OKA-1 MANIFOLDS

ANTONIO ALARCÓN AND FRANC FORSTNERIČ

ABSTRACT. We introduce and study a new class of complex manifolds, Oka-1 manifolds, characterized by the property that holomorphic maps from any open Riemann surface to the manifold satisfy the Runge approximation and the Weierstrass interpolation conditions. We prove that every complex manifold which is dominable at most points by spanning tubes of complex lines in affine spaces is an Oka-1 manifold. In particular, a manifold dominable by $\mathbb{C}^n$ at most points is an Oka-1 manifold. By using this criterion, we provide many examples of Oka-1 manifolds among compact complex surfaces, including all Kummer surfaces and all elliptic K3 surfaces. We show that the class of Oka-1 manifolds is invariant under Oka-1 maps inducing a surjective homomorphism of fundamental groups; this includes holomorphic fibre bundles with connected Oka fibres. In another direction, we prove that every bordered Riemann surface admits a holomorphic map with dense image in any connected complex manifold. The analogous result is shown for holomorphic Legendrian immersions in an arbitrary connected complex contact manifold.

CONTENTS

1. INTRODUCTION

The study of holomorphic curves is a perennial subject in complex and algebraic geometry. In this paper, we introduce and investigate a new class of complex manifolds characterized

Date: 13 February 2024.
2010 *Mathematics Subject Classification.* Primary 32E30, 32H02, 32Q56; secondary 14H55.
Key words and phrases. Riemann surface; complex curve; Oka-1 manifold; Oka manifold; K3 surface.

by the property that they admit plenty of holomorphic curves parameterized by any open Riemann surface. Here is the precise definition of this class of manifolds.

Definition 1.1. A connected complex manifold X with a Riemannian distance function dist_X is an *Oka-1 manifold* if for any open Riemann surface R, Runge compact set K in R, discrete sequence $a_i \in R$ without repetitions, continuous map $f : R \to X$ which is holomorphic on a neighbourhood of $K \cup \bigcup_i \{a_i\}$, number $\epsilon > 0$, and integers $k_i \in \mathbb{N} = \{1, 2, \ldots\}$ there is a holomorphic map $F : R \to X$ which is homotopic to f and satisfies

(1) $\sup_{p \in K} \mathrm{dist}_X(F(p), f(p)) < \epsilon$ and
(2) F agrees with f to order k_i in the point a_i for every i.

If condition (1) can be satisfied then X has the Oka-1 property with approximation. If in addition condition (2) holds with $k_i = 1$ then X has the Oka-1 property with approximation and interpolation. A complex manifold X is Oka-1 if every component of X is such.

Recall that a compact set K in a Riemann surface R is said to be Runge if $R \setminus K$ does not have any relatively compact connected component.

The definition of an Oka-1 manifold is motivated by classical results for holomorphic functions on open Riemann surfaces due to Runge [49], Weierstrass [54], Behnke and Stein [6], and Florack [18]. The term is inspired by the notion of an Oka manifold, which developed from the Oka–Grauert theory. A complex manifold X is an Oka manifold if it satisfies the approximation and interpolation conditions, analogous to those in Definition 1.1, for maps $S \to X$ from any Stein manifold S. (See [26, Chapter 5] and [23, 29].) Thus, every Oka manifold is also an Oka-1 manifold. The converse fails in general. For example, there is a discrete set $A \subset \mathbb{C}^2$ whose complement is not dominable by $\mathbb{C}^2$ (see [48] or [26, Sect. 4.7]), hence it fails to be Oka, but it is Oka-1 by a general position argument (see Corollary 2.9).

In this paper, we investigate the class of Oka-1 manifolds by combining techniques from complex analysis and complex and algebraic geometry.

Here are some immediate observations. If X is an Oka-1 manifold then for every point $x \in X$ and tangent vector $v \in T_x X$ there exists an entire map $f : \mathbb{C} \to X$ with $f(0) = x$ and $f'(0) = v$. Hence, the Kobayashi pseudometric of X vanishes identically and every bounded plurisubharmonic function on X is constant, i.e., X is Liouville. It is even strongly Liouville; see Corollary 7.5. Assuming that X is connected, it admits holomorphic maps with dense images from any open Riemann surface, in particular, from $\mathbb{C}$. This implies that the class of compact Kähler or projective Oka-1 manifolds is conjecturally related to several important classes of complex manifolds studied in the literature, such as the special manifolds in the sense of Campana [9, 10]; see the discussion by Campana and Winkelmann in [11].

The conditions in Definition 1.1 easily imply that a homotopy from the initial map f to a holomorphic map F can be chosen to consist of maps $R \to X$ which are holomorphic on a neighbourhood of $K \cup \bigcup_i \{a_i\}$ and agree with f to order k_i at a_i for every i. Note however that the axiom of Oka-1 manifolds does not include the parameteric case concerning families of maps $R \to X$. While the parameteric Oka property follows from the basic Oka property [26, Proposition 5.15.1], the proof uses the basic Oka principle for maps from Stein manifolds of arbitrary dimension, and it does not apply to Oka-1 manifolds. One can of course introduce and study the class of complex manifolds satisfying the parameteric Oka-1 property, but most of the techniques developed in this paper do not apply to this case.

We now describe our main results, deferring the precise statements to individual sections.

In Section 2 we introduce a geometric sufficient condition on a complex manifold to be Oka-1; see Theorem 2.2. This condition concerns dominability by spanning tubes of complex lines in affine spaces $\mathbb{C}^n$. It holds on any complex manifold X which is dominable by $\mathbb{C}^n$ at every point in a Zariski open domain, so every such manifold is Oka-1; see Corollary 2.5. In particular, a connected algebraic manifold which is algebraically dominable by $\mathbb{C}^n$ is Oka-1. Dominability by tubes of lines is a considerably weaker condition than any of the sufficient conditions in the theory of Oka manifolds, and is the first known condition implying an Oka-type property which is purely local on the target manifold.

After the preparatory Sections 3–5, Theorem 2.2 is proved in Section 6 as a special case of Theorem 6.1. The proof reveals several features of independent interest. In particular, Proposition 2.7 shows that, to establish the Oka-1 property of a complex manifold X, it suffices to show that every holomorphic map $K \to X$ from a neighbourhood of a compact set K with piecewise smooth boundary in an open Riemann surface R can be approximated by holomorphic maps $L = K \cup D \to X$, where $D \subset R$ is any compact disc attached to K along a boundary arc. This resembles the convex approximation property, CAP, characterizing the class of Oka manifolds (see [26, Section 5]). Here, we only need to approximate maps from one-dimensional domains which, however, may be topologically nontrivial.

In Section 7 we study functorial properties of the class of Oka-1 manifolds. Among the main results of the section are Theorem 7.6, which shows that the class of Oka-1 manifolds is invariant under Oka maps inducing a surjective homomorphism of fundamental groups, and Corollary 7.9 which gives the same conclusion for Oka-1 maps; see Definition 7.7. In particular, if $h : X \to Y$ is a holomorphic fibre bundle with a connected Oka fibre, then X is an Oka-1 manifold if and only if Y is an Oka-1 manifold. Recall that Oka maps preserve the class of Oka manifolds; see [23, Theorem 3.15]. We show by examples that Oka-1 manifolds are in general not open or closed in smooth families of manifolds. We also introduce the class LSAP of complex manifolds having the so-called *local spray approximation property*; see Definition 7.11. This condition is local, it holds on every Oka manifold and is invariant under Oka maps, it implies the conclusion of Proposition 2.7 and hence the Oka-1 property, and it seems to have nontrivial functorial properties that remain to be fully explored.

The question of holomorphic dominability of complex surfaces by $\mathbb{C}^2$ was studied in the seminal paper [8] by Buzzard and Lu. In Section 8 we combine their results, and some extensions obtained by inspection of their proofs, with the analytic methods developed in this paper to summarize what we know about which such surfaces are Oka-1. In particular, we show that the class of Oka-1 manifolds contains most compact complex surfaces of Kodaira dimension $-\infty$, all Kummer surfaces and elliptic K3 surfaces, and many elliptic surfaces of Kodaira dimension 1. It turns out that for many classes of compact complex surfaces, the conditions of being Oka, Oka-1, dominable by $\mathbb{C}^2$, and having a Zariski dense entire line $\mathbb{C} \to X$ are equivalent. We expect that this is a low dimensional phenomenon and that the gaps between these conditions increase with the dimension of the manifold.

In Section 9 we discuss the conjecture that every rationally connected projective manifold is an Oka-1 manifold. Evidence that this may hold true is given by the results of Campana and Winkelmann [11], who constructed holomorphic lines $\mathbb{C} \to X$ with given jets through any given sequence of points in a rationally connected manifold. As explained in Sect. 9, an affirmative answer to our Conjecture 9.1 follows from Proposition 2.7 and a theorem of Gournay [33, Theorem 1.1.1]; however, we could not understand the details of his proof.

Finally, in Section 10 we prove that for every connected complex manifold X and open bordered Riemann surface M there exist holomorphic curves $M \to X$ passing through any given sequence of points in X. Essentially the same proof, together with the main result of the paper [22], gives the analogous statement for holomorphic Legendrian immersions from bordered Riemann surfaces to any connected complex contact manifold. The existence of proper holomorphic maps, immersions, and embeddings from bordered Riemann surfaces to certain noncompact complex manifolds was studied in [15, 28].

The results in this paper also hold for holomorphic maps from open 1-dimensional complex spaces, since every such space is normalized by an open Riemann surface.

2. A COMPLEX MANIFOLD DENSELY DOMINABLE BY TUBES OF LINES IS OKA-1

In this section, we introduce a geometric condition implying the Oka-1 property. It is based on the notions of a tree and a tube of complex lines, and of dominability by such tubes. It holds in particular on any complex manifold X which is dominable by $\mathbb{C}^{\dim X}$ at most points.

An affine complex line in $\mathbb{C}^n$ is a set of the form $\Lambda = \{a + tv : t \in \mathbb{C}\} = a + \mathbb{C}v$, where $a \in \mathbb{C}^n$ and $v \in \mathbb{C}^n \setminus \{0\}$ is a *direction vector* of Λ.

Definition 2.1. A *tree of lines* in $\mathbb{C}^n$ is a connected set $\Lambda = \bigcup_{i=1}^k \Lambda_i$ whose *branches* Λ_i are affine complex lines with linearly independent direction vectors $v_i \in \mathbb{C}^n$. The tree Λ is *spanning* if $k = n$; equivalently, if the direction vectors $v_1, \ldots, v_k$ are a basis of $\mathbb{C}^n$. An open connected neighbourhood $T \subset \mathbb{C}^n$ of a tree of lines Λ is a *tube of lines* around Λ if T is a union of affine translates of Λ. Such a tube T is *spanning* if the tree Λ is spanning.

A complex manifold X is said to be *dominable* by a complex manifold Z at a point $x \in X$ if there exists a holomorphic map $F : Z \to X$ and a point $z \in Z$ such that $F(z) = x$ and the differential $dF_z : T_z Z \to T_x X$ is surjective; this requires that $\dim Z \geq \dim X$. The classical notion of dominability refers to the case $Z = \mathbb{C}^n$ with $n = \dim X$.

We denote by $\mathcal{H}^k = \mathcal{H}^k_{X,g}$ the k-dimensional Hausdorff measure on a manifold X with respect to a Riemannian metric g on X; see Federer [17] or Morgan [47] for this notion.

Here is our first main result. Note that the choice of the Riemannian metric g on X is irrelevant in the condition $\mathcal{H}^{2n-1}(E) = 0$ used in the theorem.

Theorem 2.2. *Let X be a complex manifold of dimension n. Assume that there is a closed subset $E \subset X$ with $\mathcal{H}^{2n-1}(E) = 0$ such that at every point $x \in X \setminus E$, X is dominable by a spanning tube of lines $T \subset \mathbb{C}^N$ (possibly depending on x). Then, X is an Oka-1 manifold.*

Remark 2.3. When the condition in the theorem holds, we say that X is *densely dominable* by (spanning) tubes of lines. If this holds with $E = \varnothing$, we say that X is *strongly dominable* by tubes of lines. Note that $\mathbb{C}^n$ itself is a spanning tube of lines.

We wish to point out that dense (and even strong) dominability by tubes of lines, or by Euclidean spaces, is a considerably weaker condition than any of the sufficient conditions used in the theory of Oka manifolds. It is the first known condition implying an Oka-type property of a complex manifold X which is local in the Hausdorff topology, in the sense that dominability is required at a single point of X at a time. For domains in $\mathbb{C}^n$, a similar condition was considered in [14, Theorem 1.12], where the main focus was on the study of domains in $\mathbb{R}^n$ satisfying a similar property with respect to minimal surfaces.

Theorem 2.2 is proved in Section 6 in a more general form; see Theorem 6.1. Sections 3–5 contain the preparatory technical lemmas. In the remainder of this section, we discuss applications of this theorem and its relationship to extant results in the literature.

The following obvious corollary was the vantage point of our investigations.

Corollary 2.4. *If Ω is a connected domain in $\mathbb{C}^n$ such that every point $z \in \Omega$ is contained in a spanning tube of lines $T_z \subset \Omega$, then Ω is an Oka-1 manifold. In particular, every spanning tube of lines in $\mathbb{C}^n$ is an Oka-1 manifold.*

A domain Ω in a complex manifold X is said to be *Zariski open* if the complement $X \setminus \Omega$ is a proper closed complex subvariety of X. This is the holomorphic analogue of the standard notion in the category of complex algebraic manifolds. Let us record several further observations which follow from Theorem 2.2.

Corollary 2.5. *Let X be a connected complex manifold of dimension n.*

(a) *If X is densely dominable by $\mathbb{C}^n$, then X is an Oka-1 manifold.*
(b) *If X is dominable by $\mathbb{C}^n$ at every point in a Zariski open subset, or if it contains a Zariski open Oka domain, then X is an Oka-1 manifold.*
(c) *If Y is a complex manifold of dimension $n = \dim X$ which is densely dominable by tubes of lines and $h : Y \to \Omega$ is a proper holomorphic map onto a Zariski open subset Ω of X, then X is an Oka-1 manifold.*
(d) *If $h : Y \to X$ is a surjective holomorphic map of compact complex manifolds of the same dimension and Y is densely dominable, then X is an Oka-1 manifold.*
(e) *A connected algebraic manifold algebraically dominable by $\mathbb{C}^n$ is an Oka-1 manifold.*

Proof. Part (a) follows directly from Theorem 2.2. To obtain (b), note that an Oka domain $\Omega \subset X$ is dominable by $\mathbb{C}^n$ with $n = \dim X$ at every point. If Ω is Zariski open in X then $A = X \setminus \Omega$ is a closed complex subvariety of X with $\mathcal{H}^{2n-1}(A) = 0$. Hence, X is densely dominable by $\mathbb{C}^n$, so it is Oka-1 by (a). To prove (c), assume that E is a closed subset of Y with $\mathcal{H}^{2n-1}(E) = 0$. Then, $E' = h(E \cup \mathrm{br}\, h)$ is a closed subset of Ω with $\mathcal{H}^{2n-1}(E') = 0$. (Here, $\mathrm{br}\, h$ denotes the branch locus of a holomorphic map h.) Since $A = X \setminus \Omega$ is a proper complex subvariety of X, $A \cup E'$ is a closed subset of X with $\mathcal{H}^{2n-1}(A \cup E') = 0$. Note that X is dominable at every point of $X \setminus A \cup E'$, so the conclusion follows from (a). Part (d) is an obvious consequence of (c). Finally, to see (e) note that if X is an algebraic manifold and $F : \mathbb{C}^n \to X$ is a dominating algebraic map then $F(\mathbb{C}^n \setminus \mathrm{br}\, F)$ is a Zariski open domain in X. Hence, X is densely dominable by $\mathbb{C}^n$, and thus an Oka-1 manifold by (a). $\square$

Examples show that parts (c) and (d) in the corollary fail in general if $\dim Y > \dim X$. However, given an Oka map $h : Y \to X$ which induces a surjective homomorphism of fundamental groups, X is Oka-1 if and only if Y is Oka-1; see Theorem 7.6. A connected Oka manifold X is strongly dominable by a holomorphic map $F : \mathbb{C}^n \to X$ with $n = \dim X$, in the sense that $F(\mathbb{C}^n \setminus \mathrm{br}\, F) = X$ (see [27]). It is not known whether every complex n-manifold which is strongly dominable by $\mathbb{C}^n$ is an Oka manifold; see [26, Section 7.1].

A comprehensive study of complex surfaces holomorphically dominable by $\mathbb{C}^2$ was made by Buzzard and Lu [8]. Inspection shows that the complex surfaces for which dominability is established in their paper are actually densely dominable by $\mathbb{C}^n$, hence Oka-1. We discuss these applications in Section 8. Among the highlights, we mention that every Kummer surface and every elliptic K3 surface is an Oka manifold; see Proposition 8.4 and Corollary 8.6.

When $\dim X = 1$, i.e., X is a Riemann surface, dominability by tubes of lines is clearly equivalent to dominability by $\mathbb{C}$, which holds if and only if X is one of the surfaces $\mathbb{CP}^1, \mathbb{C}, \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, or a torus. These are precisely the Riemann surfaces which are Oka manifolds; see [26, Corollary 5.6.4]. Summarizing, we have the following observation.

