Oka tubes in holomorphic line bundles

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Abstract Let (E,h) be a semipositive hermitian holomorphic line bundle on a compact complex manifold X with $\dim X > 1$. Assume that for each point $x \in X$ there exists a divisor $D \in |E|$ in the complete linear system determined by E whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property. Then, the disc bundle $\Delta_h(E) = \{e \in E : |e|_h < 1\}$ is an Oka manifold while $E \setminus \overline{\Delta_h(E)} = \{e \in E : |e|_h > 1\}$ is a Kobayashi hyperbolic domain. In particular, the zero section of E admits a basis of Oka neighbourhoods $\{|e|_h < c\}$ with $c > 0$. We show that this holds if X is a projective space, a complex Grassmannian, or a product of Grassmannians. This phenomenon contributes to the heuristic principle that Oka properties are related to metric positivity of complex manifolds.

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1. Introduction

A complex manifold Y is called an *Oka manifold* if holomorphic maps $S \to Y$ from any Stein manifold S satisfy the Oka principle with approximation on compact holomorphically convex subsets of S and interpolation on closed complex subvarieties of S; see [16, Definition 5.4.1 and Theorem 5.4.4]. This is a central holomorphic flexibility notion in complex geometry, and it is of major interest to find new examples of Oka manifolds. A complex manifold Y is an *Oka-1 manifold* [2] if these properties hold for maps $S \to Y$ from any open Riemann surface S. Every complex homogeneous manifold is Oka (see Grauert [22] and [16, Proposition 5.6.1]). Many further examples were given by Gromov [30] and others; see the surveys in [16, 18].

In this paper, we describe a new phenomenon in Oka theory, relating the Oka property of tubes in hermitian holomorphic line bundles on compact Oka manifolds with the curvature properties of the metric. In particular, we show that disc bundles in many Griffiths semipositive holomorphic line bundles are Oka manifolds. This holds for semipositive ample line bundles on projective spaces (see Theorem 1.1), Grassmannians (see Proposition 4.4), and their products (see Corollary 4.9). Further examples can be found in Section 4. Our main result, Theorem 1.5, establishes this phenomenon for any polarised manifold (X,E) with the polarised density property, see Definition 1.7. An important ingredient in the proofs are the recent results of the second named author [40], who found large classes of Oka manifolds given as complements of closed holomorphically sets in $\mathbb{C}^n$ and in Stein manifolds with the density property.

Let $\pi : E \to X$ be a holomorphic line bundle on a connected compact complex manifold X, and let h be a hermitian metric on E. Denote by $|e|_h$ the norm of $e \in E$. We are interested in conditions on X and the hermitian line bundle (E,h) which ensure that the disc bundle

$$\Delta_h(E) = \{e \in E : |e|_h < 1\} \tag{1.1}$$

is an Oka manifold. In particular, when does the zero section $E(0) = \{e \in E : |e|_h = 0\}$ admit a basis of open Oka neighbourhoods? It turns out that these questions are related to semipositivity of the metric h, hence to the existence of nontrivial holomorphic sections $X \to E$.

We begin with some immediate observations. The total space E is Oka if and only if the base X is Oka [16, Theorem 5.6.5]. Since $\Delta_h(E)$ admits a holomorphic retraction onto the zero section $E(0) \cong X$, if $\Delta_h(E)$ is Oka then X is Oka [16, Proposition 5.6.8]. For any $c > 0$ the disc bundle $\Delta_{h,c}(E) = \{|e|_h < c\}$ is biholomorphic to $\Delta_h(E)$ by a dilation in the fibres, so an affirmative answer to the first question implies the same for the second one. The answers to both questions are negative for any hermitian metric h on the trivial line bundle $E = X \times \mathbb{C}$. In this case, $\Delta_h(E)$ is contained in $X \times c\Delta$ for some $c > 0$, where $\Delta \subset \mathbb{C}$ is the unit disc. This manifold admits a bounded plurisubharmonic function coming from $c\Delta$ which is nonconstant on every open subset, so it cannot contain any Oka domain [16, Proposition 7.1.9]. The same argument applies to trivial vector bundles of higher rank on a compact complex manifold.

A more subtle analysis is tied to the curvature of the metric h, which determines the geometric shape of the disc bundle (1.1). The curvature of h is the $(1,1)$-form on X given by

$$\mathfrak{i}\Theta_h = -\mathfrak{i}\,\partial\overline{\partial}\log h = -\frac{1}{2}dd^c \log h, \quad \mathfrak{i} = \sqrt{-1}$$

(see (2.3)). A hermitian holomorphic line bundle (E, h) is *positive* if $\mathfrak{i}\Theta_h$ is a positive $(1,1)$-form, and *semipositive* if $\mathfrak{i}\Theta_h \geq 0$. A holomorphic line bundle E is positive if it admits a hermitian metric with positive curvature. The disc bundle $\Delta_h(E)$ is a Hartogs domain in E, and the Levi form of its boundary is the hermitian form determined by $dd^c \log h$ (see Proposition 2.3). Hence, the metric negativity $\mathfrak{i}\Theta_h < 0$ at $x_0 \in X$ is equivalent to $\Delta_h(E)$ being strongly pseudoconvex over a neighbourhood of x_0, so it is not Oka. (Indeed, a domain with a strongly pseudoconvex boundary point admits a nonconstant bounded plurisubharmonic function, hence it cannot be Oka; see [16, Proposition 7.1.9].) If on the other hand $\mathfrak{i}\Theta_h \geq 0$ then $\Delta_h(E)$ is pseudoconcave, and we will show that it is an Oka manifold in many cases of interest.

We begin by considering line bundles on the simplest compact Oka manifolds, the projective spaces $\mathbb{CP}^n$. The isomorphism classes of holomorphic line bundles on a complex space X are in bijective correspondence with the elements of the Picard group $\mathrm{Pic}(X) = H^1(X, \mathscr{O}^*)$. For projective spaces, $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$ is a free cyclic group generated by the *hyperplane section bundle* $\mathscr{O}_{\mathbb{CP}^n}(1)$ (see Griffiths and Harris [29] or Wells [53]). It is customary to write $\mathscr{O}_{\mathbb{CP}^n}(k)$ for the k-th tensor power of $\mathscr{O}_{\mathbb{CP}^n}(1)$. The dual $\mathbb{U} = \mathscr{O}_{\mathbb{CP}^n}(-1)$ of $\mathscr{O}_{\mathbb{CP}^n}(1)$ is the *universal bundle*; see [53, p. 17, Example 2.6]. The line bundle $\mathscr{O}_{\mathbb{CP}^n}(k)$ is *positive* resp. *negative* according to whether $k > 0$ or $k < 0$. It admits a hermitian metric whose curvature is k-times the Fubini–Study form on $\mathbb{CP}^n$ (see Example 2.4).

Theorem 1.1. *Given a positive holomorphic line bundle $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ on $\mathbb{CP}^n$ $(n \geq 1,\ k \geq 1)$ and a semipositive hermitian metric h on E (i.e., $\mathfrak{i}\Theta_h \geq 0$), the following assertions hold.*

(a) *The punctured disc bundle $\Delta^*_h(E) = \{e \in E : 0 < |e|_h < 1\}$ is an Oka manifold, and the disc bundle $\Delta_h(E) = \{e \in E : |e|_h < 1\}$ is an Oka-1 manifold.*
(b) *If $n \geq 2$ or $E = \mathscr{O}_{\mathbb{CP}^n}(1)$ then the disc bundle $\Delta_h(E)$ is an Oka manifold.*
(c) *The domain $D_h(E) = E \setminus \overline{\Delta_h(E)} = \{e \in E : |e|_h > 1\}$ is Kobayashi hyperbolic and has pseudoconvex boundary $bD_h(E) = \{|e|_h = 1\}$.*

For a negative holomorphic line bundle $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ $(n \geq 1,\ k \leq -1)$ and a seminegative hermitian metric h on E ($\mathfrak{i}\Theta_h \leq 0$), the following assertions hold.

(a') *The domain $\Delta^*_h(E)$ is Kobayashi hyperbolic and pseudoconvex along $\{|e|_h = 1\}$.*
(b') *The domain $D_h(E) = E \setminus \overline{\Delta_h(E)}$ is Oka.*

These results hold if the metric h is continuous and semipositive (resp. seminegative) in the weak sense. They also hold for the restrictions of these bundles to any affine Euclidean chart in $\mathbb{CP}^n$.

With (E,h) as in part (b) of the theorem, the circle bundle $\{e \in E : |e|_h = 1\}$ splits E into a relatively compact Oka domain $\{|e|_h < 1\}$ and a hyperbolic domain $\{|e|_h > 1\}$. A phenomenon of this type was first observed by Forstnerič and Wold [21] who showed that, under a mild assumption on an unbounded closed convex set $K \subset \mathbb{C}^n$ $(n > 1)$, its interior $\mathring{K}$ is Kobayashi hyperbolic while its complement $\mathbb{C}^n \setminus K$ is an Oka domain.

Note that the natural projection $\Delta_h(E) \to \mathbb{CP}^n$ in Theorem 1.1 is a holomorphic submersion and a topological fibre bundle, the base and the total space are Oka manifolds in case (b), yet its fibres are Kobayashi hyperbolic. In particular, it is not an Oka map (see Definition 2.8) since the fibres of an Oka map are Oka manifolds (see [18, Proposition 3.14]). We now show that this phenomenon does not occur in holomorphic fibre bundles.

Proposition 1.2. *If $E \to X$ is a holomorphic fibre bundle on a connected complex manifold X whose fibre Y is Kobayashi hyperbolic with $\dim Y > 0$, then E is not an Oka manifold.*

Proof. Let $\pi : \widetilde{X} \to X$ be the universal covering. The pullback bundle $\pi^* E \to \widetilde{X}$ has the same fibre Y. Since Y is hyperbolic, there are no nontrivial holomorphic maps to its holomorphic automorphism group $\mathrm{Aut}(Y)$ (see Kobayashi [35, Theorem 5.4.5]), so this is a flat bundle. Since $\widetilde{X}$ is simply connected, it follows that the bundle $\pi^* E \to \widetilde{X}$ is trivial, isomorphic to $\widetilde{X} \times Y$ (see Royden [47, Corollary 1]). This manifold is not Oka due to the hyperbolic factor Y. Since the natural map $\pi^* E \to E$ is a holomorphic covering map and the class of Oka manifolds is invariant under such maps (see [16, Proposition 5.6.3]), the manifold E is not Oka. □

Theorem 1.1 is proved in Section 3; here is an outline. If $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ with $k > 0$, then for any hermitian metric h on E the restriction of the disc bundle $\Delta_h(E)$ to any affine Euclidean chart in $\mathbb{CP}^n$ is a Hartogs domain Ω in $\mathbb{C}^{n+1}$ whose radius grows at least linearly (see Example 2.4). If h is semipositive then Ω is pseudoconcave (see Proposition 2.3 (iii')). By Proposition 3.1, such a domain is Oka if $n \geq 2$. This result and the localization theorem for Oka manifolds [38, Theorem 1.4] are the key to the proof of part (b). Part (a) is seen by an explicit analysis of the hyperplane section bundle $\mathscr{O}_{\mathbb{CP}^n}(1)$, using that complements of compact polynomially convex sets in $\mathbb{C}^{n+1}$ $(n \in \mathbb{N})$ are Oka (see Kusakabe [40, Corollary 1.3]), and that the relevant properties of these tubes are preserved under tensor powers (see Proposition 2.1 and Corollary 2.2). When passing to the hermitian dual bundle (E^*, h^*), positivity and negativity get reversed and the punctured disc bundle $\Delta^*_h(E)$ is biholomorphic to the outer tube $D_{h^*}(E^*) = \{h^* > 1\}$ of the dual bundle, which gives part (b'). Parts (c) and (a') follow from Grauert's result on blowing down exceptional varieties [23, Satz 1, p. 341]; see Remark 1.10.

