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Embedded complex curves in the affine plane

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Abstract

This paper brings several contributions to the classical Forster–Bell–Narasimhan conjecture and the Yang problem concerning the existence of proper, almost proper, and complete injective holomorphic immersions of open Riemann surfaces in the affine plane $\mathbb{C}^2$ satisfying interpolation and hitting conditions. We also show that every compact Riemann surface contains a Cantor set whose complement admits a proper holomorphic embedding in $\mathbb{C}^2$, and every connected domain in $\mathbb{C}^2$ admits complete, everywhere dense, injectively immersed complex discs. The focal point of the paper is a lemma saying for every compact bordered Riemann surface, M, closed discrete subset E of $\mathring{M} = M \setminus bM$, and compact subset $K \subset \mathring{M} \setminus E$ without holes in $\mathring{M}$, any $\mathscr{C}^1$ embedding $f : M \hookrightarrow \mathbb{C}^2$ which is holomorphic in $\mathring{M}$ can be approximated uniformly on K by holomorphic embeddings $F : M \hookrightarrow \mathbb{C}^2$ which map $E \cup bM$ out of a given ball and satisfy some interpolation conditions.

Keywords Riemann surface · Complex curve · Complete holomorphic embedding

Mathematics Subject Classification Primary 32E10 · 32E30 · 32H02; secondary 14H55

1 Introduction and main results

This paper contributes to the following three interesting topics in global complex geometry, the main focus being on the interrelationship between them:

- The classical Forster–Bell–Narasimhan conjecture (see [11, 19]) asking whether every open Riemann surface admits a *proper* holomorphic embedding in $\mathbb{C}^2$. The general case is still open; for positive results, see the surveys in [20, Sects. 9.10–9.11] and [16].

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- Yang's problem [50, 51] concerning the existence of *complete* bounded complex submanifolds of $\mathbb{C}^n$; see the up-to-date comprehensive survey [2].
- The existence of *dense* holomorphic curves in complex manifolds; see Forstnerič and Winkelmann [27, 47] and [6, Sect. 10].

The focal point of the paper is the following lemma, which is proved in Sect. 2.

Lemma 1.1 *Let M be a compact bordered Riemann surface with boundary of class $\mathscr{C}^s$ for some $s > 1$. Given a $\mathscr{C}^1$ embedding $f : M \hookrightarrow \mathbb{C}^2$ which is holomorphic on $\mathring{M} = M \backslash bM$, a compact set $K \subset \mathring{M}$ without holes, a compact polynomially convex set $L \subset \mathbb{C}^2$ such that $f(M \backslash \mathring{K}) \cap L = \varnothing$, finite sets $A = \{\alpha_1, \ldots, \alpha_l\} \subset \mathring{M} \setminus K$ and $B = \{\beta_1, \ldots, \beta_l\} \subset \mathbb{C}^2 \setminus L$, a closed discrete set $E \subset \mathring{M}$ such that $E \cap (A \cup K) = \varnothing$, and numbers $\epsilon > 0$ and $r > 0$, there is a holomorphic embedding $F : M \hookrightarrow \mathbb{C}^2$ satisfying the following conditions:*

(a) *$F(bM \cup E) \cap r\overline{\mathbb{B}} = \varnothing$. (Here, $\mathbb{B}$ denotes the unit ball of $\mathbb{C}^2$.)*
(b) *$F(M \setminus \mathring{K}) \cap L = \varnothing$.*
(c) *$\sup_{x \in K} |F(x) - f(x)| < \epsilon$.*
(d) *F agrees with f to a given finite order at a given finite set of points in $K \setminus f^{-1}(B)$.*
(e) *$F(\alpha_j) = \beta_j$ for $j = 1, \ldots, l$.*

We may assume that the surface M in the lemma is a closed domain in a compact Riemann surface, R, whose boundary bM consists of real analytic Jordan curves, and M has no connected component without boundary. (See Stout [43, Theorem 8.1] and note that any conformal diffeomorphism of M onto such a domain is of class $\mathscr{C}^1(M)$; see [7, Theorem 1.10.10].) A map $F : M \to \mathbb{C}^2$ is said to be holomorphic if it extends to a holomorphic map from an open neighbourhood of M in the ambient Riemann surface. A *hole* of a compact set in an open surface is a relatively compact connected component of its complement. See Remark 1.11 concerning the validity of the hypotheses of the lemma.

Lemma 1.1 is based on techniques developed by Wold [48, 49] and Forstnerič and Wold [28] for constructing proper holomorphic embeddings of bordered Riemann surfaces in $\mathbb{C}^2$. In their constructions, some boundary points of the given surface M are sent to infinity where they remain at all subsequent steps. Our proof of Lemma 1.1 uses a similar construction with additional precision, but the points at infinity are finally brought back to $\mathbb{C}^2$. Thus, the main novelty of Lemma 1.1, and of the related Lemma 4.2, is that we keep the entire Riemann surface M as an embedded complex curve in $\mathbb{C}^2$ while pushing its boundary and the discrete set $E \subset \mathring{M}$ (or a countable family of discs in Lemma 4.2) arbitrarily far towards infinity.

These two lemmas lead to new existence, approximation, interpolation, and hitting theorems for complete injectively immersed complex curves in $\mathbb{C}^2$, or in domains of $\mathbb{C}^2$, satisfying additional global conditions such as being proper, almost proper, or dense, which are presented in the sequel. At the same time, they give a simpler proof of a number of known results. Our lemmas reduce proofs of these applications to formal induction schemes without the need of dealing with the technical issues at every step. A similar role is played by the Riemann–Hilbert method (see [17, 22] and [7, Chapter 6]), but when the target is a complex surface that technique, unlike ours, tends to introduce self-intersections which cannot be removed since a generic immersion from a Riemann surface has transverse double points.

As a first illustration of the constructions that can be carried out using Lemma 1.1, we establish the following interpolation result for almost proper injective holomorphic immersions of bordered Riemann surfaces in $\mathbb{C}^2$. It is proved in Sect. 3.

Theorem 1.2 *Let M be a compact bordered Riemann surface with $\mathscr{C}^s$ boundary for some $s > 1$, and let $f : M \hookrightarrow \mathbb{C}^2$ be a $\mathscr{C}^1$ embedding which is holomorphic on $\mathring{M}$. Given a* *compact set $K \subset \mathring{M}$ without holes, a discrete sequence $\alpha_j \in \mathring{M} \setminus K$ without repetition which clusters only on bM, a sequence $\beta_j \in \mathbb{C}^2$ without repetition, and a number $\epsilon > 0$, there is an almost proper injective holomorphic immersion $F : \mathring{M} \hookrightarrow \mathbb{C}^2$ satisfying the following conditions:*

(i) $\sup_{x \in K} |F(x) - f(x)| < \epsilon$.
(ii) *F agrees with f to a given finite order at a given finite set in $K \setminus f^{-1}(\{\beta_j : j = 1, 2, \dots\})$.*
(iii) *$F(\alpha_j) = \beta_j$ for all $j = 1, 2, \dots$.*

In particular, F can be chosen such that $F(\mathring{M})$ is everywhere dense in $\mathbb{C}^2$.

If the sequence $\beta_j \in \mathbb{C}^2$ is closed and discrete, then there is a proper holomorphic embedding $F : \mathring{M} \hookrightarrow \mathbb{C}^2$ satisfying conditions (i)-(iii).

Recall that a continuous map $f : X \to Y$ of topological spaces is said to be *almost proper* if for every compact set $K \subset Y$ the connected components of $f^{-1}(K)$ are all compact. Given a compact bordered Riemann surface M as in Theorem 1.2, one cannot hit an arbitrary countable subset of $\mathbb{C}^2$ by proper holomorphic maps $\mathring{M} \to \mathbb{C}^2$, but this can be done by almost proper maps. In fact, almost proper maps are in some sense the best class of holomorphic maps $\mathring{M} \to \mathbb{C}^2$ that can hit any given countable subset of $\mathbb{C}^2$.

The last part of Theorem 1.2 implies the following known result concerning the Forster–Bell–Narasimhan Conjecture; see Globevnik [30] in the case of the disc, and [28, Corollaries 1.2 and 1.3] by Forstnerič and Wold and [35, Theorem 1] by Kutzschebauch et al. for an arbitrary M. We state it with additional precision concerning approximation and interpolation.

Corollary 1.3 *Given a holomorphic embedding $f : M \hookrightarrow \mathbb{C}^2$ of a compact bordered Riemann surface M, a compact set $K \subset \mathring{M}$ without holes, and closed discrete sequences $\alpha_j \in \mathring{M}$ and $\beta_j \in \mathbb{C}^2$ without repetitions such that $K \cap \{\alpha_j : j \in \mathbb{N}\} = \varnothing$, we can approximate f uniformly on K by proper holomorphic embeddings $F : \mathring{M} \hookrightarrow \mathbb{C}^2$ satisfying $F(\alpha_j) = \beta_j$ for all $j = 1, 2, \dots$.*

We give a unified proof of Theorem 1.2 and Corollary 1.3, based on Lemma 1.1, and we supply some details related to [35, Lemma 2.2]; see Remark 2.1. The analogous result for algebraic curves in $\mathbb{C}^2$ is a special case of [24, Theorem 1.3]; see also [20, Theorem 4.17.1].

Recall that an immersed submanifold $\varphi : Z \to \mathbb{R}^n$ is said to be *complete* if the Riemannian metric on Z, obtained by pulling back the Euclidean metric on $\mathbb{R}^n$ via φ, is a complete metric; equivalently, for every proper path $\gamma : [0, 1) \to Z$ the path $\varphi \circ \gamma : [0, 1) \to \mathbb{R}^n$ has infinite Euclidean length. It is obvious that every almost proper immersion $\varphi : Z \to \mathbb{R}^n$ is complete, hence the immersions F in Theorem 1.2 are complete. It seems that this gives the first examples of a specific bordered Riemann surface, other than the disc, admitting a complete nonproper injective holomorphic immersion into $\mathbb{C}^2$. The construction of complete injectively immersed complex lines $\mathbb{C} \hookrightarrow \mathbb{C}^2$ with dense images was given by the authors in [5].

The technique of bringing back the points at infinity also applies in conjunction with [29, Lemma 3.1], thereby yielding an analogue of Lemma 1.1 for circle domains with countably many boundary components in the Riemann sphere $\mathbb{CP}^1$; see Lemma 4.2. This gives a simpler proof of the theorem of Forstnerič and Wold [29, Theorem 1.1] saying that every circle domain in $\mathbb{CP}^1$ embeds properly holomorphically in $\mathbb{C}^2$; see Theorem 4.1. (For domains with finitely many boundary components, this was proved by Globevnik and Stensønes [31].) As indicated in [29, p. 500], the analogous result likely holds for circle domains in tori.

Nothing seems to be known about this problem for domains in compact Riemann surfaces of genus > 1, where the main problem is to find a suitable initial embedding of the uniformized surface into $\mathbb{C}^2$. On the other hand, in Sect. 5 we use Lemma 1.1 to prove the following hitting theorem for almost proper injective holomorphic immersions in $\mathbb{C}^2$ from domains obtained by removing countably many pairwise disjoint closed discs from any compact Riemann surface.

Theorem 1.4 *Let R be a compact Riemann surface and $\Omega = R\setminus\bigcup_{i=0}^{\infty} D_i$ be an open domain in R whose complement is the union of countably many pairwise disjoint closed discs D_i with $\mathscr{C}^s$ boundaries for some $s > 1$. Given a $\mathscr{C}^1$ embedding $f : M_k = R\setminus\bigcup_{i=0}^{k} \mathring{D}_i \hookrightarrow \mathbb{C}^2$ for some $k \geq 0$ that is holomorphic on the open bordered surface $\mathring{M}_k = R \setminus \bigcup_{i=0}^{k} D_i$, a compact set $K \subset \Omega$, a number $\epsilon > 0$, and a countable set $B \subset \mathbb{C}^2$, there is an almost proper (hence complete) injective holomorphic immersion $F : \Omega \hookrightarrow \mathbb{C}^2$ such that*

(i) $\sup_{x\in K} |F(x) - f(x)| < \epsilon$,
(ii) *F agrees with f to a given finite order at a given finite set of points in Ω, and*
(iii) $B \subset F(\Omega)$.

In particular, there exists an almost proper (hence complete) injective holomorphic immersion $\Omega \hookrightarrow \mathbb{C}^2$ with everywhere dense image.

Lemma 1.1 can also be combined with the method developed by Forstnerič [21] for constructing complete bounded embedded holomorphic null curves in $\mathbb{C}^3$ with Cantor ends, as well as complete bounded minimal surfaces in $\mathbb{R}^3$ and some other related types of surfaces with Cantor ends. In this way we obtain the following result proved in Sect. 6.

Theorem 1.5 *If R is a compact Riemann surface and $B \subset \mathbb{C}^2$ is a countable subset, there exist a Cantor set $C \subset R$ and an almost proper injective holomorphic immersion $F : R\setminus C \hookrightarrow \mathbb{C}^2$ whose image contains B. If B is closed and discrete in $\mathbb{C}^2$ then F can be chosen to be a proper holomorphic embedding. Hence, every compact Riemann surface contains a Cantor set whose complement admits a proper holomorphic embedding in $\mathbb{C}^2$.*

The Cantor sets which arise in the proof of Theorem 1.5 are small modifications of the standard Cantor set in the plane, and they have almost full measure in a surrounding domain. The last statement in Theorem 1.5 generalizes a recent result by Di Salvo and Wold [16, Theorem 1.1], who constructed a Cantor set of large measure in $\mathbb{CP}^1$ whose complement admits a proper holomorphic embedding in $\mathbb{C}^2$. The first examples of Cantor sets in $\mathbb{CP}^1$ whose complements embed properly in $\mathbb{C}^2$ were given by Orevkov [37], and Di Salvo [15] showed that Orevkov's construction also yields examples having Hausdorff dimension zero.

So far, we have been talking about (almost) proper injective holomorphic immersions in $\mathbb{C}^2$. However, Lemma 1.1 can also be applied to the construction of complete injectively immersed holomorphic curves in more general domains in $\mathbb{C}^2$, at the cost of losing control of their conformal structure and in some case of almost properness.

To motivate this line of developments, we recall that Yang [50, 51] asked in 1977 whether there exist complete bounded complex submanifolds of a complex Euclidean space $\mathbb{C}^n$ of dimension > 1. The Yang problem has been a focus of interest in the last decades; we refer to the recent survey [2]. It is an open problem whether for every compact bordered Riemann surface M as in Lemma 1.1 there is a complete holomorphic embedding $\mathring{M} \hookrightarrow \mathbb{C}^2$ with bounded image; see [8, Problem 1.5]. In fact, given such a Riemann surface M other than the closed disc, all known complete holomorphic embeddings $\mathring{M} \hookrightarrow \mathbb{C}^2$ are proper in $\mathbb{C}^2$ (see [28, Corollary 1.2]) or else the complex structure of the embedded surface may change. By using Lemma 1.1, we construct complete embedded complex curves with a given smooth structure in any pseudoconvex Runge domain of $\mathbb{C}^2$ as in the following theorem. In this case one clearly cannot control the complex structure of the examples.