Corollary 2.6. *For a Riemann surface X the following properties are pairwise equivalent.*

- *X is an Oka manifold.*
- *X is an Oka-1 manifold.*
- *X is dominable by $\mathbb{C}$.*
- *X is not Kobayashi hyperbolic.*
- *X is one of the Riemann surfaces $\mathbb{CP}^1, \mathbb{C}, \mathbb{C}^*$, or a torus.*

The proof of Theorem 2.2, given in Section 6, shows that the following ostensibly weaker approximation and interpolation conditions imply Oka-1 properties (see Remark 6.2). We shall say that a map $f : K \to X$ is holomorphic on a compact set K if it is the restriction to K of a holomorphic map on a neighbourhood of K in the ambient manifold.

Proposition 2.7. *Let X be a connected complex manifold.*

(a) *Assume that for any open Riemann surface R and pair of compact sets $K \subset L$ in R with piecewise smooth boundaries such that $D = L \setminus \mathring{K}$ is a disc attached to K along an arc $\alpha \subsetneq bD$, every holomorphic map $f : K \to X$ can be approximated uniformly on K by holomorphic maps $\tilde{f} : L \to X$. Then, X has the Oka-1 property with approximation.*

(b) *If in addition the map $\tilde{f}$ in part (a) can be chosen such that it agrees with f at a given finite set of points in K, then X has the Oka-1 property with approximation and interpolation. If in addition jet interpolation is possible then X is an Oka-1 manifold.*

The conditions in the proposition obviously imply the analogous conditions if $L \setminus \mathring{K}$ is a union of annuli. Indeed, attaching an annulus along a boundary component of K amounts to successively attaching a pair of discs. This is the noncritical case in the proof of Theorem 2.2 (see Case 1), which holds by Lemma 5.1 if X is densely dominable by tubes of lines.

Remark 2.8. If X is a complex n-dimensional manifold and E is a closed subset of X with $\mathcal{H}^{2n-2}(E) = 0$, then a generic holomorphic map $f : M \to X$ from a compact bordered Riemann surface avoids E. Indeed, take a holomorphic submersion $F : M \times \mathbb{B}^N \to X$ as in (5.6), with $F(\cdot\,, 0) = f$, where $\mathbb{B}^N$ denotes the unit ball in $\mathbb{C}^N$. Since $\dim_{\mathbb{R}} M \times \mathbb{B}^N = 2N+2$, the condition $\mathcal{H}^{2n-2}(E) = 0$ implies $\mathcal{H}^{2N}(F^{-1}(E)) = 0$ by Fubini's theorem, and hence the set of parameters $t \in \mathbb{B}^N$ for which the range of the map $f_t = F(\cdot\,, t) : M \to X$ intersects

E has zero $2N$-dimensional measure. Likewise, if $\mathcal{H}^{2n-1}(E) = 0$, the same argument shows that a generic holomorphic map $f : M \to X$ satisfies $f(bM) \cap E = \varnothing$, and in this case f can be chosen to agree with a given holomorphic map $M \to X$ to a given finite order at finitely many given points of $\mathring{M} = M \setminus bM$. By using this argument inductively in the proof of Theorem 2.2 we obtain the following corollary.

Corollary 2.9. *Let X be an Oka-1 manifold of dimension n. If E is a closed subset of X with $\mathcal{H}^{2n-2}(E) = 0$, then $X \setminus E$ is an Oka-1 manifold. This holds in particular if E is a closed complex subvariety of codimension at least two in X.*

The hypothesis $\mathcal{H}^{2n-2}(E) = 0$ is optimal. Indeed, the corollary fails in general if E is a complex hypersurface. For example, the complement in $\mathbb{CP}^n$ of $2n+1$ hyperplanes in general position is Kobayashi hyperbolic by Green's theorem [35], and so is the complement of a very general hypersurface of sufficiently high degree; see Brotbek [7]. There is no analogue of Corollary 2.9 for Oka manifolds where even the question of removability of a point is an open problem, and closed discrete sets in $\mathbb{C}^n$ are not removable in general.

The jet transversality theorem shows that a generic holomorphic map $M \to X$ from a compact bordered Riemann surface to an arbitrary complex manifold X is an immersion if $\dim X > 1$ and an injective immersion if $\dim X > 2$; see [26, Section 8.8]. Using this fact inductively in the proof of Theorem 2.2 gives the following corollary.

Corollary 2.10. *If X is an Oka-1 manifold of dimension $n > 1$ then holomorphic maps $F : R \to X$ in Definition 1.1 can be chosen to be immersions (injective immersions if $n > 2$) if the interpolation conditions allow it. If a complex manifold X of dimension > 1 has the Oka-1 property with approximation, then every open Riemann surface R admits a holomorphic immersion $R \to X$ (injective immersion if $\dim X > 2$) with everywhere dense image.*

Corollary 2.10 clearly fails if the manifold X is Brody hyperbolic. On the other hand, it was shown by Forstnerič and Winkelmann [24] that every connected complex manifold X admits a holomorphic disc $\mathbb{D} \to X$ hitting any given sequence in X, so there are holomorphic discs in X with everywhere dense images. (See also Kollár [41].) In Section 10 we generalize this to maps from any bordered Riemann surface (see Theorem 10.1), and also from some other open Riemann surfaces with more complicated topology (see Corollary 10.3).

Further examples of Oka-1 manifolds are discussed in Section 8 where we focus on complex surfaces. Note however that a compact complex manifold may be dominable by tubes of lines but not contain any rational curves. Examples include tori $\mathbb{C}^n/\Gamma$, where Γ is a discrete group of translations on $\mathbb{C}^n$. These are complex homogeneous manifolds and hence Oka manifolds (see [26, Proposition 5.6.1] due to Grauert [34]). By Corollary 2.4 and Proposition 7.4, a torus contains many domains which are dominable by tubes of lines. Another such example are Hopf manifolds. Every Hopf manifold is an unramified quotient of $\mathbb{C}^n \setminus \{0\}$ $(n > 1)$ by a cyclic group, so it is an Oka manifold [26, Corollary 5.6.11]. Like tori, Hopf surfaces contain many Oka-1 domains.

Problem 2.11. Let $T \subset \mathbb{C}^n$ be a spanning tube of lines for some $n \geq 2$ (see Definition 2.1).

(a) Does there exist a dominating holomorphic map $\mathbb{C}^n \to T$?
(b) Is T an Oka manifold?

(c) Is there a compact Oka-1 manifold which is not an Oka manifold?

We expect that the answers to questions (a) and (b) are negative, while the answer to (c) is positive in dimension > 1. Natural candidates may be the K3 surfaces.

Remark 2.12. Consider $\mathbb{CP}^1$ with the coordinate $z \in \mathbb{C} \cup \{\infty\}$. We claim that the extended (spanning) tube of rational curves

$$\widetilde{T} = \big\{(z, w) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : |z| < 1 \text{ or } |w| < 1\big\}$$

is an Oka surface, and hence by [26, Theorem 5.5.1 (e)] there is a surjective holomorphic map $\mathbb{C}^2 \to \widetilde{T}$. Indeed, the complement of the tree of rational curves $\Lambda = \{z = 0\} \cup \{w = 0\}$ in $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ equals $\mathbb{C}^2$ with the complex coordinates $(1/z, 1/w)$, and $K = X \setminus \widetilde{T}$ is the closed bidisc $\{|1/z| \le 1,\ |1/w| \le 1\}$, which is polynomially convex in $X \setminus \Lambda \cong \mathbb{C}^2$. Hence,

$$\widetilde{T} \setminus \Lambda = (X \setminus K) \setminus \Lambda = (X \setminus \Lambda) \setminus K = \mathbb{C}^2 \setminus K$$

is an Oka surface by Kusakabe's theorem [43, Theorem 1.2 and Corollary 1.3] (see also [32]). The same argument applies to any translation of the tree Λ within the tube $\widetilde{T}$. Clearly, there are translates $\Lambda_2, \Lambda_3 \subset \widetilde{T}$ of $\Lambda = \Lambda_1$ with $\bigcap_{i=1}^3 \Lambda_i = \varnothing$. Hence, $\widetilde{T} = \bigcup_{i=1}^3 \widetilde{T} \setminus \Lambda_i$ is a union of Zariski open Oka domains $\widetilde{T} \setminus \Lambda_i$, so it is an Oka manifold by Kusakabe's localization theorem [44, Theorem 1.4]. This argument clearly fails in dimensions $n > 2$.

Let us say a few words about the proof of Theorem 2.2.

In Section 3 we collect some basic definitions and observations concerning trees and tubes of complex lines in affine spaces $\mathbb{C}^n$.

In Section 4 we obtain the first main lemma used in the proof (see Lemma 4.1), which pertains to the situation described in Proposition 2.7. More precisely, let K be a compact domain with piecewise smooth boundary in an open Riemann surface R, and let $D \subset R$ be a compact disc attached to K along a boundary arc $\alpha = K \cap D = bK \cap bD$ such that the set $L = K \cup D$ has piecewise smooth boundary. Given a spanning tube of lines $T \subset \mathbb{C}^n$ and a holomorphic map $f : K \to \mathbb{C}^n$ such that $f(\alpha) \subset T$, we show that f can be approximated uniformly on K by holomorphic maps $\tilde{f} : L \to \mathbb{C}^n$ such that $\tilde{f}(D) \subset T$. The analogous result holds for local holomorphic sprays of maps $K \to \mathbb{C}^n$ sending α to T; see Remark 4.2.

In Section 5 we use Lemma 4.1 and methods from Oka theory to prove Lemma 5.1, which provides the noncritical case in the proof of Theorem 2.2. With $K \subset R$ as above, this concerns the approximation of a holomorphic map $f : K \to X$ by holomorphic maps $\tilde{f} : L \to X$, where $L \Supset K$ is a compact set with piecewise smooth boundary such that $L \setminus \mathring{K}$ is the union of finitely many pairwise disjoint compact annuli. In addition, $\tilde{f}$ can be chosen to agree with f to a given finite order at a given finite set of points in K.

Using Lemma 5.1, we obtain Theorem 2.2 in Section 6 as a special case of Theorem 6.1.

In the paper, we shall frequently use the Mergelyan approximation theorem on compact sets in Riemann surfaces with interpolation at finitely many points, both for functions and for manifold-valued maps. We recall the relevant notions and terminology. Given a compact set K in a complex manifold R, a map $f : K \to X$ to another complex manifold is said to be of class $\mathscr{A}(K)$ if f is continuous on K and holomorphic on the interior $\mathring{K}$ of K. A map is said to be holomorphic on K if it is holomorphic on a neighbourhood of K; this class is denoted by $\mathscr{O}(K)$. A compact set K in R is said to have the *Mergelyan property* if every function $f \in \mathscr{A}(K)$ is a uniform limit on K of functions in $\mathscr{O}(K)$. If R is an open Riemann surface then this holds in particular if K is Runge in R (see [19, Corollary 7]). The following result, which we state for reader's convenience, is [4, Theorem 1.13.1]; see also [19, Sec. 7.2].

Theorem 2.13. *Assume that K is a compact set with the Mergelyan property in a Riemann surface R. Given a complex manifold X, a map $f : K \to X$ of class $\mathscr{A}(K)$, and finitely many points $a_i \in K$ $(i = 1, \ldots, m)$, we can approximate f as closely as desired uniformly on K by holomorphic maps $\tilde{f} : U \to X$ on open neighbourhoods $U \subset R$ of K such that $\tilde{f}(a_i) = f(a_i)$ for every $i = 1, \ldots, m$. In addition, $\tilde{f}$ can be chosen to agree with f to any given finite order in the points $a_i \in \mathring{K} = K \setminus bK$.*

3. TREES AND TUBES OF LINES

The notions of a tree and a tube of (affine complex) lines in $\mathbb{C}^n$ was introduced in Definition 2.1. In this section we collect some observations which will be used in the sequel.

We can enumerate the branches Λ_i of a tree $\Lambda = \bigcup_{i=1}^k \Lambda_i$ so that for each $i \geq 2$, the branch Λ_i intersects the subtree $\Lambda^{i-1} = \bigcup_{j=1}^{i-1} \Lambda_j$ in a single point. The intersections are transverse (normal crossings) due to linear independence of direction vectors of the branches. Several branches may intersect at the same point; we call Λ a *regular tree* if this does not happen. Note that a tree with k branches is regular if and only of it has exactly $k-1$ singular points (simple nodes). Our definition of a tree of lines is similar to that of a *tree of rational curves* in a complex manifold X (see [13, Definition 4.23] or [40]). However, an addition which is important in the proof of Theorem 2.2 is that the direction vectors of the branches of a tree of lines are linearly independent, and they are a basis of $\mathbb{C}^n$ if and only if the tree is spanning.

By a linear change of coordinates on $\mathbb{C}^n$ we can map the direction vectors $v_1, \ldots, v_k$ of a tree Λ to the first k standard unit vectors $e_1, \ldots, e_k$, where $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 on the i-th spot. Hence, it will suffice to consider trees of lines in coordinate directions.

Example 3.1. Let $z = (z_1, \ldots, z_n)$ be complex coordinates on $\mathbb{C}^n$. For each $i = 1, \ldots, n$ let

$$\Lambda_i = \{z \in \mathbb{C}^n : z_j = 0 \text{ for all } j \in \{1, \ldots, n\} \setminus \{i\}\} = \mathbb{C}e_i \tag{3.1}$$

be the coordinate axis in the z_i direction.

(a) The union $\Lambda = \bigcup_{i=1}^k \Lambda_i$ of coordinate axes is a tree. It is a spanning tree if and only if $k = n$, and is a regular tree if and only if $n = 2$.

(b) Let $a_1, \ldots, a_k \in \mathbb{C}$ for $1 \leq k < n$ be complex numbers. The set

$$\Lambda = \Lambda_n \cup \bigcup_{i=1}^k \big(a_i e_n + \Lambda_i\big) \tag{3.2}$$

is called a *simple tree* or a *comb*, and Λ_n is the *stem* (or the *handle*) of Λ. It is spanning if and only if $k = n-1$, and is regular if and only if the numbers a_i are pairwise distinct. Every tree of length ≤ 3 is a simple tree in suitable affine coordinates, but most trees of length > 3 are not simple.

Lemma 3.2. *For every tree of lines in $\mathbb{C}^n$ there is an affine change of coordinates which maps it to a tree of the form*

$$\Lambda = \mathbb{C}e_n \cup \bigcup_{j=1}^{l} \Lambda^j, \tag{3.3}$$

where each Λ^j is a tree in coordinate directions such that $\Lambda^j \cap \mathbb{C}e_n = a_j e_n$ for some numbers $a_1, \ldots, a_l \in \mathbb{C}$ (not necessarily distinct). A tree (3.3) *is said to be in* normal form.

Proof. Pick any branch of a given tree and map it to $\mathbb{C}e_n$ by an affine transformation which maps the direction vectors of the branches to coordinate vectors. Let $a_1, \ldots, a_l \in \mathbb{C}$ be such that $a_i e_n$ are the singular points of the new tree Λ. Then, Λ satisfies the lemma. $\square$

Note that an affine linear transformation of $\mathbb{C}^n$ maps a tree of lines Λ to another tree of lines Λ', and it maps a tube T around Λ to a tube T' around Λ'.

It will be convenient to use *polydisc tubes*. Let $\Delta^k \subset \mathbb{C}^k$ denote the unit polydisc. The polydisc tube of radius $r > 0$ around the coordinate axis $\Lambda_n = \mathbb{C}e_n$ is defined by

$$\mathscr{T}_r(\Lambda_n) = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : z' \in r\Delta^{n-1}\}.$$

For any affine complex line $\Lambda \subset \mathbb{C}^n$ there is an affine unitary change of coordinates $U : \mathbb{C}^n \to \mathbb{C}^n$ mapping Λ onto Λ_n, and we take $\mathscr{T}_r(\Lambda) = U^{-1}(\mathscr{T}_r(\Lambda_n))$. If Λ is a tree of lines in the normal form (3.3), then the polydisc tube $\mathscr{T}_r(\Lambda)$ is defined to be the union of polydisc tubes of the same radius r around its branches.

Remark 3.3. One can consider trees of lines in $\mathbb{C}^n$ having more than n branches. However, examples show that a spanning tree with more than n branches need not contain a spanning tree with n branches, such as those considered above. Our proof of Theorem 2.2 does not apply if we assume dominability by spanning trees with more than n branches.

4. Extending a holomorphic map across a bump taking values in a tube

The following lemma will be used in the proof of Theorem 2.2.

Lemma 4.1. *Assume that K is a compact domain with piecewise smooth boundary in an open Riemann surface R, and D is a compact topological disc with piecewise smooth boundary in R such that $\alpha = D \cap K = bK \cap bD$ is an arc and the compact set $L = K \cup D$ has piecewise smooth boundary. Let $f = (f_1, \ldots, f_n) : K \to \mathbb{C}^n$ be a map of class $\mathscr{A}(K)$ and $T \subset \mathbb{C}^n$ be a spanning tube of lines such that $f(\alpha) \subset T$. Then we can approximate f as closely as desired uniformly on K and interpolate it to any given finite order at a given finite set of points in $\mathring{K}$ by holomorphic maps $\tilde{f} : K \cup D \to \mathbb{C}^n$ such that $\tilde{f}(D) \subset T$.*

Proof. Recall that $\Delta^n \subset \mathbb{C}^n$ denotes the unit polydisc centred at the origin. We shall first prove the lemma under the following additional assumptions on f and T:

(a) $f(\alpha) \subset r\Delta^n$ for some $r > 0$, and
(b) $T \subset \mathbb{C}^n$ is the polydisc tube of radius r around a tree of lines Λ in the normal form (3.3). (Recall that the polydisc tube $\mathscr{T}_r(\Lambda)$ is the union of polydisc tubes around its branches.)