We now proceed towards our main results. Recall that a holomorphic vector field on a complex manifold X is said to be *complete* if its flow exists for all complex values of time, so it forms a complex one-parameter group of holomorphic automorphisms of X. The following class of complex manifolds was introduced by Varolin [52]; see also [16, Definition 4.10.1].

Definition 1.3. A complex manifold X has the *density property* if every holomorphic vector field on X can be approximated uniformly on compacts by sums and commutators of complete holomorphic vector fields on X.

Every Stein manifold X with the density property has infinite dimensional automorphism group (hence $\dim X > 1$), and it is an elliptic Oka manifold (see [16, Proposition 5.6.23]). The fact that the Euclidean spaces $\mathbb{C}^n$, $n > 1$, have the density property was discovered by Andersén and Lempert [4]. Most complex Lie groups and complex homogeneous manifolds have the density property. Surveys can be found in [16, Chapter 4], [19], and [41].

Theorem 1.4. *Assume that X is a compact complex submanifold of $\mathbb{CP}^n$ such that for the affine charts $U_i \cong \mathbb{C}^n$ covering $\mathbb{CP}^n$ the Stein manifold $X \cap U_i$ has the density property for every $i = 0, \ldots, n$. Let $E \cong \mathscr{O}_{\mathbb{CP}^n}(k)$ with $k \geq 1$ be a positive holomorphic line bundle on $\mathbb{CP}^n$ endowed with a hermitian metric h satisfying $\mathrm{i}\Theta_h|_{TX} \geq 0$. Then the disc bundle $\Delta_h(E)|_X = \{e \in E|_X : |e|_h < 1\}$ is an Oka manifold while $D_h(E)|_X = \{e \in E|_X : |e|_h > 1\}$ is a Kobayashi hyperbolic domain in $E|_X$ with pseudoconvex boundary $\{e \in E|_X : |e|_h = 1\}$.*

An example satisfying Theorem 1.4 is the hyperquadric hypersurface

$$X = \big\{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : z_0^2 + z_1^2 + \cdots + z_n^2 = 0\big\}, \;\; n \geq 3.$$

The intersection of X with any affine chart $z_i \neq 0$ is the complexified sphere in $\mathbb{C}^n$, which is a Danielewski manifold and has the density property (see Kaliman and Kutzschebauch [34]). This *null quadric* plays a major role in the theory of minimal surfaces; see [3]. Another example is the Plücker embedding of a Grassmannian of dimension > 1; see Example 4.3.

Denote by $|E|$ the complete linear system of divisors on X associated to a holomorphic line bundle $E \to X$ (see e.g. [29]). The divisors in $|E|$ are the zero sets of nontrivial holomorphic sections of E. The following is our main result.

Theorem 1.5. *Let E be a holomorphic line bundle on a compact complex manifold X. Assume that for each point $x \in X$ there exists a divisor $D \in |E|$ whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property. Given any semipositive hermitian metric h on E, the disc bundle $\Delta_h(E)$ (1.1) is an Oka manifold while $D_h(E) = E \setminus \overline{\Delta_h(E)}$ is a Kobayashi hyperbolic domain with pseudoconvex boundary $bD_h(E) = \{|e|_h = 1\}$. In particular, the zero section of E admits a basis of Oka neighbourhoods $\Delta_{h,c}(E) = \{|e|_h < c\}$ with $c > 0$.*

A holomorphic line bundle E on a compact complex manifold X is called *basepoint-free* if the intersection of the divisors in $|E|$ is empty. If this holds, there is a holomorphic map $\Phi : X \to \mathbb{CP}^n$ for some $n \in \mathbb{N}$ (see (3.3)) such that E is isomorphic to the pullback $\Phi^*\mathscr{O}_{\mathbb{CP}^n}(1)$ of the hyperplane section bundle (see [31, Theorem II.7.1]). Hence, E admits a semipositive hermitian metric obtained by pulling back a positive metric on $\mathscr{O}_{\mathbb{CP}^n}(1)$ (see Example 2.4). This gives the following metric-free corollary to Theorem 1.5.

Corollary 1.6. *Let E be a holomorphic line bundle on a compact complex manifold X. If for each point $x \in X$ there exists a divisor $D \in |E|$ whose complement is a Stein neighbourhood of x with the density property, then the zero section of E admits a basis of Oka neighbourhoods.*

Theorems 1.4 and 1.5 are proved in Section 3. In Section 4 we give several examples. To simplify the discussion, we recall the following notions. A holomorphic line bundle E on a compact complex manifold X is called *ample* if some positive tensor power $E^{\otimes k}$ ($k > 0$) is *very ample*, meaning that holomorphic sections of $E^{\otimes k}$ provide an embedding of X into a projective space by a map of the form (3.3). The Kodaira embedding theorem [37] implies that a positive line bundle is ample; conversely, every ample line bundle admits a hermitian metric that makes it a positive line bundle. A *polarised manifold* (X, E) is a pair of a compact complex manifold X and an ample line bundle E on X. Note that such X is necessarily projective, and every projective manifold admits an ample line bundle. We introduce the following notion.

Definition 1.7. (a) A polarised manifold (X, E) has the *polarised density property* if for each point $x \in X$ there exists a divisor $D \in |E|$ whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property.

(b) A compact projective manifold X has the polarised density property if (X, E) has the polarised density property for every ample line bundle E on X.

Recall that a Stein manifold with the density property is an Oka manifold (see [16, Proposition 5.6.23]). Hence, if (X, E) has the polarised density property, then X is an Oka manifold by the localization theorem for Oka manifolds (see Kusakabe [38, Theorem 1.4]).

It is easily seen that every holomorphic line bundle satisfying the condition of Theorem 1.5 is ample (see Proposition 4.1). Hence, Theorem 1.5 can be equivalently stated as follows.

Theorem 1.8. *If (X, E) is a polarised manifold with the polarised density property, then for any semipositive hermitian metric h on E the disc bundle $\Delta_h(E)$ (1.1) is an Oka manifold while the domain $D_h(E) = E \setminus \overline{\Delta_h(E)}$ is Kobayashi hyperbolic.*

Theorem 1.1 says that the projective space $\mathbb{CP}^n$ of dimension $n > 1$ has the polarised density property. In Section 4 we prove the following results on this topic.

- If (X, E) has the polarised density property then so does $(X, E^{\otimes k})$ for every $k > 1$ (see Proposition 4.2).
- Every complex Grassmannian of dimension > 1 has the polarised density property (see Proposition 4.4).
- If (X_1, E_1) and (X_2, E_2) have the polarised density property then so does their exterior tensor product $(X_1 \times X_2, E_1 \boxtimes E_2)$ (see Proposition 4.5).
- If (X, E) has the polarised density property, then $(X \times \mathbb{CP}^n, E \boxtimes \mathscr{O}_{\mathbb{CP}^n}(k))$ $(n, k > 0)$ also has the polarised density property (see Proposition 4.6).
- Recall that a *rational manifold* is a projective manifold birationally isomorphic to a projective space. If $X_1, \ldots, X_m$ $(m \geq 2)$ are rational manifolds such that every X_i with $\dim X_i > 1$ has the polarised density property, then their product $X = X_1 \times X_2 \times \cdots \times X_m$ also has the polarised density property (see Proposition 4.8).

So far we have only discussed line bundles. One may ask what can be said about the Oka properties of (semi) positive hermitian vector bundles (E, h) of rank > 1 on an Oka manifold X. In particular, when is the tube $\{e \in E : |e|_h < 1\}$ Oka? Its boundary $\{|e|_h = 1\}$ is strongly pseudoconvex in the fibre direction and pseudoconcave in the remaining directions; see [14, Proposition 6.2]. We are not aware of any example of an Oka domain whose boundary fails to be pseudoconcave. For the same reason, we do not know anything about these questions if the hermitian metric has mixed signature. On the other hand, we obtain the following analogue of Theorem 1.1 (b') for any Griffiths seminegative hermitian vector bundle (see Griffiths [26, 28] and Definition 2.5) of rank > 1 (possibly trivial) on an Oka manifold.

Theorem 1.9. *If (E, h) is a Griffiths seminegative hermitian holomorphic vector bundle of rank > 1 on a (not necessarily compact) Oka manifold X, then $D_h(E) = \{e \in E : |e|_h > 1\}$ is an Oka domain with pseudoconcave boundary $bD_h(E) = \{e \in E : |e|_h = 1\}$.*

Remark 1.10. If (E, h) is a Griffiths seminegative holomorphic vector bundle on a complex manifold X, then the function $\phi(e) = |e|_h^2$ is plurisubharmonic on E (see Proposition 2.6 and Remark 2.7). If in addition the metric h is Griffiths negative then ϕ is strongly plurisubharmonic on $E \setminus E(0)$. In the latter case, with X compact, the zero section $E(0)$ is the maximal compact complex submanifold of E, which can be blown down to a point (see Grauert [23, Satz 1, p. 341]). This gives a Stein space $\widetilde{E}$, which is typically singular at the blown-down point, such that the image of the tube $\{|e|_h < c\}$ is a relatively compact domain in $\widetilde{E}$ for any $c > 0$, and the tube $\{0 < |e|_{\tilde{h}} < 1\} \subset E$ is Kobayashi hyperbolic for any hermitian metric $\tilde{h}$ on E.

Remark 1.11. The proofs of Theorems 1.1, 1.4, 1.5, and 1.9, given in Section 3, show that these results hold also for continuous hermitian metrics. Indeed, the basic relationship between semipositivity or seminegativity of the hermitian metric and the eigenvalues of the Levi form of the norm function remains in place (see Remark 2.7).

As an application of our results, we show in Section 5 that the Oka properties of tube domains in holomorphic vector bundles $E \to X$ on a compact complex manifold X imply the existence of holomorphic maps $S \to E$ from any Stein manifold S with $\dim S < \dim E$ having the cluster set either in the zero section $E(0)$ (when E is a positive line bundle; see Theorem 5.2) or at infinity (when E is a Griffiths negative vector bundle; see Theorem 5.1).

Our results contribute to the heuristic principle that Oka properties are related to metric positivity of complex manifolds while holomorphic rigidity properties, such as Kobayashi hyperbolicity, are related to metric negativity. Examples of this principle are discussed in [18, Sect. 11]; let us recall the most important ones and mention some new ones.

Beginning on the rigidity side, a hermitian manifold with holomorphic sectional curvature bounded above by a negative constant is Kobayashi hyperbolic; see Grauert and Reckziegel [24], whose result generalizes the Ahlfors–Schwarz lemma [1], and the results by Wu and Yau [54, 55], Tosatti and Yang [50], Diverio and Trapani [12], and Broder and Stanfield [10], among several others. Furthermore, every compact complex manifold of Kodaira general type is volume hyperbolic [36], and hence no such manifold is Oka.

On the flexibility side, every compact Kähler manifold with semipositive holomorphic bisectional curvature is Oka; see [18, Theorem 11.4], which follows from the classification of such manifolds by Mori [45] and Siu and Yau [48] (for positive bisectional curvature, when they are projective spaces) and Mok [44] in the semipositive case. As for not necessarily Kähler metrics, if (X, h) is a compact connected hermitian manifold whose holomorphic bisectional curvature is semipositive everywhere and positive at a point, then X is a projective space (see Ustinovskiy [51, Corollary 0.3]), which is Oka. Every compact Kähler manifold with positive holomorphic sectional curvature is rationally connected and projective (see Yang [56]), hence an Oka-1 manifold (see Alarcón and Forstnerič [2, Corollary 9.2]). It is not known whether every such manifold is Oka. A recent result of Matsumura [43, Theorem 1.3] implies that a projective manifold with semipositive holomorphic sectional curvature is the total space of a holomorphic fibre bundle over an Oka manifold with a projective rationally connected fibre enjoying the corresponding semipositivity. By [16, Theorem 5.6.5] the problem whether every such manifold is Oka reduces to the rationally connected case. Hence, the main next step is to better understand the relationship between (semi) positivity of holomorphic sectional curvature and the Oka property for rationally connected projective manifolds.