Theorem 1.6 *Let $D \subset \mathbb{C}^2$ be a pseudoconvex Runge domain and B be a countable subset of D. On every open Riemann surface S there are a domain M, which is diffeotopic to S, and a complete, almost proper, injective holomorphic immersion $F : M \hookrightarrow D$ such that $B \subset F(M)$. If in addition the set B is closed in D and discrete, then $F : M \hookrightarrow D$ can be chosen to be a complete proper holomorphic embedding.*

Theorem 1.6 is proved in Sect. 7. The special case when $D = \mathbb{C}^2$ and $B = \varnothing$, guaranteeing the existence of properly embedded complex curves in $\mathbb{C}^2$ with arbitrary topology, was established in 2013 by Alarcón and López [9, Theorem 4.5]. This showed that there is no topological restriction to the Forster–Bell–Narasimhan conjecture. (For embeddings in $\mathbb{C}\times\mathbb{C}^*$ and $(\mathbb{C}^*)^2$, see Ritter [38, 39], Lárusson and Ritter [36], and Remark 2.3.) In the special case when $D = \mathbb{B}$ is the open unit ball and $B \subset \mathbb{B}$ is closed and discrete, Theorem 1.6 was proved by Alarcón and Globevnik [8]. Likewise, when $D = \mathbb{B}$ or $D = \mathbb{C}^2$ and S is of finite topology, it was established by the authors in [5], except for the almost properness condition. For arbitrary pseudoconvex Runge domains D in $\mathbb{C}^2$, Theorem 1.6 also generalizes and simplifies the proofs of some hitting results for (not necessarily complete) properly embedded complex curves in D due to Forstnerič, Globevnik, and Stensønes [23] and Alarcón [1]. Adapting the arguments in [5, 8, 9] to the use of labyrinths of compact sets in pseudoconvex Runge domains, constructed by Charpentier and Kosiński in [13], leads to a proof of Theorem 1.6 in the case when B is closed in D and discrete, or S is finitely connected. The proof of Theorem 1.6 that we give here, based on Lemma 1.1 and using the labyrinths from [13], is considerably simpler and provides the general case of the theorem.

In Sect. 8 we establish the following analogue of Theorem 1.6 in which we do not impose any condition whatsoever on the given connected domain in $\mathbb{C}^2$; the cost being not to guarantee almost properness of the obtained immersion.

Theorem 1.7 *Let $X \subset \mathbb{C}^2$ be a connected domain and B be a countable subset of X. Given an open Riemann surface S, there are a domain $M \subset S$, which is diffeotopic to S, and a complete injective holomorphic immersion $F : M \hookrightarrow X$ such that $B \subset F(M)$. In particular, F can be chosen to have everywhere dense image in X.*

All similar results in the literature pertain to special domains in $\mathbb{C}^2$; see the discussion below Theorem 1.6 and the survey [2]. On the other hand, omitting the injectivity condition in dimension two, it was shown by Forstnerič and Winkelman [27, 47] that every connected complex manifold X with $\dim X > 1$ admits an immersed holomorphic disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \to X$ with dense image; if $\dim X > 2$ then the immersion can be chosen injective. Recently the analogous result was obtained by the authors for any bordered Riemann surface and for some other classes of open Riemann surfaces [6, Theorem 10.1].

Let d_{H} denote the Hausdorff distance between subsets of Euclidean spaces. The following corollary, which follows by inspecting the proof of Theorem 1.7, shows that every embedded holomorphic disc is arbitrarily close to a complete one in the Hausdorff distance.

Corollary 1.8 *Given a $\mathscr{C}^1$ embedding $G : \overline{\mathbb{D}} \hookrightarrow \mathbb{C}^2$ which is holomorphic on $\mathbb{D}$, a compact set $K \subset \mathbb{D}$, and a number $\epsilon > 0$, there is a complete injective holomorphic immersion $F : \mathbb{D} \to \mathbb{C}^2$ such that $|F - G| < \epsilon$ on K and $d_{\mathrm{H}}(\overline{F(\mathbb{D})}, G(\overline{\mathbb{D}})) < \epsilon$.*

We do not know whether the injective immersion F in Corollary 1.8 can be chosen to extend continuously to $\overline{\mathbb{D}}$ or to satisfy $|F - G| < \epsilon$ on $\mathbb{D}$. In particular, it remains an open question whether F can be chosen such that $F(\mathbb{D})$ is bounded by a Jordan curve. All these tasks can be carried out if one allows the map F to have double points; see [3, 4, 45].

Insisting on the almost properness condition, we also establish the following result, which is obtained by a slight modification of the proof of Theorem 1.7. Again, we do not impose any condition whatsoever on the given connected domain X in $\mathbb{C}^2$.

Theorem 1.9 *Let $X \subset \mathbb{C}^2$ be a connected domain and B be a countable subset of X. On every open Riemann surface S there exists a connected, relatively compact domain M such that M has the same topological genus as S and there is a complete almost proper, injective holomorphic immersion $F : M \to X$ with $B \subset F(M)$. In particular, F can be chosen to have everywhere dense image in X.*

Our method of proof does not allow to ensure that the domain M in Theorem 1.9 is homeomorphic to the given open Riemann surface S. In particular, we cannot control its ends set, which could be more complicated than that of S.

Remark 1.10 (*On completeness*) By a minor modification of the proofs, using that any two metrics on a compact space are comparable, we can ensure that the injective holomorphic immersions F obtained in Theorems 1.2, 1.4, 1.5, 1.6, 1.7, and 1.9 are complete with respect to any given Riemannian metric (not necessarily complete or the Euclidean one) on the target domain $\mathbb{C}^2$, $D \subset \mathbb{C}^2$, or $X \subset \mathbb{C}^2$, respectively.

Remark 1.11 (*On the hypotheses in Lemma* 1.1) It is not known whether every compact bordered Riemann surface embeds holomorphically in $\mathbb{C}^2$. Here is a way to obtain such surfaces. Any compact Riemann surface, R, admits a holomorphic immersion $f : R \to \mathbb{CP}^2$ in the projective plane with finitely many simple double points $f(a_j) = f(b_j)$, where $a_j \neq b_j$ for $j = 1, \ldots, m$ (see Griffiths and Harris [32]). Given a complex line $\Lambda \subset \mathbb{CP}^2$, the punctured Riemann surface $R' = R \backslash (f^{-1}(\Lambda) \cup \{b_1, \ldots, b_m\})$ is injectively immersed in $\mathbb{CP}^2 \backslash \Lambda \cong \mathbb{C}^2$, and hence any compact domain in R' is embedded in $\mathbb{C}^2$. There is considerable freedom in the above choices, showing that most domains with smooth boundary in any compact Riemann surface satisfy Lemma 1.1. However, we are not aware of suitable results in the literature on controlling the location of double points in an immersed compact Riemann surface in $\mathbb{CP}^2$, and it seems an open problem whether one could put all punctures in the above construction in an arbitrarily small disc around any given point of R. If this were true, then one could embed the interior of any finite bordered Riemann surface properly holomorphically in $\mathbb{C}^2$.

2 Proof of Lemma 1.1

The proof involves four main steps: (1) using a holomorphic automorphism of $\mathbb{C}^2$ to satisfy the interpolation conditions in (e), (2) exposing and sending to infinity a point in each boundary component of M, (3) pushing the boundary bM and the discrete set E out of a given ball by a holomorphic automorphism of $\mathbb{C}^2$ (see condition (a)), and (4) bringing back the points at infinity. The first three steps are obtained by following and augmenting the proofs of [28, Corollary 1.2] and [35, Lemma 2.2], while the last step uses a new idea. For the sake of readability we give a complete exposition, beginning with preliminaries.

Denote the coordinates on $\mathbb{C}^2$ by $z = (z_1, z_2)$, and let $\pi_i : \mathbb{C}^2 \to \mathbb{C}$ for $i = 1, 2$ denote the projection $\pi_i(z_1, z_2) = z_i$. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{B}$ the open unit ball in $\mathbb{C}^2$.

Let $f : M \hookrightarrow \mathbb{C}^2$ and the sets $B, L \subset \mathbb{C}^2$ be as in Lemma 1.1, and let $c_1, \ldots, c_{l'} \in K \setminus f^{-1}(B)$ denote the points at which we must fulfil the interpolation condition (d). By Mergelyan theorem (see [18, Theorem 16]) we can approximate f in the $\mathscr{C}^1(M)$ topology by a holomorphic map $\tilde{f} : U \to \mathbb{C}^2$ from an open neighbourhood $U \subset R$ of M which agrees with f to a given finite order k at every point $c_1, \ldots, c_{l'}$. Assuming that the approximation is close enough and up to shrinking U around M, we may assume that $\tilde{f} : U \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding satisfying $\tilde{f}(U \setminus \mathring{K}) \cap L = \varnothing$. By a standard transversality argument we can also ensure that $B \cap \tilde{f}(U) = \varnothing$. We replace f by $\tilde{f}$ and drop the tilde.

Finally, by a small perturbation of the map f, keeping the above conditions in place, we can ensure that the set $L \cup f(M)$ is polynomially convex in $\mathbb{C}^2$. Indeed, by Stolzenberg's theorem [41] the polynomial hull of $L \cup f(bM)$ is the union of this set with complex curves having their boundaries in $L \cup f(bM)$, and we can arrange that there are no such curves besides $f(M)$. Here is an explicit way of doing this. Choose a compact domain $M' \subset U$ containing M in its interior such that M is a strong deformation retract of M'. We may assume that the function $f_1 = \pi_1 \circ f \in \mathscr{O}(U)$ is nonconstant on each component of U. We approximate $f_2 = \pi_2 \circ f$ on M (with interpolation at the points $c_1, \ldots, c_{l'}$) by a smooth function $\hat{f}_2$ on M' which is holomorphic on $\mathring{M}'$ and does not extend holomorphically across any boundary point of M'. Set $\hat{f} = (f_1, \hat{f}_2)$. Take a point $p \in bM'$ at which $df_1(p) \neq 0$. (Note that almost every point of bM' is such.) Locally at the image point $q = \hat{f}(p) \in \mathbb{C}^2$ we can represent the complex curve $\Sigma' = \hat{f}(M')$ with smooth boundary $b\Sigma' = \hat{f}(bM')$ as a graph over the first coordinate such that the graphing function is holomorphic on the local projection of Σ' but does not extend holomorphically past the point $q_1 = \pi_1(q) = f_1(p)$. It follows that Σ' is not contained in any complex curve containing q in the interior, since such a curve would provide a holomorphic extension of the graphing function to a neighbourhood of q_1. We claim that $L \cup \Sigma'$ is polynomially convex. Indeed, by Stolzenberg [41] the set $\widehat{L \cup b\Sigma'} \setminus (L \cup b\Sigma')$ is a pure one-dimensional closed complex subvariety which is closed in $\mathbb{C}^2 \setminus (L \cup b\Sigma')$. If this subvariety has an irreducible component Λ which is not contained in Σ', then $\overline{\Lambda}$ must contain a connected component C of $b\Sigma'$, and the boundary uniqueness theorem (see Chirka [14, Proposition 1, p. 258]) shows that $\Sigma' \cup \Lambda$ is a complex curve in $\mathbb{C}^2$ containing C, contradicting the choice of $\hat{f}$. This proves the claim. Since M is holomorphically convex in M', Rossi's local maximum modulus principle (see Rosay [40] for a simple proof) implies that $L \cup \hat{f}(M)$ is also polynomially convex. Furthermore, for every compact set $K \subset \mathring{M}$ without holes such that $\hat{f}(M \setminus \mathring{K}) \cap L = \varnothing$ the set $L \cup \hat{f}(K)$ is polynomially convex. We now replace f by a map satisfying all the stated conditions.

(1) *Fulfilling condition (e) in the lemma.* We have arranged above that the set $L \cup f(M)$ is polynomially convex in $\mathbb{C}^2$. Since K has no holes in $\mathring{M}$ and $f(M \setminus \mathring{K}) \cap L = \varnothing$, the set

$$L' := L \cup f(K) \tag{2.1}$$

is also polynomially convex (see the argument above). Recall that

$$A = \{\alpha_1, \ldots, \alpha_l\} \subset \mathring{M} \setminus K \quad \text{and} \quad B = \{\beta_1, \ldots, \beta_l\} \subset \mathbb{C}^2 \setminus L'.$$

By the choice of f we have that $B \cap f(U) = \varnothing$. By [20, Proposition 4.15.3] there is a holomorphic automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^2)$ which approximates the identity map on L', it agrees with the identity to a given finite order k at the points $f(c_1), \ldots, f(c_{l'}) \in f(K) \subset L'$, and it satisfies $\Phi(f(\alpha_j)) = \beta_j$ for $j = 1 \ldots, l$. Replacing f by $\Phi \circ f$ we may thus assume that f fulfills condition (e) in the lemma and the other properties remain in place. We now add to A the given finite set of points in $\mathring{M}$ at which we shall interpolate the map to a given finite order in the subsequent steps of the proof, and we suitably enlarge the set $K \subset \mathring{M}$ so that it has no holes and contains this new bigger set A, while the discrete set $E \subset \mathring{M}$ remains in $\mathring{M} \setminus (A \cup K)$. This will ensure that the final map F will satisfy condition (e).

(2) *Exposing boundary points.* We follow the exposition in [28, Sect. 4]. On each boundary curve $C_j \subset bM$ we choose a point a_j and attach to M a smooth embedded arc $\gamma_j \subset U$ such that $\gamma_j \cap M = \{a_j\}$, the intersection of bM and γ_j is transverse at a_j, and the arcs $\gamma_1, \ldots, \gamma_m$ are pairwise disjoint. Let b_j denote the other endpoint of γ_j. On the image side, we choose smoothly embedded pairwise disjoint arcs $\lambda_1, \ldots, \lambda_m \subset \mathbb{C}^2 \setminus L'$, where L' is given by (2.1), such that for every $j = 1, \ldots, m$ we have that $\lambda_j \cap f(M) = f(a_j)$, λ_j agrees with $f(\gamma_j)$ near the endpoint $f(a_j)$, the other endpoint p_j of λ_j satisfies

$$|\pi_2(p_j)| > \sup\{|\pi_2(z)| : z \in L'\},$$

the set $f(M) \cup \bigcup_{i=1}^m \lambda_i$ intersects the complex line

$$\Lambda_j = \pi_2^{-1}(\pi_2(p_j)) = \mathbb{C} \times \{\pi_2(p_j)\} \tag{2.2}$$

only at the point p_j, and the tangent vector to λ_j at p_j has nonvanishing second component. (See [28, Fig. 2, p. 109] where the projection π_1 is used in place of π_2.)