These conditions on f and T imply that $f(\alpha) \subset r\Delta^n \subset T$. Indeed, $r\Delta^n$ is contained in the tube of radius r around the stem $\Lambda_n = \mathbb{C}e_n$ of the tree T.

By Mergelyan theorem we may assume that f is holomorphic on a neighbourhood of K in R. Whenever invoking Runge or Mergelyan theorem, we shall also interpolate the given map in the given finite set of points in $\mathring{K}$ without mentioning it again (see Theorem 2.13).

For simplicity of exposition, we first consider the case when Λ is a simple tree (a comb) of the form (3.2). We begin by explaining how to choose the first $n-1$ components of the new map $\tilde{f} = (\tilde{f}', \tilde{f}_n) = (\tilde{f}_1, \ldots, \tilde{f}_n)$; the last component $\tilde{f}_n$ will be determined in the final step. The general case when Λ is of the form (3.3) will be obtained by induction on n.

Let $\beta = bD \setminus \alpha$ be the complementary arc to α in bD. Pick a closed topological disc $\Delta_0 \subset D$ such that $\Delta_0 \cap \alpha = \varnothing$ and $\Delta_0 \cap bD$ is an arc contained in β. We extend the first component f_1 from K to $K \cup \Delta_0$ by setting $f_1 = 0$ on Δ_0. By Runge theorem we can approximate f_1 on $K \cup \Delta_0$ by a holomorphic function $\tilde{f}_1$ on $L = K \cup D$ such that $|\tilde{f}_1| < r$ holds on $\alpha \cup \Delta_0$; see (a). Hence, there is a closed disc $\Delta_1 \subset D$ such that

(i$_1$) $\Delta_1 \cap (\alpha \cup \Delta_0) = \varnothing$,
(ii$_1$) $\overline{D \setminus \Delta_1}$ is the union of two disjoint discs and $b\Delta_1 \cap \beta$ consists of two disjoint arcs, and
(iii$_1$) $|\tilde{f}_1| < r$ holds on $\overline{D \setminus \Delta_1}$.

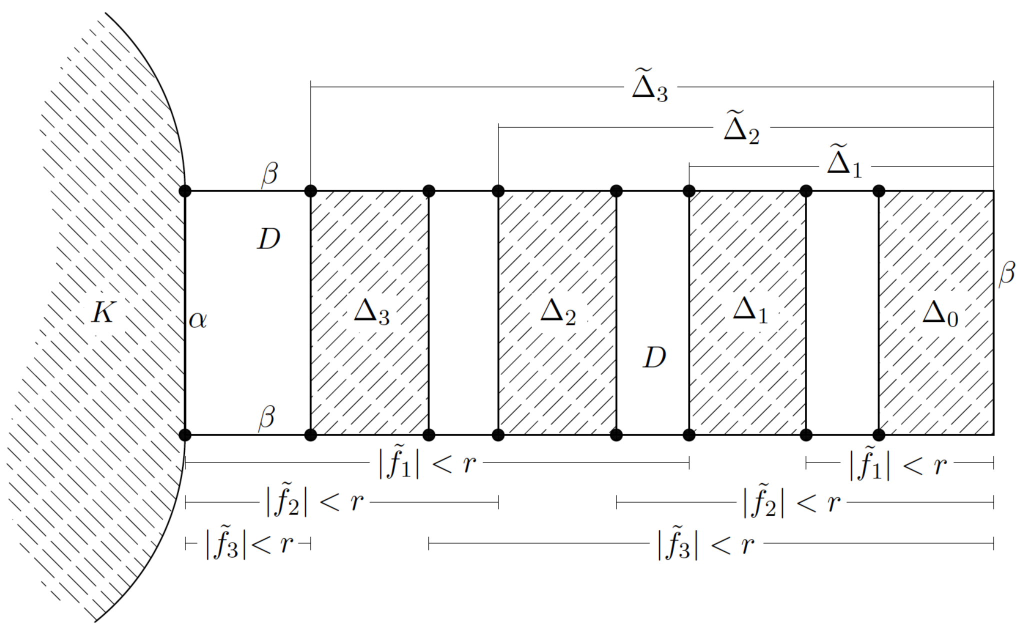


Figure 4.1. Proof of Lemma 4.1 – the special case

See Figure 4.1. Condition (iii$_1$) holds if the disc Δ_1 satisfying conditions (i$_1$) and (ii$_1$) is chosen large enough. Indeed, by increasing Δ_1 the set $\overline{D \setminus \Delta_1}$ shrinks to $\alpha \cup \Delta_0$, and we have that $|\tilde{f}_1| < r$ on $\alpha \cup \Delta_0$. Note that $K \cap \Delta_1 = \varnothing$, and hence the set $K \cup \Delta_1$ is Runge in an open neighbourhood of $L = K \cup D$.

If $n = 2$, we proceed to the final argument explaining how to choose the last component $\tilde{f}_n$. Assume now that $n > 2$. Let $\widetilde{\Delta}_1$ denote the union of Δ_1 and the component of $D \setminus \Delta_1$ containing Δ_0, so $\widetilde{\Delta}_1 \subset D$ is a closed disc disjoint from α (see Fig. 4.1). We extend the second component f_2 of f to the set $K \cup \widetilde{\Delta}_1$ by taking $f_2 = 0$ on $\widetilde{\Delta}_1$ and then apply Runge theorem on $K\cup\widetilde\Delta_1$ to find a holomorphic function $\tilde f_2$ on L such that $|\tilde f_2|<r$ holds on $\alpha\cup\widetilde\Delta_1$; see (a). Hence, there is a disc $\Delta_2\subset D$ such that

(i$_2$) $\Delta_2\cap(\alpha\cup\widetilde\Delta_1)=\varnothing$,

(ii$_2$) $\overline{D\setminus\Delta_2}$ is the union of two disjoint discs and $b\Delta_2\cap\beta$ consists of two disjoint arcs, and

(iii$_2$) $|\tilde f_2|<r$ holds on $\overline{D\setminus\Delta_2}$.

As in the first step, condition (iii$_2$) holds if the disc Δ_2 satisfying (i$_2$) and (ii$_2$) is chosen big enough within $D\setminus(\alpha\cup\widetilde\Delta_1)$. Let $\widetilde\Delta_2$ denote the union of Δ_2 and the component of $D\setminus\Delta_2$ containing $\widetilde\Delta_1$, so $\widetilde\Delta_2\subset D$ is a closed disc disjoint from α. See Figure 4.1.

If $n=3$, we proceed to the last step. If on the other hand $n>3$, we repeat the same argument with the component f_3 to obtain a holomorphic function $\tilde f_3$ on L such that $|\tilde f_3|<r$ holds on $\alpha\cup\widetilde\Delta_2$. We then pick a disc $\Delta_3\subset D$ which is disjoint from $\alpha\cup\widetilde\Delta_2$ such that $D\setminus\Delta_3$ is the union of two disjoint discs, one of them containing $\widetilde\Delta_2$, and $|\tilde f_3|<r$ holds on $D\setminus\Delta_3$. Let $\widetilde\Delta_3$ denote the union of Δ_3 and the component of $D\setminus\Delta_3$ containing Δ_2. See Figure 4.1.

Clearly we can continue inductively in order to approximate the first $n-1$ components $f_1,\ldots,f_{n-1}$ of f by holomorphic functions $\tilde f_1,\ldots,\tilde f_{n-1}$ on L such that

$$|\tilde f_i|<r\ \text{ holds on }\ \overline{D\setminus\Delta_i}\ \text{ for } i=1,\ldots,n-1. \tag{4.1}$$

We now extend the last component f_n to the Runge compact set $K'=K\cup\bigcup_{i=0}^{n-1}\Delta_i$ by setting $f_n=a_i$ on Δ_i for $i=0,1,\ldots,n-1$, where $a_0=0$ and the numbers $a_i\in\mathbb{C}$ for $i=1,\ldots n-1$ are as in (3.2) with $k=n-1$. Using Runge theorem we approximate f_n on K' by a holomorphic function $\tilde f_n$ on $L=K\cup D$ such that

$$|\tilde f_n-a_i|<r\ \text{ holds on }\ \Delta_i\ \text{ for } i=0,1,\ldots,n-1. \tag{4.2}$$

Conditions (4.1) and (4.2) imply that the map $\tilde f=(\tilde f',\tilde f_n)=(\tilde f_1,\ldots,\tilde f_n)$ sends the disc D into the tube T. Indeed, on the disc Δ_i for $i=1,\ldots,n-1$ all components of $\tilde f'$ except perhaps $\tilde f_i$ are smaller than r in absolute value while $|\tilde f_n-a_i|<r$, so $\tilde f(\Delta_i)$ is contained in the polydisc tube of radius r around the affine line $a_ne_n+\Lambda_i\subset\Lambda$. On the other hand, on $D\setminus\bigcup_{i=1}^{n-1}\Delta_i$ all components of $\tilde f'$ are smaller than r, so its image by $\tilde f$ is contained in the polydisc tube of radius r around the stem $\Lambda_n=\mathbb{C}e_n\subset\Lambda$. Note also that

$$|\tilde f_j|<r \text{ holds on } \alpha\cup\Delta_0 \text{ for all } j=1,\ldots,n. \tag{4.3}$$

This proves the lemma (under the assumptions (a) and (b) on f and T) if Λ is a simple tree. Keeping the assumptions (a) and (b) in place, we now let Λ be any tree of the form (3.3) with subtrees Λ^j for $j=1,\ldots,l$. We proceed by induction on n. The result clearly holds for $n=2$ since in this case every tree is a comb. Assume inductively that $n>2$ and the result (including the condition (4.3)) holds in dimensions $<n$. Let Σ^j denote the coordinate subspace of $\mathbb{C}^{n-1}\times\{0\}$ spanned by the direction vectors of the affine lines in the tree Λ^j. By renumbering the coordinates we may assume that Σ^1 is spanned by the coordinate directions $1,\ldots,n_1$ for some $n_1<n$. Pick a disc $\Delta_0\subset D$ as above. By the inductive hypothesis, we can approximate the component functions f_i for $i=1,\ldots,n_1$ by functions $\tilde f_i\in\mathscr{O}(K\cup D)$ satisfying $|\tilde f_i|<r$ on $\alpha\cup\Delta_0$ such that the map $\tilde f^1=(\tilde f_1,\ldots,\tilde f_{n_1}):K\cup D\to\mathbb{C}^{n_1}$ send D into the polydisc tube $T^1\subset\mathbb{C}^{n_1}$ of radius r around the tree Λ^1. Choose a disc $\Delta_1\subset D$ satisfying conditions (i$_1$)–(ii$_1$) above, and with condition (iii$_1$) replaced by

(iii$_1$) $|\tilde f_i| < r$ holds on $\overline{D\setminus\Delta_1}$ for $i=1,\ldots,n_1$.

If $l=1$ (so $n_1=n-1$), we proceed to the last step. Otherwise, we repeat the same procedure for the second subtree Λ^2. We may assume that the subspace $\Sigma^2\subset\mathbb{C}^n$ associated to Λ^2 consists of coordinate directions $n_1+1,\ldots,n_2$ for some $n_2<n$. Let $\widetilde\Delta_1$ denote the union of Δ_1 and the component of $D\setminus\Delta_1$ containing Δ_0. We extend the components f_i of f for $i=n_1+1,\ldots,n_2$ to the set $K\cup\widetilde\Delta_1$ by taking $f_i=0$ on $\widetilde\Delta_1$ and apply the inductive hypothesis to find holomorphic functions $\tilde f_i$ on L such that $\tilde f^2=(\tilde f_{n_1+1},\ldots,\tilde f_{n_2}):L\to\mathbb{C}^{n_2}$ maps D into the polydisc tube $T^2\subset\mathbb{C}^{n_2}$ of radius r around the tree Λ^2, and $|\tilde f_i|<r$ holds on $\alpha\cup\widetilde\Delta_1$ for all $i=n_1+1,\ldots,n_2$. Hence, there is a disc $\Delta_2\subset D$ such that conditions (i$_2$)–(ii$_2$) hold, and (iii$_2$) is replaced by

(iii$_2$) $|\tilde f_i|<r$ holds on $\overline{D\setminus\Delta_2}$ for $i=n_1+1,\ldots,n_2$.

Clearly this process continues inductively. In the last step, we choose $\tilde f_n$ as in (4.2) with $n-1$ replaced by l. We see as before that the holomorphic map $\tilde f=(\tilde f^1,\ldots,\tilde f^l,\tilde f_n):L\to\mathbb{C}^n$ sends D into T. This closes the induction step and completes the proof of the lemma under the additional assumptions (a) and (b) made at the beginning of the proof.

It remains to prove the general case of the lemma with the only assumption that $f(\alpha)\subset T$, where T is a tube around an arbitrary spanning tree of lines Λ. By Lemma 3.2, for every point $p\in\alpha$ there are an open neighbourhood $\alpha_p\subset bK$ of p, a tube $T_p\subset T$ containing a translate of Λ through $f(p)$ such that $f(\alpha_p)\subset T_p$, and an affine isomorphism $U_p:\mathbb{C}^n\to\mathbb{C}^n$ such that $U_p(f(p))=0$, $U_p(T_p)$ is a tube of radius $r_p>0$ around a spanning tree of the form (3.3), and $U_p(f(\alpha_p))\subset r_p\Delta^n$. In other words, the assumptions (a) and (b) hold for the arc α_p, the map $U_p\circ f$, the tube $U_p(T_p)$, and the number r_p. This allows us to subdivide the arc α into a finite union $\alpha=\bigcup_{j=1}^s\alpha_j$ of closed subarcs lying back-to-back such that the above conditions hold on each α_j for an affine linear change of coordinates U_j on $\mathbb{C}^n$, tube $T_j\subset U_j(T)$, and number $r_j>0$. Let p_{j-1} and p_j denote the endpoints of α_j such that $p_j=\alpha_j\cap\alpha_{j+1}$ for $j=1,\ldots,s-1$. Choose an embedded arc $\gamma_j\subset D$ connecting the point $p_j=\alpha_j\cap\alpha_{j+1}$ to a point $q_j\in\beta=bD\setminus\alpha$ so that these arcs are pairwise disjoint, and hence they subdivide D into the union $D=\bigcup_{j=1}^s D_j$ of discs satisfying $D_j\cap D_{j+1}=\gamma_j$ for $j=1,\ldots,s-1$ and $D_j\cap D_k=\varnothing$ if $|j-k|>1$. For notational reasons we also set $\gamma_0=p_0$ and $\gamma_s=p_s$. (See Figure 4.2.) We extend f to the arc γ_j as the constant map $f(p)=f(p_j)$

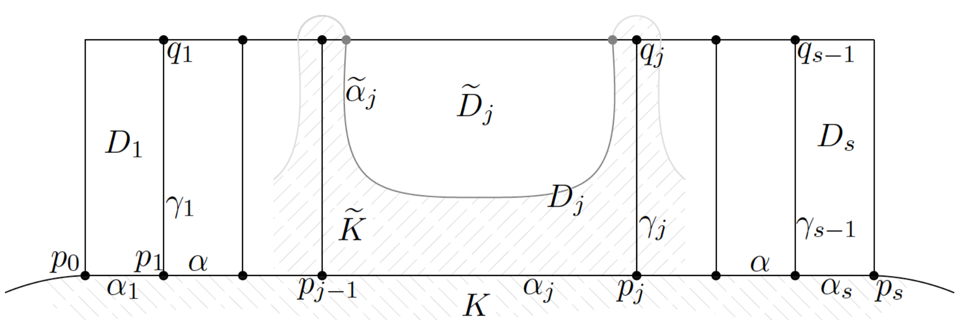


Figure 4.2. Proof of Lemma 4.1 – the general case

for each $p\in\gamma_j$. By Mergelyan theorem we can approximate f as closely as desired on the compact set $S=K\cup\bigcup_{j=1}^{s-1}\gamma_j$ by a holomorphic map defined on an open neighbourhood V

of S. Choose a small compact neighbourhood $\widetilde{K} \subset V$ of S with smooth boundary such that $\overline{D \setminus \widetilde{K}} = \bigcup_{j=1}^{s} \widetilde{D}_j$ is the union of pairwise disjoint closed discs $\widetilde{D}_j \subset D_j$, and $\widetilde{\alpha}_j = b\widetilde{D}_j \cap \widetilde{K}$ is an arc close enough to $\gamma_{j-1} \cup \alpha_j \cup \gamma_j$ such that

$$U_j(f(\widetilde{\alpha}_j)) \subset r_j \Delta^n \subset T_j \quad \text{holds for } j = 1, \ldots, s.$$

(See Figure 4.2.) It remains to apply the already established special case of the lemma to successively extend the map (with approximation on $\widetilde{K}$ and interpolation in the given finitely many points in $\mathring{K}$) across each of the discs $\widetilde{D}_1, \ldots, \widetilde{D}_s$. Since $L = K \cup D \subset \widetilde{K} \cup \bigcup_{j=1}^{s} \widetilde{D}_j$, this completes the proof. □

Remark 4.2. In the proof of Lemma 5.1 we shall use a version of Lemma 4.1 for a certain holomorphic map $K \times \overline{\mathbb{B}}^N \to \mathbb{C}^n$ for some $N \in \mathbb{N}$, where $\mathbb{B}^N$ is the unit ball of $\mathbb{C}^N$. We see by inspection that the same proof applies by using the Oka–Weil theorem instead of the Runge theorem. This will be used in the proof of Lemma 5.1.