2. Preliminaries

In this section, we recall the necessary notions and tools, and we prepare some results which will be used in the proofs given in the following section.

A holomorphic line bundle $E \to X$ is given on some open covering $\{U_i\}_i$ of X by a 1-cocycle of nonvanishing holomorphic functions $\phi_{i,j} : U_{i,j} = U_i \cap U_j \to \mathbb{C}^*$. A point $(x, t) \in U_j \times \mathbb{C}$ with $x \in U_{i,j}$ is identified in E with $(x, \phi_{i,j}(x)t) \in U_i \times \mathbb{C}$. A holomorphic section $f : X \to E$ is given by a 1-cochain $f_i \in \mathscr{O}(U_i)$ satisfying $f_i = \phi_{i,j} f_j$ on $U_{i,j}$. A hermitian metric h on E is given on any holomorphic line bundle chart $(x, t) \in U_i \times \mathbb{C}$ by $h(x, t) = h_i(x)|t|^2$, where the positive functions $h_i : U_i \to (0, +\infty)$ satisfy the compatibility conditions

$$h_i(x)|\phi_{i,j}(x)|^2 = h_j(x) \quad \text{for } x \in U_{i,j}. \tag{2.1}$$

The curvature of the metric h is the $(1,1)$-form on X given on each chart U_i by

$$\Theta_h = -\partial\overline{\partial}\log h_i = -\partial\overline{\partial}\log h = \frac{\mathrm{i}}{2} dd^c \log h. \tag{2.2}$$

(The second equality holds since $\partial\overline{\partial}\log|t|^2 = 0$ on $t \neq 0$.) The bundle (E,h) is said to be positive (resp. negative) if the real $(1,1)$-form

$$\mathrm{i}\Theta_h = -\mathrm{i}\,\partial\overline{\partial}\log h = -\frac{1}{2} dd^c \log h \tag{2.3}$$

on X is positive (resp. negative). Similarly we define semipositivity and seminegativity. It is obvious that the restriction of a (semi) positive line bundle $E \to X$ to a complex submanifold $Y \subset X$ is (semi) positive. If (E',h') is another hermitian holomorphic line bundle on X given on the same open covering $\{U_i\}_i$ by the 1-cocyle $\phi'_{i,j}$, then the tensor product line bundle $E \otimes E'$ is given by the 1-cocycle $\phi_{i,j}\phi'_{i,j} \in \mathscr{O}(U_{i,j},\mathbb{C}^*)$. If f and f' are holomorphic section of E and E', respectively, given by 1-cochains $f_i, f'_i \in \mathscr{O}(U_i)$, then $f \otimes f'$ is a holomorphic section of $E \otimes E'$ given by the 1-cochain $f_i f'_i \in \mathscr{O}(U_i)$. If a hermitian metric h' on E' is given by functions $h'_i : U_i \to (0,\infty)$, then the product metric $h \otimes h'$ on $E \otimes E'$ is defined by the collection $h_i h'_i : U_i \to (0,\infty)$. From (2.2) we see that

$$\Theta_{h\otimes h'} = \Theta_h + \Theta_{h'}.$$

Hence, the product of semipositive metrics is semipositive, and is positive if one of the metrics is positive. For $k \in \mathbb{Z}$ we denote by $E^{\otimes k}$ the k-th tensor power of E, given by the 1-cocycle $\phi^k_{i,j}$. If h is a hermitian metric on E given by functions $h_i(x)$ (2.1), then the metric $h^{\otimes k}$ on $E^{\otimes k}$ is given by the functions $h_i(x)^k$ for $x \in U_i$. The hermitian dual bundle (E^*,h^*) is naturally isomorphic to (E^{-1},h^{-1}), where we omitted the tensor product sign. From (2.2) we see that

$$\Theta_{h^{\otimes k}} = k\,\Theta_h \quad \text{for all } k \in \mathbb{Z}.$$

Conversely, if $E = L^{\otimes k}$ $(k \neq 0)$ and h is a hermitian metric on E given in charts $U_i \subset X$ by positive functions h_i, then $h = \tilde{h}^{\otimes k}$ where $\tilde{h}$ is a hermitian metric on the line bundle L defined by the collection of functions $\tilde{h}_i = h_i^{1/k} : U_i \to (0,\infty)$.

Proposition 2.1. *Let (E,h) be a hermitian holomorphic line bundle on a complex manifold X.*

- (i) *For every $k \in \mathbb{N}$ there is a surjective fibre preserving holomorphic map $\Psi_k : E \to E^{\otimes k}$ such that $\Psi_k(E(0)) = E^{\otimes k}(0)$ and the maps $\Psi_k : \Delta^*_h(E) \to \Delta^*_{h^{\otimes k}}(E^{\otimes k})$ and $\Psi_k : D_h(E) \to D_{h^{\otimes k}}(E^{\otimes k})$ are unbranched k-sheeted holomorphic coverings.*
- (ii) *The punctured disc bundle $\Delta^*_h(E)$ is fibrewise biholomorphic to the outer tube $D_{h^*}(E^*) = \{h^* > 1\}$ in the dual bundle (E^*,h^*).*

Proof. If $E \to X$ is given by a 1-cocyle $\phi_{i,j} \in \mathscr{O}^*(U_{i,j})$, then $E^{\otimes k}$ is given by the 1-cocyle $\phi^k_{i,j}$. Denote by $\Phi_{i,j}(x,t_j) = (x,\phi_{i,j}(x)t_j)$ the transition maps in E and by $\Phi^k_{i,j}(x,t_j) = (x,\phi_{i,j}(x)^k t_j)$ the associated transition maps in $E^{\otimes k}$. We define the map Ψ_k on any chart $U_i \times \mathbb{C}$ by $\Psi_k(x,t_i) = (x,t_i^k)$. Since $t_i = \phi_{i,j}(x)t_j$ for $x \in U_{i,j}$, we have that

$$(\Psi_k \circ \Phi_{i,j})(x,t_j) = \Psi_k(x,\phi_{i,j}(x)t_j) = (x,\phi_{i,j}(x)^k t_j^k) = (\Phi^k_{i,j} \circ \Psi_k)(x,t_j),$$

showing that $\Psi_k : E \to E^{\otimes k}$ is a well-defined k-sheeted covering projection which is branched along $E(0)$, and $\Psi_k : E \setminus E(0) \to E^{\otimes k} \setminus E^{\otimes k}(0)$ is an unbranched k-sheeted covering. From the definition of the metric $h^{\otimes k}$ on $E^{\otimes k}$ it follows that $\Psi_k : \Delta^*_h(E) \to \Delta^*_{h^{\otimes k}}(E^{\otimes k})$ and $\Psi_k : D_h(E) \to D_{h^{\otimes k}}(E^{\otimes k})$ are unbranched holomorphic coverings. This proves (i).

Part (ii) is seen as follows. Compactifying each fibre $E_x \cong \mathbb{C}$ ($x \in X$) with the point at infinity yields a holomorphic fibre bundle $\widehat E \to X$ with fibre $\mathbb{CP}^1$ having a well-defined ∞-section $E(\infty) \cong X$. Set $\widetilde E = \widehat E \setminus E(0) \to X$. If $t \in \mathbb{C}$ is a coordinate on a fibre E_x then $u = t^{-1}$ is a coordinate on $\widetilde E_x$, and the transition functions between the u-coordinates are $\phi_{i,j}^{-1} = 1/\phi_{i,j}$. Hence, $(\widetilde E, h^{-1})$ is a hermitian holomorphic line bundle on X with zero section $\widetilde E(0) = E(\infty)$ which is naturally isomorphic to the dual line bundle (E^*, h^*). Under this identification, the identity map on $\widehat E$ induces a fibre preserving biholomorphism

$$\mathcal{I} : E \setminus E(0) \to E^* \setminus E^*(0) \tag{2.4}$$

mapping $\Delta^*_h(E)$ onto $D_{h^*}(E^*) = \{h^* > 1\}$ and $D_h(E)$ onto $\Delta^*_{h^*}(E^*)$. □

Corollary 2.2. *Let (E, h) be a hermitian holomorphic line bundle on a complex manifold X.*

(i) *If the punctured disc bundle $\Delta^*_{h^{\otimes k}}(E^{\otimes k})$ is Oka for some $k \in \mathbb{N}$ then it is Oka for all $k \in \mathbb{N}$, and in such case the disc bundle $\Delta_{h^{\otimes k}}(E^{\otimes k})$ is Oka-1 for all $k \in \mathbb{N}$.*
(ii) *$\Delta^*_h(E)$ is Oka (resp. hyperbolic) if and only if $D_{h^*}(E^*)$ is Oka (resp. hyperbolic).*

Proof. All claims except the second statement in part (i) follow from Proposition 2.1 and the fact that both the class of Oka manifolds and the class of hyperbolic manifolds are invariant under covering projections. If $\Delta^*_h(E)$ is Oka, it is the image of a strongly dominating holomorphic map $\mathbb{C}^{n+1} \to \Delta^*_h(E)$ with $n = \dim X$ (see [17]). Thus, the disc bundle $\Delta_h(E)$ is densely dominable by $\mathbb{C}^{n+1}$, and hence an Oka-1 manifold by [2, Corollary 2.5 (b)]. □

Recall that a real function f of class $\mathscr{C}^2$ on a complex manifold X is plurisubharmonic if $dd^c f \geq 0$ and strongly plurisubharmonic if $dd^c f > 0$. Both conditions generalize to upper semicontinuous functions with values in $[-\infty, +\infty)$ (see Grauert and Remmert [25]). The curvature formula (2.3) for a hermitian metric h leads to the following observation, which we record for reference. (See also Proposition 2.6 for vector bundles of higher rank.)

Proposition 2.3. *Let h be a hermitian metric of class $\mathscr{C}^2$ on a holomorphic line bundle $E \to X$. The following conditions are equivalent.*

(i) *The curvature of h is seminegative:* $\mathrm{i}\Theta_h \leq 0$.
(ii) *The function* $\log h$ *is plurisubharmonic on E.*
(iii) *The disc bundle $\Delta_h(E) = \{h < 1\}$ is pseudoconvex along $b\Delta_h(E) = \{h = 1\}$.*

Furthermore, if $\mathrm{i}\Theta_h < 0$ *then h is strongly plurisubharmonic on $E \setminus E(0)$. Likewise, the following conditions are equivalent.*

(i') *The curvature of h is semipositive:* $\mathrm{i}\Theta_h \geq 0$.
(ii') *The function* $-\log h$ *is plurisubharmonic on $E \setminus E(0)$.*
(iii') *The disc bundle $\Delta_h(E) = \{h < 1\}$ is pseudoconcave along $b\Delta_h(E) = \{h = 1\}$.*

Proof. The equivalence (i) ⇔ (ii) is an immediate consequence of the curvature formula (2.3). Assume now that U is a Stein domain in X such that $E|_U \cong U \times \mathbb{C}$ is a trivial line bundle. On this chart we have $h(x,t) = \xi(x)|t|^2$ for some positive $\mathscr{C}^2$ function ξ on U, and

$$\Delta_h(E)|_U = \{(x,t) \in U \times \mathbb{C} : h(x,t) < 1\} = \{(x,t) \in U \times \mathbb{C} : |t|^2 \mathrm{e}^{\log \xi(x)} < 1\} \tag{2.5}$$

is a Hartogs domain. If $\log h$ is plurisubharmonic (condition (ii) holds) then so is h, and hence $\Delta_h(E)|_U$ is pseudoconvex. The converse is also well-known and easily seen: if the Hartogs domain (2.5) is pseudoconvex then $\log \xi$ is plurisubharmonic on U, and hence $\log h$

is plurisubharmonic on $E|_U$. This proves (ii) $\Leftrightarrow$ (iii). If $\mathrm{i}\Theta_h < 0$ then $\log \xi$ and hence ξ are strongly plurisubharmonic, so h is strongly plurisubharmonic on $E \setminus E(0)$. The equivalences (i') $\Leftrightarrow$ (ii') $\Leftrightarrow$ (iii') are proved in the same way and we leave out the details. $\square$