We now modify f, keeping it fixed on a neighbourhood $U_1 \subset U$ of M and extending it to a smooth diffeomorphism $\gamma_j \to \lambda_j$ for every $j = 1, \ldots, m$ such that $f(b_j) = p_j$. Set

$$S = M \cup \bigcup_{j=1}^{m} \gamma_j \subset R.$$

Applying Mergelyan theorem (see [18, Theorem 16]), we can approximate f as closely as desired in $\mathscr{C}^1(S)$ by a holomorphic map $\tilde f : V \to \mathbb{C}^2$ on an open neighbourhood $V \subset R$ of S such that $\tilde f$ agrees with f to a given finite order k at the points in the finite set $A \subset \mathring{M}$ defined in the previous step, and $\tilde f$ agrees with f to the second order at the endpoints a_j and b_j of γ_j for $j = 1, \ldots, m$. If the approximation is close enough and up to shrinking V around S, the map $\tilde f : V \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding satisfying

$$\tilde f(V \setminus \mathring{K}) \cap L = \varnothing. \tag{2.3}$$

Furthermore, for every $j = 1, \ldots, m$ the complex line Λ_j (2.2) intersects the embedded complex curve $\tilde f(V)$ only at the point $p_j = \tilde f(b_j) = f(b_j)$ and the intersection is transverse.

We have now arrived at the main point of the exposing of points technique. By [28, Theorem 2.3] there is a conformal diffeomorphism

$$\tau : M \to \tau(M) \subset V \tag{2.4}$$

such that for every $j = 1, \ldots, m$ we have that $\tau(a_j) = b_j$, τ maps a small neighbourhood $U_j \subset M$ of the point $a_j \in bM$ in a thin tube around the arc γ_j, τ agrees with the identity map to a given order k at the points of the finite set $A \subset \mathring{M}$, and τ is arbitrarily $\mathscr{C}^1$ close to the identity map on $M \setminus \bigcup_{j=1}^m U_j$. We can choose τ to have any finite order of smoothness on M; for technical reasons which will become apparent in the next step we shall assume that it is of class $\mathscr{C}^3(M)$. The map

$$h = \tilde f \circ \tau : M \hookrightarrow \mathbb{C}^2 \tag{2.5}$$

is then a $\mathscr{C}^3$ embedding which is holomorphic on $\mathring{M}$, its image $h(M)$ is a compact domain with $\mathscr{C}^3$ boundary in the embedded complex curve $\tilde f(V) \subset \mathbb{C}^2$, and τ can be chosen such that for each $j = 1, \ldots, m$ the complex line Λ_j (2.2) intersects $h(M)$ only at $p_j = f(b_j)$. Assuming as we may that τ is close enough to the identity on K, (2.3) implies

$$h(M \setminus \mathring{K}) \cap L = \varnothing.$$

Let g be a rational shear on $\mathbb{C}^2$ of the form

$$g(z_1, z_2) = \left(z_1 + \sum_{j=1}^{m} \frac{\rho\, \mathrm{e}^{\mathrm{i}\theta_j}}{z_2 - \pi_2(p_j)} + P(z_2), z_2 \right) \tag{2.6}$$

where $\rho > 0$, $\theta_j \in \mathbb{R}$, and $P(z_2)$ is a holomorphic polynomial chosen such that the function $\sum_{j=1}^{m} \frac{\rho\, \mathrm{e}^{\mathrm{i}\theta_j}}{z_2 - \pi_2(p_j)} + P(z_2)$ vanishes to order k at the point $\pi_2(f(a))$ for every $a \in A$, and P vanishes at every point $\pi_2(p_j)$ for $j = 1, \ldots, m$. By taking the constant $\rho > 0$ in (2.6) sufficiently small, the polynomial P can be chosen such that $|P|$ is as small as desired on the compact set $\pi_2(L')$ where L' is given by (2.1). This gives a holomorphic embedding

$$g \circ \tilde{f} : V \setminus \{b_1, \ldots, b_m\} \hookrightarrow \mathbb{C}^2 \tag{2.7}$$

with simple poles at the points $b_1, \ldots, b_m$. Similarly, we have a $\mathscr{C}^3$ embedding

$$g \circ h = g \circ \tilde{f} \circ \tau : M' = M \setminus \{a_1, \ldots, a_m\} \hookrightarrow \mathbb{C}^2 \tag{2.8}$$

which is holomorphic on $\mathring{M}$, it approximates the embedding h (2.5) as closely as desired on K provided that the constant $\rho > 0$ in (2.6) is chosen small enough, it satisfies

$$(g \circ h)(M' \setminus \mathring{K}) \cap L = \varnothing, \tag{2.9}$$

it agrees with h (and hence with f) to order k at the points of the finite set A, and the map $g \circ h$ sends the points $a_1, \ldots, a_m$ to infinity. More precisely, recalling that $bM = \bigcup_{j=1}^{m} C_j$, for every $j = 1, \ldots, m$ the set

$$\sigma_j = (g \circ h)(C_j \setminus \{a_j\}) \subset \mathbb{C}^2 \tag{2.10}$$

is a properly embedded curve of class $\mathscr{C}^3$, diffeomorphic to $\mathbb{R}$, which is asymptotic to a line at every end (see [49, Lemma 2] for the details), the first coordinate projection $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ maps σ_j to a proper curve $\tilde{\sigma}_j = \pi_1(\sigma_j) \subset \mathbb{C}$, and $\pi_1 : \sigma_j \to \tilde{\sigma}_j$ is a diffeomorphism near infinity. Furthermore, the numbers $\theta_j \in \mathbb{R}$ in (2.6) can be chosen such that the projected curves $\tilde{\sigma}_j$ have different asymptotic directions, and for every sufficiently big number $s > 0$ the set $\mathbb{C} \setminus \big(s\overline{\mathbb{D}} \cup \bigcup_{j=1}^{m} \tilde{\sigma}_j \big)$ has no bounded connected components. These choices are independent of the number $\rho > 0$, which can be chosen arbitrarily small.

With the curves σ_j given by (2.10) and the discrete set $E \subset \mathring{M}$ as in the lemma, we define

$$\Gamma = \bigcup_{j=1}^{m} \sigma_j \subset \mathbb{C}^2 \quad \text{and} \quad E' = (g \circ h)(E) \subset \mathbb{C}^2 \setminus \Gamma. \tag{2.11}$$

Since E only clusters on bM, the set E' only clusters on Γ, so $E' \cup \Gamma$ is closed and the projection $\pi_1 : E' \cup \Gamma \to \mathbb{C}$ is proper. Furthermore, since the curves σ_j are asymptotic to lines at infinity, for any $\mathbb{C}$-linear projection $\pi'_1 : \mathbb{C}^2 \to \mathbb{C}$ sufficiently close to π_1 the projection $\pi'_1 : E' \cup \Gamma \to \mathbb{C}$ is still proper. By a general position argument, π'_1 can be chosen such that

$$\pi'_1 : E' \cup \Gamma \to \mathbb{C} \text{ is proper}, \ \pi'_1 : E' \to \mathbb{C} \text{ is injective, and } \pi'_1(E') \cap \pi'_1(\Gamma) = \varnothing. \tag{2.12}$$

By a linear change of coordinates on $\mathbb{C}^2$ we may assume that this holds for π_1. Furthermore, if π_1 was not changed much, then by our conditions on Γ there is a number $s_0 \geq 0$ such that

$$\text{the domain } \mathbb{C} \setminus (s\overline{\mathbb{D}} \cup \pi_1(\Gamma)) \text{ has no holes for } s \geq s_0. \tag{2.13}$$

Remark 2.1 The last two conditions in (2.12) are not discussed in [35, proof of Lemma 2.2]. Without them, we are unable to complete the proof of Lemma 2.2 [the problem appears in the construction of a shear ψ in (2.16)]. Indeed, we do not know how to prove [35, Lemma 2.2] without assuming that the linear projections $\mathbb{C}^2 \to \mathbb{C}$ sufficiently close to π_1 are proper on Γ.

(3) *Pushing $E' \cup \Gamma$ out of the ball $r\overline{\mathbb{B}}$.* We shall find an automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^2)$ sending the set $E' \cup \Gamma$ out of the given ball $r\overline{\mathbb{B}}$. This is accomplished by the following lemma based on [49, Lemma 1] and [35, Lemma 2.2]. In light of Remark 2.1, we include a proof.

Lemma 2.2 *Let E' and Γ be as in (2.11), satisfying conditions (2.12) and (2.13) for the projection $\pi_1(z_1, z_2) = z_1$. Given a compact polynomially convex set $L \subset \mathbb{C}^2$ with $L \cap (E' \cup \Gamma) = \varnothing$ and numbers $r > 0$ (big) and $\epsilon > 0$ (small), there is an automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^2)$ satisfying the following conditions:*

(i) $|\Phi(z) - z| < \epsilon$ *for all* $z \in L$,
(ii) $\Phi(E' \cup \Gamma) \subset \mathbb{C}^2 \setminus r\overline{\mathbb{B}}$, *and*
(iii) Φ *agrees with the identity map to a given order k at given points $q_1, \ldots, q_l \in L$.*

Proof We shall obtain Φ as a composition $\Phi = \phi \circ \psi$ of two automorphisms, where ϕ will do the main job and ψ will be a shear (2.16) taking care of things at infinity.

Choose a compact polynomially convex set $L' \subset \mathbb{C}^2$ containing L in its interior such that $L' \cap (E' \cup \Gamma) = \varnothing$ and a number $\epsilon' > 0$ to be specified later. Let $s_0 \geq 0$ be as in (2.13). Pick $s \geq s_0$ such that $L' \subset s\mathbb{D} \times \mathbb{C}$ and set

$$\widetilde{E} = E' \cap (s\overline{\mathbb{D}} \times \mathbb{C}) \quad \text{and} \quad \widetilde{\Gamma} = \Gamma \cap (s\overline{\mathbb{D}} \times \mathbb{C}). \tag{2.14}$$

We move $\widetilde{\Gamma}$ out of the ball $r\overline{\mathbb{B}}$ by an isotopy of embeddings of class $\mathscr{C}^3$ within the set $\mathbb{C}^2 \setminus L'$. Since $\widetilde{\Gamma}$ is a union of smooth embedded pairwise disjoint arcs, the union of L' with the image of $\widetilde{\Gamma}$ at every stage of the isotopy is polynomially convex by Stolzenberg [42]. Hence, [25, Theorem 2.1] due to Forstnerič and Løw furnishes an automorphism $\phi_1 \in \mathrm{Aut}(\mathbb{C}^2)$ satisfying

(i') $|\phi_1(z) - z| < \epsilon'$ for all $z \in L'$, and
(ii') $\phi_1(\widetilde{\Gamma}) \cap r\overline{\mathbb{B}} = \varnothing$.

(The proof of [25, Theorem 2.1] relies on the Andersén–Lempert theorem in the version given by Forstnerič and Rosay [26]; see also [20, Theorem 4.9.2].)

Since the discrete set $\widetilde{E}$ only clusters on $\widetilde{\Gamma}$ (see (2.14)), condition (i') implies that $\widetilde{E} = \widetilde{E}_1 \cup \widetilde{E}_2$ where $\phi_1(\widetilde{E}_1) \cap r\overline{\mathbb{B}} = \varnothing$ and the set $\widetilde{E}_2 = \widetilde{E} \setminus \widetilde{E}_1$ is finite. The compact set $\widetilde{E}_1 \cup \widetilde{\Gamma} \cup L'$ is polynomially convex by [35, Lemma 2.3]. Therefore, [20, Proposition 4.15.3] furnishes an automorphism $\phi_2 \in \mathrm{Aut}(\mathbb{C}^2)$ which is arbitrarily close to the identity map on $\phi_1(\widetilde{E}_1 \cup \widetilde{\Gamma} \cup L')$ and maps the finite set $\phi_1(\widetilde{E}_2)$ into $\mathbb{C}^2 \setminus r\overline{\mathbb{B}}$. If the approximation is close enough then the automorphism $\phi_2 \circ \phi_1 \in \mathrm{Aut}(\mathbb{C}^2)$ satisfies the conditions

(i'') $|(\phi_2 \circ \phi_1)(z) - z| < \epsilon'$ for all $z \in L'$, and
(ii'') $(\phi_2 \circ \phi_1)(\widetilde{E} \cup \widetilde{\Gamma}) \cap r\overline{\mathbb{B}} = \varnothing$.

We now correct $\phi_2 \circ \phi_1$ so that the above conditions are preserved and the interpolation condition (iii) holds. Since $\phi_2 \circ \phi_1$ is close to the identity on L', its k-jet at each point q_j

is close to the k-jet of the identity map. To satisfy (iii) we take $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ where a suitable automorphism $\phi_3 \in \mathrm{Aut}(\mathbb{C}^2)$ is obtained by [20, proof of Theorem 4.9.2, p. 140], which relies on the jet-interpolation theorem for holomorphic automorphisms [20, Corollary 4.15.5, p. 174]. Note that ϕ_3 can be chosen arbitrarily close to the identity map on any given compact set in $\mathbb{C}^2$ if the k-jet of $\phi_2 \circ \phi_1$ at each point q_j is close enough to the k-jet of the identity map. Hence, assuming that $\epsilon' > 0$ is chosen small enough, the automorphism $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ satisfies condition (i), condition

$$\phi(\widetilde{E} \cup \widetilde{\Gamma}) \cap r\overline{\mathbb{B}} = \varnothing, \tag{2.15}$$

and condition (iii). Recall the notation (2.14) and set

$$E'' = E' \setminus \widetilde{E} \quad \text{and} \quad \Gamma' = \Gamma \setminus \widetilde{\Gamma}.$$

The problem now is that $\phi(E'' \cup \Gamma')$ may intersect the ball $r\overline{\mathbb{B}}$. These intersections are removed by precomposing ϕ with a shear $\psi \in \mathrm{Aut}(\mathbb{C}^2)$ of the form

$$\psi(z_1, z_2) = (z_1, z_2 + \xi(z_1)) \tag{2.16}$$

for a suitably chosen entire function $\xi : \mathbb{C} \to \mathbb{C}$ which is close to 0 on the disc $s\overline{\mathbb{D}}$. The idea is explained in [35, proof of Lemma 2.2] (based on [12, Lemma 2.2]); however, some additions to their argument are necessary in light of Remark 2.1. Note that $\phi^{-1}(r\overline{\mathbb{B}})$ is a compact polynomially convex set. Let $s \geq s_0$ be as above, and pick $s_1 > s$ such that

$$\pi_1(\phi^{-1}(r\overline{\mathbb{B}})) \subset s_1\mathbb{D}. \tag{2.17}$$

The shear ψ of the form (2.16) should be chosen such that

$$\psi(E' \cup \Gamma) \cap \phi^{-1}(r\overline{\mathbb{B}}) = \varnothing. \tag{2.18}$$

By (2.17) this does not impose any condition on the function ξ on $\mathbb{C} \setminus s_1\mathbb{D}$, while on $s\overline{\mathbb{D}}$ it suffices to take ξ close enough to 0 in view of (2.15). It is explained in [49, Lemma 1] how to determine ξ on the curves $\pi_1(\Gamma) \cap (s_1\mathbb{D} \backslash s\mathbb{D})$ such that $\psi(\Gamma) \cap \phi^{-1}(r\overline{\mathbb{B}}) = \varnothing$. To find an entire function ξ on $\mathbb{C}$ such that the above holds, one uses Mergelyan theorem (see [18, Theorem 16]) and condition (2.13). Since the set E' only clusters on Γ, there are at most finitely many points $Q = \{e_1, \ldots, e_i\} \subset E'$ such that $\psi(e_j) \in \phi^{-1}(r\overline{\mathbb{B}})$ for $j = 1, \ldots, i$, i.e., condition (2.18) fails only at these points. In view of the last two conditions in (2.12) we can redefine ξ at the points $\pi_1(e_1), \ldots, \pi_1(e_i) \in s_1\mathbb{D} \backslash s\overline{\mathbb{D}}$, approximating the previously chosen function sufficiently closely on the polynomially convex set $\left[\pi_1((E' \backslash Q) \cup \Gamma) \cap (s_1\overline{\mathbb{D}})\right] \cup s\overline{\mathbb{D}} \subset \mathbb{C}$, so that the new shear map ψ satisfies (2.18). The interpolation of the identity at the given points $q_1, \ldots, q_l \in L$ is achieved by choosing ξ such that it vanishes to order k at every point $\pi_1(q_j) \in \pi_1(L) \subset s\mathbb{D}$ for $j = 1, \ldots, l$. The automorphism $\Phi = \phi \circ \psi \in \mathrm{Aut}(\mathbb{C}^2)$ then satisfies Lemma 2.2. □

