

REMOVING SINGULARITIES OF MINIMAL SURFACES BY ISOTOPIES

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ABSTRACT. Given an open Riemann surface M , we show that the branch points and the complete ends of finite total curvature of a conformal minimal surface $M \rightarrow \mathbb{R}^n$, $n \geq 3$, can be removed by an isotopy through such surfaces. The analogous result holds for null holomorphic curves $M \rightarrow \mathbb{C}^n$.

1. INTRODUCTION

Let M be a connected open Riemann surface. Recall that an immersion $u = (u_1, \dots, u_n) : M \rightarrow \mathbb{R}^n$ ($n \geq 3$) is *conformal* (angle preserving) if and only if its $(1, 0)$ -differential $\partial u = (\partial u_1, \dots, \partial u_n)$ (the $\mathbb{C}$ -linear part of the differential $du = \partial u + \bar{\partial} u$) satisfies the nullity condition

$$(1.1) \quad (\partial u_1)^2 + \dots + (\partial u_n)^2 = 0.$$

(See e.g. [18] or [3].) A conformal immersion $u : M \rightarrow \mathbb{R}^n$ parametrises a minimal surface in $\mathbb{R}^n$ with the Euclidean metric ds^2 if and only if it is harmonic, if and only if ∂u is a holomorphic 1-form. Assuming that this holds and choosing a nowhere vanishing holomorphic 1-form θ in M , we have $2\partial u = f\theta$ where $f : M \rightarrow \mathbb{C}^n$ is a holomorphic map with values in the punctured null quadric $\mathbf{A}_* = \mathbf{A} \setminus \{0\}$, where

$$(1.2) \quad \mathbf{A} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + \dots + z_n^2 = 0\}.$$

Given any point $x_0 \in M$, we recover u from its Weierstrass data $f\theta$ by

$$(1.3) \quad u(x) = u(x_0) + \Re \int_{x_0}^x 2\partial u = u(x_0) + \Re \int_{x_0}^x f\theta, \quad x \in M.$$

Here, $\Re$ denotes the real part. Conversely, a holomorphic map $f : M \rightarrow \mathbf{A}_*$ satisfying $\Re \oint_{\gamma} f\theta = 0$ for every closed curve $\gamma \subset M$ (that is, $\Re(f\theta)$ is exact on M) determines a conformal minimal immersion $u : M \rightarrow \mathbb{R}^n$ by the above formula. The *generalised Gauss map*, or simply the *Gauss map*, of u is the holomorphic map

$$(1.4) \quad \mathcal{G}(u) : M \rightarrow Q = \{[z_1 : \dots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \dots + z_n^2 = 0\}$$

given by

$$(1.5) \quad \mathcal{G}(u)(p) = [\partial u_1(p) : \dots : \partial u_n(p)], \quad p \in M.$$

In this paper, we consider minimal surfaces with isolated singularities. One type of singularities are branch points. Let $u : M \rightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ map with rank 2 at some point. Denote by $\text{br}(u) \subsetneq M$ the set of points $x \in M$ at which u is not an immersion, i.e., $\text{rank } du_x < 2$. If the immersion $u : M \setminus \text{br}(u) \rightarrow \mathbb{R}^n$ is conformal and harmonic, then ∂u is a continuous $(1, 0)$ -form on M which is holomorphic on $M \setminus \text{br}(u)$ and satisfies $\{\partial u = 0\} = \text{br}(u)$. By a theorem of Radó [19] (see also [20, Theorem 3.4.17]), ∂u extends holomorphically to M , so $\text{br}(u)$ is a closed discrete subset of M and u is harmonic on M . The points of $\text{br}(u)$ are called *branch points* of u , and u is said to be a *branched conformal minimal surface*; see e.g. [18, Ch. 6] or [3, Remark 2.3.7]. Branch points of minimal surfaces are not removable by small deformations

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[3, Remark 3.12.6]. Our first result is that they are removable by isotopies. It is proved in Section 3; see the more precise statement in Theorem 3.1.

Theorem 1.1. *Given a branched conformal minimal surface $u : M \rightarrow \mathbb{R}^n$, there is an isotopy of branched conformal minimal surfaces $u_t : M \rightarrow \mathbb{R}^n$, $t \in [0, 1]$, such that $u_0 = u$ and u_1 is an immersion everywhere on M , that is, $\text{br}(u_1) = \emptyset$. Furthermore, we can choose the isotopy such that for each $t \in [0, 1]$ the Gauss map $\mathcal{G}(u_t)$ (1.5) of u_t equals $\mathcal{G}(u)$ in their common domain of definition $M \setminus (\text{br}(u_t) \cup \text{br}(u))$.*

By an *isotopy*, we mean a family of maps depending continuously on a parameter $t \in [0, 1]$. The space

$$\text{CMI}_{\text{br}}(M, \mathbb{R}^n)$$

of branched conformal minimal surfaces $M \rightarrow \mathbb{R}^n$ is endowed with the compact-open topology and contains the subspace $\text{CMI}(M, \mathbb{R}^n)$ of conformal minimal immersions $M \rightarrow \mathbb{R}^n$. Recall that $u \in \text{CMI}_{\text{br}}(M, \mathbb{R}^n)$ is said to be *nonflat* if and only if $u(M)$ is not contained in an affine plane of $\mathbb{R}^n$; equivalently, the image of the map $f = 2\partial u/\theta : M \rightarrow \mathbf{A}$ is not contained in a ray of $\mathbf{A}$ (1.2). Also, u is called *full* if and only if $f(M)$ is not contained in a proper linear subspace of $\mathbb{C}^n$. (See Definition 2.5.2 and Lemma 2.5.3 in [3, p. 106].) Note that the second assertion in Theorem 1.1 implies that if the given surface u is nonflat (resp. full) then the isotopy u_t ($t \in [0, 1]$) can be chosen to consist of nonflat (resp. full) surfaces.

Another important type of isolated singularities of minimal surfaces are *complete ends of finite total curvature*. The Gaussian curvature of a smooth immersed surface $u : M \rightarrow \mathbb{R}^n$ is a function $K : M \rightarrow \mathbb{R}$ whose value at $p \in M$ is the Gauss curvature of the Riemannian metric u^*ds^2 at p . If $u \in \text{CMI}(M, \mathbb{R}^n)$ is a minimal surface then K assumes values in $\mathbb{R}_- = (-\infty, 0]$, and the total curvature is the number $\text{TC}(u) = \int_M K dA \in [-\infty, 0]$, where dA is the area measure determined by u^*ds^2 . (See [3, Sect. 2.6].) We say that u is of *finite total curvature* if $\text{TC}(u) > -\infty$. A minimal surface $u : M \rightarrow \mathbb{R}^n$ is said to be *complete* if the metric u^*ds^2 induces a complete distance function on M . If M is a bordered Riemann surface with compact closure $\overline{M}$, $P \subset M$ is a compact subset, and $u : \overline{M} \setminus P \rightarrow \mathbb{R}^n$ is a complete conformal minimal surface of finite total curvature, then P is a finite set by a theorem of Huber [15] (see also [3, Theorem 2.6.4]), ∂u extends to a meromorphic 1-form on M with a pole of order ≥ 2 at every point of P by the Chern–Osseman theorem [7] (see also [3, Theorem 4.1.1]), u is proper at every end $p \in P$, and its asymptotic behaviour at p is described by the Jorge–Meeks theorem [16] (see also [3, Theorem 4.1.3]). Conversely, a nontrivial meromorphic 1-form $\phi = (\phi_1, \dots, \phi_n)$ on an open Riemann surface M (such ϕ is called an *abelian differential*) has a closed discrete polar locus $P(\phi) \subset M$. If ϕ satisfies the nullity condition (1.1) and has vanishing real periods on closed curves in $M' = M \setminus P(\phi)$, then it determines a conformal minimal surface $u : M' \rightarrow \mathbb{R}^n$ by $u(x) = \Re \int^x \phi$ with a complete end of finite total curvature at each point of $P(\phi)$. Let