Example 2.4 (Special hermitian line bundles on projective spaces)**.** Let $z = (z_0, z_1, \dots, z_n)$ be Euclidean coordinates on $\mathbb{C}^{n+1}$ and $[z] = [z_0 : z_1 : \cdots : z_n]$ the associated homogeneous coordinates on $\mathbb{CP}^n$. On the affine chart $U_i = \{[z] \in \mathbb{CP}^n : z_i \neq 0\} \cong \mathbb{C}^n$ $(i = 0, 1, \dots, n)$ we have the affine coordinates $z^i = (z_0/z_i, \dots, z_n/z_i)$, where the term $z_i/z_i = 1$ omitted. Fix $k \in \mathbb{Z}$ and define a hermitian metric h on $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ by

$$h([z], t) = \frac{|t|^2}{(1 + |z^i|^2)^k} = \frac{|z_i|^{2k}}{|z|^{2k}} |t|^2 \quad \text{for } [z] \in U_i \text{ and } t \in \mathbb{C}. \tag{2.6}$$

The transition functions on $\mathscr{O}_{\mathbb{CP}^n}(k)$ are $\phi_{i,j}([z]) = (z_j/z_i)^k$ (see [53, p. 18]). In view of (2.1) we see that $h = \tilde{h}^{\otimes k}$, where $\tilde{h}$ is the metric on $\mathscr{O}_{\mathbb{CP}^n}(1)$ given by (2.6) with $k = 1$. It follows from (2.3) and (2.6) that $\mathrm{i}\Theta_h = k\,\mathrm{i}\,\partial\overline{\partial} \log\big(|z|^2\big)$, which is k-times the Fubini–Study form. Identifying U_i with $\mathbb{C}^n$, the disc tube of the bundle $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ with the metric (2.6), restricted to U_i, is given by

$$\Delta_h(E)|_{U_i} = \big\{(z, t) \in \mathbb{C}^n \times \mathbb{C} : |t| < (1 + |z|^2)^{k/2}\big\}. \tag{2.7}$$

This is a Hartogs domain whose radius is of order $|z|^k$ as $|z| \to \infty$. Since any two hermitian metrics on E are comparable, the disc bundle of any hermitian metric on E grows at this rate.

We now recall the notions of Griffiths (semi) positivity and Griffiths (semi) negativity of a hermitian holomorphic vector bundle (E, h) of arbitrary rank $r \geq 1$ on a complex manifold X of dimension n (see Griffiths [26, 28]). The hermitian metric h on E is given in any local frame $(e_1, \cdots, e_r)$ by a hermitian matrix function $h = (h_{\lambda\mu})$ with

$$h_{\lambda\mu}(x) = (e_\mu(x), e_\lambda(x))_h \quad \text{for } \lambda, \mu = 1, \dots, r.$$

Its connection matrix θ_h and the curvature form Θ_h are given in any local holomorphic frame by

$$\theta_h = h^{-1}\partial h, \qquad \Theta_h = \overline{\partial}\,\theta_h = -h^{-1}\partial\overline{\partial} h + h^{-1}\partial h \wedge h^{-1}\overline{\partial} h.$$

(See [11, Chapter V] or [53, Chapter III].) For a line bundle these equal $\theta_h = h^{-1}\partial h = \partial \log h$ and $\Theta_h = -\partial\overline{\partial} \log h$ (cf. (2.2)). In local holomorphic coordinates $z = (z_1, \dots, z_n)$ on X and a local frame $(e_1, \dots, e_r)$ on E, we can identify the curvature tensor

$$\mathrm{i}\Theta_h = \sum_{\substack{i,j=1,\dots,n \\ \lambda,\mu=1,\dots,r}} c_{ij\lambda\mu} dz_i \wedge d\bar{z}_j \cdot e_\lambda^* \otimes e_\mu$$

with the hermitian form on $TX \otimes E$ given by

$$\widetilde{\Theta}_h(\xi \otimes v) = \sum_{\substack{i,j=1,\dots,n \\ \lambda,\mu=1,\dots,r}} c_{ij\lambda\mu} \xi_i \bar{\xi}_j v_\lambda \bar{v}_\mu.$$

The following notions are due to Griffiths [26, 28]; see also [6] and [11, Chapter VII].

Definition 2.5. Let $E \to X$ be a holomorphic vector bundle. A hermitian metric h on E is *Griffiths semipositive* (resp. *Griffiths seminegative*) if $\widetilde{\Theta}_h(\xi \otimes v) \geq 0$ (resp. $\widetilde{\Theta}_h(\xi \otimes v) \leq 0$) for all $\xi \in T_xX$ and $v \in E_x$ $(x \in X)$. If there is strict inequality for all $\xi \in T_xX \setminus \{0\}$ and $v \in E_x \setminus \{0\}$ $(x \in X)$ then the metric is *Griffiths positive* (resp. *Griffiths negative*).

For line bundles, Griffiths positivity (resp. negativity) coincides with the previous definition. The following proposition explains the connection between Griffiths seminegativity of a hermitian metric and plurisubharmonicity of the associated squared norm function. The equivalences between (i), (ii) and (iii) can be found in Raufi [46, Sect. 2]. The equivalences (ii) ⇔ (iv) and (iii) ⇔ (v) are obvious. For the last statement, see [14, Proposition 6.2].

Proposition 2.6. *For a hermitian metric h on a holomorphic vector bundle $E \to X$ the following conditions are equivalent:*

(i) *The metric h is Griffiths seminegative.*
(ii) *For any local holomorphic section u of E, the function $|u|_h^2$ is plurisubharmonic.*
(iii) *For any local holomorphic section u of E, the function $\log|u|_h^2$ is plurisubharmonic.*
(iv) *The squared norm function $\phi(e) = |e|_h^2$ is plurisubharmonic on E.*
(v) *The function $\log\phi(e) = \log|e|_h^2$ is plurisubharmonic on E.*

If h is Griffiths negative then the function ϕ in (iv) is strongly plurisubharmonic on $E \setminus E(0)$.

Remark 2.7. The conditions (ii) and (iii) in Proposition 2.6 are equivalent also for a *continuous* hermitian metric on a holomorphic vector bundle, and they can be used to define Griffiths seminegativity and semipositivity for a not necessarily smooth hermitian metric (see [46, Definition 1.2 and Sect. 2]). The equivalences (ii) ⇔ (iv) and (iii) ⇔ (v) are obvious in the continuous case as well. For a metric of class $\mathscr{C}^2$, the relationship between the eigenvalues of the curvature form and those of the Levi form of the squared norm function can be found in Griffiths [27, p. 426]; see also the summary in [14, Proposition 6.2]. In the notation used in the latter paper, Griffiths seminegativity can be expressed by $s(e) = 0$ for every $e \in E \setminus E(0)$.

In the final part of this section we recall some notions from Oka theory and a couple of results which are frequently used in the sequel. We begin by recalling the notion of Oka property of a holomorphic map and of *Oka map*; see [16, Definitions 7.4.1 and 7.4.7] where this is called the *parametric Oka property with approximation and interpolation*, abbreviated POPAI.

Definition 2.8. A holomorphic map $\pi : Y \to Z$ of reduced complex spaces has the *Oka property* if holomorphic maps $f : X \to Z$ from any Stein manifold X satisfy the parametric h-principle for liftings $F : X \to Y$ with $\pi \circ F = f$. The map $\pi : Y \to Z$ is an *Oka map* if it satisfies the Oka property and is a topological (Serre) fibration.

More precisely, the Oka property of the map $\pi : Y \to Z$ means that every continuous lifting $F_0 : X \to Y$ of a given holomorphic map $f : X \to Z$ is homotopic through liftings of f to a holomorphic lifting $F : X \to Y$. Furthermore, if F_0 is holomorphic on a compact $\mathscr{O}(X)$-convex subset $K \subset X$ and on a closed complex subvariety $X' \subset X$, then the homotopy of liftings $F_t : X \to Y$ $(t \in [0,1])$ can be chosen such that every map F_t is holomorphic on $K \cup X'$, it agrees with F_0 on X', and it approximates F_0 uniformly on K (and uniformly in the parameter $t \in [0,1]$). Finally, the analogous conditions hold for any continuous family of holomorphic maps $f_p : X \to Z$ depending on a parameter p in a compact Hausdorff space.

For a holomorphic submersion $\pi : Y \to Z$, the basic Oka property implies the parametric Oka property (see [16, Theorem 7.4.3]). If $\pi : Y \to Z$ is an Oka map of complex manifolds with Z connected then π is a surjective submersion, its fibres are Oka manifolds (see [18, Proposition 3.14]), and Y is an Oka manifold if and only if Z is an Oka manifold (see [18, Theorem 3.15]).

The following result is due to Kusakabe [40, Lemma 5.1]; see also [18, Proposition 3.18].

Proposition 2.9. *Assume that for every point y in a complex manifold Y there exist complex manifolds $Z_1, \ldots, Z_k$ and holomorphic submersions $\pi_j : Y \to Z_j$ $(j = 1, \ldots, k)$ enjoying the Oka property such that $T_yY = \sum_{j=1}^k \ker(d\pi_j)_y$. Then Y is an Oka manifold.*

An unbounded closed set S in a complex manifold Y is called holomorphically convex (or $\mathscr{O}(Y)$-convex) if S is the union of an increasing sequence of compact $\mathscr{O}(Y)$-convex sets.

Definition 2.10 (Definition 4.1 in [40])**.** Let $\pi : Y \to Z$ be a holomorphic submersion. A closed subset S of Y is called a *family of compact holomorphically convex sets* if the restriction $\pi|_S : S \to Z$ is proper and each point of Z admits an open neighbourhood $U \subset Z$ such that the set $S \cap \pi^{-1}(U)$ is $\mathscr{O}(\pi^{-1}(U))$-convex.

The following is a special case of [40, Theorem 4.2] which is used in this paper.

Theorem 2.11. *Let $\pi : Y \to Z$ be a holomorphic fibre bundle whose fibre is a Stein manifold with the density property, and let $S \subset Y$ be a family of compact holomorphically convex sets. Then the restriction $\pi|_{Y\setminus S} : Y \setminus S \to Z$ enjoys the Oka property.*

In [40, Theorem 4.2] it is assumed that the map $\pi : Y \to Z$ is a holomorphic submersion and each point of Z admits an open neighborhood $U \subset Z$ such that $\pi^{-1}(U)$ is Stein and the restriction $\pi^{-1}(U) \to U$ enjoys the fibred density property. When $\pi : Y \to Z$ is a holomorphic fibre bundle, the latter condition clearly holds if the fibre is Stein and has the density property.

3. Proofs of the main results

In this section, we prove Theorems 1.1, 1.4, 1.5, and 1.9. We also obtain Theorem 3.3.

Proof of Theorem 1.1. We begin by considering the hyperplane section bundle $E = \mathscr{O}_{\mathbb{CP}^n}(1)$. The total space E can be identified with $\mathbb{CP}^{n+1} \setminus \{0\}$, where $0 \in \mathbb{C}^{n+1}$ is an affine chart in $\mathbb{CP}^{n+1}$, such that the zero section $E(0)$ is the hyperplane at infinity $\mathbb{CP}^n = \mathbb{CP}^{n+1} \setminus \mathbb{C}^{n+1}$ and the fibres of the projection $\pi : E \to \mathbb{CP}^n$ are the punctured complex lines through the origin $0 \in \mathbb{C}^{n+1}$, with the added point at infinity. Let h be a semipositive hermitian metric on E, $\mathrm{i}\Theta_h \geq 0$. Proposition 2.3 shows that the function $1/h$ is plurisubharmonic on $E \setminus E(0) = \mathbb{C}^{n+1} \setminus \{0\}$. Clearly, this function tends to infinity at $E(0)$ and tends to 0 at the origin $0 \in \mathbb{C}^{n+1}$, so it extends to a plurisubharmonic exhaustion function on $\mathbb{C}^{n+1}$. Therefore, the set $K = \{1/h \leq 1\} = \{h \geq 1\}$ is a compact polynomially convex neighbourhood of the origin (see Stout [49, Theorem 1.3.11]). Note that $\Delta_h^*(E) = \mathbb{C}^{n+1} \setminus K$, which is Oka by [40, Corollary 1.3] (see also [20, Theorem 1.2]), $\Delta_h(E) = \mathbb{CP}^{n+1} \setminus K$, which is Oka by [18, Corollary 5.2], and $D_h(E) = \mathring{K} \setminus \{0\}$, which is a bounded domain in $\mathbb{C}^{n+1}$ and hence Kobayashi hyperbolic. Finally, if H is a complex hyperplane in $\mathbb{CP}^{n+1}$ then $\mathbb{CP}^{n+1} \setminus (H \cup K)$ is Oka by [21, Theorem 1.3]. This shows that for any affine chart $\mathbb{C}^n \cong U \subset \mathbb{CP}^n$ the restricted disc bundle $\Delta_h(E)|_U$ is Oka as well. This proves the theorem for $E = \mathscr{O}_{\mathbb{CP}^n}(1)$.