(4) Bringing back the points at infinity. Recall that $\tilde{f} : V \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding satisfying (2.3) and $g \circ \tilde{f} : V \backslash \{b_1, \ldots, b_m\} \hookrightarrow \mathbb{C}^2$ is given by (2.7). Furthermore, $\tau : M \to \tau(M) \subset V$ is a conformal diffeomorphism in (2.4) and $h = \tilde{f} \circ \tau$ (2.5). The compact set $K \subset \mathring{M}$ is holomorphically convex in $\mathring{M}$, and in view of (2.9) the compact set

$$\widetilde{L} := L \cup (g \circ h)(K) \subset \mathbb{C}^2 \tag{2.19}$$

is polynomially convex (for the details, see [48] or [20, proof of Theorem 4.14.6, p. 168]). Let $\Phi \in \mathrm{Aut}(\mathbb{C}^2)$ be given by Lemma 2.2 with L replaced by $\widetilde{L}$, the set $E' = (g \circ h)(E)$, and with interpolation to order k (see condition (iii)) on the finite set $f(A) = (g \circ h)(A) \subset (g \circ h)(K) \subset \widetilde{L}$. Consider the holomorphic embedding

$$\widetilde{F} := \Phi \circ g \circ \tilde{f} : V \setminus \{b_1, \ldots, b_m\} \hookrightarrow \mathbb{C}^2.$$

The image of $\widetilde{F}$ is an embedded complex curve in $\mathbb{C}^2$ containing the image of the embedding

$$\widetilde{F} \circ \tau = \Phi \circ g \circ h : M \setminus \{a_1, \ldots, a_m\} \hookrightarrow \mathbb{C}^2.$$

By the construction, $\widetilde{F} \circ \tau$ satisfies Lemma 1.1 except that the points $a_1, \ldots, a_m \in bM$ are sent to infinity. We now bring them back to $\mathbb{C}^2$ as follows. On every curve $\tau(C_j) \subset \tau(bM)$ we choose a proper closed arc $I_j \subsetneq \tau(C_j)$ containing the point $\tau(a_j) = b_j$ in its interior. Let v be a smooth vector field on R along the set $I = \bigcup_{j=1}^m I_j$ which is transverse to $\tau(bM)$ and points to the interior of $\tau(M)$. Mergelyan theorem allows us to approximate v by a holomorphic vector field on a neighbourhood of $\tau(M)$ in V, still denoted v, which vanishes to order k at every point of A. (Note that the tangent bundle of V is trivial, so we may think of v as a function.) The flow ψ_t of v for small values of $|t|$, with $\psi_0 = \mathrm{Id}$, exists on a neighbourhood of $\tau(M)$ in V and consists of biholomorphic maps whose k-jet at every point of A agrees with the k-jet of the identity map. Since v points to the interior of $\tau(M)$ along I, for small $t > 0$ the closed domain $\psi_t(\tau(M)) \subset V$ (which is conformally diffeomorphic to M) does not contain any of the points $b_1, \ldots, b_m$, and hence for such t the embedding

$$F = \Phi \circ g \circ \tilde{f} \circ \psi_t \circ \tau : M \hookrightarrow \mathbb{C}^2$$

satisfies the conclusion of Lemma 1.1.

Remark 2.3 (A) The same proof shows that Lemma 1.1 also holds if the holomorphic map $f : M \to \mathbb{C}^2$ has finitely many branch points in $\mathring{M}$; see [28, Theorem 1.1] for the details.

(B) The proof of Lemma 1.1 can be adapted to give an analogous result for embeddings of bordered Riemann surfaces in $\mathbb{C} \times \mathbb{C}^*$. In this case, the compact set $L \subset \mathbb{C} \times \mathbb{C}^*$ should be holomorphically convex in $\mathbb{C} \times \mathbb{C}^*$, and condition (i) in the lemma should be replaced by asking that $F(E \cup bM)$ lies outside the cylinder $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le r,\ 1/r \le |z_2| \le r\}$ for a given $r > 1$. An inductive application of this lemma yields the analogue of Corollary 1.3 for proper holomorphic embeddings $\mathring{M} \hookrightarrow \mathbb{C} \times \mathbb{C}^*$ (see Ritter [38, Theorem 4]).

3 Proof of Theorem 1.2

We begin with some reductions. As in Lemma 1.1, we assume as we may that M is a closed smoothly bounded domain in a compact Riemann surface R. Fix K, α_j, β_j, and ϵ as in the statement of Theorem 1.2. Also, fix finitely many points $c_1, \ldots, c_l$ in $K \setminus f^{-1}(\{\beta_j : j = 1, 2, \ldots\})$ for the interpolation condition (ii), as well as a number $k \in \mathbb{N}$ for the interpolation order. By Mergelyan theorem in the $\mathscr{C}^1$ topology with interpolation at the points $c_1, \ldots, c_l$ (see [18, Theorem 16]), we may assume that f is given by a holomorphic embedding $f : U \hookrightarrow \mathbb{C}^2$ on an open neighbourhood $U \subset R$ of M.

Choose a smoothly bounded compact domain $K_0 \subset \mathring{M}$ which is a strong deformation retract of M such that $K \subset \mathring{K}_0$. By renumbering the points α_j and β_j, we may assume that $\alpha_1, \ldots, \alpha_j \in K_0$ and $\alpha_i \notin K_0$ for all $i > j$. Applying Lemma 1.1 we find a holomorphic embedding $f_0 : M \hookrightarrow \mathbb{C}^2$ which approximates f as closely as desired on K, it agrees with f to order k at the points $c_1, \ldots, c_l$, and it satisfies $f_0(\alpha_i) = \beta_i$ for $i = 1, \ldots, j$. We now add the points $\{\alpha_1, \ldots, \alpha_j\}$ to the finite set $\{c_1, \ldots, c_l\}$ and drop them from the list of α-points in the theorem. Likewise, we drop $\beta_1, \ldots, \beta_j$ from the list of β-points.

We take f_0 as our new starting map and K_0 as the new initial compact set in the theorem. Set $K_{-1} = \varnothing$ and $\epsilon_0 = \epsilon/2$. We may assume without loss of generality that $\beta_j \neq 0$ for all $j = 1, 2, \ldots$, and that there is $r_0 > 0$ such that

$$f_0(M) \cap r_0\overline{\mathbb{B}} = \varnothing. \tag{3.1}$$

Choose a sequence $0 < r_0 < r_1 < r_2 < \cdots$ with $\lim_{j\to\infty} r_j = +\infty$. Fix a point $\alpha_0 \in K_0 \setminus f_0^{-1}(\{\beta_j : j = 1, 2, \ldots\})$ and set $\beta_0 = f_0(\alpha_0)$. We shall inductively construct a sequence of triples $X_j = \{f_j, K_j, \epsilon_j\}$, $j \in \mathbb{N}$, where

- $f_j : M \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding,
- $K_j \subset \mathring{M}$ is a smoothly bounded compact domain which is a strong deformation retract of M, and
- $\epsilon_j > 0$ is a number,

such that

$$\bigcup_{j\in\mathbb{N}} K_j = \mathring{M} \tag{3.2}$$

and the following conditions hold for all $j \in \mathbb{N}$:

(1_j) $K_{j-1} \cup \{\alpha_i : i = 0, \ldots, j\} \subset \mathring{K}_j$ and $\{\alpha_i : i > j\} \cap K_j = \varnothing$.
(2_j) $\sup_{x\in K_{j-1}} |f_j(x) - f_{j-1}(x)| < \epsilon_j$.
(3_j) $\epsilon_j < \epsilon_{j-1}/2$ and every holomorphic map $\varphi : \mathring{M} \to \mathbb{C}^2$ with $|\varphi - f_{j-1}| < 2\epsilon_j$ on K_{j-1} is an embedding on K_{j-2}.
(4_j) $f_j(\alpha_i) = \beta_i$ for all $i \in \{0, \ldots, j\}$.
(5_j) f_j agrees with f_{j-1} to order k at c_i for all $i \in \{1, \ldots, l\}$.
(6_j) $f_j(M \setminus \mathring{K}_j) \cap r_j\overline{\mathbb{B}} = \varnothing$.
(7_j) $f_j(M \setminus \mathring{K}_{j-1}) \cap \min\{r_{j-1}, |\beta_j|/2\}\overline{\mathbb{B}} = \varnothing$.

Assuming the existence of such a sequence, the proof of Theorem 1.2 is completed as follows. Conditions (1_j), (2_j), (3_j), and (3.2) ensure that there exists a limit map

$$F = \lim_{j\to\infty} f_j : \mathring{M} \to \mathbb{C}^2$$

which is an injective holomorphic immersion and satisfies

$$\sup_{x\in K_{j-1}} |F(x) - f_{j-1}(x)| < 2\epsilon_j \quad \text{for all } j \in \mathbb{N}. \tag{3.3}$$

This implies condition (i) in the theorem; recall that $f_0 = f$ and $2\epsilon_1 < \epsilon_0 < \epsilon$. Conditions (4_j) and (5_j) ensure that $F(\alpha_i) = \beta_i$ for all $i = 1, 2, \ldots$ and F agrees with f to order k at c_i for all $i \in \{1, \ldots, l\}$; so, (ii) and (iii) hold as well. Now, (3.3) and condition (6_j) guarantee that

$$\inf_{x\in bK_j} |F(x)| > r_j - 2\epsilon_{j+1} > r_j - \epsilon_0 \quad \text{for all } j \in \mathbb{N}. \tag{3.4}$$

Since the increasing sequence of compact sets K_j exhausts $\mathring{M}$ (see (3.2)) and we have that $\lim_{j\to\infty} r_j = +\infty$, it follows that $F : \mathring{M} \hookrightarrow \mathbb{C}^2$ is an almost proper map. Likewise, (3.3) and (7_j) give that

$$\inf_{x\in K_j\setminus\mathring{K}_{j-1}} |F(x)| > \min\left\{r_{j-1}, \frac{|\beta_j|}{2}\right\} - \epsilon_0 \quad \text{for all } j \in \mathbb{N}. \tag{3.5}$$

If the sequence $\beta_j \in \mathbb{C}^2$ is closed and discrete, we have that $\lim_{j\to\infty} |\beta_j| = +\infty$, and hence

$$\lim_{j\to\infty} \min\left\{r_{j-1}, \frac{|\beta_j|}{2}\right\} = +\infty.$$

This, (3.2), (3.5), and conditions (1_j) imply that in this special case the injective immersion $F : \mathring{M} \hookrightarrow \mathbb{C}^2$ which we constructed is a proper map, hence an embedding, thereby proving the final assertion of the theorem.

Let us explain the induction. The basis is given by the triple $X_0 = \{f_0, K_0, \epsilon_0\}$; observe that it meets conditions (1_0), (4_0), and (6_0), see (3.1), while (2_0), (3_0), (5_0), and (7_0) are void. For the inductive step we fix $j \in \mathbb{N}$ and assume that we have a triple $X_{j-1} = \{f_{j-1}, K_{j-1}, \epsilon_{j-1}\}$ enjoying (1_{j-1}), (4_{j-1}), and (6_{j-1}). By the Cauchy estimates, there is a number $\epsilon_j > 0$ so small that (3_j) holds true. Recall that $\beta_j \neq 0$ and set

$$L = \min\left\{r_{j-1}, \frac{|\beta_j|}{2}\right\} \overline{\mathbb{B}}.$$

By (6_{j-1}), we have that

$$f_{j-1}(M \setminus \mathring{K}_{j-1}) \cap L = \varnothing.$$

Choose a number $r > r_j$ so large that

$$f_{j-1}(M) \subset (r - \epsilon_j)\mathbb{B}. \tag{3.6}$$

Lemma 1.1 then applies to the embedding f_{j-1}, the compact set K_{j-1} (it has no holes since it is a strong deformation retract of M), the compact polynomially convex set $L \subset \mathbb{C}^2$, the singletons $\{\alpha_j\} \subset \mathring{M}\setminus K_{j-1}$ (see (1_{j-1})) and $\{\beta_j\} \subset \mathbb{C}^2\setminus L$, the closed discrete set $E = \{\alpha_i : i > j\} \subset \mathring{M}$ (note that $E \cap (K_{j-1} \cup \{\alpha_j\}) = \varnothing$ by (1_{j-1})), and the numbers $\epsilon_j > 0$ and $r > 0$, furnishing us with a holomorphic embedding $f_j : M \hookrightarrow \mathbb{C}^2$ satisfying (2_j), (5_j), (7_j), and the following conditions:

(a) $f_j(bM \cup \{\alpha_i : i > j\}) \cap r\overline{\mathbb{B}} = \varnothing$.
(b) $f_j(\alpha_j) = \beta_j$ and $f_j(\alpha_i) = f_{j-1}(\alpha_i)$ for all $i \in \{1, \ldots, j-1\}$.
(c) $f_j(K_{j-1}) \subset r\mathbb{B}$.

The first part of condition (b) is ensured by Lemma 1.1-(e), while the second part is granted by Lemma 1.1-(d). Condition (c) is implied by (2_j) and (3.6).