5. Extending a holomorphic map across an annulus

By using Lemma 4.1 and gluing methods from Oka theory, we now prove the following lemma, which is the main ingredient in the proof of Theorems 2.2 and 6.1. It provides the so-called noncritical case in the construction of holomorphic maps $R \to X$.

Lemma 5.1. *Let X be a complex manifold of dimension n with a complete distance function dist_X. Assume that $\Omega \subset X$ is an open subset and $E \subset \Omega$ is a closed subset with $\mathcal{H}^{2n-1}(E) = 0$ such that X is dominable by a tube of lines at every point $x \in \Omega \setminus E$. Let R be an open Riemann surface and $K \subset L$ be compact Runge sets in R with piecewise smooth boundaries such that $K \subset \mathring{L}$ and K is a strong deformation retract of L. Assume that $f : K \to X$ is a map of class $\mathscr{A}(K)$ such that $f(bK) \subset \Omega$. Given a finite set $A = \{a_1, \ldots, a_m\} \subset \mathring{K}$ and numbers $\epsilon > 0$ and $k \in \mathbb{N}$, there is a holomorphic map $\tilde{f} : L \to X$ satisfying*

- (i) $\sup_{p \in K} \mathrm{dist}_X(\tilde{f}(p), f(p)) < \epsilon$, *and*
- (ii) *$\tilde{f}$ agrees with f to order k at every point of A.*

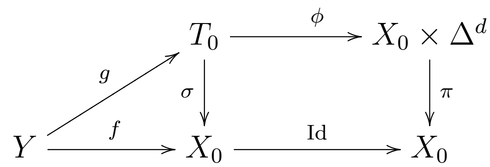
If Ω is dominable by a tube of lines at every point $x \in \Omega \setminus E$ then $\tilde{f}$ can be chosen such that

- (iii) $\tilde{f}(L \setminus \mathring{K}) \subset \Omega$.

Note that the condition that X be dominable at a point $x \in \Omega \subset X$ is weaker than the condition that Ω be dominable at x. In the former case, a map which is dominating at $x \in \Omega$ may have range in X, while in the latter case it must lie in Ω.

Proof. The conditions imply that the compact set $D = L \setminus \mathring{K}$ is a union of finitely many pairwise disjoint annuli. It suffices to consider the case when D is a single annulus since the same construction can be performed independently on each of them.

Before proceeding, we recall the following implication of the implicit function theorem. Given a holomorphic map $\sigma : T \to X$ of complex manifolds which is a submersion at a point $t_0 \in T$, there are open neighbourhoods $t_0 \in T_0 \subset T$, $\sigma(t_0) \in X_0 \subset X$ and a biholomorphic map $\phi : T_0 \to X_0 \times \Delta^d$ with $\dim X + d = \dim T$ such that $\sigma|_{T_0} = \pi \circ \phi$, where $\pi : X_0 \times \Delta^d \to X_0$ is the projection $\pi(x,t) = x$. We shall call such a triple (T_0, σ, X_0) a *submersion chart*. (When $\dim T = \dim X$, this says that σ is locally biholomorphically at a point of maximal rank.) Given a submersion chart (T_0, σ, X_0), every holomorphic map $f : Y \to X_0$ lifts to a holomorphic map $g : Y \to T_0$ such that $\sigma \circ g = f$.



Given $l \in \mathbb{N}$ we write $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z} = \{0, 1, \dots, l-1\}$. The assumptions on $E \subset \Omega \subset X$ and the general position argument in Remark 2.8 imply that, after a small perturbation of f which is fixed to the given order k in the points in $A \subset \mathring{K}$, we have that $f(bK) \subset \Omega \setminus E$. Hence, we can subdivide the closed Jordan curve bK into a finite union of compact subarcs $\{\alpha_i : i \in \mathbb{Z}_l\}$ lying back-to-back and satisfying the following conditions.

(A1) α_i and α_{i+1} have a common endpoint p_{i+1} and are otherwise disjoint for every $i \in \mathbb{Z}_l$.
(A2) $\bigcup_{i \in \mathbb{Z}_l} \alpha_i = bK$.
(A3) For every $i \in \mathbb{Z}_l$ there are a spanning tube of lines $T_i \subset \mathbb{C}^{n_i}$ for some $n_i \geq \dim X$, a holomorphic map $\sigma_i : T_i \to X$, a neighbourhood $U_i \subset \Omega \setminus E$ of $f(\alpha_i)$, and an open subset $\omega_i \subset T_i$ such that $\sigma_i(\omega_i) = U_i$ and the triple $(\omega_i, \sigma_i, U_i)$ is a submersion chart.

Condition (A3) can be achieved if the arcs α_i are chosen sufficiently short.

Let p_i and p_{i+1} denote the endpoints of α_i, ordered so that $p_{i+1} = \alpha_i \cap \alpha_{i+1}$ for each $i \in \mathbb{Z}_l$. Choose an embedded arc $\gamma_i \subset D$ connecting the point p_i to a point $q_i \in bL$ so that these arcs are pairwise disjoint, they intersect $bD = bK \cup bL$ only in the respective endpoints p_i and q_i, and these intersections are transverse. Note that the compact set

$$S = K \cup \bigcup_{i \in \mathbb{Z}_l} \gamma_i \tag{5.1}$$

is admissible for Mergelyan approximation. Recall that $f(\alpha_i) \subset U_i$ by condition (A3). We extend f as a constant map to each arc γ_i having the value $f(p_i)$, and we use Mergelyan theorem (see Theorem 2.13) to approximate the resulting map $f : S \to X$ uniformly on S by a holomorphic map $V \to X$ on a neighbourhood $V \subset R$ of S, which we still denote by f. Assuming that the approximation is close enough, we have that

$$f(\gamma_i \cup \alpha_i \cup \gamma_{i+1}) \subset U_i \ \text{ holds for each } i \in \mathbb{Z}_l. \tag{5.2}$$

Let $\widetilde{S} \subset V$ be a thin compact neighbourhood of S with smooth boundary and set

$$\widetilde{K} = L \cap \widetilde{S} \subset V. \tag{5.3}$$

In light of (5.2) and $U_i \subset \Omega$ (see (A3)), the set $\widetilde{S}$ can be chosen such that

$$f(\widetilde{K} \setminus \mathring{K}) \subset \Omega \setminus E.$$

Furthermore, we can choose $\widetilde{S}$ (and hence $\widetilde{K}$) such that the set

$$\overline{L \setminus \widetilde{K}} = \bigcup_{i \in \mathbb{Z}_l} D_i \tag{5.4}$$

is the union of pairwise disjoint compact discs D_i with piecewise smooth boundaries, and for each $i \in \mathbb{Z}_l$ the arc $\widetilde{\alpha}_i = \overline{bD_i \cap \mathring{L}}$ is so close to the arc $\gamma_i \cup \alpha_i \cup \gamma_{i+1}$ that (5.2) implies

$$f(\widetilde{\alpha}_i) \subset U_i \quad \text{for all } i \in \mathbb{Z}_l. \tag{5.5}$$

The complementary arc $bD_i \setminus \widetilde{\alpha}_i$ lies in bL. See Figure 5.1.

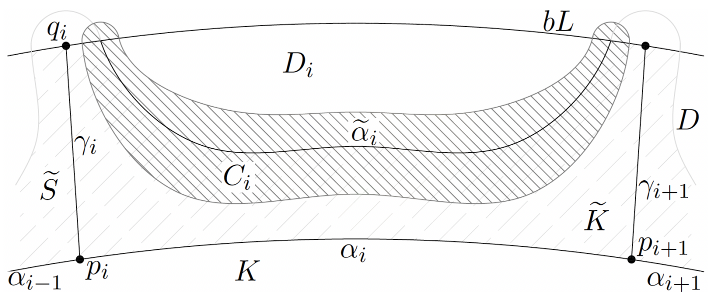


Figure 5.1. Sets is the proof of Lemma 5.1

By standard methods, using flows of holomorphic vector fields and up to shrinking the neighbourhood V around $\widetilde{K}$ (5.3) if necessary, we can find a holomorphic map

$$F : V \times \mathbb{B}^N \to X \ \text{ for some } N \geq \dim X \tag{5.6}$$

satisfying the following conditions (see [26, Lemma 5.10.4]):

(F1) $F(\cdot\,, 0) = f$,
(F2) $F(\cdot\,, t)$ agrees with f to order k at each point of $A \subset \mathring{K}$ for every $t \in \mathbb{B}^N$, and
(F3) the partial derivative $\frac{\partial}{\partial t}\big|_{t=0} F(p, t) : \mathbb{C}^N \to T_{f(p)}X$ is surjective for all $p \in V \setminus A$.

Such F is called a (local) holomorphic spray with the core f which is dominating on $V \setminus A$. (Here, $A = \{a_1, \ldots, a_m\}$ is the finite set given in the statement.)

For each $i \in \mathbb{Z}_l$ we pick a compact smoothly bounded disc neighbourhood $C_i \Subset V$ of the arc $\widetilde{\alpha}_i$ such that $f(C_i) \subset U_i$ (see (5.5)) and $C_i \cap A = \varnothing$. Hence for some $r \in (0, 1)$ we have that $F(C_i \times r\overline{\mathbb{B}}^N) \subset U_i$. Replacing the map $F(\cdot\,, t)$ by $F(\cdot\,, rt)$ for each $t \in \mathbb{B}^N$, we may assume that this holds for $r = 1$. Furthermore, C_i can be chosen such that $D_i \setminus \mathring{C}_i$ is a closed disc attached to C_i as in Lemma 4.1; see Figure 5.1.

Since $(\omega_i, \sigma_i, U_i)$ is a submersion chart for the map $\sigma_i : T_i \to X$ (see condition (A3)), there exists for each $i \in \mathbb{Z}_l$ a holomorphic map $g_i : C_i \times \overline{\mathbb{B}}^N \to \omega_i \subset T_i \subset \mathbb{C}^{n_i}$ such that

$$F = \sigma_i \circ g_i \ \text{ holds on } C_i \times \overline{\mathbb{B}}^N. \tag{5.7}$$

Since $D_i \setminus \mathring{C}_i$ is a closed disc attached to C_i along an arc, Lemma 4.1 and Remark 4.2 show that we can approximate the map g_i as closely as desired uniformly on $C_i \times \overline{\mathbb{B}}^N$ by a holomorphic map $\tilde{g}_i : (C_i \cup D_i) \times \overline{\mathbb{B}}^N \to T_i \subset \mathbb{C}^{n_i}$. Define the map G_i by

$$G_i = \sigma_i \circ \tilde{g}_i : (C_i \cup D_i) \times \overline{\mathbb{B}}^N \to X \ \text{ for } i \in \mathbb{Z}_l. \tag{5.8}$$

It follows from (5.7) that G_i approximates F uniformly on $C_i \times \overline{\mathbb{B}}^N$ for every $i \in \mathbb{Z}_l$.

Recall that the spray F is dominating on $V \setminus A$ and $C_i \cap A = \varnothing$ for every $i \in \mathbb{Z}_l$. Hence, there is a number $r \in (0, 1)$ depending on F such that, if the approximations are close enough, we can glue F and the sprays G_i $(i \in \mathbb{Z}_l)$ into a holomorphic spray $\widetilde{F} : L \times r\mathbb{B}^N \to X$ which approximates F on $\widetilde{K} \times r\mathbb{B}^N$, and it agrees with F (and hence with $f = F(\cdot\,, 0)$) to order k at every point of $A \times r\mathbb{B}^N$ (see (F2)). We refer to [26, Propositions 5.8.1 and 5.9.2] for this gluing technique. The holomorphic map $\tilde{f} = \widetilde{F}(\cdot\,, 0) : L \to X$ then satisfies the conclusion of Lemma 5.1 provided that the approximations made in the proof are close enough.

It remains to justify (iii). As before, we may assume after a small perturbation of f that $f(bK) \subset \Omega \setminus E$. There is a subdivision of bK into arcs α_i as in (A2), open neighbourhoods $U_i \subset \Omega \setminus E$ of their images $f(\alpha_i)$, and holomorphic maps $\sigma_i : T_i \to \Omega$ from spanning tubes of lines $T_i \subset \mathbb{C}^{n_i}$ satisfying condition (A3). We can choose the set $\widetilde{K}$ in (5.3), neighbourhoods $C_i \subset V$ of the arcs $\widetilde{\alpha}_i = bD_i \cap L$ in (5.5), and the spray F in (5.6) such that $F(C_i \times \overline{\mathbb{B}}^N) \subset \Omega$ holds for every $i \in \mathbb{Z}_l$. From (5.8) and $\sigma_i(T_i) \subset \Omega$ it follows that each spray G_i (5.8) has range in Ω. An inspection of the gluing method (see [26, Propositions 5.8.1 and 5.9.2]) shows that the spray $\widetilde{F} : L \times r\mathbb{B}^N \to X$ obtained by gluing F with the G_i's satisfies

$$\widetilde{F}((C_i \cup D_i) \times \bar{r}\mathbb{B}^N) \subset G_i((C_i \cup D_i) \times \overline{\mathbb{B}}^N) \subset \Omega \quad \text{for every } i \in \mathbb{Z}_l.$$

Since $L = \widetilde{K} \cup \bigcup_{i \in \mathbb{Z}_l} D_i$ (see (5.4)) and the holomorphic map $\tilde{f} = \widetilde{F}(\cdot\,, 0) : L \to X$ approximates f on $\widetilde{K}$, it follows that $\tilde{f}(L \setminus \mathring{K}) \subset \Omega$, so condition (iii) in the lemma holds. □

6. Proof of Theorem 2.2

In this section we prove the following result, which includes Theorem 2.2 as a special case with $\Omega = X$ (compare with Definition 1.1 of an Oka-1 manifold).

Theorem 6.1. *Let X be a complex manifold endowed with a complete distance function dist_X, and let $\Omega \subset X$ be a domain which is densely dominable by tubes of lines (see Remark 2.3). Assume that R is an open Riemann surface, K is a compact Runge set in R, $a_i \in R$ is a discrete sequence without repetitions, and $f : R \to X$ is a continuous map which is holomorphic on an open neighbourhood of $K \cup \bigcup_i \{a_i\}$ and satisfies $f(R \setminus \mathring{K}) \subset \Omega$. Given $\epsilon > 0$ and positive integers $k_i \in \mathbb{N}$, there is a holomorphic map $F : R \to X$ which is homotopic to f and satisfies the following conditions:*

(i) $\sup_{p \in K} \mathrm{dist}_X(F(p), f(p)) < \epsilon$,
(ii) $F(R \setminus \mathring{K}) \subset \Omega$, *and*
(iii) *F agrees with f to order k_i in the point a_i for every i.*

Proof. Call $A = \{a_i\}_{i \in \mathbb{N}}$, and let $U \subset R$ be an open neighbourhood of $K \cup A$ such that the given map f is holomorphic on U. Pick a smooth strongly subharmonic Morse exhaustion function $\rho : R \to \mathbb{R}_+$ and an increasing sequence $0 < c_1 < c_2 < \cdots$ diverging to infinity such that the compact sets $K_j = \{\rho \le c_j\}$ for $j = 1, 2, \ldots$ satisfy the following conditions:

(A) $K \subset K_1 \subset U$ and $A \cap K_1 \subset K$.
(B) The number c_j is a regular value of ρ and the smooth level set $\{\rho = c_j\} = bK_j$ is disjoint from A for $j = 1, 2, \ldots$.
(C) For every $j = 1, 2, \ldots$ the set $D_j = \{c_j \le \rho \le c_{j+1}\} = K_{j+1} \setminus \mathring{K}_j$ contains at most one critical point of ρ or at most one point of A, but not both.

The construction of such a function ρ and a sequence c_j is standard, using the fact that Morse critical points are isolated. Condition (A) is achieved by choosing ρ and c_1 such that the set

$K_1 = \{\rho \le c_1\}$ is sufficiently close to K; this is possible since K is Runge in R. Note that each $K_j = \{\rho \le c_j\}$ is a smoothly bounded Runge compact domain in R and the sequence $K_1 \Subset K_2 \Subset \cdots \Subset \bigcup_{j=1}^{\infty} K_j = R$ is a normal exhaustion of R.

