the homogeneity of a variety under the action of the group which they generate. Makar–Limanov and Kaliman developed in [116] sophisticated algebraic techniques to compute this invariant for special classes of affine varieties containing the Koras–Russell cubic. Refinements of these techniques by Dubouloz, Moser, Jauslin and Poloni enabled the construction of pairs of exotic affine 3-folds failing the Zariski Cancellation Problem (see Problem 11.1), the construction of holomorphically trivial deformations of pairwise non-isomorphic algebraic exotic affine 3-folds [59], and a full description of the group of algebraic automorphisms of the Koras–Russell cubic KR [60, 139]. The latter group is infinite dimensional like the group  $\text{Aut}_{\text{alg}}(\mathbb{C}^3)$ , but it acts on KR with precisely four orbits, in contrast to the transitivity of the action of  $\text{Aut}_{\text{alg}}(\mathbb{C}^3)$  on  $\mathbb{C}^3$ .

In general, a characterisation of affine spaces is considered very useful if it can be used to solve the following Zariski Cancellation Problem.

**PROBLEM 11.1.** Assuming that  $X$  is an affine algebraic variety such that  $X \times \mathbb{C}^k$  is isomorphic to  $\mathbb{C}^{n+k}$  for some  $k \in \mathbb{N}$ , is  $X$  isomorphic to the affine space  $\mathbb{C}^n$ ? The same question for  $X$  a Stein manifold and the word isomorphic replaced by biholomorphic.

For  $n = 1$  the affirmative answer is not difficult, while for  $n = 2$  it is a deep result due to Fujita [93] and Miyanishi and Sugie [138]. The case  $n \geq 3$  is completely open. In the holomorphic case dimension  $n = 1$  is easy, and the problem is open in higher dimensions.

Since the density property of a Stein manifold is a precise way of saying that the group of its automorphisms is big, the following problem of Tóth and Varolin [170] is very natural.

PROBLEM 11.2. Is every Stein manifold with the density property which is diffeomorphic to  $\mathbb{R}^{2n}$  also biholomorphic to  $\mathbb{C}^n$ ?

This question remains unsolved. An affirmative answer would yield (following Derksen and Kutzschebauch [53]) a nonlinearizable holomorphic action of  $\mathbb{C}^*$  on  $\mathbb{C}^3$ , and we would know that holomorphic  $\mathbb{C}^*$ -actions on  $\mathbb{C}^n$  are linearizable if and only if  $n \leq 2$ . As it stands, the case  $n = 3$  is unsolved, and it can only be solved with a good characterisation of  $\mathbb{C}^3$  in hand. Incidentally, the linearization problem was the main motivation of Makar–Limanov for introducing his invariant. The most spectacular use of the Makar–Limanov invariant was the proof by Kaliman et al. [108] that every algebraic  $\mathbb{C}^*$ -actions on  $\mathbb{C}^3$  is linearizable. This invariant was the crucial tool in proving that all potential counterexamples to linearization (like the Koras–Russell 3-fold) are non-isomorphic to  $\mathbb{C}^3$ .

It is still unknown whether an affine algebraic variety which is biholomorphic to  $\mathbb{C}^n$  is also algebraically isomorphic to  $\mathbb{C}^n$ . This is known as *Zaidenberg’s problem* [186]. If Problem 11.2 of Tóth and Varolin has an affirmative answer, then the Koras–Russell cubic KR is a counterexample to Zaidenberg’s problem. Indeed, Leuenberger [133] showed that KR has the density property (see Example (5) in Subsec. 2.1), and it is known to be diffeomorphic to  $\mathbb{R}^6$ .

One can naturally generalise the cancellation problem as follows.

PROBLEM 11.3. Let  $X$  and  $Y$  be affine algebraic manifolds such that  $X \times \mathbb{C}$  is algebraically isomorphic to  $Y \times \mathbb{C}$ . Does it follow that  $X$  is isomorphic to  $Y$ ? The analogous problem for Stein manifolds and biholomorphisms.