Note that (b) and (4_{j-1}) ensure (4_j). Finally, in view of (a), (c), and the fact that $r > r_j$, we can choose a smoothly bounded compact domain $K_j \subset \mathring{M}$, being a strong deformation retract of M, such that (1_j) and (6_j) hold true. Indeed, by (a) we can first take a smoothly bounded compact domain $K'_j \subset \mathring{M}$ which is a strong deformation retract of M such that $K_{j-1} \cup \{\alpha_i : i = 0, \ldots, j\} = K_{j-1} \cup \{\alpha_j\} \subset \mathring{K}'_j$ and

$$f_j(M \setminus \mathring{K}'_j) \cap r\overline{\mathbb{B}} = \varnothing. \tag{3.7}$$

If the set $E' = E \cap K'_j = \{\alpha_i : i > j\} \cap K'_j$ is empty, then we simply choose $K_j = K'_j$. Otherwise, E' is a finite set (recall that the sequence $\alpha_i \in \mathring{M}$ only clusters on bM) and we can choose K'_j so that $E' \subset \mathring{K}'_j\setminus K_{j-1}$; see (1_{j-1}). Let $\Omega_1, \ldots, \Omega_m$ denote the connected components of $K'_j \setminus \mathring{K}_{j-1}$; these are smoothly bounded compact annuli since $K_{j-1} \subset \mathring{K}'_j$ is a strong deformation retract of K'_j. Fix $i \in \{1, \ldots, m\}$. Conditions (a), (c), and (3.7) imply that the set $\Omega'_i = \Omega_i \cap f_j^{-1}(\mathbb{C}^2 \setminus r\overline{\mathbb{B}})$ is disjoint from bK_{j-1} and it contains $E' \cap \Omega_i$ as well as an open neighbourhood of $\Omega_i \cap bK'_j$. Since Ω'_i is open in Ω_i, these properties and the maximum principle show that Ω'_i is path connected, and hence we can choose a smooth Jordan arc $\gamma_i \subset \Omega'_i \setminus \{\alpha_j\}$ containing $E' \cap \Omega_i$, having an endpoint in $\Omega_i \cap bK'_j$ and otherwise disjoint from bK'_j. Set $\gamma = \bigcup_{i=1}^m \gamma_i$. Note that $K_{j-1} \cup \{\alpha_j\} \subset \mathring{K}'_j \setminus \gamma$, K_{j-1} is a strong deformation retract of $\mathring{K}'_j \backslash \gamma$, $\{\alpha_i : i > j\} \cap K'_j \backslash \gamma = \varnothing$, and $f_j(M \backslash (\mathring{K}'_j \backslash \gamma)) \cap r\overline{\mathbb{B}} = \varnothing$. It is clear that every sufficiently large smoothly bounded compact domain $K_j \subset \mathring{K}'_j \setminus \gamma$ which is a strong deformation retract of M satisfies conditions (1_j) and (6_j). This closes the induction.

Note that at each step of the induction we are allowed to choose the compact domain $K_j \subset \mathring{M}$ as large as desired under the only restriction imposed by the second part of condition (1_j). So, since the sequence $\alpha_j \in \mathring{M}$ is closed and discrete, we can proceed in such a way that condition (3.2) is satisfied. This completes the proof of Theorem 1.2.

4 Proper embeddings of circle domains in $\mathbb{CP}^1$ into $\mathbb{C}^2$

Recall that a circle domain in $\mathbb{CP}^1$ is an open domain of the form

$$\Omega = \mathbb{CP}^1 \setminus \bigcup_{i=0}^{\infty} D_i \tag{4.1}$$

where D_i are pairwise disjoint closed round discs. By the uniformization theorem of He and Schramm [33], every domain of the form (4.1), where D_i are pairwise disjoint closed topological discs (homeomorphic images of $\overline{\mathbb{D}}$), is conformally equivalent to a circle domain. We give a simpler proof of the following result [29, Theorem 1.1] due to Forstnerič and Wold.

Theorem 4.1 *Every circle domain in $\mathbb{CP}^1$ embeds properly holomorphically into $\mathbb{C}^2$.*

We shall use the following analogue of Lemma 1.1 adapted to this situation.

Lemma 4.2 *Let Ω be a circle domain (4.1) in $\mathbb{CP}^1$, and let $k \in \mathbb{Z}_+$. Given a $\mathscr{C}^1$ embedding $f : M_k = \mathbb{CP}^1 \setminus \bigcup_{i=0}^k \mathring{D}_i \hookrightarrow \mathbb{C}^2$ which is holomorphic in $\mathring{M}_k$, a compact set $K \subset \Omega$ which is $\mathscr{O}(\mathring{M}_k)$-convex, a compact polynomially convex set $L \subset \mathbb{C}^2$ such that $f(M_k \backslash \mathring{K}) \cap L = \varnothing$, points $\alpha \in \Omega \backslash K$ and $\beta \in \mathbb{C}^2 \setminus L$, and a number $r > 0$, we can approximate f as closely as desired uniformly on K by a holomorphic embedding $F : M_k \hookrightarrow \mathbb{C}^2$ which agrees with f at finitely many given points in K and satisfies*

$$F\Big(bM_k \cup \bigcup_{i=k+1}^{\infty} D_i\Big) \subset \mathbb{C}^2 \setminus r\overline{\mathbb{B}}, \quad F(M_k \setminus \mathring{K}) \cap L = \varnothing, \quad \textit{and} \quad F(\alpha) = \beta.$$

This lemma is obtained by combining [29, proof of Lemma 3.1] with the proof of Lemma 1.1 in Sect. 2. The only difference in [29, Lemma 3.1] when compared to the technique used in the earlier papers [28, 48, 49] is that the conformal diffeomorphism $\tau : M \to \tau(M) \subset V$ in (2.4) is chosen such that it maps M onto a domain with piecewise smooth boundary in the ambient Riemann surface V. (In the context of Lemma 4.2, we apply this argument to the bordered Riemann surface $M = M_k$.) The finitely many corner points of $\tau(bM)$ are mapped to infinity by the embedding $\tilde{f}$; see (2.5). The main point of this change is to ensure that the first coordinate projection $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$, restricted to the image of M, is injective near infinity; this enables one to find a shear ψ (2.16) in Step (3) of the proof of Lemma 1.1 such that the resulting automorphism $\Phi = \phi \circ \psi \in \mathrm{Aut}(\mathbb{C}^2)$ maps all the discs D_i out of the given ball $r\overline{\mathbb{B}}$. It is clear that the method in step (4) of the proof of Lemma 1.1, using the flow ψ_t of a suitably chosen holomorphic vector field on V, still applies to a domain with corners, so we can bring the points at infinity back to $\mathbb{C}^2$. Finally, we approximate the new conformal diffeomorphism $\psi_t \circ \tau$ sufficiently closely by a smooth conformal diffeomorphism from M onto its image in the ambient surface V, thereby removing the corners. We leave further details to the reader.

Proof of Theorem 4.1 Let $\Omega \subset \mathbb{CP}^1$ be the circle domain (4.1). Let $\mathbb{B}$ denote the unit ball of $\mathbb{C}^2$. Set $M_0 = \mathbb{CP}^1 \backslash \mathring{D}_0$; this is a closed disc. Choose a holomorphic embedding $f_0 : M_0 \hookrightarrow \mathbb{C}^2$ and a compact, smoothly bounded, $\mathscr{O}(\Omega)$-convex set $K_0 \subset \Omega$. Its $\mathscr{O}(\mathring{M}_0)$-convex hull is the union of K_0 with at most finitely many smoothly bounded open discs in $\mathring{M}_0$, each containing a disc from the family $\{D_i\}_{i\in\mathbb{N}}$. Hence, there are finitely many discs $D_{j(1)}, \ldots, D_{j(k_1)}$ such that, setting $M_1 = M_0 \backslash \bigcup_{i=1}^{k_1} \mathring{D}_{j(i)}$, the set K_0 is $\mathscr{O}(\mathring{M}_1)$-convex. Let $J_1 = \mathbb{N} \backslash \{j(1), j(2), \ldots, j(k_1)\}$. By Lemma 4.2 we can approximate the embedding $f_0|_{M_1}$ as closely as desired uniformly on K_0 by a holomorphic embedding $f_1 : M_1 \hookrightarrow \mathbb{C}^2$ such that

$$f_1\big(bM_1 \cup \bigcup_{i\in J_1} D_i\big) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}.$$

Since $\mathring{M}_1 \setminus \bigcup_{i\in J_1} D_i = \Omega$, there is a compact, smoothly bounded, $\mathscr{O}(\Omega)$-convex set $K_1 \subset \Omega$ with $K_0 \subset \mathring{K}_1$ such that $f_1(M_1 \backslash \mathring{K}_1) \subset \mathbb{C}^2 \backslash \overline{\mathbb{B}}$. By the same argument as above, we can find finitely many discs $D_{j(k_1+1)}, \ldots, D_{j(k_2)}$ from the given family such that, setting

$$M_2 = M_1 \setminus \bigcup_{i=k_1+1}^{k_2} \mathring{D}_{j(i)} = M_0 \setminus \bigcup_{i=1}^{k_2} \mathring{D}_{j(i)},$$

the set K_1 is $\mathscr{O}(\mathring{M}_2)$-convex. Let $J_2 = \mathbb{N} \backslash \{j(1), j(2), \ldots, j(k_2)\}$. By Lemma 4.2 applied to the embedding $f_1|_{M_2}$ and the polynomially convex set $L = \overline{\mathbb{B}} \subset \mathbb{C}^2$ we can approximate f_1 on K_1 by a holomorphic embedding $f_2 : M_2 \hookrightarrow \mathbb{C}^2$ such that

$$f_2\big(bM_2 \cup \bigcup_{i\in J_2} D_i\big) \subset \mathbb{C}^2 \setminus 2\overline{\mathbb{B}} \quad \text{and} \quad f_2(M_2 \setminus \mathring{K}_1) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}.$$

Hence, there is a smoothly bounded $\mathscr{O}(\Omega)$-convex set $K_2 \subset \Omega$ with $K_1 \subset \mathring{K}_2$ such that

$$f_2(M_2 \setminus \mathring{K}_2) \subset \mathbb{C}^2 \setminus 2\overline{\mathbb{B}}.$$

Continuing inductively, we obtain the following:

(1) an increasing sequence $K_0 \subset K_1 \subset K_2 \subset \cdots$ of compact, smoothly bounded, $\mathscr{O}(\Omega)$-convex domains with $K_j \subset \mathring{K}_{j+1}$ for each $j \in \mathbb{Z}_+$ and $\bigcup_{j=0}^{\infty} K_j = \Omega$,
(2) a decreasing sequence of circle domains $M_0 \supset M_1 \supset \cdots$ such that $\bigcap_{i=0}^{\infty} M_i \supset \overline{\Omega}$ and K_j is $\mathscr{O}(\mathring{M}_{j+1})$-convex for each $j = 0, 1, 2, \ldots$, and
(3) a sequence of holomorphic embeddings $f_j : M_j \hookrightarrow \mathbb{C}^2$ such that for every $j \in \mathbb{N}$, the map f_j approximates f_{j-1} uniformly on K_{j-1} as closely as desired and it satisfies

$$f_j\big(bM_j \cup \bigcup_{i\in J_j} D_i\big) \subset \mathbb{C}^2 \setminus j\overline{\mathbb{B}} \quad \text{and} \quad f_j(M_j \setminus \mathring{K}_{j-1}) \subset \mathbb{C}^2 \setminus (j-1)\overline{\mathbb{B}}. \tag{4.2}$$

Here, $J_j \subset \mathbb{N}$ is such that $\mathring{M}_j \setminus \bigcup_{i\in J_j} D_i = \Omega$. The second condition in (4.2) gives

$$f_j(K_j \setminus \mathring{K}_{j-1}) \subset f_j(M_j \setminus \mathring{K}_{j-1}) \subset \mathbb{C}^2 \setminus (j-1)\overline{\mathbb{B}} \quad \text{for all} \quad j = 1, 2, \ldots. \tag{4.3}$$

Assuming as we may that the approximation of f_{j-1} by f_j is sufficiently close at every step, these conditions clearly imply that the sequence f_j converges uniformly on compacts in Ω to a proper holomorphic embedding $f = \lim_{j\to\infty} f_j : \Omega \hookrightarrow \mathbb{C}^2$. □

Remark 4.3 To prove Theorem 1.4 in this special case, we modify the above construction so that for each $j = 1, 2, \ldots$ we first pick a point $\alpha_j \in \Omega \setminus K_{j-1}$ and then choose the next embedding $f_j : M_j \hookrightarrow \mathbb{C}^2$ so that it approximates f_{j-1} on K_{j-1}, it satisfies

$$f_j(\alpha_j) = \beta_j \in B \quad \text{and} \quad f_j\big(bM_j \cup \bigcup_{i\in J_j} D_i\big) \subset \mathbb{C}^2 \setminus j\overline{\mathbb{B}},$$

and f_j agrees with f_{j-1} at the previously chosen points $\alpha_1, \ldots, \alpha_{j-1} \in K_{j-1}$ so that $f_j(\alpha_i) = \beta_i \in B$ holds for $i = 1, \ldots, j$. We then pick the next compact, smoothly bounded, $\mathscr{O}(\Omega)$-convex set $K_j \subset \Omega$ such that

$$K_{j-1} \cup \{\alpha_j\} \subset \mathring{K}_j \quad \text{and} \quad f_j(bK_j) \subset \mathbb{C}^2 \setminus j\overline{\mathbb{B}}. \tag{4.4}$$

(However, we cannot fulfil condition (4.3) due to the interpolation condition $f_j(\alpha_j) = \beta_j$, since there is no assumption on the set $B = \{\beta_j\} \subset \mathbb{C}^2$.) By choosing the set $K_j \subset \Omega$ big enough at every step to ensure that $\bigcup_{j=1}^\infty K_j = \Omega$, condition (4.4) ensures that the limit holomorphic embedding $f = \lim_{j\to\infty} f_j : \Omega \hookrightarrow \mathbb{C}^2$ is almost proper.

The general case of Theorem 1.4 is proved in the following section.

Remark 4.4 A geometric disc in a Riemann surface R is the image of a round disc in the universal covering space $\widetilde{R} \in \{\mathbb{CP}^1, \mathbb{C}, \mathbb{D}\}$ of R. A circle domain in R is a domain all of whose complementary connected components are closed geometric disks and points (punctures). By He and Schramm [33, Theorem 0.2], every open Riemann surface with finite genus and at most countably many ends is conformally equivalent to a circle domain Ω in a compact Riemann surface R. If Lemma 4.2 could be proved for such domains without punctures, it would follow that any such domain embeds properly holomorphically in $\mathbb{C}^2$.

5 Proof of Theorem 1.4

We shall need the following generalization of [29, Lemma 2.2] to domains in an arbitrary compact Riemann surface R.

Lemma 5.1 *Assume that R is a compact Riemann surface of genus ν and Ω is a connected open domain in R of the same genus ν. Given a closed set L in R which is a union of connected components of $R \setminus \Omega$ and an open set $V \subset R$ containing L, there exist finitely many pairwise disjoint, smoothly bounded closed discs $\overline{\Delta}_i \subset V$ $(i = 1, \ldots, m)$ such that*

$$L \subset \bigcup_{i=1}^m \Delta_i \quad \textit{and} \quad \bigcup_{i=1}^m b\Delta_i \subset \Omega. \tag{5.1}$$

Proof Let $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^\infty K_j = \Omega$ be an exhaustion by smoothly bounded compact connected sets with $K_j \subset \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$. By choosing K_1 big enough, every set K_j has the same genus ν as R, and hence $R \setminus K_j$ is the union of finitely many open discs $\mathcal{U}_j = \{U_1^j, \ldots, U_{m(j)}^j\}$ with pairwise disjoint closures. We claim that for j large enough there are discs $\Delta_1, \ldots, \Delta_m \in \mathcal{U}_j$ satisfying (5.1). Indeed, if this is not the case, there is a decreasing sequence of closed discs $\overline{U_{k(j)}^j} \supset \overline{U_{k(j+1)}^{j+1}}$ such that $U_{k(j)}^j \cap L \neq \varnothing$ and $\overline{U_{k(j)}^j} \cap (R \setminus V) \neq \varnothing$ for each j; but then $\bigcap_{j=1}^\infty \overline{U_{k(j)}^j}$ would be a complementary component of Ω which is contained in L and intersects $R \setminus V$, a contradiction. □

Given a compact subset $L \subset R \setminus \Omega$ and open smoothly bounded discs $\Delta_i \subset R$ $(i = 1, \ldots, m)$ with pairwise disjoint closures satisfying (5.1), we shall say that the set

$$\Gamma = \bigcup_{i=1}^m b\Delta_i \subset \Omega$$

is a *surrounding system* for L, or simply that Γ surrounds L. The set

$$c(\Gamma) = \Omega \setminus \bigcup_{i=1}^m \overline{\Delta}_i \tag{5.2}$$

is called the *core component* of $\Omega \setminus \Gamma$. Note that if $\Gamma \subset \Omega$ surrounds L and $\delta : [0, 1) \to \Omega$ is a path such that $\delta(0) \in c(\Gamma)$ and δ has a limit point in L, then $\delta([0, 1)) \cap \Gamma \neq \varnothing$.