$$\text{CMI}_s(M, \mathbb{R}^n)$$

denote the space of conformal minimal immersions $u : M \setminus P \rightarrow \mathbb{R}^n$, where $P = P(u)$ is a closed discrete subset of M and ∂u is meromorphic on M with an effective pole at every point $p \in P$ and no other zeros or poles. (The subscript *s* stands for singularities.) With θ as above, we have $2\partial u = f\theta$ where f is a meromorphic map on M with values in $\mathbf{A}_*$ (1.2) whose polar locus is P . (See Subsect. 2.2.) We can view f as a holomorphic map to the complex submanifold

$$(1.6) \quad Y = \mathbf{A}_* \cup Q$$

of $\mathbb{C}^n \cup \mathbb{CP}^{n-1} = \mathbb{CP}^n$, where Q is the hyperquadric in (1.4). The topology on $\text{CMI}_s(M, \mathbb{R}^n)$ is defined by the condition that a sequence $u_j \in \text{CMI}_s(M, \mathbb{R}^n)$ converges to $u \in \text{CMI}_s(M, \mathbb{R}^n)$ if and only if $f_j = 2\partial u_j/\theta : M \rightarrow Y$ converges to $f = 2\partial u/\theta : M \rightarrow Y$ in the compact-open topology and there is a point $x_0 \in M \setminus P(f)$ such that $u_j(x_0)$ converges to $u(x_0)$ in $\mathbb{R}^n$. Note that $\text{CMI}_s(M, \mathbb{R}^n)$ contains the subspace $\text{CMI}(M, \mathbb{R}^n)$ with its usual compact-open topology. We denote by

$$\text{CMI}_{\text{br}}^f(M, \mathbb{R}^n) \subset \text{CMI}_{\text{br}}(M, \mathbb{R}^n), \quad \text{CMI}_s^f(M, \mathbb{R}^n) \subset \text{CMI}_s(M, \mathbb{R}^n), \quad \text{CMI}^f(M, \mathbb{R}^n) \subset \text{CMI}(M, \mathbb{R}^n)$$

the corresponding subspaces of full maps.

The order of the pole of f at $p \in P$ is the local intersection number of f with Q at p , which is positive, so a pole cannot be removed by a small deformation of u . Our second main result is that complete ends of finite total curvature of full minimal surfaces can be removed by an isotopy. It is proved in Sect. 4.

Theorem 1.2. *Let M be an open Riemann surface and $n \geq 3$ an integer. For any $u \in \text{CMI}_s^f(M, \mathbb{R}^n)$ there is an isotopy $u_t \in \text{CMI}_s^f(M, \mathbb{R}^n)$, $t \in [0, 1]$, such that $u_0 = u$ and u_1 is defined everywhere on M , that is, $u_1 \in \text{CMI}^f(M, \mathbb{R}^n)$.*

We wish to explain the reason for a somewhat different assumption on the initial minimal surface u in Theorem 1.1 (where u need not be full) and in Theorem 1.2, where u is assumed to be full. The proof of Theorem 1.1 amounts to finding a path of holomorphic abelian differentials ω_t ($t \in [0, 1]$) on M with values in $\mathbf{A}$ (1.2) and vanishing real periods such that $\omega_0 = 2\partial u$ and ω_1 has no zeros. Integrating these abelian differentials by the Weierstrass formula (1.3) gives an isotopy of conformal minimal surfaces $u_t(x) = \Re \int^x \omega_t$ satisfying the conclusion of Theorem 1.1. The proof is accomplished in two steps. In the first step (see Proposition 3.2 (b)) we find a path of nontrivial abelian differentials $\varpi_t = h_t \omega_0$ ($t \in [0, 1]$) with values in $\mathbf{A}$, where h_t is a path of meromorphic functions on M , satisfying $\varpi_0 = \omega_0$ (that is, $h_0 \equiv 1$) but without paying attention to the period conditions. This does not require fullness. In the second step we find a path of nowhere vanishing holomorphic functions ξ_t on M , with $\xi_0 \equiv 1$, such that the path $\omega_t = \xi_t \varpi_t$ ($t \in [0, 1]$) satisfies that each ω_t has vanishing real periods; see Proposition 2.2. This does not required fullness either; see Remark 3.3. The second assertion in Theorem 1.1 is granted in this construction since each ω_t is of the form $\omega_t = 2\xi_t h_t \partial u$; see (1.5). Only Runge approximation for functions into $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is required for this task. The proof of Theorem 1.2 follows the same scheme but now the path of abelian differentials ω_t on M is constructed to be nowhere vanishing and such that ω_1 is holomorphic on M (that is, it has no poles). In this case, the period problem is considerably more delicate than in the context of Theorem 1.1, and in order to solve it we use Runge approximation for maps into the Oka manifold Y in (1.6). This forces us to ask that the given surface u in Theorem 1.2 be full, and prevents us to preserve the Gauss map along the isotopy.

Going further, the proofs of Theorems 1.1 and 1.2 show that the two results can be combined, that is, we can push both the branch points and the ends of finite total curvature out of the surface by an isotopy. Let

$$\text{CMI}_{\text{br},s}(M, \mathbb{R}^n) \supset \text{CMI}_{\text{br},s}^f(M, \mathbb{R}^n)$$

denote the set of conformal minimal immersions $u : M \setminus \mathcal{E}_u \rightarrow \mathbb{R}^n$, where

$$(1.7) \quad \mathcal{E}_u = \mathcal{E}_u^0 \cup \mathcal{E}_u^\infty = \text{br}(u) \cup P(u)$$

is a (possibly empty) closed discrete subset of M (depending on u) such that ∂u is meromorphic on M with the zero set $\mathcal{E}_u^0 = \text{br}(u)$ (the branch locus of u) and the polar locus $\mathcal{E}_u^\infty = P(u)$ (the set of complete ends of finite total curvature of u), and its subset of full maps. We call $\mathcal{E}_u$ the *singular locus* of u . The topology on $\text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$ is determined in the same way as on its subspace $\text{CMI}_s(M, \mathbb{R}^n)$.

The following is our third main result.

Corollary 1.3. *Let M be an open Riemann surface and $n \geq 3$ an integer. For every $u \in \text{CMI}_{\text{br},s}^f(M, \mathbb{R}^n)$ there is an isotopy $u_t \in \text{CMI}_{\text{br},s}^f(M, \mathbb{R}^n)$, $t \in [0, 1]$, such that $u_0 = u$ and u_1 is an immersion defined everywhere on M , that is, $u_1 \in \text{CMI}^f(M, \mathbb{R}^n)$.*

Every flat conformal minimal immersion is isotopic to a nonflat one [3, Theorem 5.7.6], hence the immersion u_1 in Theorem 1.1 can be chosen nonflat whenever one does not insist on the condition on the Gauss map. Furthermore, by a recent result of Vrhovnik [21], every nonflat conformal minimal immersion $M \rightarrow \mathbb{R}^n$, $n \geq 3$, is isotopic to a proper one, which can be chosen an immersion with simple double points if

$n = 4$ and an embedding if $n \geq 5$. Therefore, the immersions u_1 in Theorems 1.1 and 1.2, and in Corollary 1.3, can be chosen of this kind.