For its dual bundle $E^* = \mathbb{U} = \mathscr{O}_{\mathbb{CP}^n}(-1)$, the universal bundle on $\mathbb{CP}^n$, parts (a') and (b') of the theorem follow immediately from the results for $E = \mathscr{O}_{\mathbb{CP}^n}(1)$ in view of Proposition 2.1 (ii). Indeed, the total space of $\mathbb{U}$ is biholomorphic to $\mathbb{C}^{n+1}$ blown up at the origin, its zero section $\mathbb{U}(0)$ is the exceptional fibre over $0 \in \mathbb{C}^{n+1}$, the fibres of the projection $\pi : \mathbb{U} \to \mathbb{CP}^n$ are the complex lines $\mathbb{C}z$ for $z \in \mathbb{C}^{n+1} \setminus \{0\}$, and $\mathbb{U} \setminus \mathbb{U}(0)$ is biholomorphic to $\mathbb{C}^{n+1} \setminus \{0\}$. If h is a seminegative hermitian metric on $\mathbb{U}$ then $1/h$ is a semipositive hermitian metric on $E = \mathbb{CP}^{n+1} \setminus \{0\}$, $K = \{h \leq 1\}$ is a compact polynomially convex neighbourhood of the origin blown up at the origin, the domain $D_h(\mathbb{U}) = \{h > 1\} = \Delta^*_{1/h}(E)$ is Oka, and the domain $\Delta^*_h(\mathbb{U}) = \{0 < h < 1\} = \{1/h > 1\} = D_{1/h}(E)$ is hyperbolic.

For the tensor powers $E^{\otimes k} = \mathscr{O}_{\mathbb{CP}^n}(k)$ with $k > 1$, parts (a) and (c) follow from the already proved result for $E = \mathscr{O}_{\mathbb{CP}^n}(1)$ and Corollary 2.2. Likewise, the proofs of (a') and (b') for $\mathbb{U}^{\otimes k} = \mathscr{O}_{\mathbb{CP}^n}(-k)$ with $k > 1$ follow from the case for $\mathbb{U}$ and Corollary 2.2.

It remains to prove part (b) for semipositive bundles (E, h) with $E = \mathscr{O}_{\mathbb{CP}^n}(k)$ when $k \geq 1$ and $n \geq 2$. The key to the proof is the following result of independent interest. The idea used in this proposition will also be applied in the proofs of Theorems 1.4 and 1.5.

Proposition 3.1. *Assume that ϕ is a positive continuous function on $\mathbb{C}^n$ $(n \geq 2)$ such that $\log\phi$ is plurisubharmonic, and there is a constant $c > 0$ such that $\phi(z) \geq c\,|z|$ holds for all $z \in \mathbb{C}^n$. Then, the following (pseudoconcave) Hartogs domain is an Oka domain:*

$$\Omega = \{(z,t) \in \mathbb{C}^n \times \mathbb{C} : |t| < \phi(z)\}. \tag{3.1}$$

Proof. Let $T : \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ denote the projection $T(z,t) = t$. Consider the closed set

$$\begin{aligned} S &= \mathbb{C}^{n+1} \setminus \Omega = \{(z,t) \in \mathbb{C}^n \times \mathbb{C} : |t| \geq \phi(z)\} \\ &= \{(z,t) \in \mathbb{C}^n \times \mathbb{C}^* : \log\phi(z) - \log|t| \leq 0\}. \end{aligned}$$

Since $\log|t|$ is harmonic on $t \in \mathbb{C}^*$, the function $\psi(z,t) = \log\phi(z) - \log|t|$ is plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}^*$. Since ϕ is assumed to grow at least linearly near infinity, the restricted projection $T|_S : S \to \mathbb{C}$ is proper. It follows that for every $r > 0$ the set

$$S_r = \{(z,t) \in S : |t| \leq r\} = \{(z,t) \in \mathbb{C}^n \times \mathbb{C}^* : \psi(z,t) \leq 0 \text{ and } \log|t| \leq \log r\} \tag{3.2}$$

is compact and $\mathscr{O}(\mathbb{C}^n \times \mathbb{C}^*)$-convex (see [49, Theorem 1.3.11]). Since the fibre of the map T is $\mathbb{C}^n$ with $n \geq 2$, which has the density property, Theorem 2.11 implies that the restricted projection $T : (\mathbb{C}^n \times \mathbb{C}^*) \setminus S \to \mathbb{C}^*$ has the Oka property. Since $S \cap \{t = 0\} = \varnothing$, the projection $T : \mathbb{C}^{n+1} \setminus S \to \mathbb{C}$ has the Oka property as well (see [39, Theorem 4.1], or use the localization principle for the Oka property of a holomorphic submersion, given by [16, Theorem 7.4.4] and originally proved in [15, Theorem 4.7]).

Since the function ϕ in (3.1) is assumed to grow at least linearly at infinity, we have that $\Lambda \cap S = \varnothing$ for every complex hyperplane $\Lambda \subset \mathbb{C}^{n+1}$ sufficiently close to $\Lambda_0 = \{t = 0\}$, and there is a path Λ_s $(s \in [0,1])$ of such hyperplanes connecting Λ_0 to Λ. For any such Λ, the set S_r in (3.2) is also $\mathscr{O}(\mathbb{C}^{n+1} \setminus \Lambda)$-convex by [18, Corollary A.5]. As $r \to \infty$ these sets exhaust S, so S is $\mathscr{O}(\mathbb{C}^{n+1} \setminus \Lambda)$-convex. Let $T_\Lambda : \mathbb{C}^{n+1} \to \mathbb{C}$ be a $\mathbb{C}$-linear projection with $(T_\Lambda)^{-1}(0) = \Lambda$. If Λ is sufficiently close to Λ_0 then the restricted projection $T_\Lambda : S \to \mathbb{C}$ is still proper. Using again Theorem 2.11, we infer that the projection $T_\Lambda : \mathbb{C}^{n+1} \setminus S \to \mathbb{C}$ has the Oka property. Applying this conclusion for two linearly independent projections and using Proposition 2.9, we see that $\mathbb{C}^{n+1} \setminus S = \Omega$ is an Oka manifold. □

We continue with the proof of part (b) in Theorem 1.1. Let E be a positive holomorphic line bundle with a semipositive hermitian metric h on $\mathbb{CP}^n$ with $n > 1$. From the equivalences (i') ⇔ (ii') ⇔ (iii') in Proposition 2.3 and (2.5), we see that the restriction of the disc bundle $\Delta_h(E)$ to any affine chart $\mathbb{C}^n \cong U \subset \mathbb{CP}^n$ is a pseudoconcave Hartogs domain of the form (3.1) with $\log\phi$ plurisubharmonic. We have seen in Example 2.4 that the function ϕ grows at least linearly near infinity. Hence, Proposition 3.1 implies that $\Delta_h(E)|_U$ is an Oka domain. Note that $\Delta_h(E)|_U$ is a Zariski open domain in $\Delta_h(E)$. Since charts of this kind cover $\Delta_h(E)$, the localization theorem for Oka manifolds (see [38, Theorem 1.4]) implies that $\Delta_h(E)$ is Oka. □

Remark 3.2. The proof of Proposition 3.1 also gives the following more general result related to [40, Theorem 1.6]. Recall that a closed subset S of a Stein manifold X is said to be $\mathscr{O}(X)$-convex if it is exhausted by an increasing sequence of compact $\mathscr{O}(X)$-convex sets.

Theorem 3.3. *Let S be a closed subset of $\mathbb{C}^n \times \mathbb{C}^*$ $(n \geq 2)$ which is $\mathscr{O}(\mathbb{C}^n \times \mathbb{C}^*)$-convex (this holds in particular if S is convex). Assume that for every complex hyperplane $\mathbb{C}^n \cong \Lambda \subset \mathbb{C}^{n+1}$ close enough to $\Lambda_0 = \mathbb{C}^n \times \{0\}$ we have that $\Lambda \cap S = \varnothing$. Then, $\mathbb{C}^{n+1} \setminus S$ is an Oka domain.*

The hypothesis that the condition $\Lambda \cap S = \varnothing$ for all hyperplanes Λ close to Λ_0 is equivalent to asking that the projective closures of Λ_0 and S do not intersect at infinity. For closed subsets S of Euclidean spaces of dimension ≥ 3, Theorem 3.3 generalizes [21, Theorem 1.1] due to Forstnerič and Wold. Indeed, the holomorphic convexity hypothesis on the set S in the latter result (where it is called E) is strictly stronger than the one in Theorem 3.3. However, Theorem 3.3 does not apply to subsets of $\mathbb{C}^2$, while the cited result [21, Theorem 1.1] does.

Proof of Theorem 1.4. Let $\mathbb{C}^n \cong U_i \subset \mathbb{CP}^n$ for $i = 0, \ldots, n$ be affine Euclidean charts covering $\mathbb{CP}^n$ such that the Stein manifold $X_i = X \cap U_i$ has the density property for every i. By the localization theorem [38, Theorem 1.4], it suffices to prove that the restricted bundle $\Delta_h(E)|_{X_i}$ is Oka for every i. There is a standard trivialization $E|_{U_i} \cong U_i \times \mathbb{C}$, and the bundle $\Delta_h(E)|_{U_i} = \{(z,t) : |t| < \phi(z)\}$ is a Hartogs domain of the form (3.1) with the function $\phi : U_i \to (0, \infty)$ growing at least linearly near infinity (see Example 2.4). Hence,

$$\Omega_i := \Delta_h(E)|_{X_i} = \{(x,t) \in X_i \times \mathbb{C} : |t| < \phi(x)\}.$$

Since $\mathrm{i}\Theta_h \geq 0$ on X, Ω_i is pseudoconcave (see the equivalence (i') $\Leftrightarrow$ (iii') in Proposition 2.3) and $\log \phi$ is plurisubharmonic on X_i. Hence, the closed set $S = E|_{X_i} \setminus \Omega_i$ is holomorphically convex in $X_i \times \mathbb{C}^*$ (see the proof of Proposition 3.1). Let $T : X_i \times \mathbb{C} \to \mathbb{C}$ denote the projection onto the second factor, $T(x,t) = t$. The above properties imply that the restricted projection $T|_S : S \to \mathbb{C}$ is proper, and S is a family of compact holomorphically convex sets in X_i with respect to T (see Definition 2.10). Since X_i is a Stein manifold with the density property, Theorem 2.11 implies that the restricted projection $T : (X_i \times \mathbb{C}) \setminus S = \Omega_i \to \mathbb{C}$ has the Oka property. We now apply the same argument with tilted projections $T_\Lambda : U_i \times \mathbb{C} \to \mathbb{C}$ defined by affine hyperplanes $\Lambda \subset U_i \times \mathbb{C} \cong \mathbb{C}^{n+1}$ sufficiently close to $\Lambda_0 = U_i \times \{0\}$. Such a hyperplane Λ is the graph of a $\mathbb{C}$-linear function $t = \xi(x)$ of $x \in U_i \cong \mathbb{C}^n$, and $\Lambda \cap (X_i \times \mathbb{C}) = \{(x,t) : x \in X_i,\ t = \xi(x)\}$. Fibres of the restricted projection $T_\Lambda : X_i \times \mathbb{C} \to \mathbb{C}$ are parallel translates of $\Lambda \cap (X_i \times \mathbb{C})$ in the vertical t-direction, so this projection is a (trivial) holomorphic fibre bundle with fibre X_i. Since ϕ grows at least linearly, the projection $T_\Lambda : S \to \mathbb{C}$ is proper if Λ is close enough to Λ_0. For such Λ, the same argument as before shows that $T_\Lambda : (X_i \times \mathbb{C}) \setminus S = \Omega_i \to \mathbb{C}$ has the Oka property. Clearly, finitely many such projections satisfy the hypotheses in Proposition 2.9, and hence Ω_i is an Oka domain. By the localization theorem [38, Theorem 1.4] this proves that $\Delta_h(E)|_X$ is Oka. The fact that the exterior tube $D_h(E)|_X$ is Kobayashi hyperbolic is seen as in the proof of Theorem 1.1 (c). □