Proof of Theorem 1.4 Let R and $\Omega = R \setminus \bigcup_{i=0}^\infty D_i$ be as in the theorem, so D_i are closed pairwise disjoint closed discs. Note that Ω has the same topological genus ν as R. Set

$$M_j = R \setminus \bigcup_{i=0}^j \mathring{D}_i \quad \text{for } \ j = 0, 1, \ldots. \tag{5.3}$$

For all $j \geq 0$ we have that $\Omega \subset \mathring{M}_j$, $M_{j+1} \cup \mathring{D}_{j+1} = M_j$, $bM_{j+1} = bM_j \cup bD_{j+1}$, and $\Omega = \bigcap_{j=0}^\infty \mathring{M}_j$. By a *surrounding system* $\Gamma \subset \Omega$ for M_j, we shall mean a surrounding system for $R \setminus \mathring{M}_j = \bigcup_{i=0}^j D_i$. Note that if $\delta : [0, 1) \to \Omega$ is a path such that $\delta(0) \in c(\Gamma)$ (see (5.2)) and $\delta([0, 1))$ has a limit point in $bM_j = \bigcup_{i=0}^j bD_i$, then $\delta([0, 1))$ intersects Γ.

Let $k \in \{0, 1, \ldots\}$, $f : M_k \hookrightarrow \mathbb{C}^2$, $K \subset \Omega$, $\epsilon > 0$, and $B \subset \mathbb{C}^2$ be as in the statement of the theorem. We shall assume without loss of generality that the set $B = \{\beta_1, \beta_2, \ldots\}$ is infinite, and for simplicity of notation we assume that $k = 0$ (the same argument will apply in the general case). Set $f_0 = f : M_0 = R \setminus \mathring{M}_0 \hookrightarrow \mathbb{C}^2$. Fix points $c_1, \ldots, c_l \in \Omega$ at which we wish to interpolate (see condition (ii) in the theorem). We may assume without loss of generality that $f_0(\{c_1, \ldots, c_l\}) \cap B = \varnothing$. Using Lemma 5.1 we find a smooth Jordan curve $\Gamma_0 \subset \Omega$ surrounding the disc $D_0 = R \setminus \mathring{M}_0$ such that $\Gamma_0 \cap \{c_1, \ldots, c_l\} = \varnothing$ and f_0 vanishes nowhere on Γ_0; the last condition is easily arranged by a small deformation. Then, choose a smoothly bounded compact connected domain $K_0 \subset \Omega$ with genus ν such that $K \cup \{c_1, \ldots, c_l\} \cup \Gamma_0 \subset \mathring{K}_0$. Pick a point $\alpha_0 \in \mathring{K}_0 \setminus f_0^{-1}(B)$ and set $\beta_0 = f_0(\alpha_0)$. We also let $K_{-1} = \varnothing$ and $\epsilon_0 = \epsilon/2$.

We shall inductively construct a sequence of tuples $T_j = \{f_j, K_j, \Gamma_j, \epsilon_j, \alpha_j\}$, $j \in \mathbb{N}$, where

- $f_j : M_j \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding,
- $K_j \subset \Omega$ is a smoothly bounded compact connected domain of genus ν,
- $\Gamma_j \subset \mathring{K}_j$ is a surrounding system for M_j,
- $\epsilon_j > 0$ is a number, and

- $\alpha_j \in \mathring{K}_j$ is a point,

such that

$$\bigcup_{j=0}^{\infty} K_j = \Omega \tag{5.4}$$

and the following conditions hold for all $j = 1, 2, \ldots$:

(1_j) $K_{j-1} \subset \mathring{K}_j$.
(2_j) $\Gamma_j \cup \{\alpha_j\} \subset \mathring{K}_j \setminus K_{j-1}$ and $K_{j-1} \cup \{\alpha_j\} \subset c(\Gamma_j)$; see (5.2).
(3_j) $\epsilon_j < \epsilon_{j-1}/2$ and every holomorphic map $\varphi : \Omega \to \mathbb{C}^2$ with $|\varphi - f_{j-1}| < 2\epsilon_j$ on K_{j-1} is an embedding on K_{j-2}. (Recall that $K_{-1} = \varnothing$.)
(4_j) $\sup_{x \in K_{j-1}} |f_j(x) - f_{j-1}(x)| < \epsilon_j$.
(5_j) $f_j(\alpha_i) = \beta_i$ for $i = 0, 1, \ldots, j$.
(6_j) f_j agrees with f_{j-1} to a given order at $c_i \in K_0$ for $i = 1, \ldots, l$.
(7_j) $f_j(\Gamma_j) \cap j\overline{\mathbb{B}} = \varnothing$.

The basis of the induction is given by $T_0 = \{f_0, K_0, \Gamma_0, \epsilon_0, \alpha_0\}$. It meets conditions (1_0), (5_0), and (7_0), while the remaining conditions are void for $j = 0$. For the inductive step, assume that we have tuples $T_0, \ldots, T_{j-1}$ satisfying the required conditions for some $j \geq 1$, and let us construct T_j. Choose $\epsilon_j > 0$ so small that (3_j) holds. Next, choose a compact set $K'_{j-1} \subset \mathring{M}_j$ without holes in $\mathring{M}_j$ such that $K_{j-1} \subset \mathring{K}'_{j-1}$. (The set K'_{j-1} need not be contained in Ω.) Pick a point $\alpha_j \in \Omega \setminus K'_{j-1}$ (this set is nonempty since $K'_{j-1} \subset \mathring{M}_j$ is compact while Ω is not relatively compact in $\mathring{M}_j$). Lemma 1.1 applied to the embedding $f_{j-1}|_{M_j} : M_j \hookrightarrow \mathbb{C}^2$, the compact set $K'_{j-1} \subset \mathring{M}_j$, the singletons $\{\alpha_j\} \subset \mathring{M}_j \setminus K'_{j-1}$ and $\{\beta_j\} \subset \mathbb{C}^2$, and the number $\epsilon_j > 0$ furnishes a holomorphic embedding $f_j : M_j \hookrightarrow \mathbb{C}^2$ satisfying (4_j)–(6_j) and

$$f_j(bM_j) \cap j\overline{\mathbb{B}} = \varnothing.$$

Hence, there is an open set $V \subset R$ containing $R \setminus \mathring{M}_j = \bigcup_{i=0}^{j} D_i$ such that

$$(K_{j-1} \cup \{\alpha_j\}) \cap \overline{V} = \varnothing \quad \text{and} \quad |f_j| > j \text{ on } V \cap M_j.$$

By Lemma 5.1 there is a surrounding system $\Gamma_j = \bigcup_{i=0}^{j} \gamma_i \subset \Omega \cap V$ for M_j such that $\gamma_i = b\Delta_i$, where $\Delta_i \subset V$ is a disc containing D_i and the closed discs $\overline{\Delta}_i$ for $i = 0, 1, \ldots, j$ are pairwise disjoint. Hence, $\Gamma_j \cap K_{j-1} = \varnothing$, $K_{j-1} \cup \{\alpha_j\} \subset c(\Gamma_j)$ (this is the second condition in (2_j)), and (7_j) holds. Finally, choose any smoothly bounded compact connected domain $K_j \subset \Omega$ containing $K_{j-1} \cup \{\alpha_j\} \cup \Gamma_j$ in its interior. Hence, K_j is of genus ν (the same as the genus of R) and conditions (1_j) and (2_j) hold. The induction may now proceed. Note that condition (5.4) can be fulfilled since we may choose $K_j \subset \Omega$ as large as desired at each step.

As in the proof of Theorem 1.2, there is a limit map $F = \lim_{j\to\infty} f_j : \bigcup_{j=0}^{\infty} K_j = \Omega \hookrightarrow \mathbb{C}^2$ which is an injective holomorphic immersion and satisfies conditions (i), (ii), and (iii) in the statement of the theorem; note that $B = F(\{\alpha_j : j \in \mathbb{N}\}) \subset F(\Omega)$. Finally, conditions (2_j)–(4_j) and (7_j) guarantee that $\inf_{x \in \Gamma_j} |F(x)| > j - \epsilon$ for all $j \in \mathbb{N}$. Since $\Gamma_j = b(c(\Gamma_j))$ for every j and $c(\Gamma_1) \Subset c(\Gamma_2) \Subset \cdots \subset \bigcup_{j\in\mathbb{N}} c(\Gamma_j) = \Omega$ is an exhaustion of Ω by connected, smoothly bounded compact domains in view of (5.4), (1_j), and (2_j), this inequality shows that the map $F : \Omega \to \mathbb{C}^2$ is almost proper. □

6 Proof of Theorem 1.5

We begin by recalling a construction of a Cantor set in a domain $\Omega_0 \subset \mathbb{C}$. In the first step, we choose a smoothly bounded compact convex domain $\Delta_0 \subset \Omega_0$. Removing from Δ_0 a suitably chosen open neighbourhood Υ_0 of the vertical straight line segment divides Δ_0 in two smoothly bounded compact convex subsets Δ_0^1 and Δ_0^2 of the same width. Next, for $j = 1, 2$ we remove from Δ_0^j an open neighborhood Υ_0^j of the horizontal straight line segment dividing Δ_0^j in two convex subsets of the same height, making sure that the two connected components of $\Delta_0^j \setminus \Upsilon_0^j$ are smoothly bounded compact convex domains. This gives a compact set

$$\Omega_1 = \Delta_0 \setminus (\Upsilon_0 \cup \Upsilon_0^1 \cup \Upsilon_0^2) \subset \mathring{\Omega}_0 \tag{6.1}$$

which is the union of four pairwise disjoint, smoothly bounded compact convex domains Ω_1^j, $j = 1, \ldots, 4$. In the second step, we repeat the same procedure for each convex compact domain Ω_1^j from the first generation, thereby getting four pairwise disjoint smoothly bounded compact convex domains in its interior. This gives a compact set $\Omega_2 \subset \mathring{\Omega}_1$ which is the union of sixteen smoothly bounded compact convex domains. Continuing inductively, we obtain a decreasing sequence of smoothly bounded compact domains

$$\Omega_1 \Supset \Omega_2 \Supset \Omega_3 \Supset \ldots \tag{6.2}$$

such that for each $i \geq 1$ the domain Ω_i consists of 4^i pairwise disjoint smoothly bounded compact convex domains. The intersection

$$C = \bigcap_{i=1}^{\infty} \Omega_i \subset \Omega_0 \tag{6.3}$$

is then a Cantor set in $\mathbb{C}$. Moreover, choosing the separating neighbourhoods sufficiently small at each step of the construction, we may ensure that $\mu(C) > \mu(\Delta_0) - \delta$ for any given $\delta > 0$, where μ denotes the 2-dimensional Lebesgue measure on $\mathbb{C}$.

We now explain the proof of Theorem 1.5. Let R be a compact Riemann surface and $B \subset \mathbb{C}^2$ be a countable set. Assume that $B = \{\beta_1, \beta_2, \ldots\}$ is infinite and $0 \notin B$. Let Ω_0 be a smoothly bounded compact convex domain in a holomorphic coordinate chart on R such that there is a holomorphic embedding $f_0 : R \setminus \mathring{\Omega}_0 \hookrightarrow \mathbb{C}^2$. Such a set and embedding f_0 always exists; we may for instance choose Ω_0 to be the complement of a small open neighbourhood of the curves in a suitable homology basis of R. Fix $\epsilon_0 > 0$, set $K_0 = R \setminus \mathring{\Omega}_0$ and $K_{-1} = \varnothing$, and assume without loss of generality that there is a number $r_0 > 0$ satisfying

$$f_0(K_0) \cap r_0 \overline{\mathbb{B}} = \varnothing. \tag{6.4}$$

(Cf. (3.1); recall that $\mathbb{B}$ is the unit ball in $\mathbb{C}^2$.) Choose a point $\alpha_0 \in \mathring{K}_0 \setminus f_0^{-1}(B)$ and set $\beta_0 = f_0(\alpha_0)$. Also choose any sequence $0 < r_0 < r_1 < r_2 < \cdots$ with $\lim_{j \to \infty} r_j = +\infty$.

Let $\Delta_0 \subset \mathring{\Omega}_0$ be a smoothly bounded compact convex domain so large that f_0 extends to a holomorphic embedding $f_0 : R \setminus \mathring{\Delta}_0 \hookrightarrow \mathbb{C}^2$ satisfying $f_0(R \setminus \mathring{\Delta}_0) \cap r_0 \overline{\mathbb{B}} = \varnothing$. Choose a positive number $\epsilon_1 < \epsilon_0/2$. An application of Mergelyan theorem in two steps (see [18, Theorem 16]) furnishes a compact set Ω_1 as in (6.1), consisting of four pairwise disjoint smoothly bounded compact convex domains, and a holomorphic embedding $\tilde{f}_0 : K_1 = R \setminus \mathring{\Omega}_1 \hookrightarrow \mathbb{C}^2$ such that

$$\tilde{f}_0(K_1 \setminus \mathring{K}_0) \cap r_0 \overline{\mathbb{B}} = \varnothing \quad \text{and} \quad \sup_{x \in K_0} |\tilde{f}_0(x) - f_0(x)| < \epsilon_1/2. \tag{6.5}$$

Indeed, we first extend f_0 to a smooth embedding $f_0 : (R\setminus\mathring{\Delta}_0) \cup E_0 \hookrightarrow \mathbb{C}^2$, where E_0 is the vertical straight line segment dividing Δ_0 in two convex subsets of the same width, such that $f_0(E_0) \cap r_0\overline{\mathbb{B}} = \varnothing$. Note that E_0 intersects $R\setminus\mathring{\Delta}_0$ only at its endpoints and the intersections are transverse. By Mergelyan theorem (see [18, Theorem 16]) we can then approximate f_0 in the $\mathscr{C}^1$ topology on $(R \setminus \mathring{\Delta}_0) \cup E_0$ by a holomorphic embedding $f_0' : (R\setminus\mathring{\Delta}_0) \cup \overline{\Upsilon}_0 \hookrightarrow \mathbb{C}^2$, where Υ_0 is a neighborhood of E_0 in Δ_0 as explained above, such that

$$f_0'((\Omega_0 \setminus \mathring{\Delta}_0) \cup \overline{\Upsilon}_0) \cap r_0\overline{\mathbb{B}} = \varnothing.$$

We then repeat the process simultaneously in the two components Δ_0^1 and Δ_0^2 of $\Delta_0\setminus\Upsilon_0$: we suitably extend f_0' to $E_0^1 \cup E_0^2$, where E_0^j, $j = 1, 2$, is the horizontal straight line segment dividing Δ_0^j in two convex subsets of the same height, and apply Mergelyan theorem to approximate f_0' in the $\mathscr{C}^1$ topology on $(R\setminus\mathring{\Delta}_0) \cup \overline{\Upsilon}_0 \cup E_0^1 \cup E_0^2$ by a holomorphic embedding $\tilde{f}_0 : \Omega_1 \hookrightarrow \mathbb{C}^2$, where Ω_1 is of the form (6.1) and $\tilde{f}_0$ satisfies (6.5) for $K_1 = R\setminus\mathring{\Omega}_1$.