An immediate consequence of Theorems 1.1, 1.2, and Corollary 1.3 is that each of the inclusions in

$$(1.8) \quad \begin{array}{ccc} & \text{CMI}_{\text{br}}^f(M, \mathbb{R}^n) & \\ \swarrow & & \searrow \\ \text{CMI}^f(M, \mathbb{R}^n) & & \text{CMI}_{\text{br},s}^f(M, \mathbb{R}^n) \\ \searrow & & \swarrow \\ & \text{CMI}_s^f(M, \mathbb{R}^n) & \end{array}$$

induces a surjection of path components. The same holds for the inclusion

$$(1.9) \quad \text{CMI}(M, \mathbb{R}^n) \hookrightarrow \text{CMI}_{\text{br}}(M, \mathbb{R}^n)$$

by Theorem 1.1. Recall that for any open Riemann surface M we have $H_1(M, \mathbb{Z}) = \mathbb{Z}^l$, $l \in \mathbb{Z}_+ \cup \{\infty\}$. The set of path components of the space $\text{CMI}^f(M, \mathbb{R}^3)$ is in bijective correspondence with the 2^l elements of the group $(\mathbb{Z}_2)^l$ (see [12, Corollary 1.6] or [3, Corollary 3.12.4]), and $\text{CMI}^f(M, \mathbb{R}^n)$ is path connected for $n \geq 4$ by [5, Theorem 6.1]. For the space of nonflat conformal minimal immersions, the same holds by [12, Corollary 1.6], and for $\text{CMI}(M, \mathbb{R}^n)$ it holds by [3, Corollary 5.7.7]. This implies the following corollary.

Corollary 1.4. *Each of the spaces in (1.8) and (1.9) is path connected for $n \geq 4$. If $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^l$, there is a surjection from $(\mathbb{Z}_2)^l$ to the set of path components of each of the spaces in (1.8) and (1.9) for $n = 3$.*

Problem 1.5. Do the inclusions in (1.8) and (1.9) induce bijections of path components? Are they weak homotopy equivalences?

The analogues of Theorems 1.1, 1.2, and Corollary 1.3 also hold, with essentially the same proofs, for generalised null curves $F : M \rightarrow \mathbb{C}^n$, $n \geq 3$. These are nonconstant meromorphic maps whose differential $\partial F = dF$ has isolated zeros and poles and assumes values in the null quadric $\mathbf{A}$ (1.2). Equivalently, given a nowhere vanishing holomorphic 1-form θ on M , we have $dF = f\theta$ where f is a holomorphic map from M to the complex subvariety $\mathbf{A} \cup Q \subset \mathbb{CP}^n$ (see (1.4)). The real and the imaginary part of any generalised null curve $M \rightarrow \mathbb{C}^n$ are elements of the space $\text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$.

Our method of proof of the main results allows not only to remove the singularities but also to move them freely within the surface. For example, in the context of Theorem 1.1, given a closed discrete subset $C \subset M$ and a bijective map of C to a subset of the branch locus $\text{br}(u)$, there is an isotopy $u_t \in \text{CMI}_{\text{br}}(M, \mathbb{R}^n)$, $t \in [0, 1]$, such that $u_0 = u$ and $\text{br}(u_1) = C$; see Corollary 3.4. An analogous statement holds in the context of Theorem 1.2; see Corollary 4.1. This follows by a straightforward modification of our proofs.

2. THE TOOLBOX

2.1. A Weierstrass interpolation theorem with parameters. In the proof of Theorem 1.1, we shall need the following parametric version of Weierstrass interpolation for finitely many points in an open Riemann surface. This is a special case of [1, Lemma 4.2] but with added approximation on a compact Runge set.

Lemma 2.1. *Let K be a compact Runge set in an open Riemann surface M and $a_j : [0, 1] \rightarrow M \setminus K$, $j = 1, \dots, k$, real analytic maps such that the points $a_1(t), \dots, a_k(t)$ are distinct for every $t \in [0, 1]$. Also let $\Lambda \subset M$ be a closed discrete subset disjoint from $\bigcup_{j=1}^k a_j([0, 1])$ and $\lambda : \Lambda \rightarrow \mathbb{N}$ a map. Given integers $n_1, \dots, n_k \in \mathbb{Z}$ and a number $\epsilon > 0$, there is a real analytic path of meromorphic functions $\{f_t\}_{t \in [0, 1]}$ on M such that for every $t \in [0, 1]$ and $j = 1, \dots, k$, the function f_t has degree n_j at $a_j(t)$ and has no other zeros or poles, $\max_{x \in K, t \in [0, 1]} |f_t(x) - 1| < \epsilon$, and $f_t - 1$ vanishes to order $\lambda(p)$ at p for every $p \in \Lambda$.*

Proof. It suffices to prove the result for $k = n_1 = 1$. This gives for each j a path of holomorphic functions $\{f_{j,t}\}_{t \in [0,1]}$ on M with a simple zero at $a_j(t)$ and no other zeros, satisfying the approximation condition on K and the interpolation conditions at points $p \in \Lambda$. The function $f_t = \prod_{j=1}^k f_{j,t}^{n_j}$ then satisfies the theorem.

The real analytic map $a = a_1 : [0, 1] \rightarrow M$ extends to a holomorphic map $a : D \rightarrow M$ from an open simply connected neighbourhood $D \subset \mathbb{C}$ of the interval $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$. Its graph $\Sigma = \{(z, a(z)) : z \in D\} \subset D \times M$ is a smooth closed complex hypersurface in the Stein surface $D \times M$. Shrinking D around $[0, 1]$ if necessary, we ensure that $\Sigma \cap (D \times (K \cup \Lambda)) = \emptyset$. Since D is contractible, we have $H^2(D \times M, \mathbb{Z}) \cong H^2(M, \mathbb{Z}) = 0$. Hence, Oka's solution of the second Cousin problem in [17] implies that every divisor on $D \times M$ is a principal divisor. Applying this to the divisor Σ gives a holomorphic function $f \in \mathcal{O}(D \times M)$ that vanishes to order 1 at every point of Σ and has no other zeros. The function $f_t = f(t, \cdot) \in \mathcal{O}(M)$ then has a simple zero at $a(t)$ and no other zeros for every $t \in D$. Since K is Runge in M and D is contractible, the inclusion $D \times K \hookrightarrow D \times M$ is homotopy equivalent to the inclusion of a finite bouquet of circles representing K in the finite or countable bouquet of circles representing M . Hence, the map $1/f : D \times K \rightarrow \mathbb{C}^*$ extends to a continuous map $D \times M \rightarrow \mathbb{C}^*$. Since $[0, 1] \times K$ is holomorphically convex in $D \times M$, the Oka principle for maps to the complex homogeneous manifold $\mathbb{C}^*$ (see Grauert [13] or [11, Theorem 5.4.4]) gives a holomorphic function $g : D \times M \rightarrow \mathbb{C}^*$ approximating $1/f$ uniformly on $[0, 1] \times K$ and such that $g - 1/f$ vanishes to order $\lambda(p)$ on $D \times \{p\}$ for every $p \in \Lambda$. (There are no topological obstructions for these interpolation conditions since the sets $D \times \{p\}$ are contractible.) Replacing f by fg gives a function satisfying the lemma provided that the approximation of $1/f$ by g was close enough on $[0, 1] \times K$. $\square$