Proof of Theorem 1.5. By the assumption there are holomorphic sections $s_0, \ldots, s_n : X \to E$ such that $X_i = \{x \in X : s_i(x) \neq 0\}$ is a Stein manifold with the density property for every $i = 0, 1, \ldots, n$ and $\bigcup_{i=0}^n X_i = X$. Consider the holomorphic map $\Phi : X \to \mathbb{CP}^n$ given by

$$\Phi(x) = [s_0(x) : s_1(x) : \cdots : s_n(x)] \in \mathbb{CP}^n, \quad x \in X. \tag{3.3}$$

(The map Φ is well-defined since $s_i(x)$ are elements of the 1-dimensional vector space $E_x \cong \mathbb{C}$ and at least one of them is nonzero for every x.) Note that Φ maps $X_i = \{s_i \neq 0\}$ to the complement of the standard i-th hyperplane $\mathbb{CP}^{n-1} \cong H_i \subset \mathbb{CP}^n$, and $\Phi^{-1}(\mathbb{CP}^n \setminus H_i) = X_i$.

Then, E is isomorphic to $\Phi^*\mathscr{O}_{\mathbb{CP}^n}(1)$, the pullback of the hyperplane section bundle (see [31, Theorem II.7.1]). For completeness, we include a simple argument. Let $\pi^* : E^* \to X$ be the dual bundle of $\pi : E \to X$, and denote by $\langle e^*, e\rangle$ the natural pairing of elements $e \in E$ and $e^* \in E^*$ over the same base point $\pi(e) = \pi^*(e^*) \in X$. Let $\mathbb{U} \to \mathbb{CP}^n$ be the universal bundle. We can identify $\mathbb{U}$ with $\mathbb{C}^{n+1}$ blown up at the origin so that the zero section $\mathbb{U}(0) \cong \mathbb{CP}^n$ is the exceptional fibre over $0 \in \mathbb{C}^{n+1}$ and the fibres of the projection $\mathbb{U} \to \mathbb{CP}^n$ are the complex lines in $\mathbb{C}^{n+1}$ through the origin. The holomorphic map $\widetilde{\Phi} : E^* \to \mathbb{C}^{n+1}$ given by

$$\widetilde{\Phi}(e^*) = \big(\langle e^*, s_i(x)\rangle\big)_{i=0}^n, \quad e^* \in E^*,\ x = \pi^*(e^*) \in X$$

maps E^*_x isomorphically onto the complex line in $\mathbb{C}^{n+1}$ determined by the point $\Phi(x) \in \mathbb{CP}^n$, so it gives a line bundle isomorphism $E^* \cong \Phi^*\mathbb{U}$. It follows that $E \cong \Phi^*\mathbb{U}^* = \Phi^*\mathscr{O}_{\mathbb{CP}^n}(1)$.

The proof can now be completed as for Theorem 1.4. The restricted bundle $E|_{X_i} \cong X_i \times \mathbb{C}$ admits a trivialization induced via the map Φ (3.3) by the standard trivialization of $\mathscr{O}_{\mathbb{CP}^n}(1)$ over $U_i = \mathbb{CP}^n \setminus H_i \cong \mathbb{C}^n$. In this trivialization, $\Delta_h(E)|_{X_i}$ is a pseudoconcave Hartogs domain of the form (3.1) in $X_i \times \mathbb{C}$. The same argument as in the proof of Theorem 1.4, using the Oka property of tilted projections $(X_i \times \mathbb{C}) \setminus \Delta_h(E)|_{X_i} \to \mathbb{C}$ which come from linear projections $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ close to the standard projection onto the second factor, shows that $\Delta_h(E)|_{X_i}$ is Oka for every $i = 0, \dots, n$. By the localization theorem, it follows that $\Delta_h(E)$ is Oka. □

Proof of Theorem 1.9. Let $\pi : E \to X$ denote the vector bundle projection and set $S = \{e \in E : |e|_h \le 1\}$. Assuming that $\operatorname{rank} E = r > 1$ and the hermitian metric h is Griffiths seminegative, we wish to prove that the exterior tube $D_h(E) = E \setminus S = \{e \in E : |e|_h > 1\}$ is an Oka manifold. Condition (iv) in Proposition 2.6 shows that the squared norm function $\phi(e) = |e|_h^2$ is plurisubharmonic on E. Hence, for each holomorphic chart $\psi : U \to \mathbb{B}^n$ from an open set $U \subset X$ onto the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ $(n = \dim X)$ and each $0 < \rho < 1$, the compact set $\{e \in S|_U : |\psi \circ \pi(e)| \le \rho\}$ is defined by plurisubharmonic functions in the Stein manifold $E|_U$, so it is $\mathscr{O}(E|_U)$-convex (see Stout [49, Theorem 1.3.11]). Since $E \to X$ is a holomorphic vector bundle of rank $r \ge 2$, its fibre $\mathbb{C}^r$ has the density property [4]. Hence, Theorem 2.11 implies that the projection $\pi : D_h(E) = E \setminus S \to X$ has the Oka property (see Definition 2.8). Since it is also a topological fibre bundle, it is an Oka map. As X is an Oka manifold, it follows that $D_h(E)$ is an Oka manifold (see [18, Theorem 3.15] saying that, if $Y \to X$ is an Oka map of complex manifolds, then Y is an Oka manifold if and only if X is an Oka manifold). □

4. Examples of line bundles satisfying Theorem 1.5

In this section, we give examples and obtain functorial properties of the class of polarised manifolds with the polarised density property (see Definition 1.7).

We first show that every holomorphic line bundle satisfying the condition of Theorem 1.5 is ample, and hence it is natural to restrict ourselves to the polarised situation from the beginning. It is easily seen that for a polarised manifold (X, E) and a divisor $D \in |E|$, the complement $X \setminus D$ is affine (and hence Stein) since E is ample.

Proposition 4.1. *Let E be a holomorphic line bundle on a compact complex manifold X. Assume that for each point $x \in X$ there exists a divisor $D \in |E|$ whose complement $X \setminus D$ is a Stein neighbourhood of x. Then E is ample.*

Proof. By the assumption, there are finitely many section $s_0, s_1, \dots, s_n : X \to E$ whose divisors $D_i = \{s_i = 0\}$ have empty intersection and $X \setminus D_i = \{s_i \neq 0\}$ is Stein for every $i = 0, 1, \dots, n$. Consider the holomorphic map $\Phi = [s_0 : \cdots : s_n] : X \to \mathbb{CP}^n$ (3.3). We have seen in the proof of Theorem 1.5 that E is isomorphic to the pullback $\Phi^* \mathscr{O}_{\mathbb{CP}^n}(1)$ of the hyperplane section bundle (cf. [31, Theorem II.7.1]). Given a point $z = [z_0 : \cdots : z_n] \in \mathbb{CP}^n$, choose $i \in \{0, \ldots, n\}$ such that $z_i \neq 0$ and note that $\Phi^{-1}(z)$ is a closed complex subvariety of X contained in $X \setminus D_i$, which is Stein. Since a Stein manifold does not contain any compact complex subvariety of positive dimension, $\Phi^{-1}(z)$ is a finite set (or empty), so Φ is a finite holomorphic map. It follows that the line bundle $E \cong \Phi^* \mathscr{O}_{\mathbb{CP}^n}(1)$ is ample (see Lazarsfeld [42, proof of Theorem 1.2.13]). □

Proposition 4.2. *If a polarised manifold (X, E) has the polarised density property, then so does every positive tensor power $(X, E^{\otimes k})$ for $k > 0$.*

Proof. If the line bundle E is given on an open cover $\{U_i\}$ of X by a 1-cocycle $\phi_{i,j}$, then a holomorphic section $f : X \to E$ is given by a collection of holomorphic functions $f_i : U_i \to \mathbb{C}$ satisfying $f_i = \phi_{i,j} f_j$ on $U_{i,j}$. Since the bundle $E^{\otimes k}$ is given by the 1-cocycle $\phi_{i,j}^k$, the collection f_i^k defines a holomorphic section $f^{\otimes k}$ of $E^{\otimes k}$. Evidently, $\{f = 0\} = \{f^{\otimes k} = 0\}$. By the assumption there are holomorphic sections $s_0, \ldots, s_n : X \to E$ such that for every $i = 0, 1, \ldots, n$ the domain $X_i = \{s_i \neq 0\}$ is a Stein manifold with the density property and $\bigcup_{i=0}^n X_i = X$. Hence, for any integer $k \geq 1$ the collection $s_0^{\otimes k}, \ldots, s_n^{\otimes k}$ of sections of $E^{\otimes k}$ shows that $(X, E^{\otimes k})$ has the polarised density property. □

Example 4.3 (Line bundles on Grassmannians)**.** Given integers $1 \leq m < n$ we denote by $G_{m,n}$ the Grassmann manifold of complex m-dimensional subspaces of $\mathbb{C}^n$. Note that $G_{1,n} = \mathbb{CP}^{n-1}$. These manifolds are complex homogeneous, and hence Oka. The Plücker embedding $P : G_{m,n} \hookrightarrow \mathbb{CP}^N$, with $N = \binom{n}{m} - 1$, sends an m-plane $\mathrm{span}(v_1, \ldots, v_m) \in G_{m,n}$ (where $v_1, \ldots, v_m \in \mathbb{C}^n$ are linearly independent vectors) to the complex line in $\mathbb{C}^{N+1}$ given by the vector $v_1 \wedge \cdots \wedge v_m \in \Lambda^m(\mathbb{C}^n) \cong \mathbb{C}^{N+1}$. The intersection of the submanifold $X = P(G_{m,n}) \subset \mathbb{CP}^N$ with an affine chart $\mathbb{C}^N \cong U \subset \mathbb{CP}^N$ is biholomorphic to $\mathbb{C}^{m(n-m)}$, which has the density property if $m(n-m) = \dim G_{m,n} > 1$. It follows that the pullback $P^* \mathscr{O}_{\mathbb{CP}^N}(1)$ of the hyperplane section bundle on $\mathbb{CP}^N$ to $G_{m,n}$ has the polarised density property and Theorem 1.4 applies to it. Every holomorphic line bundle on $G_{m,n}$ is obtained from a line bundle on $\mathbb{CP}^N$, and the pullback map $P^* : \mathrm{Pic}(\mathbb{CP}^N) \to \mathrm{Pic}(G_{m,n})$ is a group isomorphism; hence, $\mathrm{Pic}(G_{m,n}) \cong \mathbb{Z}$ (see [9, Example 1.1.4 (3)] or [13, Lemma 11.1]). The pullback of the universal bundle $\mathbb{U} = \mathscr{O}_{\mathbb{CP}^N}(-1)$ is isomorphic to the determinant bundle of the universal bundle on $G_{m,n}$, and it generates $\mathrm{Pic}(G_{m,n})$. Write $\mathscr{O}_{G_{m,n}}(k)$ for the $(-k)$-th tensor power of this generator. Thus, $\mathscr{O}_{G_{m,n}}(1) = P^* \mathscr{O}_{\mathbb{CP}^N}(1)$. A line bundle $E \to G_{m,n}$ is positive (resp. negative) if $E \cong \mathscr{O}_{G_{m,n}}(k)$ for some $k > 0$ (resp. $k < 0$). The above observation for $\mathscr{O}_{G_{m,n}}(1)$ and Proposition 4.2 imply the following.