Given a compact set $L \subset R$ and a pair of continuous maps $f, g : L \to X$, we write

$$d_L(f,g) = \max_{p \in L} \mathrm{dist}_X(f(p), g(p)).$$

Set $K_0 = K$, $f_0 = f|_K$, and $\epsilon_0 = \epsilon/2$, where K and $\epsilon > 0$ are given in the statement. By the assumption, there is a closed subset $E \subset \Omega$ with $\mathcal{H}^{2n-1}(E) = 0$ such that Ω is dominable by a spanning tube of lines at every point of $\Omega \setminus E$. By Remark 2.8, there is a holomorphic map $f_1 : K_1 \to X$ which is ϵ_0-close to f on K_1, it agrees with f to order k_i at every point $a_i \in A \cap K_1$, and it satisfies

$$f_1(K_1 \setminus \mathring{K}_0) \subset \Omega \quad \text{and} \quad f_1(bK_1) \subset \Omega \setminus E. \tag{6.1}$$

Set $D_0 = K_1 \setminus \mathring{K}_0$. We shall inductively construct a sequence of holomorphic maps $f_j : K_j \to X$ and numbers $\epsilon_j > 0$ satisfying the following conditions for each $j = 1, 2, \ldots$:

(a$_j$) $d_{K_{j-1}}(f_j, f_{j-1}) < \epsilon_{j-1}$.
(b$_j$) $f_j(D_{j-1}) \subset \Omega$ and $f_j(bK_j) \subset \Omega \setminus E$. (Here, $D_{j-1} = K_j \setminus \mathring{K}_{j-1}$ is given by (C).)
(c$_j$) f_j agrees with f to order k_i at every point $a_i \in A \cap K_j$.
(d$_j$) f_j is homotopic to $f|_{K_j} : K_j \to X$ by a homotopy mapping $K_j \setminus \mathring{K}$ to Ω.
(e$_j$) $\epsilon_j < \frac{1}{2}\min\{\epsilon_{j-1}, \mathrm{dist}_X(f_j(D_{j-1}), X \setminus \Omega)\}$. (If $\Omega = X$ then the second number under min is treated as $+\infty$.)

The beginning of the induction is given by the map f_1 found above and a number ϵ_1 satisfying (e$_1$). Assume that for some $j \in \mathbb{N}$ we have found maps f_k and numbers ϵ_k for $k = 1, \ldots, j$, and let us explain how to find the next pair $(f_{j+1}, \epsilon_{j+1})$. We distinguish cases.

Case 1: D_j does not contain any critical point of ρ nor any point of A. In this case, a map $f_{j+1} : K_{j+1} \to X$ satisfying (a$_{j+1}$)–(c$_{j+1}$) is given by Lemma 5.1. Assuming as we may that the approximation in (a$_{j+1}$) is close enough, $f_{j+1}|_{K_j}$ is homotopic to f_j by a homotopy staying close to f_j, hence mapping $K_j \setminus \mathring{K}$ to Ω. Since K_j is a strong deformation retract of K_{j+1} and $f(R \setminus \mathring{K}) \subset \Omega$, we obtain a homotopy from f_{j+1} to f satisfying (d$_{j+1}$). We then choose a number $\epsilon_{j+1} > 0$ satisfying (e$_{j+1}$), thereby completing the induction step in this case.

Case 2: D_j contains a critical point p of ρ. Our assumptions imply that such a point p is unique, it is contained in the interior $\mathring{D}_j = \mathring{K}_{j+1} \setminus K_j$, and $D_j \cap A = \varnothing$. There is a compact smoothly embedded arc $\lambda \subset bK_j \cup \mathring{D}_j = \mathring{K}_{j+1} \setminus \mathring{K}_j$ passing through p and having endpoints in bK_j such that λ intersects bK_j only in these endpoints and the intersections are transverse, and the set $K_j \cup \lambda$ is a strong deformation retract of K_{j+1}. (A discussion of the possible changes of topology depending on the Morse index of ρ at p can be found in [4, Section 1.4].) Condition (d$_j$) implies that we can extend f_j smoothly across the arc λ such that $f_j(\lambda) \subset \Omega$ and $f_j : K_j \cup \lambda \to X$ is homotopic to $f|_{K_j \cup \lambda}$ by a homotopy mapping $(K_j \cup \lambda) \setminus \mathring{K}$ to Ω. We apply Mergelyan theorem (see Theorem 2.13) to approximate f_j on $K_j \cup \lambda$ by a holomorphic map $\tilde{f}_j : V \to X$ on a neighbourhood $V \subset K_{j+1}$ of $K_j \cup \lambda$ such that $\tilde{f}_j(bK_j \cup \lambda) \subset \Omega$ and $\tilde{f}_j$ agrees with f_j to the given order k_i at all points $a_i \in A \cap K_j$. Then, there are a compact smoothly bounded neighbourhood $K' \subset V$ of $K_j \cup \lambda$ such that $\tilde{f}_j(K' \setminus \mathring{K}_j) \subset \Omega$ and $K_{j+1} \setminus K'$ is a finite union of annuli. Applying Lemma 5.1 as in Case 1 to the pair of sets $K' \subset K_{j+1}$ and the map $\tilde{f}_j$ gives a holomorphic map $f_{j+1} : K_{j+1} \to X$ satisfying conditions (a$_{j+1}$)–(d$_{j+1}$). Finally, we pick a number ϵ_{j+1} satisfying condition (e$_{j+1}$).

Case 3: D_j contains a point $a_i \in A$. Our assumptions imply that such a point is unique and D_j does not contain any critical point of ρ. Hence, D_j is a finite union of annuli and K_j is a strong deformation retract of K_{j+1}. Let $U \subset R$ be the open set on which the initial continuous map $f : R \to X$ is holomorphic. Pick a closed disc $\Delta \subset U \cap \mathring{D}_j$ containing the point a_i in its interior. Choose a smooth embedded arc $\lambda \subset bK_j \cup \mathring{D}_j \setminus \mathring{\Delta}$ connecting a point $p \in bK_j$ to a point $q \in b\Delta$ such that λ intersects bK_j and $b\Delta$ only at p and q, respectively, and the intersections are transverse. The set $\widetilde{K}_j = K_j \cup \Delta \cup \lambda$ is then a strong deformation retract of K_{j+1} and an admissible set for Mergelyan approximation. Since $f(R \setminus \mathring{K}) \subset \Omega$, condition (d$_j$) implies that $f_j(bK_j) \subset \Omega$ and $f(a_i) \in \Omega$ lie in the same connected component of Ω. Hence we can extend f_j from K_j to a smooth map $\tilde{f}_j : \widetilde{K}_j \to X$ which equals f on Δ and $\tilde{f}_j$ is homotopic to $f|_{\widetilde{K}_j}$ by a homotopy mapping $\widetilde{K}_j \setminus \mathring{K}$ to Ω. We can now complete the proof as in Case 2, first approximating the extended map $\tilde{f}_j : \widetilde{K}_j \to X$ by a holomorphic map on a neighbourhood of $\widetilde{K}_j$ and then applying Lemma 5.1 to find the next holomorphic map $f_{j+1} : K_{j+1} \to X$ satisfying (a$_{j+1}$) –(d$_{j+1}$). Finally we choose $\epsilon_{j+1} > 0$ satisfying (e$_{j+1}$).

This completes the induction step in all cases. The theorem now follows by verifying that the limit map $F = \lim_{j\to\infty} f_j : R \to X$ exists and satisfies the stated conditions. This is an obvious consequence of conditions (a$_j$)–(e$_j$) whose verification is left to the reader. □

Remark 6.2. It is obvious that the proof of Theorem 6.1 also gives Proposition 2.7. Indeed, Case 1 (the noncritical case) in the proof holds by the Oka-1 property with approximation and interpolation in the finitely many points in the sublevel set $K_j = \{\rho \le c_j\}$, while the proof of Cases 2 and 3 only uses the Mergelyan theorem (see Theorem 2.13).

7. Functorial properties of Oka-1 manifolds

In this section we study the behaviour of the Oka-1 property under certain natural operations in the category of complex manifolds, and its relationship to other flexibility properties of complex manifolds studied in the literature. A survey of these issues for the smaller class of Oka manifolds can be found in [26, Chapter 7] and [23, 30].

It is obvious from the definition that an increasing union of Oka-1 manifolds is an Oka-1 manifold. The proof of the following simple observation is left to the reader.

Proposition 7.1. *The product $Z = X \times Y$ is an Oka-1 manifold if and only if X and Y are Oka-1 manifolds.*

Next, we look at the following problem. Let X and Y be connected complex manifolds. Under which conditions on a surjective holomorphic map $h : X \to Y$ is X an Oka-1 manifold if (and only if) Y is an Oka-1 manifold?

Our first result in this direction concerns unramified holomorphic covering projections.

Proposition 7.2. *Let $h : X \to Y$ be a holomorphic covering projection.*

(a) *If Y is an Oka-1 manifold, then X is an Oka-1 manifold.*
(b) *If X has the Oka-1 property for complex lines $\mathbb{C} \to X$, then so does Y.*

Proof. To prove part (a), assume that Y is an Oka-1 manifold. Let $K \subset L = K \cup D$ be compact sets in an open Riemann surface R as in Proposition 2.7, and let $f : K \to X$ be a holomorphic map. Then, the projection $g = h \circ f : K \to Y$ (see (7.1)) can be approximated uniformly on K by holomorphic maps $\tilde{g} : L \to Y$, with interpolation in given finitely many points $a_1, \ldots, a_m \in K$. We may assume that there is at least one point a_i in each connected component of K. If the approximation is close enough then $\tilde{g}$ is homotopic to g on K, and hence it lifts to a unique holomorphic map $\tilde{f} : L \to X$ which agrees with f at the points $a_1, \ldots, a_m$ and approximates f on K. Hence, X is Oka-1 by Proposition 2.7.

This argument fails in the opposite direction since a map $K \to Y$ need not lift to a map $K \to X$, unless the set K is simply connected. In the latter case, every holomorphic map $K \to Y$ admits a holomorphic lift $K \to X$. To prove (b), assume that X has the Oka-1 property for complex lines $\mathbb{C} \to X$. Let $K \subset \mathbb{C}$ be a smoothly bounded Runge compact set, hence simply connected, and let $g : K \to Y$ be a holomorphic map. Since K is simply connected, g lifts to a holomorphic map $f : K \to X$. By the assumption we can approximate f as closely as desired by a holomorphic map $\tilde{f} : \mathbb{C} \to X$ which agrees with f to a given order k in the points $a_1, \ldots, a_m \in K$. The holomorphic map $\tilde{g} := h \circ \tilde{f} : \mathbb{C} \to Y$ then approximates g on K and agrees with g to order k in the points $a_1, \ldots, a_m$. Applying this argument inductively on an increasing sequence of discs exhausting $\mathbb{C}$ gives (b). ☐

Problem 7.3. If $h : X \to Y$ is a holomorphic covering projection and X is an Oka-1 manifold, is Y an Oka-1 manifold?

On the other hand, since a tube of lines is simply connected, we have the following observation concerning the sufficient condition for Oka-1 manifolds in Theorem 2.2.

Proposition 7.4. *If $h : X \to Y$ is a holomorphic covering space, then X is dominable by tubes of lines if and only Y is dominable by tubes of lines. The same holds for dense and strong dominabilities by tubes of lines.*

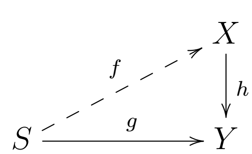
Recall that a complex manifold X is said to be *Liouville* if it carries no nonconstant negative plurisubharmonic functions, and *strongly Liouville* if the universal covering space of X is Liouville. As remarked in the introduction, every Oka-1 manifold is Liouville. Proposition 7.2 gives the following stronger conclusion.

Corollary 7.5. *Every Oka-1 manifold is strongly Liouville.*

Next, we show that the class of Oka-1 manifolds is invariant under Oka maps with connected fibres. We recall this notion, referring to [26, Sect. 7.4] and [23, Sect. 3.6] for a more detailed presentation.

A holomorphic map $h : X \to Y$ of complex manifolds is said to enjoy the *parametric Oka property with approximation and interpolation* (POPAI) if for every holomorphic map $g : S \to Y$ from a Stein manifold S, any continuous lifting $f_0 : S \to X$ is homotopic through liftings of g to a holomorphic lifting $f : S \to X$ as in the following diagram,

(7.1)



with approximation on a compact $\mathscr{O}(S)$-convex subset of S and interpolation on a closed complex subvariety of S on which f_0 is holomorphic. Furthermore, the analogous conditions must hold for families of maps $g_p : S \to Y$ depending continuously on a parameter p in a compact Hausdorff space; see [26, Definitions 7.4.1 and 7.4.7] for the details.

A holomorphic map $h : X \to Y$ is said to be an *Oka map* if it enjoys POPAI and is a Serre fibration (see [46, 25] and [26, Definition 7.4.7]). Assuming that Y is connected, such a map is necessarily a surjective submersion and its fibres are Oka manifolds (see [23, Proposition 3.14]). In particular, the constant map $X \to point$ is an Oka map if and only if X is an Oka manifold. More generally, if $h : X \to Y$ is an Oka map then X is an Oka manifold if and only if Y is an Oka manifold (see [23, Theorem 3.15]).

Theorem 7.6. *Let $h : X \to Y$ be an Oka map between connected complex manifolds.*

(a) *If Y is an Oka-1 manifold then X is an Oka-1 manifold.*
(b) *If X is an Oka-1 manifold and the homomorphism $h_* : \pi_1(X) \to \pi_1(Y)$ of fundamentals groups is surjective, then Y is an Oka-1 manifold.*
(c) *If $h : X \to Y$ is a holomorphic fibre bundle projection with a connected Oka fibre, then X is an Oka-1 manifold if and only if Y is an Oka-1 manifold.*

Proof. Fix a pair of compact sets $K \subset L = K \cup D$ in an open Riemann surface R as in Proposition 2.7, where D is a disc attached to K along a boundary arc $\alpha = bD \cap bK$. Also, let A be a finite subset of K and $k \in \mathbb{N}$.

Proof of (a). Let $f_0 : K \to X$ be a holomorphic map. Since Y is an Oka-1 manifold, the holomorphic map $h \circ f_0 : K \to Y$ can be approximated uniformly on K by holomorphic maps $g : L \to Y$ with interpolation to order k in the points of A. Assuming that the approximation is close enough, we see as in [23, proof of Theorem 3.15] that there is a holomorphic map $f_1 : K \to X$ which is uniformly close to f_0 on K, it agrees with f_0 to order k in the points of A, and it satisfies $h \circ f_1 = g$ on K; i.e., f_1 is a lifting of g, see (7.1). (The construction of f_1 uses a holomorphic family of holomorphic retractions on the fibres of h, provided by [23, Lemma 3.16].) Since $g : L \to Y$ is a holomorphic map, f_1 is a holomorphic lifting of g over K, and $h : X \to Y$ is an Oka map, we can approximate f_1 (and hence f_0) uniformly on K by holomorphic maps $f : L \to X$ satisfying $h \circ f = g$ such that f agrees with f_1 to order k in the points of A. Hence, Proposition 2.7 shows that X is an Oka-1 manifold.

Proof of (b). Note that each connected component of K has the homotopy type of a finite bouquet of circles. The assumption that the homomorphism $h_* : \pi_1(X, x) \to \pi_1(Y, h(x))$ is surjective for some (and hence for all) $x \in X$ therefore implies that every holomorphic map $g : K \to Y$ lifts to a continuous map $f_0 : K \to X$ such that $h \circ f_0 = g$. Since h is an Oka map, we can homotopically deform f_0 to a holomorphic map $f_1 : K \to X$ satisfying $h \circ f_1 = g$. Since X is an Oka-1 manifold, we can approximate f_1 as closely as desired uniformly on K by a holomorphic map $\tilde{f} : L \to X$ which agrees with f_1 to order k in the points of A. The holomorphic map $\tilde{g} = h \circ \tilde{f} : L \to Y$ then approximates g uniformly on K, and it agrees with g to order k in the points of A. Hence, Y is an Oka-1 manifold by Proposition 2.7.

Proof of (c). This follows from (a) and (b) by noting that a holomorphic fibre bundle map $h : X \to Y$ is an Oka map if and only if its fibre is an Oka manifold [26, Corollary 7.4.8 (i)], and if the fibre of h is connected then $h_* : \pi_1(X, x) \to \pi_1(Y, h(x))$ is surjective. ☐

We now introduce the notion of an Oka-1 map by analogy with Oka maps.

Definition 7.7. A holomorphic map $h : X \to Y$ of complex manifolds is an *Oka-1 map* if

(i) h is a Serre fibration, and
(ii) given an open Riemann surface R, a holomorphic map $g : R \to Y$, and a continuous lifting $f_0 : R \to X$ of g which is holomorphic on a neighbourhood of a compact Runge subset $K \subset R$, we can deform f_0 through liftings of g to a holomorphic lifting $f : R \to X$ which approximates f_0 as closely as desired on K and agrees with f_0 to any given finite order in given finitely many points of K (see (7.1)).

Note that the constant map $X \to point$ is an Oka-1 map if and only if X is an Oka-1 manifold. Obviously, every Oka map is also an Oka-1 map, but the converse fails at least for maps with noncompact fibres. We have the following analogue of [23, Proposition 3.14].

Proposition 7.8. *An Oka-1 map $h : X \to Y$ to a connected complex manifold Y is a surjective submersion and its fibres are Oka-1 manifolds.*

Proof. It follows from the definition of an Oka-1 map that for every holomorphic disc $g : \Delta \to Y$ and point $x \in h^{-1}(g(0)) \in X$ there is a holomorphic lifting $f : \Delta \to X$ with $h \circ f = g$ and $f(0) = x$. Hence, every tangent vector $v \in T_{g(0)}Y$ lies in the image of the differential $dh_x : T_xX \to T_{g(0)}Y$, so h is a submersion. Since any pair of points in Y lie in the image of a holomorphic disc $\Delta \to Y$, h is surjective. Hence, every fibre $h^{-1}(y)$ $(y \in Y)$ is a closed complex submanifold of X. Applying the definition of an Oka-1 map to liftings of constant maps $R \to y \in Y$ shows that $h^{-1}(y)$ is an Oka-1 manifold for every $y \in Y$. $\square$

An inspection of the proof of Theorem 7.6 shows that it holds if $h : X \to Y$ is an Oka-1 map, so we obtain the following corollary.