2.2. Abelian differentials and complex cones. Let M be a Riemann surface. An abelian differential $\omega = (\omega_1, \dots, \omega_n)$ on M with values in $\mathbb{C}^n$ (whose components ω_i are meromorphic 1-forms on M) is said to be *nontrivial* if it is not identically zero, and is said to be *full* if its range is not contained in a proper linear subspace of $\mathbb{C}^n$. A nontrivial abelian differential determines a divisor (ω) on M defined as follows. Let $\zeta : U \rightarrow \mathbb{C}$ be a local holomorphic coordinate around a point $p \in M$ with $\zeta(p) = 0$. In this coordinate, $\omega = f(\zeta)d\zeta$ where $f = (f_1, \dots, f_n)$ is a meromorphic map on U . Let $k(p) \in \mathbb{Z}$ be the unique integer such that $\zeta^{-k(p)}f(\zeta)$ is holomorphic near $\zeta = 0$ and nonvanishing at $\zeta = 0$ (i.e., $f_i(0) \neq 0$ for some $i \in \{1, \dots, n\}$). Then, $(\omega) = \sum_{p \in M} k(p)p$. Its support $\text{supp}(\omega) = \{p \in M : k(p) \neq 0\}$ is a closed discrete subset of M . Likewise, the divisors of zeros and poles of ω are, respectively,

$$(2.1) \quad (\omega)_0 = \sum_{p \in M, k(p) > 0} k(p)p, \quad (\omega)_\infty = \sum_{p \in M, k(p) < 0} (-k(p))p,$$

hence $(\omega) = (\omega)_0 - (\omega)_\infty$. The support of $(\omega)_0$ and $(\omega)_\infty$ is the zero set and the polar set of ω , respectively.

A complex cone in $\mathbb{C}^n$ is a closed analytic subvariety $A \subset \mathbb{C}^n$ such that $\zeta A \subset A$ for every $\zeta \in \mathbb{C}$. By a theorem of Chow [9] (see also Chirka [8, p. 74, Remark]), such A is the common zero set of finitely many homogeneous polynomials on $\mathbb{C}^n$. An abelian differential $\omega = (\omega_1, \dots, \omega_n)$ on M is said to have values in A if in any local holomorphic coordinate ζ on M we have $\omega = f(\zeta)d\zeta$, where f is a meromorphic map with values in A . Such f can be seen as a holomorphic map in the projective closure of A .

2.3. A parametric interpolation theorem for multipliers with control of periods. The following approximation result with interpolation for multiplier functions is an extension of [2, Theorem 4.1]; see also [3, Theorem 5.3.1] and [4, Theorem 2.1].

Proposition 2.2. *Assume that M is an open Riemann surface, $K \subset M$ is a compact smoothly bounded Runge domain, and $\{C_j : j \in I \subset \mathbb{N}\}$ is a collection of smoothly embedded oriented Jordan curves in M determining a homology basis of M such that*

- $\bigcup_{j \in J} C_j$ is a Runge compact set in M for every finite set $J \subset I$, and

- each curve C_j contains a nontrivial arc $\tilde{C}_j$ disjoint from C_i for all $i \in I \setminus \{j\}$.

Set $C = \bigcup_{j \in I} C_j$ and $I^K = \{j \in I : C_j \subset K\}$. Let $\sigma_a : [0, 1] \rightarrow K$ ($a = 1, \dots, \alpha \in \mathbb{N}$) be a finite collection of analytic Jordan arcs with pairwise disjoint graphs in $[0, 1] \times M$, $r \in \mathbb{N}$ an integer, $\Lambda \subset M$ a closed discrete subset, and $\lambda : \Lambda \rightarrow \mathbb{N}$ a map. Set $\Sigma = \bigcup_{a=1}^{\alpha} \sigma_a([0, 1]) \subset K$ and assume that $\Lambda \cap \Sigma = \emptyset = C \cap (\Lambda \cup \Sigma)$. Let $n \in \mathbb{N}$, let θ_t ($t \in [0, 1]$) be a continuous family of $\mathbb{C}^n$ -valued full abelian differentials on M with the polar set P_t (see Subsect. 2.2), set $P = \bigcup_{t \in [0, 1]} P_t$, and assume that $C \cap P = \emptyset$. Also let $\varphi_t \in \mathcal{O}(K \cup \Lambda)$ ($t \in [0, 1]$) be a continuous family of holomorphic functions with no zeros on a neighbourhood of $K \cup \Lambda$, and $\mathbf{q}_j : [0, 1] \rightarrow \mathbb{C}^n$ ($j \in I$) a collection of continuous maps such that

$$\int_{C_j} \varphi_t \theta_t = \mathbf{q}_j(t) \quad \text{for every } j \in I^K \text{ and } t \in [0, 1].$$

Then, the family φ_t may be approximated uniformly on $[0, 1] \times K$ by continuous families of holomorphic functions $\tilde{\varphi}_t : M \rightarrow \mathbb{C}^*$ ($t \in [0, 1]$) satisfying the following conditions:

- (a) $\int_{C_j} \tilde{\varphi}_t \theta_t = \mathbf{q}_j(t)$ for every $j \in I$ and $t \in [0, 1]$.
- (b) $\tilde{\varphi}_t - \varphi_t$ vanishes to order r at $\sigma_a(t)$ for every $a \in \{1, \dots, \alpha\}$ and $t \in [0, 1]$.
- (c) $\tilde{\varphi}_t - \varphi_t$ vanishes to order $\lambda(p)$ at p for every $p \in \Lambda$ and $t \in [0, 1]$.

Furthermore, if φ_0 extends to a holomorphic function $M \rightarrow \mathbb{C}^*$ such that $\int_{C_j} \varphi_0 \theta_0 = \mathbf{q}_j(0)$ for all $j \in I$, then the homotopy $\tilde{\varphi}_t$ can be chosen with $\tilde{\varphi}_0 = \varphi_0$.

The novelties with respect to [2, Theorem 4.1] are the interpolation conditions (b) and (c), and the fact that the abelian differentials θ_t are allowed to have poles in the complement of C . This proposition will be used for various tasks in the proofs of our main results. In particular, it will be applied in the proof of Theorem 1.2 to preserve the residues when moving the poles. We shall explain the necessary modifications of [2, proof of Theorem 4.1] which ensure these extra conditions. The same arguments apply word by word in the more general framework when K is a Runge admissible set (see [2, Def. 3.1]) and the multipliers φ_t are of class $\mathcal{A}(K)$; this generalisation is well understood and we shall not discuss it here. The key to the proof of Proposition 2.2 is the following extension of [2, Lemma 3.2], which will also play a crucial role in the proof of Theorem 1.2.