Note that $K_0 \subset \mathring{K}_1$ and choose a point $\alpha_1 \in \mathring{K}_1\setminus K_0$. Arguing as in the proof of Theorem 1.2, we may use Lemma 1.1 to obtain a holomorphic embedding $f_1 : K_1 \hookrightarrow \mathbb{C}^2$ satisfying the following conditions:

(a) $\sup_{x\in K_0} |f_1(x) - \tilde{f}_0(x)| < \epsilon_1/2$. Hence, $\sup_{x\in K_0} |f_1(x) - f_0(x)| < \epsilon_1$ by (6.5).
(b) $f_1(\alpha_i) = \beta_i$ for $i = 0, 1$.
(c) $f_1(bK_1) \cap r_1\overline{\mathbb{B}} = \varnothing$.
(d) $f_1(K_1 \setminus \mathring{K}_0) \cap \min\{r_0, |\beta_1|/2\}\overline{\mathbb{B}} = \varnothing$.

We repeat this procedure inductively, following the recursive construction of a Cantor set C in Ω_0 described above; see (6.2) and (6.3). In this way, we construct a sequence of tuples $T_j = \{f_j, K_j, \epsilon_j, \alpha_j\}$, $j \in \mathbb{N}$, where

- $K_j = R\setminus\Omega_j$, where Ω_j is a domain in Ω_0 consisting of 4^j pairwise disjoint smoothly bounded compact convex domains,
- $f_j : K_j \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding,
- $\epsilon_j > 0$ is a number, and
- $\alpha_j \in \mathring{K}_j \setminus K_{j-1}$ is a point,

such that T_j satisfies $K_{j-1} \subset \mathring{K}_j$ and conditions (2_j)–(4_j), (6_j), and (7_j) in the proof of Theorem 1.2 for all $j \in \mathbb{N}$ (with $\mathring{M}$ replaced by K_{j-1} in (3_j)), and we have that

$$C = R \setminus \bigcup_{j\geq 0} K_j = \bigcap_{j\geq 0} \Omega_j$$

is a Cantor set in R. As in the proof of Theorem 1.2, there is a limit map

$$F = \lim_{j\to\infty} f_j : R \setminus C = \bigcup_{j\geq 0} K_j \to \mathbb{C}^2$$

which is an almost proper injective holomorphic immersion and satisfies

$$B = F(\{\alpha_j : j \geq 1\}) \subset F(R \setminus C).$$

Moreover, the map $F : R \setminus C \to \mathbb{C}^2$ is proper, and hence a proper holomorphic embedding, if the given set B is closed in $\mathbb{C}^2$ and discrete. This concludes the proof of Theorem 1.5.

7 Proof of Theorem 1.6

Let S be an open Riemann surface. Fix an exhaustion

$$S_0 \Subset S_1 \Subset S_2 \Subset \cdots \Subset \bigcup_{j\ge 0} S_j = S \tag{7.1}$$

of S by connected, smoothly bounded, compact domains without holes in S such that S_0 is a closed disc and the Euler characteristic of $S_j \setminus \mathring{S}_{j-1}$ equals 0 or -1 for every $j = 1, 2, \ldots$ (see [9, Lemma 4.2]). Also set $D_{-1} = D_0 = \varnothing$ and let

$$D_1 \Subset D_2 \Subset \cdots \Subset \bigcup_{j\ge 1} D_j = D \tag{7.2}$$

be an exhaustion of D by smoothly bounded, polynomially convex, strongly pseudoconvex, Runge domains [20, Sect. 2.3]. Assume that the set $B = \{\beta_1, \beta_2, \ldots\}$ is infinite, set $m_0 = 0$, and for each $j \in \mathbb{N}$ denote by m_j the unique integer such that

$$\beta_j \in \overline{D}_{m_j+1} \setminus \overline{D}_{m_j}. \tag{7.3}$$

Fix $\epsilon_0 > 0$ and set $K_0 = S_0$ and $K_{-1} = \varnothing$. Let $f_0 : K_0 \hookrightarrow D$ be a holomorphic embedding with $f_0(K_0) \subset D_1$; recall that K_0 is a disc. Choose $\alpha_0 \in \mathring{K}_0$ such that $f_0(\alpha_0) \notin B$ and set $\beta_0 = f_0(\alpha_0) \in D_1$. We shall construct a sequence $X_j = \{K_j, f_j, \epsilon_j, \alpha_j\}$, $j \in \mathbb{N}$, where

- $K_j \subset S$ is a connected, smoothly bounded, compact domain without holes in S,
- $f_j : K_j \hookrightarrow D$ is a holomorphic embedding,
- $\epsilon_j > 0$ is a positive number, and
- $\alpha_j \in \mathring{K}_j \setminus K_{j-1}$ is a point,

such that the following conditions hold for all $j \in \mathbb{N}$:

(1_j) $K_{j-1} \Subset K_j \subset S_j$ and K_j is diffeotopic to S_j.
(2_j) $\sup_{x\in K_{j-1}} |f_j(x) - f_{j-1}(x)| < \epsilon_j$.
(3_j) $\epsilon_j < \epsilon_{j-1}/2$ and every holomorphic map $\varphi : K_{j-1} \to \mathbb{C}^2$ with $|\varphi - f_{j-1}| < 2\epsilon_j$ on K_{j-1} is an embedding on K_{j-2}.
(4_j) $f_j(\alpha_i) = \beta_i$ for all $i \in \{0, \ldots, j\}$.
(5_j) $f_j(bK_i) \cap \overline{D}_i = \varnothing$ for all $i \in \{0, \ldots, j\}$.
(6_j) $f_j(K_j \setminus \mathring{K}_{j-1}) \cap \overline{D}_{j-1} \cap \overline{D}_{m_j} = \varnothing$. (See (7.3) for the definition of m_j.)
(7_j) $\mathrm{length}(f_j \circ \gamma) > 1$ for every path $\gamma : [0, 1] \to K_j$ with $\gamma(0) \in K_{j-1}$ and $\gamma(1) \in bK_j$.

Assume for a moment that such a sequence exists. Conditions (7.1) and (1_j) imply that

$$M = \bigcup_{j\in\mathbb{N}} K_j \subset S \tag{7.4}$$

is a domain in S that is diffeotopic to S. In particular, there is a complex structure J on S such that the open Riemann surface (S, J) is biholomorphic to M. (For details in a similar setting, see [3, proof of Theorem 1.4 (b) and Corollary 1.5].) By (1_j), (2_j), (3_j), and the maximum principle, there is a limit map

$$F = \lim_{j\to\infty} f_j : M \to D,$$

which is an injective holomorphic immersion and satisfies

$$\sup_{x\in K_{j-1}} |F(x) - f_{j-1}(x)| < 2\epsilon_j, \quad j \in \mathbb{N}. \tag{7.5}$$

We claim that if each $\epsilon_j > 0$ is chosen sufficiently small then F satisfies the conclusion of Theorem 1.6. Indeed, if every $\epsilon_j > 0$ is small enough then (7.4), (7.5), and conditions (1_j) and (7_j) guarantee that $\operatorname{length}(F \circ \gamma) = +\infty$ for every proper path $\gamma : [0, 1) \to M$, and hence F is complete. Likewise, (7.5) and conditions (5_j) imply that $F(bK_j) \cap \overline{D}_j = \varnothing$ for all $j \in \mathbb{N}$ whenever the ϵ_j's are sufficiently small, and hence $F : M \to D$ is an almost proper map in view of (7.2), (7.4), and conditions (1_j). Indeed, if $Q \subset D$ is compact and we take $m \in \mathbb{N}$ so large that $Q \subset D_m$, then $F^{-1}(Q) \cap bK_j = \varnothing$ for every $j > m$, hence all components of $F^{-1}(Q)$ are compact in M. Conditions (4_j) give that $B = \{\beta_1, \beta_2, \ldots\} \subset F(M)$. Finally, if each $\epsilon_j > 0$ is chosen sufficiently small then $F(K_j \backslash \mathring{K}_{j-1}) \cap \overline{D}_{j-1} \cap \overline{D}_{m_j} = \varnothing$ for all $j \in \mathbb{N}$ by (7.5) and conditions (6_j), while if the given set $B \subset D$ is closed and discrete then $\lim_{j \to \infty} \min\{j - 1, m_j\} = +\infty$; see (7.2) and (7.3). These conditions imply that $F : M \hookrightarrow D$ is a proper map, and hence a proper holomorphic embedding, provided that B is closed in D and discrete. Therefore, F satisfies the conclusion of the theorem.

To complete the proof, it remains to explain the induction. The basis is given by the tuple $X_0 = \{K_0, f_0, \epsilon_0, \alpha_0\}$; it satisfies (1_0), (4_0), (5_0), and (6_0), while the remaining conditions are void for $j = 0$. Fix $j \in \mathbb{N}$ and assume that we have a tuple $X_{j-1} = \{K_{j-1}, f_{j-1}, \epsilon_{j-1}, \alpha_{j-1}\}$ fulfilling conditions (1_{j-1}), (4_{j-1}), (5_{j-1}), and (6_{j-1}). We distinguish two cases.

Case 1: The Euler characteristic of $S_j \setminus \mathring{S}_{j-1}$ *is* 0. In this case, (7.1) and (1_{j-1}) imply that $K_{j-1} \subset \mathring{S}_j$ and K_{j-1} is a strong deformation retract of S_j. Choose an integer $d > \max\{j, m_j + 1\}$ so large that $f_{j-1}(K_{j-1}) \subset D_d$ and set $d' = \min\{j - 1, m_j\} < d$. Assume without loss of generality that $\beta_j \notin f_{j-1}(K_{j-1})$. By (5_{j-1}) and (7.3) we have that

$$f_{j-1}(bK_{j-1}) \cup \{\beta_j\} \subset D_d \setminus \overline{D}_{d'}. \tag{7.6}$$

Pick a point $a \in bK_{j-1}$ and attach to K_{j-1} a smooth embedded arc $\eta \subset \mathring{S}_j$ such that $\eta \cap K_{j-1} = \{a\}$ and the intersection of bK_{j-1} and η is transverse at a. On the image side, choose a smoothly embedded arc

$$\lambda \subset D_d \setminus \overline{D}_{d'} \tag{7.7}$$

such that λ agrees with $f_{j-1}(\eta)$ near the endpoint $f_{j-1}(a)$, $\lambda \cap f_{j-1}(K_{j-1}) = f_{j-1}(a)$, and the other endpoint of λ equals β_j. Let α_j denote the other endpoint of η and extend f_{j-1} to a smooth diffeomorphism $\eta \to \lambda$ such that $f_{j-1}(\alpha_j) = \beta_j$. By Mergelyan theorem (see [18, Theorem 16]) we may assume in view of (5_{j-1}), (7.6), and (7.7) that there is a connected, smoothly bounded, compact domain $K \subset S$ without holes such that

(i) $\alpha_j \in K_{j-1} \cup \eta \Subset K \subset \mathring{S}_j$,
(ii) K_{j-1} is a strong deformation retract of K, and
(iii) $f_{j-1} : K \hookrightarrow D_d$ is a holomorphic embedding satisfying

$$f_{j-1}(\alpha_j) = \beta_j \quad \text{and} \quad f_{j-1}(\overline{K \setminus K_{j-1}}) \cap \overline{D}_{d'} = \varnothing.$$

It follows that K is a strong deformation retract of S_j as well. By Charpentier and Kosiński [13, Lemma 2.4] there is a compact polynomially convex set $\Gamma \subset D_{d+1} \setminus \overline{D}_d$ whose connected components are holomorphically contractible (for example, convex) such that $\overline{D}_d \cup \Gamma$ is polynomially convex and $\operatorname{length}(\sigma) > 1$ for every path $\sigma : [0, 1] \to D \backslash \Gamma$ with $\sigma(0) \in \overline{D}_d$ and $\sigma(1) \in D \backslash D_{d+1}$. It follows that the compact set

$$L = \overline{D}_{d'} \cup \Gamma = (\overline{D}_{j-1} \cap \overline{D}_{m_j}) \cup \Gamma$$

is also polynomially convex, and $f_{j-1}(\overline{K \backslash K_{j-1}}) \cap L = \varnothing$ by (iii). Lemma 1.1 furnishes a holomorphic embedding $f_j : K \hookrightarrow \mathbb{C}^2$ satisfying the following conditions:

(a) $f_j(bK) \cap \overline{D}_{d+1} = \varnothing$.
(b) $f_j(\overline{K \setminus K_{j-1}}) \cap L = \varnothing$.
(c) $\sup_{x \in K_{j-1} \cup \eta} |f_j(x) - f_{j-1}(x)| < \epsilon_j$ (note that $K_{j-1} \cup \eta$ has no holes in $\mathring{K}$).
(d) $f_j(\alpha_i) = f_{j-1}(\alpha_i) = \beta_i$ for all $i \in \{0, \dots, j\}$ (see (iii) and (4_{j-1})).
(e) $f_j(bK_i) \cap \overline{D}_i = \varnothing$ for all $i \in \{0, \dots, j-1\}$ (see (5_{j-1})).

Here, $\epsilon_j > 0$ is so small that condition (3_j) holds and

$$f_j(K_{j-1} \cup \eta) \subset D_d \tag{7.8}$$

(see (iii) and (c)). Note that (7.8) ensures that the point $\alpha_j \in \eta$ lies in the connected component of $f_j^{-1}(D_d)$ containing K_{j-1}. This, the maximum principle, and conditions (i) and (a) guarantee the existence of a connected, smoothly bounded, compact domain $K_j \subset \mathring{K}$ without holes in S satisfying (1_j),

$$\alpha_j \in \mathring{K}_j, \quad f_j(bK_j) \cap \overline{D}_{d+1} = \varnothing, \quad \text{and} \quad f_j(K_j) \subset D. \tag{7.9}$$

Condition (c) implies (2_j); (d) ensures (4_j); (e), (7.9), and $d > j$ give (5_j); (b) implies (6_j); and (b), (7.8), (7.9), and the properties of Γ ensure (7_j). This closes the induction in this case.