Corollary 7.9. *If $h : X \to Y$ is an Oka-1 map of connected complex manifolds then the conclusion of Theorem 7.6 holds.*

However, we do not know the answer to the following question.

Problem 7.10. Let $h : X \to Y$ be a holomorphic fibre bundle whose fibre is an Oka-1 manifold. Is h an Oka-1 map?

The proof of the analogous affirmative result for Oka manifolds and Oka maps (see [26, Theorem 5.6.5] and [23, Theorem 3.15]) does not apply.

One might wonder why is the conclusion of Theorem 7.6 (b) weaker for Oka-1 manifolds than for Oka manifolds, where the condition on the homotopy groups is unnecessary. The reason is that Oka manifolds are characterized by the convex approximation property (CAP), which refers to holomorphic maps from bounded convex sets in complex Euclidean spaces to the given manifold; see [26, Definition 5.4.3 and Theorem 5.4.4].

We now introduce an approximation condition on a complex manifold which implies that it is Oka-1; see Definition 7.11. This is a version of the *convex approximation property*, CAP (see [26, Definition 5.4.3]), applied to dominating holomorphic sprays on discs with images in a given open subset of X. It encapsulates the condition which is needed to glue a pair of sprays in the proof of Lemma 5.1. The condition is invariant under Oka maps and under dominating holomorphic maps, but we do not know whether it characterizes Oka-1 manifolds.

Let us call a pair of compact topological discs $D \subset D' \subset \mathbb{C}$ a *special pair* if both discs have piecewise smooth boundaries and $D' \setminus \mathring{D}$ is a disc attached to D along a boundary arc. A holomorphic map $F : D \times \mathbb{B}^N \to X$ to a complex manifold X is called a *holomorphic spray* of maps $D \to X$ with the core $f = F(\cdot\,, 0)$. Such a spray is said to be *dominating* if the partial derivative $\frac{\partial}{\partial t}\big|_{t=0} F(z,t) : \mathbb{C}^N \to T_{f(z)}X$ is surjective for all $z \in D$. (Dominating sprays were used in the proof of Lemma 5.1, see (5.6).)

Definition 7.11. A complex manifold X has the *local spray approximation property*, LSAP, at point $x \in X$ if there is an open neighbourhood $V \subset X$ of x satisfying the following condition. Given a special pair of compact discs $D \subset D' \subset \mathbb{C}$ and a dominating holomorphic spray $F : D \times \mathbb{B}^N \to V$, there is a number $r = r(F) \in (0,1)$ such that F can be approximated as closely as desired uniformly on $D \times r\overline{\mathbb{B}}^N$ by holomorphic maps $G : D' \times r\overline{\mathbb{B}}^N \to X$.

A manifold X has LSAP if the above condition holds at every point $x \in X$.

Remark 7.12. (A) The sprays in Definition 7.11 can either be defined over open neighbourhoods of the compact discs D and D', or else they could be continuous over the closed disc and holomorphic over its interior. Our applications of LSAP work in both cases since the gluing of sprays on Cartan pairs works in both cases (see [26, Section 5.9]).

(B) Since every holomorphic spray $F : D \times \mathbb{B}^N \to X$ can be extended to a dominating spray by adding additional parameters, thereby increasing the dimension N and shrinking the ball if necessary (see [26, Lemma 5.10.4] for a more general result), LSAP is equivalent to the condition that every holomorphic spray of discs (not necessarily dominating) can be approximated by a spray on a bigger disc D' as in Definition 7.11.

An inspection of the proof of Lemma 5.1 shows that if X is dominable by a tube of lines (or by $\mathbb{C}^n$) at $x \in X$, then X satisfies LSAP at x. Furthermore, if X has LSAP at every point $x \in \Omega \setminus E \subset X$, using the notation of Lemma 5.1, then the conclusion of the said lemma holds. This observation and the proofs of Theorems 6.1 and 7.6 imply the following results.

Proposition 7.13. (a) *A spanning tube of complex lines in $\mathbb{C}^n$ has LSAP.*

(b) *If a complex manifold X satisfies LSAP at every point $x \in X \setminus E$ in the complement of a closed subset $E \subset X$ with $\mathcal{H}^{2\dim X - 1}(E) = 0$, then X is an Oka-1 manifold. In particular, a complex manifold with LSAP is an Oka-1 manifold.*

(c) *A complex manifold which is densely dominable by manifolds having LSAP is Oka-1. In particular, if $f : X \to Y$ is a surjective holomorphic submersion and X is an LSAP manifold, then Y is an LSAP manifold.*

(d) *A holomorphic fibre bundle $X \to Y$ with an LSAP fibre is an Oka-1 map.*

Proof. Part (a) follows by inspecting the proof of Lemma 4.1. Parts (b) and (c) follow from the proof of Theorem 6.1, with condition LSAP replacing the use of Lemma 5.1 as explained above. Part (d) follows from the proof of Theorem 7.6. The details are similar to the proof that a holomorphic fibre bundle with Oka fibre is an Oka map (see [26, Corollary 7.4.8]), except that we also use localization as in the proof of Theorem 6.1. Note also that a lifting $f : D \to X$ of a holomorphic map $g : D \to Y$ in a holomorphic fibre bundle $h : X \to Y$ corresponds to a holomorphic section of the pullback bundle $g^*X \to D$. □

Another interesting question is whether the set of Oka-1 manifolds is open or closed in a smooth family of complex manifolds. By [30, Corollary 5] (see also [26, Corollary 7.3.3]), compact complex surfaces that are Oka (and hence Oka-1) can degenerate to a surface that is not strongly Liouville, and hence is not an Oka-1 manifold by Corollary 7.5. This shows that the property of being Oka-1 is not closed in families of compact complex manifolds. Concerning families of open manifolds, in [23, Section 10] there is an example of a holomorphic submersion $h : X \to \mathbb{C}$ from a Stein 3-fold X such that h is a trivial holomorphic fibre bundle with fibre $\mathbb{C}^2$ over $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, while the fibre X_0 over $0 \in \mathbb{C}$ is the product $\Delta \times \mathbb{C}$ which is not Liouville, and hence not Oka-1.

There are immediate examples showing that the property of being Oka or Oka-1 is not open in families of noncompact complex manifolds. For example, one can consider the family $h : X = \{(z, w) \in \mathbb{C}^2 : |zw| < 1\} \to \mathbb{C}$ whose fibre over any $z \in \mathbb{C}^*$ is the disc, while the fibre over $z = 0$ is $\mathbb{C}$. On the other hand, we are not aware of an example of an isolated Oka or Oka-1 fibre in a smooth family of compact complex manifolds.

8. Oka-1 manifolds among compact complex surfaces

In this section, we summarize what we know about which compact complex surfaces are Oka-1. Our discussion is based on the Enriques–Kodaira classification of such surfaces (see Barth et al. [5, Table 10, p. 244]), combined with Corollary 2.5 and the results of Buzzard and Lu [8] on holomorphic dominability. The analogous analysis concerning compact Oka surfaces can be found in the paper [30] by Forstnerič and Lárusson and in [26, Section 7.3]. Comparing the two lists, we shall see that for many classes of compact complex surfaces with Kodaira dimension < 2 the properties of being Oka, Oka-1, and dominable by $\mathbb{C}^2$, are pairwise equivalent. The main exceptions are the K3 surfaces and the elliptic fibrations. In the K3 class, Kummer surfaces and the elliptic K3 surfaces are Oka-1 (see Proposition 8.4 and Corollary 8.6), but it is not known whether any or all of them are Oka manifolds.

The most important invariant of a compact complex manifold X is its Kodaira dimension $\kappa_X \in \{-\infty, 0, 1, \ldots, n = \dim X\}$. Let $K_X = \Lambda^n T^* X$ denote the canonical line bundle of X, and for each integer $m \in \mathbb{N}$ let $P_m(X) = h^0(K_X^{\otimes m})$ denote the dimension of the complex vector space of holomorphic sections of the m-th tensor power of K_X. Then, $k = \kappa_X$ is the integer such that $P_m(X)$ grows like m^k as $m \to +\infty$, where $k = -\infty$ means that $K_X^{\otimes m}$ only admits the trivial (zero) section for every m. (See [5, p. 29].) By Kodaira's pioneering work [38] and its extensions (see Carlson and Griffiths [12] and Kobayashi and Ochiai [37]), a compact complex manifold X which is holomorphically or even just meromorphically dominable by $\mathbb{C}^{\dim X}$ satisfies $\kappa_X < \dim X$. Manifolds with the maximal Kodaira dimension $\kappa_X = \dim X$ are said to be *of general type*, and they cannot be Oka since they are not dominable by $\mathbb{C}^{\dim X}$. Conjecturally no such manifold is Oka-1 either, since it is believed that any holomorphic line $\mathbb{C} \to X$ in a manifold of general type is contained in a proper complex subvariety of X. This conjecture of Lang [45] from 1986 seems to be still open.

In the sequel and unless stated otherwise, X denotes a compact complex surface with Kodaira dimension $\kappa \in \{-\infty, 0, 1\}$. A complete list of such surfaces, classified according to the value of κ, can be found in the monograph by Barth et al. [5, Table 10, p. 244]. There are 10 classes, and every compact complex surface has a minimal model (obtained by blowing down all -1 rational curves) in exactly one of these classes. This minimal model is unique up to isomorphisms, except for surfaces with minimal models in the classes 1 and 3.

A *fibration* $f : X \to C$ of a complex surface X onto a complex curve C is a proper surjective holomorphic map with connected fibres. A fibration is *relatively minimal* if there are no -1 curves on any fibre. (Any such curve is rational and can be blown down.) A fibration is said to be *elliptic* if the general fibre $X_p = f^{-1}(p)$ is an elliptic curve (a complex torus). Such a surface can have any Kodaira dimension $\kappa \in \{-\infty, 0, 1\}$. A compact complex surface X is an *elliptic surface* if it admits an elliptic fibration $X \to C$ onto an elliptic curve. In this connection, we recall (see [5, Theorem 15.4, p. 127]) that if X is a compact complex surface, $f : X \to C$ is a fibration without singular fibres, and the curve C is either $\mathbb{CP}^1$ or elliptic, then f is a holomorphic fibre bundle.

8.1. **Kodaira dimension $\kappa = -\infty$.** Rational surfaces (including nonminimal ones) are Oka, and hence Oka-1. A ruled surface is Oka-1 if and only if its base is $\mathbb{CP}^1$ or an elliptic curve. Theorem 8.1 covers surfaces of class VII if the global spherical shell conjecture holds true.

Let us go through the list and justify these claims.

Every smooth *rational surface* is obtained by repeatedly blowing up a minimal rational surface. The minimal rational surfaces are the projective plane $\mathbb{CP}^2$, which is Oka, and the Hirzebruch surfaces Σ_r for $r \in \mathbb{Z}_+$. The latter are holomorphic $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$, so they are Oka by [26, Theorem 5.6.5]. Repeated blowups preserve the Oka property for surfaces in this class by [26, Proposition 6.4.6], so nonminimal rational surfaces are also Oka.

A *ruled surface* is the total space of a holomorphic fibre bundle $X \to C$ with fibre $\mathbb{CP}^1$ over a compact curve C (see [5, p. 189]). By [26, Theorem 5.6.5] such a surface X is Oka if and only if the base curve C is Oka, which holds if and only if C is $\mathbb{CP}^1$ or a quotient of $\mathbb{C}$ (see [26, Corollary 5.6.4]). By Theorem 7.6 (c), X is Oka-1 if and only if C is $\mathbb{CP}^1$ or a quotient of $\mathbb{C}$. Note that minimal ruled surfaces over $\mathbb{CP}^1$ are just the Hirzebruch surfaces.

Class VII comprises the nonalgebraic compact complex surfaces of Kodaira dimension $\kappa = -\infty$. Minimal surfaces of class VII fall into several mutually disjoint classes. For second Betti number $b_2 = 0$, we have *Hopf surfaces* and *Inoue surfaces*. For $b_2 \geq 1$, there are *Enoki surfaces*, *Inoue-Hirzebruch surfaces*, and *intermediate surfaces*; together they form the class of *Kato surfaces*. Conjecturally, all surfaces with $\kappa = -\infty$ and $b_2 \geq 1$ admit a global spherical shell, i.e., a neighbourhood of the 3-sphere $S^3 \subset \mathbb{C}^2$ holomorphically embedded into X so that the complement is connected. If this *global spherical shell conjecture* holds true then every minimal surface of class VII with $b_2 \geq 1$ is a Kato surface. The conjecture was proved in the cases $b_2 \in \{1, 2, 3\}$ by Teleman in the respective papers [51, 52, 53]. Assuming that the global spherical shell conjecture holds true, the following result gives a complete description of Oka-1 surfaces in class VII. For the corresponding description of Oka surfaces in this class, see [30, Theorem 4] or [26, Theorem 7.3.2].

Theorem 8.1. *Minimal Hopf surfaces and minimal Enoki surfaces are Oka, and hence Oka-1. Inoue surfaces, Inoue–Hirzebruch surfaces, and intermediate surfaces, minimal or blown up, are not strongly Liouville, and hence not Oka or Oka-1.*

Recall that every Oka-1 manifold is strongly Liouville by Corollary 7.5.

In summary, we have the following corollary concerning surfaces with $\kappa_X = -\infty$.

Corollary 8.2. *Modulo the global spherical shell conjecture, the following conditions are equivalent for every minimal compact complex surface X with $\kappa_X = -\infty$:*

$$\textit{Oka} \iff \textit{dominable by } \mathbb{C}^2 \iff \textit{Oka-1} \iff \textit{not strongly Liouville}.$$

8.2. **Kodaira dimension.** $\kappa = 0$. Bielliptic surfaces, Kodaira surfaces, and tori are Oka, and hence Oka-1. Elliptic and Kummer K3 surfaces are Oka-1 manifolds, but we do not know whether any or all of them are Oka manifolds. Again, we proceed case by case.

Tori (unramified quotients of $\mathbb{C}^2$) are complex homogeneous manifolds, hence Oka (see [26, Proposition 5.6.1]) and therefore Oka-1. In fact, more can be said about compact tori.

Proposition 8.3. *A complex surface Y bimeromorphic to a compact torus is densely dominable by $\mathbb{C}^2$, and hence an Oka-1 manifold.*

Proof. Let Γ be a lattice of rank 4 in $\mathbb{C}^2$, that is, a free abelian subgroup $\Gamma \cong \mathbb{Z}^4$ of $\mathbb{C}^2$. Given finitely many points $P = \{p_1, \ldots, p_m\}$ in the torus $X = \mathbb{C}^2/\Gamma$, the complement $X \setminus P$ is universally covered by $\mathbb{C}^2 \setminus \widetilde{\Gamma}$, where $\widetilde{\Gamma} = \bigcup_{i=1}^m (a_i + \Gamma)$ and $a_i \in \mathbb{C}^2$ are points mapped to p_i under the quotient projection. Buzzard and Lu showed in [8, Proposition 4.1] that the discrete set $\widetilde{\Gamma}$ is tame in $\mathbb{C}^2$. Hence, $\mathbb{C}^2 \setminus \widetilde{\Gamma}$ is an Oka manifold by [26, Proposition 5.6.17], and its unramified quotient $X \setminus P$ is an Oka manifold by [26, Proposition 5.6.3]. Any compact complex surface Y which is bimeromorphic to X admits a dominating holomorphic map from such a complement $X \setminus P$ with a Zariski dense image in Y, so Y is densely dominable by $\mathbb{C}^2$. The conclusion now follows from Corollary 2.5 (a). □

Most tori are not elliptic. The elliptic compact 2-tori form a 3-dimensional family in the 4-dimensional family of tori. A generic 2-torus does not contain any compact complex curves.

According to [5, p. 245], every *bielliptic surface* and every *primary Kodaira surface* is the total space of a holomorphic fibre bundle with torus fibre over a torus, so it is Oka by [26, Theorem 5.6.5] . A *secondary Kodaira surface* is a proper unramified holomorphic quotient of a primary Kodaira surface, so it is Oka by [26, Proposition 5.6.3]. They are elliptic fibrations over $\mathbb{CP}^1$ with $b_1(X) = 1$ and with nontrivial canonical bundle.

A *K3 surface* is a simply connected surface ($b_1 = 0$) with trivial canonical bundle, hence $\kappa = 0$. We refer to Barth et al. [5, Chapter VIII] and Huybrechts [36] for a detailed treatment of such surfaces; the basic description in [8, Section 4.2] will suffice for our needs. The class of K3 surfaces includes Kummer surfaces, which form a dense codimension 16 family in the moduli space of K3 surfaces, and the elliptic K3 surfaces, which form a dense codimension one family in the moduli space. All elliptic fibrations in the K3 class are ramified. It is not known whether any or all K3 surfaces are Oka. Here we will show that every Kummer surface and every elliptic K3 surface is an Oka-1 manifold.