Lemma 2.3. *In Proposition 2.2, assume in addition that $I^K = \{1, \dots, l\}$ for some $l \in \mathbb{N}$, write $C^K = \bigcup_{j=1}^l C_j$, and for each $t \in [0, 1]$ let $\mathcal{P}^t = (\mathcal{P}_1^t, \dots, \mathcal{P}_l^t) : \mathcal{C}(C^K) \rightarrow (\mathbb{C}^n)^l$ denote the period map whose j -th component ($j = 1, \dots, l$) is given by*

$$(2.2) \quad \mathcal{P}_j^t(g) = \int_{C_j} g \varphi_t \theta_t, \quad g \in \mathcal{C}(C_j).$$

Then there are a convex neighbourhood $D \subset \mathbb{C}$ of $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$ and a nowhere vanishing holomorphic function $\Xi : D \times \mathbb{C}^N \times M \rightarrow \mathbb{C}^$ with $\Xi(\cdot, 0, \cdot) \equiv 1$ satisfying the following conditions:*

- (a) $\Xi(t, \zeta, \cdot) - 1$ vanishes to order r at $\sigma_a(t)$ for every $a \in \{1, \dots, \alpha\}$, $t \in [0, 1]$, and $\zeta \in \mathbb{C}^N$.
- (b) $\Xi(t, \zeta, \cdot) - 1$ vanishes to order $\lambda(p)$ at p for every $p \in \Lambda$, $t \in [0, 1]$, and $\zeta \in \mathbb{C}^N$.
- (c) *For every $t \in [0, 1]$ the map*

$$\mathbb{C}^N \ni \zeta \mapsto \mathcal{P}^t(\Xi(t, \zeta, \cdot)) \in (\mathbb{C}^n)^l$$

has maximal rank equal to ln at $\zeta = 0$.

Proof. Up to enlarging K slightly, we may assume that $\Sigma := \bigcup_{a=1}^{\alpha} \sigma_a([0, 1]) \subset \mathring{K}$. Using [2, Lemma 2.1] as in [2, proof of Lemma 3.2] (see also [6, proof of Proposition 3.1]) we obtain for each $j \in \{1, \dots, l\}$ an

integer $N_j \geq n$ and continuous functions $g_{j,k} : C_j \rightarrow \mathbb{C}$ ($k = 1, \dots, N_j$) with the support on the arc $\tilde{C}_j$ such that the function $h_j : \mathbb{C}^{N_j} \times C_j \rightarrow \mathbb{C}^*$ given by

$$h_j(\zeta_j, p) = \prod_{k=1}^{N_j} e^{\zeta_{j,k} g_{j,k}(p)}, \quad \zeta_j = (\zeta_{j,1}, \dots, \zeta_{j,N_j}) \in \mathbb{C}^{N_j}, \quad p \in C_j$$

satisfies the following period domination condition:

$$(2.3) \quad \frac{\partial}{\partial \zeta_j} \mathcal{P}_j^t(h_j(\zeta_j, \cdot)) \Big|_{\zeta_j=0} : T_0 \mathbb{C}^{N_j} \longrightarrow \mathbb{C}^n \text{ is surjective for every } t \in [0, 1].$$

Recall that $\Sigma \cap (C \cup \Lambda) = \emptyset$. Choose a small smoothly bounded convex neighbourhood $D \subset \mathbb{C}$ of $[0, 1]$ such that every analytic arc $\sigma_a : [0, 1] \rightarrow \mathring{K}$ ($a \in \{1, \dots, \alpha\}$) extends to a holomorphic map $\sigma_a : D \rightarrow \mathring{K}$, and set $\Sigma' = \bigcup_{a=1}^{\alpha} \sigma_a(D)$. Let $\delta_a(t) = (t, \sigma_a(t)) \in D \times M$ for $t \in D$. By choosing the domain $D \supset [0, 1]$ small enough, we have that $\overline{\Sigma'} \subset \mathring{K} \setminus (C \cup \Lambda)$ and $\{\delta_a(D) : a = 1, \dots, \alpha\}$ is a family of pairwise disjoint closed complex curves in the Stein surface $D \times M$. Set $\Delta = \bigcup_{a=1}^{\alpha} \delta_a(D) \subset D \times \Sigma'$. We extend each function $g_{j,k} : C_j \rightarrow \mathbb{C}$ ($j \in \{1, \dots, l\}$, $k \in \{1, \dots, N_j\}$) by 0 to $\Sigma' \cup \Lambda \cup (C^K \setminus C_j)$ and view it as a continuous map $g_{j,k} : D \times (\Sigma' \cup \Lambda \cup C^K) \rightarrow \mathbb{C}$ given by $g_{j,k}(t, \cdot) = g_{j,k}$ for all $t \in D$. Note that $g_{j,k}$ vanishes on $\Delta \cup (D \times \Lambda)$. It is clear that $g_{j,k}$ extends to a continuous function on $D \times M$ that vanishes on a neighbourhood of the divisor $\Delta \cup (D \times \Lambda)$. Since the compact set $[0, 1] \times C^K \subset D \times M$ is holomorphically convex and C^K is a union of curves, Mergelyan's theorem shows that we can approximate $g_{j,k}$ uniformly on $[0, 1] \times C^K$ by a holomorphic function on a neighbourhood of $[0, 1] \times C^K$ in $D \times M$, which we still denote $g_{j,k}$. Next, a standard recursive application of the Oka–Weil theorem with jet interpolation enables us to approximate $g_{j,k}$ uniformly on $[0, 1] \times C^K$ by a holomorphic function $\tilde{g}_{j,k} \in \mathcal{O}(D \times M)$ vanishing to any given order on each connected component of $\Delta \cup (D \times \Lambda)$. (These components are $\delta_a(D)$ ($a = 1, \dots, \alpha$) and $D \times \{p\}$ for $p \in \Lambda$.) In particular, $\tilde{g}_{j,k}$ can be chosen such that $\tilde{g}_{j,k}(t, \cdot)$, $t \in [0, 1]$, vanishes to order r at the point $\sigma_a(t)$ for all $a \in \{1, \dots, \alpha\}$, and it vanishes to order $\lambda(p)$ at every point $p \in \Lambda$. Set

$$\Xi(t, \zeta, p) = \prod_{j=1}^l \prod_{k=1}^{N_j} e^{\zeta_{j,k} \tilde{g}_{j,k}(t, p)}, \quad t \in D, \quad \zeta = (\zeta_1, \dots, \zeta_l) \in \mathbb{C}^{N_1} \times \dots \times \mathbb{C}^{N_l}, \quad p \in M.$$

Setting $N = \sum_{j=1}^l N_j \geq nl$ and identifying $\mathbb{C}^N = \mathbb{C}^{N_1} \times \dots \times \mathbb{C}^{N_l}$, it is clear that $\Xi : D \times \mathbb{C}^N \times M \rightarrow \mathbb{C}^*$ is holomorphic and satisfies $\Xi(\cdot, 0, \cdot) \equiv 1$ and conditions (a) and (b) in the lemma. Moreover, (c) is guaranteed by (2.3) whenever the approximation of each $g_{j,k}$ by $\tilde{g}_{j,k}$ on $[0, 1] \times C^K$ is close enough. $\square$

Proof of Proposition 2.2. Choose a normal exhaustion

$$K = K_0 \subset K_1 \subset K_2 \subset \dots \subset \bigcup_{i=0}^{\infty} K_i = M$$

by smoothly bounded Runge compact domains such that, setting $I^i = \{j \in I : C_j \subset K_i\}$ for $i = 0, 1, 2, \dots$ (note that I^i is finite and $I^i \subset I^{i+1}$ for every $i \geq 0$), the following conditions hold for every $i \in \mathbb{N}$:

- $I^i \setminus I^{i-1}$ is either empty or a singleton.
- The compact set $K_{i-1} \cup \bigcup_{j \in I^i} C_j$ is Runge in M and admissible in the sense of [2, Def. 3.1].