Case 2: The Euler characteristic of $S_j \setminus \mathring{S}_{j-1}$ *equals* -1. In this case, there is a smooth Jordan arc $E \subset \mathring{S}_j \setminus \mathring{K}_{j-1}$, transversely attached with its two endpoints to bK_{j-1} and otherwise disjoint from K_{j-1}, such that $K_{j-1} \cup E$ is a strong deformation retract of S_j. Given $\epsilon > 0$, an application of Mergelyan theorem (see [18, Theorem 16]) furnishes a connected, smoothly bounded, compact domain $K \subset \mathring{S}_j$, without holes in S, such that $K_{j-1} \cup E \subset \mathring{K}$ and K is a strong deformation retract of S_j, and a holomorphic embedding $g : K \hookrightarrow D$ such that $\sup_{x \in K_{j-1}} |g(x) - f_{j-1}(x)| < \epsilon$, $g(\alpha_i) = \beta_i$ for all $i \in \{0, \dots, j-1\}$ (see (4_{j-1})), $g(bK_i) \cap \overline{D}_i = \varnothing$ for all $i \in \{0, \dots, j-1\}$, and $g(K \setminus \mathring{K}_{j-1}) \cap \overline{D}_{j-1} = \varnothing$ (see (5_{j-1})). See [1, p. 216, Case 1] for the details in a very similar situation. This reduces the proof of the inductive step to Case 1. This closes the induction and completes the proof of Theorem 1.6.

8 Proof of Theorems 1.7 and 1.9

For simplicity of exposition we shall prove these results in the case when the open Riemann surface S is a disc, say, $S = 2\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 2\}$. In particular, the domain $M \subset S$ in Theorem 1.7 must be a disc, while the one in Theorem 1.9 must be a planar domain. The general cases are seen by combining the proof in these special cases with the procedure to prescribe the topology in the proof of Theorem 1.6; we leave the details to interested readers.

Proof of Theorem 1.7 Let $X \subset \mathbb{C}^2$ and $B = \{\beta_j\}_{j \in \mathbb{N}} \subset X$ be as in the statement. Let us assume that $S = 2\mathbb{D}$. Set $D_0 = \overline{\mathbb{D}}$ and choose a holomorphic embedding $f_0 : D_0 \to X$. Assume that $f_0(0) \notin B$ and set $\alpha_0 = 0$, $\beta_0 = f(\alpha_0)$, and $D_{-1} = \varnothing$. Fix a number $\epsilon_0 > 0$. We shall inductively construct a sequence of smoothly bounded closed discs $D_j \subset 2\mathbb{D}$, points $\alpha_j \in \mathring{D}_j$, holomorphic embeddings $f_j : D_j \to X$, and numbers $\epsilon_j > 0$ satisfying the following conditions for all $j \in \mathbb{N} = \{1, 2, \dots\}$:

(i_j) $D_{j-1} \subset \mathring{D}_j$.
(ii_j) $|f_j - f_{j-1}| < \epsilon_j$ on D_{j-1}.
(iii_j) $f_j(\alpha_k) = \beta_k$ for all $k \in \{0, \dots, j\}$.

(iv$_j$) $\operatorname{dist}_{f_j}(0, bD_j) > j$, where $\operatorname{dist}_{f_j}$ denotes the distance function on D_j associated to the Riemannian metric induced on D_j by the Euclidean one in $\mathbb{C}^2$ via the embedding f_j.
(v$_j$) $\epsilon_j < \epsilon_{j-1}/2$ and every holomorphic map $\varphi : D_{j-1} \to \mathbb{C}^2$ such that $|\varphi - f_{j-1}| < 2\epsilon_j$ on D_{j-1} is an embedding and satisfies $\varphi(D_{j-1}) \subset X$ and $\operatorname{dist}_\varphi(0, bD_{j-1}) > j-1$.

The basis of the induction is provided by the already chosen disc $D_0 = \overline{\mathbb{D}}$, point $\alpha_0 = 0 \in \mathring{D}_0$, holomorphic embedding $f_0 : D_0 \to X$, and number $\epsilon_0 > 0$. They meet conditions (i$_0$), (iii$_0$), and (iv$_0$), while (ii$_0$) and (v$_0$) are void. For the inductive step, fix $j \in \mathbb{N}$ and assume that we have suitable objects D_k, α_k, f_k, and ϵ_k satisfying (i$_k$), (iii$_k$), and (iv$_k$) for all $k \in \{0, \ldots, j-1\}$. By (iv$_{j-1}$) we can choose a number $\epsilon_j > 0$ so small that (v$_j$) is satisfied. Reasoning as in Case 1 in the proof of Theorem 1.6, we may assume that there are a smoothly bounded closed disc $\Sigma \subset 2\mathbb{D}$ and a point $\alpha_j \in \Sigma$ such that f_{j-1} extends to a holomorphic embedding $f_{j-1} : \Sigma \to X$ with

$$\alpha_j \in \mathring{\Sigma} \supset D_{j-1} \text{ and } f_{j-1}(\alpha_j) = \beta_j. \tag{8.1}$$

Moreover, Σ can be chosen as close to D_{j-1} as desired.

Since Σ is a disc and $f_{j-1} : \Sigma \to \mathbb{C}^2$ is holomorphic, the compact set $f_{j-1}(\Sigma) \subset X$ is polynomially convex in $\mathbb{C}^2$ by a theorem of Wermer [46] (see also Stolzenberg [42] and Alexander [10]), so it admits a basis of open neighbourhoods which are smoothly bounded, strongly pseudoconvex, and Runge in $\mathbb{C}^2$. (Indeed, a compact polynomially convex set $K \subset \mathbb{C}^n$ is the zero set of a smooth plurisubharmonic exhaustion function $\rho \geq 0$ on $\mathbb{C}^n$ which is strongly plurisubharmonic on $\mathbb{C}^n$ K [44, Theorem 1.3.8], and every sublevel set $\{\rho < c\}$ for $c > 0$ of such a function is a strongly pseudoconvex Runge domain in $\mathbb{C}^n$ by [34, Theorem 4.3.4].) Let $U_1 \Subset U_2 \Subset X$ be a pair of such relatively compact neighbourhoods of $f_{j-1}(\Sigma)$ in X. By [13, Lemma 2.4] there is a compact polynomially convex set $\Gamma \subset U_2 \setminus \overline{U}_1$ whose connected components are holomorphically contractible such that $\overline{U}_1 \cup \Gamma$ is polynomially convex and $\operatorname{length}(\gamma) > 1$ for every path $\gamma : [0,1] \to X \backslash \Gamma$ with $\gamma(0) \in \overline{U}_1$ and $\gamma(1) \in X \backslash \overline{U}_2$. Since D_{j-1} is a Runge compact in $\mathring{\Sigma}$ and $f_{j-1}(\Sigma) \cap \Gamma = \varnothing$, Lemma 1.1 furnishes a holomorphic embedding $f_j : \Sigma \hookrightarrow \mathbb{C}^2$ satisfying the following conditions:

(a) $f_j(b\Sigma) \cap \overline{U}_2 = \varnothing$; recall that U_2 is compact.
(b) $f_j(\Sigma \setminus \mathring{D}_{j-1}) \cap \Gamma = \varnothing$.
(c) $|f_j - f_{j-1}| < \epsilon_j$ on a smoothly bounded closed disc Σ' with $D_{j-1} \cup \{\alpha_j\} \Subset \Sigma' \Subset \Sigma$.
(d) $f_j(\alpha_k) = f_{j-1}(\alpha_k)$ for all $k \in \{0, \ldots, j\}$.

Further, choosing $\epsilon_j > 0$ sufficiently small, condition (c) implies that

$$f_j(D_{j-1}) \subset f_j(\Sigma') \subset U_1; \tag{8.2}$$

recall that U_1 is an open neighbourhood of $f_{j-1}(\Sigma)$. Since the domain U_2 is Runge in $\mathbb{C}^2$, this and conditions (a) and (c) guarantee the existence of a smoothly bounded closed disc $D_j \subset \mathring{\Sigma} \subset 2\mathbb{D}$, with $\Sigma' \subset \mathring{D}_j$ (this implies (i$_j$)), satisfying $f_j(D_j) \subset X$ and $f_j(bD_j) \cap \overline{U}_2 = \varnothing$. This, (8.2), (b), and the properties of Γ imply $\operatorname{dist}_{f_j}(D_{j-1}, bD_j) > 1$, while (c)=(ii$_j$) and (v$_j$) ensure that $\operatorname{dist}_{f_j}(0, bD_{j-1}) > j-1$, so (iv$_j$) holds. Finally, (d), (iii$_{j-1}$), and (8.1) imply (iii$_j$). This closes the induction.

Note that $M = \bigcup_{j \in \mathbb{N}} D_j \subset 2\mathbb{D}$ is an open disc which is diffeotopic to $S = 2\mathbb{D}$. By conditions (i$_j$), (ii$_j$), and (v$_j$), there is a limit holomorphic map $F = \lim_{j \to \infty} f_j : M \to \mathbb{C}^2$ such that $|F - f_{j-1}| < 2\epsilon_j$ for all $j \in \mathbb{N}$. So, by (v$_j$), F has range in X and is a complete injective immersion. Finally, conditions (iii$_j$) ensure that $B \subset F(M)$. $\square$

Proof of Corollary 1.8 Let G, K, and ϵ be as in the statement. By Mergelyan theorem (see [18, Theorem 16]) there is a holomorphic embedding $f_0 : \overline{\mathbb{D}} \to \mathbb{C}^2$ with $|f_0 - G| < \epsilon/2$ on $\overline{\mathbb{D}}$, hence $d_{\mathrm{H}}(f_0(\overline{\mathbb{D}}), G(\overline{\mathbb{D}})) < \epsilon/2$. Let $X \subset \mathbb{C}^2$ be an $\epsilon/2$-tubular neighbourhood of $f_0(\overline{\mathbb{D}})$. It follows that $d_{\mathrm{H}}(\overline{X}, G(\overline{\mathbb{D}})) < \epsilon$. By the proof of Theorem 1.7 there are an open disc M with $\mathbb{D} \Subset M \Subset 2\mathbb{D}$ and a complete injective holomorphic immersion $h : M \to X$ such that $h(M)$ is everywhere dense in X, i.e., $\overline{h(M)} = \overline{X}$. We then have that $d_{\mathrm{H}}(\overline{h(M)}, G(\overline{\mathbb{D}})) < \epsilon$. Furthermore, an inspection of the proof of Theorem 1.7 shows that we can choose M so close to $\mathbb{D}$ that there is a holomorphic diffeomorphism $\phi : \mathbb{D} \to M$ satisfying $|h \circ \phi - f_0| < \epsilon/2$ on the compact set $K \subset \mathbb{D}$. It is clear that $F = h \circ \phi$ satisfies the conclusion of the corollary. $\square$

Proof of Theorem 1.9 Let $B = \{\beta_j\}_{j \in \mathbb{N}} \subset X \subset \mathbb{C}^2$ be as in the statement and $S = 2\mathbb{D}$. Let $D_0, f_0, \alpha_0, \beta_0, D_{-1}$, and ϵ_0 be as in the proof of Theorem 1.7. Choose an exhaustion

$$\varnothing = K_0 \Subset K_1 \Subset K_2 \Subset \cdots \subset \bigcup_{j \in \mathbb{N}} K_j = X \tag{8.3}$$

of X by compact domains. We shall inductively construct an increasing sequence of connected, smoothly bounded compact domains $D_j \subset 2\mathbb{D}$, as well as sequences of points $\alpha_j \in \mathring{D}_j$, holomorphic embeddings $f_j : D_j \to X$, and numbers $\epsilon_j > 0$ satisfying conditions (i$_j$)–(v$_j$) in the proof of Theorem 1.7 and also the following one for all $j \in \mathbb{N} = \{1, 2, \ldots\}$:

(vi$_j$) $f_j(bD_j) \cap K_j = \varnothing$.

(Unlike in the proof of Theorem 1.7, D_j need not be a disc.) Note that (vi$_0$) holds true. For the inductive step, fix $j \in \mathbb{N}$ and assume that we have suitable objects D_k, a_k, f_k, and ϵ_k satisfying (i$_k$), (iii$_k$), (iv$_k$), and (vi$_k$) for all $k \in \{0, \ldots, j-1\}$. Choose $\epsilon_j > 0$ so small that (v$_j$) holds. Reasoning as in the proof of Theorem 1.7 we may assume that f_{j-1} extends to a holomorphic embedding $f_{j-1} : \Sigma \to X$ on a connected, smoothly bounded compact domain $\Sigma \subset 2\mathbb{D}$ such that $D_{j-1} \subset \mathring{\Sigma}$, D_{j-1} is a strong deformation retract of Σ, and there is a point $\alpha_j \in \mathring{\Sigma}$ with $f_{j-1}(\alpha_j) = \beta_j$. By a small perturbation of the map f_{j-1} keeping the above conditions in place, we can ensure in addition that $f_{j-1}(\Sigma)$ is polynomially convex in $\mathbb{C}^2$; see the argument in the first part of the proof of Lemma 1.1 based on Stolzenberg's theorem [41]. We can therefore choose a pair of smoothly bounded, relatively compact, pseudoconvex domains $U_1 \Subset U_2 \Subset X$ which are Runge in $\mathbb{C}^2$ such that $f_{j-1}(\Sigma) \subset U_1$. We place a suitable labyrinth $\Gamma = \Gamma_j$ in $U_2 \setminus \overline{U}_1$ and choose a holomorphic embedding $f_j : \Sigma \to \mathbb{C}^2$ as in the proof of Theorem 1.7 with condition (a) replaced by

(a') $f_j(b\Sigma) \cap (K_j \cup \overline{U}_2) = \varnothing$.

In particular, conditions (b)–(d) and (8.2) in the proof of Theorem 1.7 are satisfied (in this case Σ' is a smoothly bounded compact domain in $\mathring{\Sigma}$ containing $D_{j-1} \cup \{\alpha_j\}$ in its interior and such that D_{j-1} is a strong deformation retract of Σ'). In view of (a') and the mentioned conditions, there is a connected, smoothly bounded compact domain $D_j \subset \mathring{\Sigma} \subset 2\mathbb{D}$ such that

$$D_{j-1} \cup \{\alpha_j\} \subset \mathring{D}_j, \ \ f_j(D_j) \subset X, \ \text{ and } f_j(bD_j) \cap (K_j \cup \overline{U}_2) = \varnothing.$$

(The last condition will be the key to ensure almost properness of the limit map; compare with the condition $f_j(bD_j) \cap \overline{U}_2 = \varnothing$ in the proof of Theorem 1.7. In general, since the domain X need not be pseudoconvex and Runge in $\mathbb{C}^2$, we cannot choose D_j such that D_{j-1} is a strong deformation retract of D_j; possibly D_j has more boundary components than D_{j-1}.) Then, conditions (i$_j$)–(vi$_j$) hold true, which closes the induction.

Set $M = \bigcup_{j \in \mathbb{N}} D_j \subset 2\mathbb{D}$, a connected relatively compact domain in $\mathbb{C}$. By the reasoning in the proof of Theorem 1.7, there is a limit map $F = \lim_{j\to\infty} f_j : M \to X$ which is a complete injective holomorphic immersion such that $B \subset F(M)$. Finally, condition (vi$_j$) ensures that F is an almost proper map provided that each number $\epsilon_j > 0$ in the inductive process is chosen sufficiently small. Indeed, by such a choice (similar to that in (v$_j$)) we can ensure that $F(bD_j) \cap K_j = \varnothing$ for all $j \in \mathbb{N}$; see (ii$_j$) and (vi$_j$). This, (i$_j$), and (8.3) imply the almost properness of $F : M \to X$. $\square$

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