Let us recall the structure of Kummer surfaces; see [5, p. 224]. Let $X = \mathbb{C}^2/\Gamma$ be a compact complex 2-torus, and let $h : \mathbb{C}^2 \to X$ be the quotient (covering) map. The involution $\mathbb{C}^2 \to \mathbb{C}^2$, $(z_1, z_2) \mapsto (-z_1, -z_2)$ descends to an involution σ on X with 16 fixed points $P = \{p_1, \ldots, p_{16}\}$. (If $\omega_1, \ldots, \omega_4 \in \mathbb{C}^2$ are the generators of the lattice Γ then $p_1, \ldots, p_{16}$ are the images under h of the 16 points $c_1\omega_1 + \cdots + c_4\omega_4$, where $c_1, \ldots, c_4 \in \{0, \frac{1}{2}\}$.) The quotient space $X/\{1, \sigma\}$ is a 2-dimensional complex space with 16 singular points $q_1, \ldots, q_{16}$. Blowing up $X/\{1, \sigma\}$ at each of these points yields a smooth compact surface Y containing pairwise disjoint smooth rational curves $C_1, \ldots, C_{16}$. (Each of them is a -2 curve.) This is the Kummer surface associated to the rank four lattice $\Gamma \subset \mathbb{C}^2$.

Let $C = \bigcup_{i=1}^{16} C_i$. Note that $Y \setminus C$ is an unramified two-sheeted quotient of $X \setminus P$. We have seen in the proof of Proposition 8.3 that $Y \setminus C$ and $X \setminus P$ are Oka manifolds. If B is a finite subset of $Y \setminus C$ and A is its saturated preimage in $X \setminus P$, then the Zariski domains $Y_0 = Y \setminus (B \cup C)$ and $X_0 = X \setminus (A \cup P)$ are manifolds of the same kind, hence Oka. If $\widetilde{Y}$ is a compact surface bimeromorphic to Y, it admits a dominating holomorphic map from such Y_0 (for some B) with a Zariski dense image in $\widetilde{Y}$, so $\widetilde{Y}$ is densely dominable by $\mathbb{C}^2$. This proves the following statement.

Proposition 8.4. *A compact complex surface bimeromorphic to a Kummer surface is densely dominable by $\mathbb{C}^2$, and hence an Oka-1 manifold.*

Next, we consider elliptic fibrations. Suppose that $f : X \to C$ is an elliptic fibration over a complex curve $C = \overline{C} \setminus P$ obtained by removing at most finitely many points P from a compact complex curve $\overline{C}$. (Such a curve C is quasi-projective.) Let $m \geq 0$ denote the cardinality of the set P. The Euler characteristic of C equals

$$\chi(C) = \chi(\overline{C}) - m = 2 - 2g(\overline{C}) - m, \tag{8.1}$$

where g denotes the genus. We assume that the fibration $f : X \to C$ has at most finitely many multiple and singular fibres. Let $n_p \in \mathbb{N}$ for $p \in C$ denote the multiplicity of the fibre $X_p = f^{-1}(p)$, so $n_p = 1$ means that the fibre is not multiple, although it may still be singular. (See [5, Chapter V] or [8, Sect. 3.2] for a detailed description of this notion.) The fibration $f : X \to C$ determines the divisor

$$D = \sum_{p \in C} \left(1 - \frac{1}{n_p}\right) p \tag{8.2}$$

with rational coefficients of degree

$$\deg D = \sum_{p \in C} \left(1 - \frac{1}{n_p}\right) \in \mathbb{Q}_+. \tag{8.3}$$

The sum is over $p \in C$ with $n_p > 1$, and $\deg D = 0$ if and only of there are no multiple fibres.

We can now state our main result concerning Oka-1 manifolds among elliptic fibrations.

Theorem 8.5. *Assume that C is a quasi-projective complex curve and $f : X \to C$ is a relatively minimal elliptic fibration with at most a finite number of multiple fibres. Let D denote the associated $\mathbb{Q}$-divisor* (8.2)*. Then, the following conditions are equivalent.*

(a) *The surface X is densely dominable by $\mathbb{C}^2$, and hence an Oka-1 manifold.*
(b) $\chi(C, D) = \chi(C) - \deg D \geq 0$. *(See* (8.1) *and* (8.3)*.)*
(c) *There is a holomorphic map $\mathbb{C} \to X$ with a Zariski dense image.*

Assuming that X is not bimeromorphic to a primary or a secondary Kodaira surface, the above conditions are also equivalent to the following:

(d) *The fundamental group $\pi_1(X)$ is a finite extension of an abelian group.*

Proof. This result is analogous to [8, Theorem 3.9] by Buzzard and Lu, except that condition (a) in their theorem, that X be dominable by $\mathbb{C}^2$, is now replaced by the stronger condition that X be densely dominable. Hence, it follows that for such fibrations these two conditions are equivalent. Condition (d) is equivalent to dominability of X by [8, Theorem 3.23]. To prove the theorem, it thus suffices to show the implication (b) $\Rightarrow$ (a).

The support $P \subset C$ of the divisor D (8.2) is a finite subset of C by the assumption. By [16, IV 9.12] the pair (C, D) has a uniformizing orbifold covering $\alpha : \widetilde{C} \to C$ where $\widetilde{C}$ is one of the Riemann surfaces $\mathbb{CP}^1, \mathbb{C}, \mathbb{D}$ according to $\chi(C, D) > 0$, $\chi(C, D) = 0$, or $\chi(C, D) < 0$. (Here, $\mathbb{D}$ is the unit disc in $\mathbb{C}$.) This means that $\alpha : \widetilde{C} \to C$ is a surjective branched holomorphic map such that $\alpha : \widetilde{C} \setminus \alpha^{-1}(P) \to C \setminus P$ is a finite covering map, and for each $p \in P$, α has ramification index n_p at every point of the fibre $\alpha^{-1}(p)$. Then, the pullback elliptic fibration $\tilde{f} : \widetilde{X} = \alpha^* X \to \widetilde{C}$ has no multiple fibres, and the natural map $\widetilde{X} \to X$ covering α is an unramified covering map. Since the properties of being dominable or densely dominable by $\mathbb{C}^2$ are invariant under covering maps, this reduces the problem to the case when $C \in \{\mathbb{CP}^1, \mathbb{C}, \mathbb{D}\}$ and $X \to C$ is an elliptic fibration without multiple fibres (see [8, Proposition 3.4].) The case $C = \mathbb{D}$ is excluded by condition (b) (see [8, proof of Theorem 3.9]), and the case $C = \mathbb{CP}^1$ reduces to $C = \mathbb{C}$ by removing a point.

It remains to show that the total space X of an elliptic fibration $f : X \to \mathbb{C}$ without multiple fibres is densely dominable. By [8, Lemma 3.8] there exists a holomorphic section $\sigma : \mathbb{C} \to X$ of the fibration f. (The proof uses the fact that every singular fibre $X_p = f^{-1}(p)$ which is not a multiple fibre admits an irreducible component of multiplicity one, so there is a local holomorphic section of X at every point. This is seen by inspecting Kodaira's list of non-multiple singular fibres, see [5, Table 3, p. 201]. A global section is then found by solving a Cousin-1 problem.) Let X_p^σ denote the union of all irreducible components of the fibre X_p which do not contain the point $\sigma(p)$ (such exist only if X_p is a singular fibre). Their union $X^\sigma = \bigcup_p X_p^\sigma$ is a closed one-dimensional complex subvariety of X. By a theorem of Kodaira [5, Proposition V-9.1, p. 206] there is a canonical fibre-preserving isomorphism $\Theta : \mathrm{Jac}(f) \to \Omega \subset X$ from the Jacobian fibration $\mathrm{Jac}(f) \to \mathbb{C}$ onto the Zariski open domain $\Omega = X \setminus X^\sigma$ in X. Recall that the fibre of the Jacobian fibration over $p \in \mathbb{C}$ is

$$\mathrm{Jac}(f)_p = \mathrm{Pic}^0(X_p) = H^1(X_p, \mathscr{O}_{X_p})/H^1(X_p, \mathbb{Z}),$$

where the inclusion $H^1(X_p, \mathbb{Z}) \hookrightarrow H^1(X_p, \mathscr{O}_{X_p})$ comes from the cohomology sequence of the exponential sheaf sequence $0 \to \mathbb{Z} \hookrightarrow \mathbb{C} \to \mathbb{C}^* \to 0$:

$$0 \to H^1(X_p, \mathbb{Z}) \to H^1(X_p, \mathscr{O}_{X_p}) \to H^1(X_p, \mathscr{O}^*_{X_p}) = \mathrm{Pic}(X_p) \to H^2(X_p, \mathbb{Z}) \to 0$$

(see [8, p. 627]). Thus, $\mathrm{Pic}^0(X_p)$ is the subgroup of the group $\mathrm{Pic}(X_p)$ consisting of holomorphic line bundles $E \to X_p$ with trivial first Chern class $0 = c_1(E) \in H^2(X_p, \mathbb{Z})$. We have that $L_p := H^1(X_p, \mathscr{O}_{X_p}) \cong \mathbb{C}$ for all $p \in \mathbb{C}$, $L = \bigsqcup_{p\in\mathbb{C}} L_p \to \mathbb{C}$ is a holomorphic line bundle, which is trivial by the Oka–Grauert principle, and $\mathrm{Jac}(f)$ is a quotient of L. Let $A \subset \mathbb{C}$ be the discrete set of points $p \in \mathbb{C}$ for which the fibre X_p is singular, and set $X' = \bigcup_{p\in A} X_p$ and $L' = \bigcup_{p\in A} L_p$. The natural quotient projection $\phi : L \to \mathrm{Jac}(f)$ is a nonramified covering map on the Zariski open set $L \setminus L'$ in L. The holomorphic map

$$h = \Theta \circ \phi : L \to \Omega = X \setminus X^\sigma$$

is then surjective, and $h : L \setminus L' \to \Omega \setminus X'$ is a nonramified covering projection. Hence, h is dominating on the Zariski open domain $\Omega \setminus X'$, so X is densely dominable by $L \cong \mathbb{C}^2$. □

Since a K3 surface has trivial fundamental group, we have the following corollary to Theorem 8.5 (see part (d) of the theorem).

Corollary 8.6. *Every elliptic K3 surface is densely dominable by $\mathbb{C}^2$, and hence Oka-1.*

By the Enriques–Kodaira classification, every compact complex surface with Kodaira dimension 0, which is not bimeromorphic to a torus or a K3 surface, is an elliptic fibration, so the question of their (dense) dominability by $\mathbb{C}^2$ is covered by Theorem 8.5.

8.3. **Kodaira dimension $\kappa = 1$.** By the Enriques–Kodaira classification, every such surface is an elliptic surface, given as the total space of an elliptic fibration $X \to C$ over an elliptic curve. These are called *properly elliptic surfaces.* The equivalence of (a) and (b) in Theorem 8.5 gives the following corollary.

Corollary 8.7. *Let $f : X \to C$ be an elliptic fibration over an elliptic curve C. Then, the elliptic surface X is densely dominable by $\mathbb{C}^2$ if and only if the fibration f has no multiple fibres. Every elliptic surface with this property is an Oka-1 manifold.*

Proof. Let D be the $\mathbb{Q}$-divisor (8.2). Since $\chi(C) = 0$, we have that

$$\chi(C, D) = \chi(C) - \deg D = -\deg D,$$

so $\chi(C, D) \geq 0$ (condition (b) in Theorem 8.5) holds if and only if $D = 0$, which means that there are no multiple fibres. □

9. A CONJECTURE ON RATIONALLY CONNECTED MANIFOLDS

A complex manifold X is said to be rationally connected if any pair of points in X is connected by a rational curve $\mathbb{CP}^1 \to X$. Among many references for rationally connected projective manifolds, we refer to the papers by Kollar et al. [39, 42] and the monographs by Kollár [40] and Debarre [13]. By [42, Theorem 2.1] several possible definitions of this class coincide. In particular, if every sufficiently general pair of points in X can be connected by an irreducible rational curve, then X is rationally connected.

There are reasons to believe that the following conjecture holds true.

Conjecture 9.1. Every compact rationally connected Kähler manifold is an Oka-1 manifold.

One indication is provided by the theorem of Campana and Winkelmann [11, Theorem 5.2] saying that every rationally connected projective manifold X admits an entire curve $\mathbb{C} \to X$ whose image contains a given countable subset of X, with prescribed finite order jets in these points [11, Corollary 5.7]. In particular, every such X contains dense entire curves $\mathbb{C} \to X$. Their construction relies on the deformation theory for rational curves in rationally connected projective manifolds, in particular, on smoothing of a comb or a tree of rational curves; see [11, Proposition 5.1 and Lemma 5.5]. This technique goes back to the seminal paper [42] by Kollár, Miyaoka, and Mori; see also Kollár [40, Theorem 7.6, p. 155]. By using these methods, we tried to show that a rationally connected projective manifold is densely dominable by tubes of lines coming from trees of rational curves, and hence is an Oka-1 manifold by Theorem 2.2. Our attempts remained inconclusive; see [2, Sect. 9].

Next, we observe that the approximation condition in Proposition 2.7 follows easily from the Runge approximation theorem for compact pseudoholomorphic curves in certain compact almost complex manifolds, due to Gournay [33, Theorem 1.1.1]. It can be verified that the conditions in his theorem hold true for rationally connected projective manifolds. Hence, if we use Gournay's theorem at face value, Proposition 2.7 implies that every projective rationally connected manifold enjoys the Oka-1 property with approximation. Some details are given in our earlier preprint [2, Sect. 9], where it is also described how interpolation could be added to Gournay's theorem, and Conjecture 9.1 is stated as [2, Theorem 9.1].

Subsequently, we tried to fully understand Gournay's proof of [33, Theorem 1.1.1], but we did not succeed. For this reason, we decided to downgrade [2, Theorem 9.1] to Conjecture 9.1. Since rationally connected manifolds are a very important and much studied class of projective manifolds, it would be of utmost interest to provide an additional explanation or an independent proof of [33, Theorem 1.1.1] at least for this class.

10. Dense holomorphic curves in an arbitrary complex manifold

In this section we prove the following result which generalizes the case $M = \Delta$ obtained by Forstnerič and Winkelmann in [24]. In the last part of the section, we give an analogous result for holomorphic Legendrian curves in complex contact manifolds; see Theorem 10.4.

Theorem 10.1. *Assume that X is a connected complex manifold endowed with a complete distance function dist_X, $\overline{M} = M \cup bM$ is a compact bordered Riemann surface, and $f : \overline{M} \to X$ is a map of class $\mathscr{A}(\overline{M})$. Given a compact subset $K \subset M$, a countable set $B \subset X$, and a number $\epsilon > 0$ there is a holomorphic map $F : M \to X$ such that*

(a) $\sup_{p\in K} \mathrm{dist}_X(F(p), f(p)) < \epsilon$,
(b) *F agrees with f to a given finite order at a given finite set of points $C \subset M$, and*
(c) *$B \subset F(M)$.*

Furthermore, F can be chosen to be an immersion if $\dim X > 1$, and to be an injective immersion if $\dim X > 2$, whenever condition (b) *allows it.*

We record the following immediate corollary.

Corollary 10.2. *If X is a complex manifold and M is a bordered Riemann surface, then there is a holomorphic map $M \to X$, which can be chosen to be an immersion if $\dim X > 1$ and an injective immersion if $\dim X > 2$, whose image is everywhere dense in X.*

It was shown by Fornæss and Stout [20, 21] that every connected complex manifold X of dimension n admits surjective holomorphic maps $\Delta^n \to X$ and $\mathbb{B}^n \to X$ from the polydisc and the ball in $\mathbb{C}^n$. Hence, to obtain density, it would suffice to prove Corollary 10.2 for maps $M \to \Delta^n$. However, there seems to be no particular advantage in this reduction, which furthermore does not give the approximation and general position result in Theorem 10.1.

Proof of Theorem 10.1. We assume without loss of generality that M is a smoothly bounded compact domain without holes in an open Riemann surface R (see Stout [50, Theorem 8.1]) and $f_0 = f$ is holomorphic on $\overline{M}$ (see Theorem 2.13). Let K, B, and ϵ be as in the statement, and assume as we may that $K_0 = K$ is a smoothly bounded compact domain which is a strong deformation retract of M, the given countable set $B = \{b_1, b_2, \ldots\}$ is infinite, $C \subset \mathring{K}_0$, and $f(C) \cap B = \varnothing$. (Here, C is the finite set for the interpolation condition in (b).) Pick a point $a_0 \in \mathring{K}_0 \setminus (C \cup f^{-1}(B))$ and set $b_0 = f_0(a_0)$. Let $\epsilon_0 = \epsilon/2$ and $K_{-1} = \varnothing$.

We shall inductively construct a sequence of holomorphic maps $f_j : \overline{M} \to X$, smoothly bounded compact domains $K_j \subset M$, numbers $\epsilon_j > 0$, and points $a_j \in M$, $j \in \mathbb{N}$, satisfying the following conditions.

(1_j) $K_{j-1} \cup \{a_j\} \subset \mathring{K}_j$ and K_j is a strong deformation retract of M.
(2_j) $\epsilon_j < \epsilon_{j-1}/2$.
(3_j) $\sup_{p \in K_{j-1}} \mathrm{dist}_X(f_j(p), f_{j-1}(p)) < \epsilon_j$.
(4_j) $f_j(a_i) = b_i$ for all $i \in \{0, \ldots, j\}$.
(5_j) f_j agrees with f to a given finite order at every point in C.