In order to ensure the latter condition for $i = 1$ we might need to replace $K = K_0$ by a slightly larger compact domain. Set $\varphi_t^0 = \varphi_t : K_0 \cup \Lambda \rightarrow \mathbb{C}^*$ for $t \in [0, 1]$. The proof consists of constructing a sequence of continuous families $\{\varphi_t^i \in \mathcal{O}(K_i \cup \Lambda)\}_{i \in \mathbb{N}}$ ($t \in [0, 1]$) of holomorphic functions without zeros on a neighbourhood of $K_i \cup \Lambda$ such that the following conditions hold for all $t \in [0, 1]$ and $i \in \mathbb{N}$:

- (A_i) φ_t^i is as close as desired to φ_t^{i-1} uniformly on $[0, 1] \times K_{i-1}$.
- (B_i) $\int_{C_j} \varphi_t^i \theta_t = q_j(t)$ holds for every $j \in I^i$.
- (C_i) $\varphi_t^i - \varphi_t$ vanishes to order r at the point $\sigma_a(t)$ for every $a \in \{1, \dots, \alpha\}$ and $t \in [0, 1]$.

(D_i) $\varphi_t^i - \varphi_t$ vanishes to order $\lambda(p)$ at every point $p \in \Lambda$.

(E_i) If φ_0 extends to a holomorphic function $M \rightarrow \mathbb{C}^*$ such that $\int_{C_j} \varphi_0 \theta_0 = q_j(0)$ for all $j \in I$, then the homotopy φ_t^i can be chosen with $\varphi_0^i = \varphi_0$.

As in the proof of [2, Theorem 4.1], if the approximation in (A_i) is close enough for every $i \in \mathbb{N}$, we obtain a limit continuous family of holomorphic functions $\tilde{\varphi}_t = \lim_{i \rightarrow \infty} \varphi_t^i : M \rightarrow \mathbb{C}^*$, $t \in [0, 1]$, satisfying Proposition 2.2. Conditions (b) and (c) are trivially guaranteed by (C_i) and (D_i).

We proceed by induction. The base is given by the family φ_t^0 ($t \in [0, 1]$). For the inductive step, we assume that we have a suitable family φ_t^{i-1} for some $i \in \mathbb{N}$ and will provide φ_t^i . We distinguish cases.

The noncritical case: $I^i = I^{i-1}$. Assume that $I^i = \{1, \dots, l \in \mathbb{N}\} \neq \emptyset$, for the proof is much simpler otherwise. Set $C^i = \bigcup_{j=1}^l C_j$ and for each $t \in [0, 1]$ consider the period map $\mathcal{P}^t : \mathcal{C}(C^i) \rightarrow (\mathbb{C}^n)^l$ defined by (2.2) with φ_t replaced by φ_t^{i-1} . By Lemma 2.3, there are a convex neighbourhood $D \subset \mathbb{C}$ of $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$ and a nowhere vanishing holomorphic function $\Xi : D \times \mathbb{C}^N \times M \rightarrow \mathbb{C}^*$ such that $\Xi(\cdot, 0, \cdot) \equiv 1$ and the following conditions hold for every $t \in [0, 1]$.

- (I) $\Xi(t, \zeta, \cdot) - 1$ vanishes to order r at $\sigma_a(t)$ for every $a \in \{1, \dots, \alpha\}$ and $\zeta \in \mathbb{C}^N$.
- (II) $\Xi(t, \zeta, \cdot) - 1$ vanishes to order $\lambda(p)$ at p for every $p \in \Lambda$ and $\zeta \in \mathbb{C}^N$.
- (III) The map $\mathbb{C}^N \ni \zeta \mapsto \mathcal{P}^t(\Xi(t, \zeta, \cdot)) \in (\mathbb{C}^n)^l$ has maximal rank equal to ln at $\zeta = 0$.

Taking into account conditions (B_{i-1})–(E_{i-1}) and using a similar argument as in the proof of Lemma 2.3 to ensure parametric interpolation, we find a continuous family of holomorphic functions $\phi_t : K_i \cup \Lambda \rightarrow \mathbb{C}^*$, $t \in [0, 1]$, on a neighbourhood of $K_i \cup \Lambda$ satisfying the following conditions:

- (i) ϕ_t is as close as desired to φ_t^{i-1} uniformly on $[0, 1] \times K_{i-1}$.
- (ii) $\phi_t - \varphi_t$ vanishes to order r at $\sigma_a(t)$ for every $a \in \{1, \dots, \alpha\}$ and $t \in [0, 1]$.
- (iii) $\phi_t - \varphi_t$ vanishes to order $\lambda(p)$ at p for every $p \in \Lambda$ and $t \in [0, 1]$.
- (iv) If φ_0 extends to a holomorphic function $M \rightarrow \mathbb{C}^*$ such that $\int_{C_j} \varphi_0 \theta_0 = q_j(0)$ for all $j \in I$, then the homotopy ϕ_t can be chosen such that $\phi_0 = \varphi_0$.

In view of condition (III), if the approximation in (i) is close enough then, arguing as in [2, proof of Lemma 4.2], the implicit function theorem furnishes a continuous path $\beta : [0, 1] \rightarrow \mathbb{C}^N$ such that $\Xi(t, \beta(t), \cdot)$ is close to 1 uniformly on K_{i-1} for all $t \in [0, 1]$ and the continuous family of holomorphic functions

$$\varphi_t^i := \Xi(t, \beta(t), \cdot) \phi_t : K_i \cup \Lambda \rightarrow \mathbb{C}^*, \quad t \in [0, 1]$$

satisfies conditions (A_i)–(E_i); in particular, in the assumptions in (iv) we can choose β with $\beta(0) = 0$. Note that (C_i) is ensured by (C_{i-1}), (I), and (ii), while (D_i) is guaranteed by (D_{i-1}), (II), and (iii).

The critical case: $I^i \neq I^{i-1}$. In this case $I^i \setminus I^{i-1} = \{j\} \subset I$. Taking into account that $C_j \cap (\Lambda \cup \Sigma \cup P) = \emptyset$ and $K_{i-1} \cup C_j$ is an admissible Runge compact set in M , the construction is reduced to the noncritical case by using Lemma 2.3 and [2, Lemma 2.3]. The details are similar to [2, proof of Lemma 4.3] and we leave them out. This completes the proof of Proposition 2.2. $\square$

3. REMOVING BRANCH POINTS

In this section we establish the following extension of Theorem 1.1 which says that we can move the branch points out of a minimal surface while keeping the poles fixed. This result and Theorem 1.2 (on moving the poles) trivially imply Corollary 1.3. We shall use the notation in (1.5) and (1.7).