Here the answer is negative in general, and additional conditions must be imposed. For example, Danielewski found that for a polynomial  $p$  with simple roots the affine surfaces  $D_n := \{(x, y, z) \in \mathbb{C}^3 : x^n y = p(x)\}$  satisfy  $D_n \times \mathbb{C} \cong D_m \times \mathbb{C}$  (unpublished preprint, 1989). Later, Fieseler [64] showed that  $D_n$  and  $D_m$  for  $n \neq m$  are not even homeomorphic by examining their fundamental group at infinity. This gives counterexamples to both the holomorphic and the algebraic cancellation. Cancellation also fails in the differentiable category. Counterexamples include some nice smooth complex algebraic varieties. Take for example a surface of Ramanujam which is contractible but not simply connected at infinity. The surface

$$S := \left\{ (x, y, z) \in \mathbb{C}^3 : \frac{(xz + 1)^3 - (yz + 1)^2 - z}{z} = 0 \right\}$$

is such an example. It is not homeomorphic to  $\mathbb{R}^4$ , but  $S \times \mathbb{C}$  is contractible and simply connected at infinity. This is easily seen from the definition of the fundamental group at infinity; however, we remark that in general for a smooth affine algebraic variety of dimension  $n \geq 3$ , contractibility implies

simple connectedness at infinity [186]. Thus,  $S \times \mathbb{R}^2$  is diffeomorphic to  $\mathbb{R}^6$ . By Miyanishi–Sugie [138] it follows that  $S \times \mathbb{C}$  is not algebraically isomorphic to  $\mathbb{C}^3$ . It remains unknown whether  $S \times \mathbb{C}$  is biholomorphic to  $\mathbb{C}^3$ .

The affine modification  $M$  of  $\mathbb{C}^4$  with divisor  $D := \mathbb{C}^3 \times \{0\} \subset \mathbb{C}^4$  along the center  $C := S \subset \mathbb{C}^3 \times \{0\}$  is another interesting example. It is given by one equation in  $\mathbb{C}^5$ :

$$M = \left\{ (x, y, z, u, v) \in \mathbb{C}^5 : uv = \frac{(xz + 1)^3 - (yz + 1)^2 - z}{z} \right\}.$$

The manifold  $M$  is known to be diffeomorphic to  $\mathbb{R}^8$  by works of Kaliman and Zaidenberg [117] and Kaliman and Kutzschebauch [109]. If  $M$  is isomorphic to  $\mathbb{C}^4$ , we have an algebraic  $\mathbb{C}^4$  in  $\mathbb{C}^5$  which is not algebraically straightenable since the defining polynomial for  $M$  has singular fibres. No such examples are known. Except for the classical case of Abhyankar–Moh–Suzuki of lines in the plane (see the discussion after Corollary 5.2), it is unknown whether a smooth algebraic hypersurface in  $\mathbb{C}^{n+1}$  which is isomorphic to  $\mathbb{C}^n$  is algebraically straightenable. If  $M$  is not isomorphic to  $\mathbb{C}^4$  but is biholomorphic to  $\mathbb{C}^4$ , it provides a counterexample to Zaidenberg’s problem. Finally, if  $M$  is not biholomorphic to  $\mathbb{C}^4$  then it is a counterexample to Problem 11.2 of Tóth and Varolin. It is still unknown whether  $M$  is isomorphic or biholomorphic to  $\mathbb{C}^4$ , which shows how difficult these questions are.

Let us finally mention some characterisations of affine spaces. Unfortunately, they are not useful for solving any of the problems we have formulated.