In addition, we shall guarantee that

(10.1) $$\bigcup_{j \in \mathbb{N}} K_j = M.$$

Assume that we have such a sequence in hand. Conditions (1_j), (2_j), (3_j), and (10.1) ensure that there is a limit holomorphic map $F = \lim_{j \to \infty} f_j : M \to X$ satisfying condition (a) in the statement of the theorem. Moreover, conditions (5_j) and (4_j) guarantee (b) and (c), respectively. So, the map F satisfies the conclusion of the theorem. Note that the final statement can be granted as explained in Remark 2.8; it suffices to take the number $\epsilon_j > 0$ sufficiently small at each step of the inductive construction.

It thus remains to explain the induction. The basis is provided by $f_0 = f$, $K_0 = K$, $\epsilon_0 = \epsilon/2$, and $a_0 \in \mathring{K}_0$; note that conditions (2_0) and (3_0) are void. For the inductive step, fix $j \in \mathbb{N}$ and assume that we have objects f_{j-1}, K_{j-1}, ϵ_{j-1}, and a_{j-1} satisfying conditions (1_{j-1}), (4_{j-1}), and (5_{j-1}). Choose any $\epsilon_j > 0$ meeting (2_j). If $b_j \in f_{j-1}(M)$ then we choose any point $a_j \in M$ with $f_{j-1}(a_j) = b_j$ and any smoothly bounded compact domain $K_j \subset M$ satisfying (1_j), and set $f_j = f_{j-1}$. Assume on the contrary that $b_j \notin f_{j-1}(M)$. Choose a point $x \in bM \subset R$ and attach to $\overline{M}$ a smooth embedded arc $\gamma \subset R$ such that $\gamma \cap \overline{M} = \{x\}$ and the intersection of bM and γ is transverse at x. Let $x' \in R \setminus \overline{M}$ denote the other endpoint of γ. Fix a point $y \in \gamma \setminus \{x, x'\}$ and extend f_{j-1} to a smooth map $f_{j-1} : \overline{M} \cup \gamma \to X$ which is holomorphic on a neighbourhood of $\overline{M}$ and satisfies $f_{j-1}(y) = b_j$. Theorem 2.13 furnishes a holomorphic map $g : V \to X$ on an open neighbourhood $V \subset R$ of $\overline{M} \cup \gamma$ such that

(i) $\sup_{p \in \overline{M} \cup \gamma} \mathrm{dist}_X(g(p), f_{j-1}(p)) < \epsilon_j/2$,
(ii) $g(a_i) = b_i$ for all $i \in \{0, \ldots, j-1\}$,
(iii) $g(y) = b_j$, and
(iv) g agrees with f to a given order at every point in C.

For (ii) and (iv) take into account (4_{j-1}) and (5_{j-1}). Now, [4, Theorem 6.7.1] (see also [31, Theorem 2.3]) furnishes a conformal diffeomorphism $\phi : \overline{M} \to \phi(\overline{M}) \subset V$ such that ϕ agrees with the identity map to a given order in the points in the finite set $C \cup \{a_0, \ldots, a_{j-1}\} \subset M$, ϕ is arbitrarily close to the identity map on K_{j-1}, and we have that

(10.2) $$y \in \phi(M).$$

In particular, ϕ can be chosen so that the map $f_j = g \circ \phi : \overline{M} \to X$ of class $\mathscr{A}(\overline{M})$ satisfies

(I) $f_j(a_i) = g(a_i)$ for all $i \in \{0, \dots, j-1\}$,
(II) $b_j \in f_j(M)$ (see (iii) and (10.2)),
(III) $\sup_{p\in K_{j-1}} \mathrm{dist}_X(f_j(p), g(p)) < \epsilon_j/2$, and
(IV) f_j agrees with g to a given order at every point in C.

By Theorem 2.13 we may assume that f_j is holomorphic on $\overline{M}$. Finally, (II) enables us to choose a point $a_j \in M$ with $f_j(a_j) = b_j$ and a smoothly bounded compact domain $K_j \subset M$ satisfying (1_j). It is then clear in view of these choices and conditions (i)–(iv) and (I)–(IV) that f_j, K_j, ϵ_j and a_j satisfy the requirements in (1_j)–(5_j). This closes the induction and completes the proof of the theorem. Note that (10.1) is guaranteed by choosing the compact domain $K_j \subset M$ sufficiently large at each step of the inductive construction. □

Combining the arguments used in [1, proofs of Theorems 1.6 and 1.8] with the proof of Theorem 10.1, one may easily establish the analogous hitting results for some classes of surfaces of infinite topology, at the cost of losing the control on the complex structure. In particular, we have the following corollary.

Corollary 10.3. *Let X be a connected complex manifold and $B \subset X$ be a countable subset. The following assertions hold.*

(i) *On every compact Riemann surface R there is a Cantor set C whose complement $M = R \setminus C$ admits a holomorphic map $F : M \to X$ with $B \subset F(M)$.*
(ii) *On every open orientable smooth surface S there is a complex structure J such that the open Riemann surface $M = (S, J)$ admits a holomorphic map $F : M \to X$ with $B \subset F(M)$.*

In both cases the holomorphic map $F : M \to X$ can be chosen to be an immersion if $\dim X > 1$ and an injective immersion if $\dim X > 2$.

We leave the details of the proof to the reader. In particular, in both cases (and up to a suitable choice of the Cantor set C in assertion (i) and of the complex structure J in assertion (ii)) there is a holomorphic map $M \to X$, which can be chosen an immersion if $\dim X > 1$ and an injective immersion if $\dim X > 2$, whose image is everywhere dense in X.

The analogue of Theorem 10.1 also holds for holomorphic Legendrian immersions in an arbitrary connected complex contact manifold (X, ξ). These are immersions which are tangential to the holomorphic contact subbundle ξ of the tangent bundle TX. The only essential difference in the proof is that we use the Mergelyan approximation theorem for Legendrian immersions $f : S \to X$ from an admissible set S in an open Riemann surface R, given by [22, Theorem 1.2]. We refer to that paper for the background on this subject. The precise result that one obtains is the following; we leave the details of the proof to the reader.

Theorem 10.4. *Assume that (X, ξ) is a connected complex contact manifold, $\overline{M} = M \cup bM$ is a compact bordered Riemann surface, and $f : \overline{M} \to X$ is a ξ-Legendrian immersion of class $\mathscr{A}^r(\overline{M})$ for some integer $r \geq 4$ (i.e., smooth of order r on $\overline{M}$ and holomorphic on M). Given a compact subset $K \subset M$, a countable set $B \subset X$, and a number $\epsilon > 0$, there is a holomorphic Legendrian immersion $F : M \to X$ satisfying the conditions in Theorem 10.1, which can be chosen injective if the interpolation conditions allow it.*

In particular, every bordered Riemann surface admits an injective holomorphic Legendrian immersion to X whose image is everywhere dense in X.

To justify the last statement in the theorem, recall that every holomorphic contact subbundle $\xi \subset TX$ is given in suitable local holomorphic coordinates at any point $p \in X$ as the kernel of the standard holomorphic contact form $\alpha = dz + \sum_{i=1}^n x_i dy_i$, on $\mathbb{C}^{2n+1}$ with $2n+1 = \dim X$, where $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, and $z \in \mathbb{C}$ are complex coordinates on $\mathbb{C}^{2n+1}$. (This is the holomorphic Darboux theorem; see Alarcón et al. [3, Appendix]). It was shown in [3] that every compact bordered Riemann surface $\overline{M}$ admits a holomorphic Legendrian embedding $g : \overline{M} \hookrightarrow (\mathbb{C}^{2n+1}, \alpha)$. Composing g by the contact isomorphism $(x, y, z) \mapsto (tx, ty, t^2 z)$ on $(\mathbb{C}^{2n+1}, \alpha)$ for a suitable $t > 0$ ensures that $g(\overline{M})$ lies in the image of the local chart, so we obtain a holomorphic Legendrian embedding $f : \overline{M} \hookrightarrow (X, \xi)$. Applying the first part of the theorem to f yields the result.

Acknowledgements. Alarcón is partially supported by the State Research Agency (AEI) via the grant no. PID2020-117868GB-I00 and the "Maria de Maeztu" Excellence Unit IMAG, reference CEX2020-001105-M, funded by MCIN/AEI/10.13039/501100011033/; and the Junta de Andalucía grant no. P18-FR-4049, Spain.

Forstnerič is supported by the European Union (ERC Advanced grant HPDR, 101053085) and grants P1-0291, J1-3005, and N1-0237 from ARRS, Republic of Slovenia.

The authors wish to thank Rafael Andrist, Kyle Broder, Frédéric Campana, Antoine Gournay, Daniel Huybrechts, János Kollár, Uroš Kuzman, Finnur Lárusson, and Thomas Peternell for consultations and helpful advice.

REFERENCES

[1] A. Alarcón and F. Forstnerič. Embedded complex curves in the affine plane. *Ann. Mat. Pura Appl. (4)*, to appear. https://doi.org/10.1007/s10231-023-01418-8.

[2] A. Alarcón and F. Forstnerič. Oka-1 manifolds. *arXiv e-prints*, 2303.15855, V3, 22. Sept. 2023. https://arxiv.org/abs/2303.15855v3.

[3] A. Alarcón, F. Forstnerič, and F. J. López. Holomorphic Legendrian curves. *Compos. Math.*, 153(9):1945–1986, 2017.

[4] A. Alarcón, F. Forstnerič, and F. J. López. *Minimal surfaces from a complex analytic viewpoint*. Springer Monographs in Mathematics. Springer, Cham, 2021.

[5] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*. Springer-Verlag, Berlin, second edition, 2004.

[6] H. Behnke and K. Stein. Entwicklung analytischer Funktionen auf Riemannschen Flächen. *Math. Ann.*, 120:430–461, 1949.

[7] D. Brotbek. On the hyperbolicity of general hypersurfaces. *Publ. Math. Inst. Hautes Études Sci.*, 126:1–34, 2017.

[8] G. T. Buzzard and S. S. Y. Lu. Algebraic surfaces holomorphically dominable by $\mathbb{C}^2$. *Invent. Math.*, 139(3):617–659, 2000.

[9] F. Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004.

[10] F. Campana. Orbifolds, special varieties and classification theory: an appendix. *Ann. Inst. Fourier (Grenoble)*, 54(3):631–665, 2004.

[11] F. Campana and J. Winkelmann. Dense entire curves in rationally connected manifolds (with an appendix by János Kollár). *Alg. Geom.*, 10(5):521–553, 2023.

[12] J. Carlson and P. Griffiths. A defect relation for equidimensinal holomorphic mappings between algebraic varieties. *Ann. of Math. (2)*, 95:557–584, 1972.

[13] O. Debarre. *Higher-dimensional algebraic geometry*. Universitext. New York, NY: Springer, 2001.

[14] B. Drinovec Drnovšek and F. Forstnerič. Flexible domains for minimal surfaces in Euclidean spaces. *J. Math. Anal. Appl.*, 517(2):15, 2023. Id/No 126653.

[15] B. Drinovec Drnovšek and F. Forstnerič. Holomorphic curves in complex spaces. *Duke Math. J.*, 139(2):203–253, 2007.

[16] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[17] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[18] H. Florack. Reguläre und meromorphe Funktionen auf nicht geschlossenen Riemannschen Flächen. *Schr. Math. Inst. Univ. Münster,*, 1948(1):34, 1948.

[19] J. E. Fornæss, F. Forstnerič, and E. Wold. Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. In *Advancements in complex analysis. From theory to practice*, pages 133–192. Cham: Springer, 2020.

[20] J. E. Fornaess and E. L. Stout. Spreading polydiscs on complex manifolds. *Amer. J. Math.*, 99(5):933–960, 1977.

[21] J. E. Fornaess and E. L. Stout. Regular holomorphic images of balls. *Ann. Inst. Fourier*, 32(2):23–36, 1982.

[22] F. Forstnerič. Mergelyan approximation theorem for holomorphic Legendrian curves. *Anal. PDE*, 15(4):983–1010, 2022.

[23] F. Forstnerič. Recent developments on Oka manifolds. *Indag. Math., New Ser.*, 34(2):367–417, 2023.

[24] F. Forstnerič and J. Winkelmann. Holomorphic discs with dense images. *Math. Res. Lett.*, 12(2-3):265–268, 2005.

[25] F. Forstnerič. Oka maps. *C. R. Math. Acad. Sci. Paris*, 348(3-4):145–148, 2010.

[26] F. Forstnerič. *Stein manifolds and holomorphic mappings (The homotopy principle in complex analysis)*, vol. 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Cham, second ed., 2017.

[27] F. Forstnerič. Surjective holomorphic maps onto Oka manifolds. In *Complex and symplectic geometry*, volume 21 of *Springer INdAM Ser.*, pages 73–84. Springer, Cham, 2017.

[28] F. Forstnerič. Embedding bordered Riemann surfaces in strongly pseudoconvex domains. *Rev. Roumaine Math. Pures Appl.*, 68(1-2):83–94, 2023.

[29] F. Forstnerič and F. Lárusson. Survey of Oka theory. *New York J. Math.*, 17A:11–38, 2011.

[30] F. Forstnerič and F. Lárusson. Holomorphic flexibility properties of compact complex surfaces. *Int. Math. Res. Not. IMRN*, 13:3714–3734, 2014.

[31] F. Forstnerič and E. F. Wold. Bordered Riemann surfaces in $\mathbb{C}^2$. *J. Math. Pures Appl. (9)*, 91(1):100–114, 2009.

[32] F. Forstnerič and E. F. Wold. Holomorphic families of Fatou-Bieberbach domains and applications to Oka manifolds. *Math. Res. Lett.*, 27(6):1697–1706, 2020.

[33] A. Gournay. A Runge approximation theorem for pseudo-holomorphic maps. *Geom. Funct. Anal.*, 22(2):311–351, 2012.

[34] H. Grauert. Approximationssätze für holomorphe Funktionen mit Werten in komplexen Rämen. *Math. Ann.*, 133:139–159, 1957.

[35] M. L. Green. Holomorphic maps into complex projective space omitting hyperplanes. *Trans. Amer. Math. Soc.*, 169:89–103, 1972.

[36] D. Huybrechts. *Lectures on $K3$ surfaces*, volume 158 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2016.

[37] S. Kobayashi and T. Ochiai. Meromorphic mappings onto compact complex spaces of general type. *Invent. Math.*, 31(1):7–16, 1975.

[38] K. Kodaira. Pluricanonical systems on algebraic surfaces of general type. *J. Math. Soc. Japan*, 20:170–192, 1968.

[39] J. Kollár. Flips, flops, minimal models, etc. In *Surveys in differential geometry. Vol. I: Proceedings of the conference on geometry and topology, held at Harvard University, Cambridge, MA, USA, April 27-29,*

1990, pages 113–199. Providence, RI: American Mathematical Society; Bethlehem, PA: Lehigh University, 1991.
[40] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergeb. Math. Grenzgeb., 3. Folge*. Berlin: Springer-Verlag, 1995.
[41] J. Kollár. Comment on: Dense entire curves... [arxiv:1905.01104], by F. Campana and J. Winkelmann. *arXiv e-prints*, 2021. https://arxiv.org/abs/2110.08171.
[42] J. Kollár, Y. Miyaoka, and S. Mori. Rationally connected varieties. *J. Algebr. Geom.*, 1(3):429–448, 1992.
[43] Y. Kusakabe. Oka properties of complements of holomorphically convex sets. *Ann. Math.*, to appear. https://arxiv.org/abs/2005.08247.
[44] Y. Kusakabe. Elliptic characterization and localization of Oka manifolds. *Indiana Univ. Math. J.*, 70(3):1039–1054, 2021.
[45] S. Lang. Hyperbolic and Diophantine analysis. *Bull. Am. Math. Soc., New Ser.*, 14:159–205, 1986.
[46] F. Lárusson. Model structures and the Oka principle. *J. Pure Appl. Algebra*, 192(1-3):203–223, 2004.
[47] F. Morgan. *Geometric measure theory. A beginner's guide*. Elsevier/Academic Press, Amsterdam, fourth edition, 2009.
[48] J.-P. Rosay and W. Rudin. Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. *Trans. Amer. Math. Soc.*, 310(1):47–86, 1988.
[49] C. Runge. Zur Theorie der Eindeutigen Analytischen Functionen. *Acta Math.*, 6(1):229–244, 1885.
[50] E. L. Stout. Bounded holomorphic functions on finite Riemann surfaces. *Trans. Amer. Math. Soc.*, 120:255–285, 1965.
[51] A. Teleman. Donaldson theory on non-Kählerian surfaces and class VII surfaces with $b_2 = 1$. *Invent. Math.*, 162(3):493–521, 2005.
[52] A. Teleman. Instantons and curves on class VII surfaces. *Ann. Math. (2)*, 172(3):1749–1804, 2010.
[53] A. Teleman. Donaldson theory in non-Kählerian geometry. In *Modern geometry. A celebration of the work of Simon Donaldson*, pages 363–392. Providence, RI: American Mathematical Society (AMS), 2018.
[54] K. Weierstrass. Zur Theorie der eindeutigen analytischen Functionen. Berl. Abh. 11-60 (1876), 1876.

ANTONIO ALARCÓN, DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA E INSTITUTO DE MATEMÁTICAS (IMAG), UNIVERSIDAD DE GRANADA, CAMPUS DE FUENTENUEVA S/N, E–18071 GRANADA, SPAIN

Email address: alarcon@ugr.es

FRANC FORSTNERIČ, FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, AND INSTITUTE OF MATHEMATICS, PHYSICS, AND MECHANICS, JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

Email address: franc.forstneric@fmf.uni-lj.si