Theorem 3.1. *Let M be an open Riemann surface and $n \geq 3$ an integer. For every $u \in \text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$ there is an isotopy $u_t \in \text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$, $t \in [0, 1]$, such that $u_0 = u$, $u_t - u$ is continuous on M (hence $\mathcal{E}_{u_t}^\infty = \mathcal{E}_u^\infty$) for all $t \in [0, 1]$, and $u_1 : M \setminus \mathcal{E}_{u_1}^\infty \rightarrow \mathbb{R}^n$ is unbranched, hence $u_1 \in \text{CMI}_s(M, \mathbb{R}^n)$. Furthermore, we can choose the isotopy such that for each $t \in [0, 1]$ the Gauss map $\mathcal{G}(u_t)$ of u_t equals $\mathcal{G}(u)$ in their common domain of definition $M \setminus (\text{br}(u_t) \cup \text{br}(u) \cup \mathcal{E}_u^\infty)$.*

The theorem says in particular that the inclusion $\text{CMI}_s(M, \mathbb{R}^n) \hookrightarrow \text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$ induces a surjection of path components. In the proof, we shall need the following result. The notion of an abelian differential with values in a complex cone $A \subset \mathbb{C}^n$ was introduced in Subsect. 2.2.

Proposition 3.2. *Assume that M is a connected open Riemann surface, $A \subset \mathbb{C}^n$ is a closed complex cone of positive dimension, and ω is a nontrivial abelian differential on M with values in A . Then there is a path of nontrivial abelian differentials $\omega_t = h_t \omega$ ($t \in [0, 1]$) with values in A , where h_t is a path of meromorphic functions on M , satisfying $\omega_0 = \omega$ (that is, $h_0 \equiv 1$) and either of the following conditions:*

- (a) $\omega_1 = h_1 \omega$ is a holomorphic 1-form on M without zeros.
- (b) h_t has no zeros for every $t \in [0, 1]$, $\omega_t - \omega = (h_t - 1)\omega$ is holomorphic on M for every $t \in [0, 1]$, and $\omega_1 = h_1 \omega$ is an abelian differential on M without zeros.

Proof. Write $(\omega) = \sum_{p \in M} k(p)p = (\omega)_0 - (\omega)_\infty$ (see Subsect. 2.2).

We first explain how to obtain a path ω_t ($t \in [0, 1]$) as in the statement satisfying condition (a). Choose a normal exhaustion $K_0 \subset K_1 \subset \dots \subset \bigcup_{i=0}^\infty K_i = M$ by compact Runge sets such that $\text{supp}(\omega) \cap K_0 = \emptyset$. Let $\text{supp}(\omega) \cap K_1 = \{p_1, \dots, p_m\}$ and set $n_j = k(p_j)$ for $j = 1, \dots, m$. For every $j = 1, \dots, m$ we choose a real analytic path $a_j : [0, 1] \rightarrow M \setminus K_0$ such that $a_j(0) = p_j$ and $a_j(1) \in M \setminus K_1$. Pick a number $\epsilon_1 > 0$. Lemma 2.1 furnishes a path $\{f_t^1\}_{t \in [0, 1]}$ of meromorphic functions on M with divisors

$$(3.1) \quad (f_t^1) = \sum_{j=1}^m n_j a_j(t), \quad t \in [0, 1],$$

such that

$$(3.2) \quad \max_{x \in K_0, t \in [0, 1]} |f_t^1(x) - 1| < \epsilon_1.$$

Note that $a_j(0) = p_j$, so the divisor $(f_0^1) = \sum_{j=1}^m n_j p_j$ is precisely the part of the divisor (ω) lying in K_1 . Hence, the 1-form $\omega' = \frac{1}{f_0^1} \omega$ has no zeros or poles on K_1 . Consider the path of abelian differentials

$$(3.3) \quad \omega_t = \frac{f_t^1}{f_0^1} \omega = f_t^1 \omega', \quad t \in [0, 1].$$

We have that $\omega_0 = \omega$, $\omega_1 = f_1^1 \omega'$, $\text{supp}(\omega_t) \cap K_0 = \emptyset$ for all $t \in [0, 1]$, $\text{supp}(\omega_1) \cap K_1 = \emptyset$, and ω_t approximates ω_0 on K_0 for all $t \in [0, 1]$. We now repeat the same procedure with the abelian differential ω_1 in order to find a path of meromorphic functions $\{f_t^2\}_{t \in [1, 2]}$ on M such that the divisor (f_1^2) agrees with the part of the divisor (ω_1) on K_2 , $\text{supp}(f_t^2) \cap K_1 = \emptyset$ for all $t \in [1, 2]$, $\text{supp}(f_2^2) \subset M \setminus K_2$, and

$$\max_{x \in K_1, t \in [1, 2]} |f_t^2(x) - 1| < \epsilon_2$$

for a given $\epsilon_2 > 0$. Set

$$\omega_t = \frac{f_t^2}{f_1^2} \omega_1, \quad t \in [1, 2].$$

Then, $\text{supp}(\omega_t) \cap K_1 = \emptyset$ for all $t \in [1, 2]$, $\text{supp}(\omega_2) \cap K_2 = \emptyset$, and ω_t approximates ω_1 uniformly on K_1 for all $t \in [1, 2]$. Continuing inductively, we obtain a path of abelian differentials ω_t , $t \in [0, \infty)$, such that $\text{supp}(\omega_t) \cap K_j = \emptyset$ for all $t \geq j$ and $j = 0, 1, \dots$. Choosing $\epsilon_j > 0$ small enough at every step, the approximation conditions ensure that $\omega_\infty = \lim_{t \rightarrow \infty} \omega_t$ is an abelian differential without zeros or poles on M . It remains to reparametrise the interval $[0, \infty)$ to $[0, 1]$. This explains part (a) of the proposition.

We now explain how to modify the above argument to obtain a path of abelian differentials ω_t , $t \in [0, 1]$, satisfying condition (b). Set $\Lambda = \text{supp}(\omega)_\infty$ and write $(\omega)_\infty = \sum_{p \in \Lambda} \lambda(p)p$. Choose $K_0 \subset K_1 \subset \dots$ as above, let $\text{supp}(\omega)_0 \cap K_1 = \{p_1, \dots, p_m\}$ and set $n_j = k(p_j) > 0$ for $j = 1, \dots, m$. Choose analytic paths

$a_j : [0, 1] \rightarrow M \setminus K_0$ ($j = 1, \dots, m$) such that $a_j(0) = p_j$, $a_j(1) \in M \setminus K_1$, and $\Lambda \cap \bigcup_{j=1}^m a_j([0, 1]) = \emptyset$. Lemma 2.1 provides a path $\{f_t^1\}_{t \in [0, 1]}$ of holomorphic functions on M satisfying (3.1), (3.2), and

$$(3.4) \quad f_t^1 - 1 \text{ vanishes to order } \lambda(p) \text{ at } p \text{ for every } p \in \Lambda.$$

Note that $(f_0^1) = \sum_{j=1}^m n_j p_j$ is the part of $(\omega)_0$ lying in K_1 , $\omega' = \frac{1}{f_0^1} \omega$ has no zeros on K_1 , and the path of abelian differentials $\omega_t = f_t^1 \omega'$ ($t \in [0, 1]$) defined as in (3.3) satisfies $\omega_0 = \omega$, $\text{supp}(\omega_t)_0 \cap K_0 = \emptyset$ for all $t \in [0, 1]$, $\text{supp}(\omega_1)_0 \cap K_1 = \emptyset$, ω_t approximates ω_0 on K_0 for all $t \in [0, 1]$, and the difference $\omega_t - \omega$ is holomorphic for all $t \in [0, 1]$ as guaranteed by (3.4). Repeating the same procedure in a recursive way as above leads to a path of abelian differentials ω_t ($t \in [0, \infty]$) that, after reparametrising $[0, \infty]$ to $[0, 1]$, satisfies the required properties. $\square$