Proposition 4.4. *Every ample holomorphic line bundle on a complex Grassmann manifold of dimension > 1 has the polarised density property. Thus, every complex Grassmannian of dimension > 1 has the polarised density property.*

To state the next result, consider a pair of polarised manifolds (X_1, E_1) and (X_2, E_2). Let $\pi_i : X_1 \times X_2 \to X_i$ for $i = 1, 2$ denote the standard projections. Then, $\pi_i^* E_i$ is a holomorphic line bundle on $X_1 \times X_2$ for $i = 1, 2$. Their tensor product

$$E = E_1 \boxtimes E_2 := (\pi_1^* E_1) \otimes (\pi_2^* E_2) \to X_1 \times X_2$$

is called the *external tensor product* of E_1 and E_2. A pair of holomorphic sections $f_i \in H^0(X_i, E_i)$ for $i = 1, 2$ defines a holomorphic section $f_1 \boxtimes f_2 \in H^0(X_1 \times X_2, E_1 \boxtimes E_2)$ by trivially extending both line bundles and sections to $X_1 \times X_2$ and taking their tensor product.

Similarly, for a pair of semipositive hermitian metrics h_i on E_i for $i = 1, 2$, the semipositive hermitian metric $h = h_1 \boxtimes h_2$ on $E_1 \boxtimes E_2$ is defined in an obvious manner by considering h_i as a hermitian metric on $\pi_i^* E_i$. Note that the restriction of $E = E_1 \boxtimes E_2$ to $X_1 \times \{x_2\}$ $(x_2 \in X_2)$ is isomorphic to E_1, and analogously for the second factor. Clearly, this operation extends to any finite number of line bundles $E_i \to X_i$, $i = 1, \ldots, m$.

Ischebeck [32] proved that if Y is a rational manifold (in particular, if Y is a projective space or a complex Grassmannian) then $\mathrm{Pic}(X \times Y) = \mathrm{Pic}(X) \times \mathrm{Pic}(Y)$, so we get all holomorphic line bundles on $X \times Y$ as external tensor products of lines bundles on X and Y.

Proposition 4.5. *If the polarised manifolds (X_1, E_1) and (X_2, E_2) have the polarised density property, then the product $(X_1 \times X_2, E_1 \boxtimes E_2)$ also has the polarised density property.*

Proof. Let $f_1, \ldots, f_n \in H^0(X_1, E_1)$ and $g_1, \ldots, g_m \in H^0(X_2, E_2)$ be holomorphic sections of the respective line bundles which satisfy the definition of the polarised density property. As explained above, we may consider both bundles and their section to be defined on $X = X_1 \times X_2$. Consider the collection of sections $f_i \boxtimes g_j \in H^0(X, E_1 \boxtimes E_2)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. For any pair of indices i, j in the given range, the set

$$U_{i,j} := \{f_i g_j \neq 0\} = \{x_1 \in X_1 : f_i(x_1) \neq 0\} \times \{x_2 \in X_2 : g_i(x_2) \neq 0\}$$

is the product of Stein manifolds with the density property, so it is Stein with the density property (see Varolin [52, p. 136, I.1]). Since the sets $U_{i,j}$ cover X, the proposition holds. □

Proposition 4.6. *If the polarised manifold (X, E) has the polarised density property, then $(X \times \mathbb{CP}^n, E \boxtimes \mathscr{O}_{\mathbb{CP}^n}(k))$ $(n > 0,\ k > 0)$ also has the polarised density property. The same is true for $(X \times G_{m,n}, E \boxtimes \mathscr{O}_{G_{m,n}}(k))$ with $1 \le m < n$ and $k > 0$.*

Proof. Since every projective space is also a Grassmannian, it suffices to consider the second case. If $\dim G_{m,n} > 1$, this follows from Propositions 4.4 and 4.5. If $\dim G_{m,n} = 1$ then $G_{m,n} = \mathbb{CP}^1$. We follow the proof of Proposition 4.5 and use that if X is a Stein manifold with the density property then $X \times \mathbb{C}$ also has the density property (see Varolin [52, p. 136, I.2]). □

Remark 4.7. Assuming that holomorphic line bundles E_1 and E_2 on a projective manifold X have the polarised density property, we do not know whether their tensor product $E_1 \otimes E_2$ has the polarised density property. Indeed, given nontrivial sections $f : X \to E_1$ and $g : X \to E_2$, the zero set of the section $f \otimes g : X \to E_1 \otimes E_2$ is $\{f = 0\} \cup \{g = 0\}$, and its complement is $\{f \neq 0\} \cap \{g \neq 0\}$. This manifold need not have the density property even if both $\{f \neq 0\}$ and $\{g \neq 0\}$ are Stein manifolds with the density property.

Recall that a projective manifold is said to be rational if it is birationally isomorphic to a projective space. Every rational curve is isomorphic to $\mathbb{CP}^1$.

Proposition 4.8. *If $X_1, \ldots, X_m$ $(m \ge 2)$ are rational manifolds such that every X_i with $\dim X_i > 1$ has the polarised density property, then their product $X_1 \times X_2 \times \cdots \times X_m$ also has the polarised density property.*

Proof. It suffices to prove the result for $m = 2$ and apply induction. By Ischebeck [32] we have that $\mathrm{Pic}(X) = \mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2)$ and each group $\mathrm{Pic}(X_i)$ is discrete. Let E be an ample line bundle on $X_1 \times X_2$. Then the restriction of E to each factor X_i $(i = 1, 2)$ is an ample line bundle E_i and $E \cong E_1 \boxtimes E_2$. If $\dim X_1 > 1$ and $\dim X_2 > 1$ then both E_1 and E_2 have the polarised density property by the assumption, and the conclusion follows from Proposition 4.5. If $\dim X_1 > 1$ and $\dim X_2 = 1$ then $X_2 \cong \mathbb{CP}^1$, $E_2 = \mathscr{O}_{\mathbb{CP}^1}(k)$ for some $k > 0$, and the conclusion follows from Proposition 4.6. The same argument applies if $\dim X_1 = 1$ and $\dim X_2 > 1$. In the remaining case, both X_1 and X_1 are isomorphic to $\mathbb{CP}^1$ and $E_i \cong \mathscr{O}_{\mathbb{CP}^1}(k_i)$ for some $k_i > 0$ $(i = 1, 2)$. Fixing a point $p = (p_1, p_2) \in X = \mathbb{CP}^1 \times \mathbb{CP}^1$ we can find a pair of holomorphic sections $f_i : \mathbb{CP}^1 \to E_i$ $(i = 1, 2)$ such that $p_i \in U_i = \{f_i \neq 0\} \cong \mathbb{C}$. Thus, $f_1 f_2$ is a section of $E \cong E_1 \boxtimes E_2$, and the set $\{(x_1, x_2) \in X : f_1(x_1)f_2(x_2) \neq 0\}$ is a neighbourhood of p isomorphic to $\mathbb{C}^2$, which has the density property. This shows that X has the polarised density property. □

Since every complex Grassmannian is a rational manifold, we have the following corollary to Propositions 4.4 and 4.8.

Corollary 4.9. *If $X = X_1 \times \cdots \times X_m$ is a product of complex Grassmannians and $\dim X > 1$, then X has the polarised density property.*

In conclusion, we pose the following open problems, from more general to more special.

Problem 4.10. Let X be a projective Oka manifold of dimension > 1 and E be an ample holomorphic line bundle on X.

(a) Is there a hermitian metric h on E such that the disc bundle $\Delta_h(E)$ is an Oka manifold? Does this hold for every semipositive hermitian metric on E?
(b) Does this hold if X is Zariski locally isomorphic to $\mathbb{C}^n$ with $n > 1$?
(c) Does this hold if X is a homogeneous rational manifold with $\dim X > 1$?

We comment on part (d) of the above problem. By the Borel–Remmert theorem [8], a rational homogeneous manifold is a product $X = X_1 \times \cdots \times X_n$ of (generalized) flag manifolds X_i $(i = 1, \ldots, n)$. Since each X_i is rational (see Borel [7]), it follows from Ischebeck's result [32] that every line bundle E on X is isomorphic to the external tensor product $E_1 \boxtimes E_2 \boxtimes \cdots \boxtimes E_n$ of line bundles $E_i \to X_i$. Note that if E is ample then so is every factor E_i. By Proposition 4.5, this reduces the problem to the case when X is a flag manifold of dimension > 1. Assume that E is an ample line bundle on X. (Line bundles of flag manifolds were studied by Andersen [5] and Iversen [33], among others). To obtain an affirmative answer, it would suffice to find a divisor $D \in |E|$ whose complement $X \setminus D$ is isomorphic to $\mathbb{C}^{\dim X}$, which has the density property. Indeed, assume that such D exists. Since X is homogeneous and $\mathrm{Pic}(X)$ is discrete, for each point $x \in X$ there exists a holomorphic automorphism φ of X such that $\varphi^* E \cong E$ and $\varphi^* D \in |\varphi^* E| = |E|$ does not contain x. Therefore, (X, E) has the polarised density property.

5. Holomorphic maps from Stein manifolds to vector bundles

Assume that (E, h) is a hermitian holomorphic vector bundle on a compact Oka manifold X. In this section, we combine the results obtained in this paper with those of Drinovec Drnovšek and Forstnerič [14] to find holomorphic maps $S \to E$ from Stein manifolds S with $\dim S < \dim E$ which are either proper or have their boundary cluster set contained in the zero section of E. The former case occurs when (E, h) is Griffiths negative and the exterior tube

$$D_h(E) = \{e \in E : |e|_h > 1\}. \tag{5.1}$$

is Oka. This holds in particular if $\mathrm{rank}\, E > 1$ (see Theorem 1.9) or if (E, h) is a negative line bundle on $\mathbb{CP}^n$ (see Theorem 1.1 (b')). The latter case occurs when (E, h) is a positive line bundle with Oka disc bundle $\Delta_h(E) = \{h < 1\}$ (1.1); sufficient conditions are given by Theorems 1.1, 1.4, 1.5, and 1.9. We begin with the former case.

Theorem 5.1. *Let (E, h) be a Griffiths negative hermitian holomorphic vector bundle on a compact complex manifold X (see Definition 2.5). Assume that S is a Stein manifold with $\dim S < \dim E$, $K \subset S$ is a compact $\mathscr{O}(S)$-convex subset, and $f_0 : S \to E$ is a continuous map which is holomorphic on a neighbourhood of K and satisfies $f_0(S \setminus \mathring{K}) \subset E \setminus E(0)$. If the domain $D_h(E)$ (5.1) is Oka, then we can approximate f_0 uniformly on K by proper holomorphic maps $f : S \to E$ homotopic to f_0. Furthermore, if $2 \dim S < \dim E$ then f can be chosen an embedding, and if $2 \dim S \le \dim E$ then f can be chosen an immersion.*

With (E, h) as in the theorem, the domain $D_h(E)$ (5.1) is Oka if E is a line bundle on $X = \mathbb{CP}^n$ (see Theorem 1.1 (b')), or if $\operatorname{rank} E > 1$ and X is an Oka manifold (see Theorem 1.9), so the result applies in these cases. If $D_h(E)$ is Oka then for every $t > 0$ the domain

$$D_{h,t} = \{e \in E : |e|_h > t\} \tag{5.2}$$

is Oka as well, since it is biholomorphic to $D_h(E) = D_{h,1}(E)$ by a fibre dilation.