It is classically known in both categories that the algebra of regular or holomorphic functions, respectively, determines the underlying space. For the affine algebraic case this holds by definition (the space is the spectrum of the algebra of regular functions), and for reduced and irreducible Stein spaces it is a classical result attributed to Remmert in his habilitation thesis; see Grauert and Remmert [100, Theorem 6, p. 184]. If one considers only the ring structure of the space of holomorphic functions, one must also impose the condition that the ring homomorphism maps  $i = \sqrt{-1}$  to itself (and not to  $-i$ ); see the appendix of the paper [105] which was published by Heisuke Hironaka under the pseudonym Hei Iss’sa. This result is known as Bers’s Theorem after Lipman Bers who first proved it for open Riemann surfaces. The fact that the  $\mathbb{C}$ -algebra of meromorphic functions determines the underlying Stein space is known as Iss’sa’s Theorem [105]. In the same paper it is shown that a field isomorphism of the spaces of meromorphic functions has to map  $i$  to  $i$  in order to induce a biholomorphism (and not an antibiholomorphism) between the Stein spaces.

In the algebraic setting, there are no applicable characterisations other than those by Miyanishi–Sugie and Ramanujam mentioned above. A recent

result of Cantat, Regeta and Xie [47] states that if  $X$  is a reduced connected affine algebraic variety whose automorphism group  $\mathrm{Aut}_{\mathrm{alg}}(X)$  is isomorphic to  $\mathrm{Aut}_{\mathrm{alg}}(\mathbb{C}^n)$  as an abstract group, then  $X$  is isomorphic to  $\mathbb{C}^n$ . Under the stronger assumption that the automorphism groups are isomorphic as ind-groups (a notion introduced by Shafarevich in [160,161]), this conclusion was obtained earlier by Kraft [120]. Another result of Andrist and Kraft [19] says that if the semigroups of self-maps  $\mathrm{End}(X)$  and  $\mathrm{End}(\mathbb{C}^n)$  are isomorphic then, up to an automorphism of the base field  $\mathbb{C}$ ,  $X$  is isomorphic to  $\mathbb{C}^n$ . (The result in [19] is proved over an algebraically closed field of arbitrary characteristic. The algebraic case over the complex numbers is already implicitly contained in [15].) In the holomorphic category, Andrist proved in [15] that if the semigroup of self-maps  $\mathrm{End}(X)$  of a complex space is abstractly isomorphic to the semigroup  $\mathrm{End}(\mathbb{C}^n)$  then  $X$  is biholomorphic or antibiholomorphic to  $\mathbb{C}^n$ . The difference in the results over the complex numbers is that, in the holomorphic case, the field automorphisms are continuous.

A result of Isaev and Kruzhilin [104], which compares to Kraft's result with the ind-group structure, says that if the automorphism group of a complex manifold  $X$  of dimension  $n$  is isomorphic as a topological group to  $\mathrm{Aut}(\mathbb{C}^n)$  then  $X$  is biholomorphic to  $\mathbb{C}^n$ . The isomorphism of topological groups implies by classical results that there is a real analytic action of the unitary affine group  $\mathbb{R}^{2n} \rtimes U_n$  on  $X$  by holomorphic automorphisms, and classifying manifolds with such isometries is not too difficult.

Another criterion comes from the work of Kutzschebauch, Lárusson and Schwarz [125]. As in the previous result, it assumes some group action on  $X$ . If a Stein manifold  $X$  carries the action of a reductive group  $G$  so that the categorical quotient is stratified biholomorphic to the quotient of a large linear action of  $G$  on  $\mathbb{C}^n$ , then  $X$  is biholomorphic to  $\mathbb{C}^n$ , and there is a biholomorphism linearizing the action. The same is true without the assumption of largeness for finite groups, as well as groups  $G$  having connected component  $\mathbb{C}^*$  [130] or  $G = \mathrm{SL}_2(\mathbb{C})$  [125].

A characterisation in the holomorphic case which is clearly worth to be explored further is due to Boc Thaler [36, Theorem IV.15].

**THEOREM 11.4.** *Let  $X$  be a Stein manifold with the density property. Then  $X$  is biholomorphic to  $\mathbb{C}^n$  if and only if  $X$  can be exhausted by Runge images of the ball.*

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