Proof. Choose a normal exhaustion $B_0 \Subset B_1 \Subset \cdots \subset \bigcup_{i=0}^{\infty} B_i = S$ by relatively compact, smoothly bounded, strongly pseudoconvex domains such that $K \subset B_0$ and the given map f_0 is holomorphic on a neighbourhood of $\overline{B}_0$. Also, choose an increasing sequence $0 < t_0 < t_1 < \cdots$ with $\lim_{i \to \infty} t_i = +\infty$. Since the hermitian metric h is Griffiths negative, the function

$$\phi : E \to [0, +\infty), \quad \phi(e) = |e|_h^2 \ \ (e \in E) \tag{5.3}$$

is strongly plurisubharmonic in $E \setminus E(0)$ (see Proposition 2.6). Clearly, ϕ is an exhaustion function on E without critical points in $E \setminus E(0)$.

Recall that $\dim S < \dim E$ and $f_0(S \setminus \mathring{K}) \subset E \setminus E(0)$ by the assumption. By [14, Theorem 1.1] we can approximate f_0 uniformly on K by a holomorphic map $\tilde f_0 : \overline{B}_0 \to E$, which is homotopic to f_0 through a family of maps sending $\overline{B}_0 \setminus \mathring{K}$ to $E \setminus E(0)$, such that $\tilde f_0(bB_0) \subset D_{h,t_0}(E)$ (see (5.2)). The homotopy condition allows us to extend $\tilde f_0$ to a continuous map $\tilde f_0 : S \to E$ satisfying $\tilde f_0(S \setminus B_0) \subset D_{h,t_0}(E)$, and the given homotopy from f_0 to $\tilde f_0$ on $\overline{B}_0$ extends to a homotopy between these two maps on all of S sending $S \setminus B_0$ to $E \setminus E(0)$.

Since the tube $D_{h,t_0}(E)$ is biholomorphic to $D_h(E)$, and hence Oka, we can apply the Oka principle in [18, Theorem 1.3] to approximate $\tilde f_0$ uniformly on $\overline{B}_0$ by a holomorphic map $f_1 : S \to E$, homotopic to $\tilde f_0$ by a homotopy as above, such that $f_1(S \setminus B_0) \subset D_{h,t_0}(E)$.

We now repeat the same procedure with the map f_1. First, we approximate f_1 on $\overline{B}_0$ by a holomorphic map $\tilde f_1 : \overline{B}_1 \to E$ such that $\tilde f_1(\overline{B}_1 \setminus B_0) \subset D_{h,t_0}(E)$ and $\tilde f_1(bB_1) \subset D_{h,t_1}(E)$. Next, we extend $\tilde f_1$ to a continuous map $\tilde f_1 : S \to E \setminus E(0)$ which agrees with the given holomorphic map $\tilde f_1$ on a neighbourhood of $\overline{B}_1$ and satisfies $\tilde f_1(S \setminus B_1) \subset D_{h,t_1}(E)$. Since the tube $D_{h,t_1}(E)$ is Oka, we can apply [18, Theorem 1.3] to approximate $\tilde f_1$ uniformly on $\overline{B}_1$ by a holomorphic map $f_2 : S \to E$ such that $f_2(S \setminus B_1) \subset D_{h,t_1}(E)$. By the same argument as in the first step, there is a homotopy connecting f_1 to f_2 sending $S \setminus \mathring{K}$ to $E \setminus E(0)$.

Continuing inductively, we find a sequence of holomorphic maps $f_i : S \to E$ for $i = 1, 2, \ldots$ such that the following conditions hold for every $i \ge 1$:

(i) f_i approximates f_{i-1} as closely as desired on $\overline{B}_{i-1}$.
(ii) $f_i(S \setminus B_{i-1}) \subset D_{h,t_{i-1}}(E)$.
(iii) f_i is homotopic to f_{i-1} through a homotopy sending $S \setminus \mathring{K}$ to $E \setminus E(0)$.

Assuming as we may that the approximation is close enough at every step, the sequence f_i converges uniformly on compacts in S to a proper holomorphic map $f : S \to E$ homotopic to the initial map f_0. (Condition (iii) is only needed to keep the induction going.) The additions in the last sentence of the theorem follow by using the well-known general position argument. We leave the obvious details to the reader. □

Assuming that (E, h) is a hermitian line bundle on X, we have seen in Section 2 that the tube $D_h(E)$ (5.2) is fibrewise biholomorphic to the punctured disc bundle $\Delta^*_{h^*}(E^*)$ in the hermitian dual bundle (E^*, h^*), and the section $E(\infty)$ in the associated $\mathbb{CP}^1$-bundle $\widehat{E} \to X$ corresponds to the zero section $E^*(0)$ of the dual bundle. Hence, Theorem 5.1 implies an analogous result for maps $S \to E^* \setminus E^*(0)$ whose cluster set lies in the zero section $E^*(0)$. However, we can prove a stronger result in this direction, allowing the initial map $S \to E^*$ to intersect the zero section is a compact set. To state the result, we recall the following notion.

A sequence $(x_j)_{j\in\mathbb{N}}$ in a topological space X is said to be divergent if for every compact set $K \subset X$ there is $j_0 \in \mathbb{N}$ such that $x_j \in X \setminus K$ for all $j \geq j_0$. Given a continuous map $f : X \to Y$ of topological space with X noncompact, its cluster set is

$$C(f) = \{y \in Y : \text{there is a divergent sequence } x_j \in X \text{ with } \lim_{j\to\infty} f(x_j) = y\}.$$

(If X is compact then $C(f) = \varnothing$.) We have the following result for maps from Stein manifolds to positive hermitian line bundles on compact Oka manifolds.

Theorem 5.2. *Assume that (E, h) is a positive hermitian holomorphic line bundle on a compact complex manifold X such that the disc bundle $\Delta_h(E)$ is Oka. Given a Stein manifold S with $\dim S \leq \dim X$, a compact $\mathscr{O}(S)$-convex set $K \subset S$, and a continuous map $f_0 : S \to E$ which is holomorphic on a neighbourhood of K and satisfies $f_0(K) \subset \Delta_h(E)$, f_0 can be approximated uniformly on K by holomorphic maps $f : S \to \Delta_h(E)$ homotopic to f_0 such that $C(f) \subset E(0)$. If in addition $2\dim S \leq \dim X$ then f can be chosen an injective immersion.*

Proof. Since the bundle (E, h) is positive, the function $\sigma = 1/h : E \to (0, +\infty)$ is strongly plurisubharmonic (see Proposition 2.3). Furthermore, $d\sigma \neq 0$ on $E \setminus E(0)$, and for any pair of numbers $0 < a < b$ the set

$$E_{a,b} = \{e \in E : a \leq \sigma(e) \leq b\} \tag{5.4}$$

is compact. Let $U \subset S$ be an open Stein domain containing K and $f_0 : S \to E$ be a continuous map which is holomorphic on U. Choose a smoothly bounded strongly pseudoconvex domain $B_0 \subset S$ such that $K \subset B_0 \subset \bar{B}_0 \subset U$ and $\bar{B}_0$ is $\mathscr{O}(S)$-convex. Recall that X, and hence E, are Oka manifolds. By the transversality theorem for holomorphic maps of Stein manifold to Oka manifolds (see [16, Corollary 8.8.7]), we may assume that the map $f_0 : U \to E$ is transverse to the zero section $E(0)$. Hence, the set

$$V_0 = \{x \in U : f_0(x) \in E(0)\}$$

is a closed complex subvariety of U which does not contain any connected component of U. The set $K \cup (\bar{B}_0 \cap V_0)$ is $\mathscr{O}(S)$-convex. Let $c_0 > 0$ be chosen such that $f_0(\overline{B}_0) \subset \{\sigma > c_0\}$. Pick numbers $c_1 > c_0$ and $\epsilon > 0$. Choose a compact $\mathscr{O}(S)$-convex set $K' \subset U$ such that

$$K \cup (\bar{B}_0 \cap V_0) \subset \mathring{K}' \quad \text{and} \quad \sigma \circ f_0 > c_1 + 1 \text{ on } bB_0 \cap K'. \tag{5.5}$$

The second condition holds if K' is a sufficiently small neighbourhood of $K \cup (\bar{B}_0 \cap V_0)$. Set $K_0 = K' \cap \bar{B}_0$. We claim that there is a holomorphic map $g : \bar{B}_0 \to E$ satisfying the following conditions for a fixed Riemannian distance function dist on E:

(i) $\mathrm{dist}(g(x), f_0(x)) < \epsilon$ for all $x \in K_0$.
(ii) $\sigma(g(x)) > \sigma(f_0(x)) - \epsilon$ for all $x \in \bar{B}_0 \setminus K_0$.
(iii) $\sigma \circ g > c_1$ on bB_0.
(iv) g is homotopic to f_0 on $\bar{B}_0$.

In the special case when $V_0 = \varnothing$ and K_0 is a compact subset of D, a map g with these properties is given by [14, Lemma 5.3], which is the main inductive step in [14, proof of Theorem 1.1]. In the case at hand, the compact set $K_0 \subset \bar{B}_0$ may intersect bB_0, but we have that $\sigma \circ f_0 > c_1 + 1$ on $bB_0 \cap K_0$ by condition (5.5). Hence, to ensure condition (iii), it suffices to apply [14, Lemma 5.2] finitely many times for points in the compact set $\{x \in bB_0 \setminus K_0 : \sigma(f_0(x)) \le c_1 + 1\}$. (The cited lemma amounts to lifting a small piece of $f_0(bB_0)$ to a higher level set of σ by a prescribed amount, while at the same time approximating f_0 on K_0 (condition (i)). This lifting procedure uses modifications involving local peak functions and gluing, and it is designed in such a way that condition (ii) can be fulfilled. Condition (iv) is built into the construction as well. The fact that the function σ is not defined on $E(0)$ is irrelevant in this proof since the compact set K_0 contains $V_0 \cap \bar{B}_0$ in its relative interior, and σ is only used on $\bar{B}_0 \setminus K_0$.)

By approximation, we may assume that g is holomorphic on a neighbourhood of $\bar{B}_0$, and we can extend it to a continuous map $g : S \to E$ homotopic to f_0. We now use the hypothesis that the disc bundle $\Delta_h(E)$ is an Oka manifold. Hence, the tube

$$\Omega_{c_1} = E(0) \cup \{\sigma > c_1\} = \Delta_{h,1/c_1}(E)$$

is Oka as well. By the Oka principle in [18, Theorem 1.3] we can approximate g uniformly on $\overline{B}_0$ by a holomorphic map $f_1 : S \to E$, homotopic to g, such that

$$f_1(S \setminus B_0) \subset \Omega_{c_1}.$$

Pick an arbitrary smoothly bounded strongly pseudoconvex domain $B_1 \subset S$ such that $\overline{B}_0 \subset B_1$ and $\overline{B}_1$ is $\mathscr{O}(S)$-convex.

Continuing inductively, we obtain a normal exhaustion of S by an increasing sequence of smoothly bounded, strongly pseudoconvex domains $B_0 \Subset B_1 \Subset \cdots \subset \bigcup_{i=0}^{\infty} B_i = S$, a sequence of continuous maps $f_i : S \to E$ $(i = 0, 1, \ldots)$, and an increasing sequence $0 < c_0 < c_1 < c_2 < \cdots$ with $\lim_{i\to\infty} c_i = +\infty$ such that for every $i = 1, 2, \ldots$ the map f_i is holomorphic on $\bar{B}_i$, it approximates f_{i-1} on $\bar{B}_{i-1}$, and it maps $\overline{B_i \setminus B_{i-1}}$ to $\Omega_{c_i} = \Delta_{h,1/c_i}(E)$. Assuming as we may that the approximation is close enough at every step, the limit map $f = \lim_{i\to\infty} f_i : X \to E$ exists, it approximates f_0 on K and is homotopic to it, and it satisfies $C(f) \subset E(0)$. We leave the obvious details of this induction to the reader. If $2 \dim S \le \dim X$ then we can additionally use the general position argument at every step of the induction to ensure that the map f is an injective immersion. □

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