Proof of Theorem 3.1. Let $u_0 \in \text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$. Then, $\omega_0 = 2\partial u_0$ is an abelian differential on M with values in the null quadric $\mathbf{A}$ (1.2) whose divisor satisfies $\text{supp}(\omega_0) = \mathcal{E}_{u_0}^0 \cup \mathcal{E}_{u_0}^\infty = \text{br}(u_0) \cup P(u_0)$ (see (1.7) and Subsect. 2.2 for the notation). Let $\{\omega_t\}_{t \in [0, 1]}$ be a path of nontrivial abelian differentials on M with values in $\mathbf{A}$, provided by Proposition 3.2 (b), so $\omega_t = h_t \omega_0$ for some meromorphic function h_t on M and $\omega_t - \omega_0$ is holomorphic for all $t \in [0, 1]$, and ω_1 has no zeros. In particular, $(\omega_t)_\infty = (\omega_0)_\infty$ for all $t \in [0, 1]$. Let $\Lambda = \text{supp}(\omega_0)_\infty$ and write $(\omega_0)_\infty = \sum_{p \in \Lambda} \lambda(p)p$. Let $\{C_j : j \in I \subset \mathbb{N}\}$ be a collection of smoothly embedded oriented Jordan curves in M determining a homology basis of M such that $\bigcup_{j \in J} C_j$ is Runge in M for every finite set $J \subset I$, each curve C_j contains a nontrivial arc $\tilde{C}_j$ disjoint from C_i for all $i \in I \setminus \{j\}$, and $C_j \cap \Lambda = \emptyset$ for all $j \in I$. Since the real part $\Re(\omega_0)$ of ω_0 is exact on M , we have that $\Re \int_{C_j} \omega_0 = 0$ for every $j \in I$. Proposition 2.2 then furnishes a path $\{\xi_t\}_{t \in [0, 1]} \subset \mathcal{O}^*(M)$ of nowhere vanishing holomorphic functions on M , with $\xi_0 = 1$, such that

$$(3.5) \quad \xi_t - 1 \text{ vanishes to order } \lambda(p) \text{ at } p \text{ for every } p \in \Lambda \text{ and } t \in [0, 1],$$

and

$$(3.6) \quad \Re \int_{C_j} \xi_t \omega_t = 0 \text{ for every } j \in I \text{ and } t \in [0, 1].$$

Since $(\omega_t)_\infty = \sum_{p \in \Lambda} \lambda(p)p$, condition (3.5) ensures that $\xi_t \omega_t - \omega_t$ is holomorphic on M , and hence so is $\xi_t \omega_t - \omega_0$ for every $t \in [0, 1]$ (recall that $\omega_t - \omega_0$ is holomorphic on M). Thus, taking into account that the curves C_j ($j \in I$) are a homology basis of M and $\Re(\omega_0)$ is exact on M , (3.6) implies that $\Re(\xi_t \omega_t)$ is exact on M as well for every $t \in [0, 1]$. Therefore, the real parts of the abelian differentials $\xi_t \omega_t$ integrate by the Weierstrass formula (1.3) to a path of conformal minimal surfaces $u_t \in \text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$ ($t \in [0, 1]$) such that u_0 is the given initial map and for every $t \in [0, 1]$ we have $\text{br}(u_t) = \text{supp}(\xi_t \omega_t)_0 = \text{supp}(\omega_t)_0$, $P(u_t) = \text{supp}(\xi_t \omega_t)_\infty = \text{supp}(\omega_t)_\infty = P(u_0)$, and $u_t - u$ is continuous on M ; recall that each ξ_t has neither zeros nor poles. In particular, $\text{br}(u_1) = \text{supp}(\omega_1)_0 = \emptyset$, and hence $u_1 \in \text{CMI}_s(M, \mathbb{R}^n)$. Finally, since $2\partial u_t = \xi_t \omega_t = 2\xi_t h_t \partial u_0$, it is clear that the Gauss map $\mathcal{G}(u_t)$ of u_t equals $\mathcal{G}(u_0)$ on $M \setminus (\text{br}(u_t) \cup \text{br}(u_0) \cup P(u_0))$ for every $t \in [0, 1]$; see (1.5). $\square$

Remark 3.3. A comment is in order regarding the use of Proposition 2.2 in the proof of Theorem 3.1. Proposition 2.2 is stated for a path of full abelian differentials θ_t on M . In our situation, ω_t takes values in the null quadric $\mathbf{A}$ (1.2) and is not assumed to be full. However, there is a $\mathbb{C}$ -linear subspace $H \subset \mathbb{C}^n$ such that ω_0 is full in H (meaning that $\omega_0/\theta : M \rightarrow H$ is full for any nowhere vanishing holomorphic 1-form θ on M), and the same is then true for every $\omega_t = h_t \omega_0$ ($t \in [0, 1]$) in the path given by the proposition. Applying Proposition 2.2 to the path ω_t with values in H gives a family of multipliers $\{\xi_t\}_{t \in [0, 1]} \subset \mathcal{O}^*(M)$ with the properties stated in the proof of Theorem 3.1.

Let us record here the following extension of Theorem 3.1 which follows by a straightforward modification of the proof.

Corollary 3.4. *If M , n , and u are as in Theorem 3.1 and $C \subset M \setminus \mathcal{E}_u^\infty$ is a (possibly empty) closed discrete subset of M that is in bijection with a subset of $\text{br}(u)$, then there is an isotopy $u_t \in \text{CMI}_{\text{br},s}(M, \mathbb{R}^n)$, $t \in [0, 1]$, such that $u_0 = u$, $u_t - u$ is continuous on M for all $t \in [0, 1]$, and $\text{br}(u_1) = C$. Furthermore, we can choose the isotopy such that for each $t \in [0, 1]$ the Gauss map $\mathcal{G}(u_t)$ of u_t equals $\mathcal{G}(u)$ in their common domain of definition $M \setminus (\text{br}(u_t) \cup \text{br}(u) \cup \mathcal{E}_u^\infty)$.*

4. REMOVING COMPLETE ENDS OF FINITE TOTAL CURVATURE

In this section, we prove Theorem 1.2. In view of Theorem 3.1, this will also yield Corollary 1.3.

Let M be an open Riemann surface and $u \in \text{CMI}_s^f(M, \mathbb{R}^n)$ for some $n \geq 3$. Denote by $P = P(u) = \{p_j\}_j \subset M$ the closed discrete set of poles of the abelian differential ∂u . We shall construct a path $P_t = \{p_j(t)\}_j \subset M$ ($t \in [0, 1]$) of closed discrete subsets such that the graphs of the paths $p_j(t)$ in $[0, 1] \times M$ are pairwise disjoint, $p_j(0) = p_j$ and $p_j(t)$ diverges to infinity in M as $t \rightarrow 1$ for every j , and an isotopy $u_t : M \setminus P_t \rightarrow \mathbb{R}^n$ of conformal minimal immersions with a complete end of finite total curvature at every point of P_t , $t \in [0, 1]$ (that is, $P(u_t) = P_t$) such that the limit $u_1 = \lim_{t \rightarrow 1} u_t : M \rightarrow \mathbb{R}^n$ exists and is a conformal minimal immersion without singularities.

Fix a holomorphic immersion $z : M \rightarrow \mathbb{C}$, which therefore provides a local holomorphic coordinate on M on a neighbourhood of any point. Choose a normal exhaustion $K_0 \subset K_1 \subset K_2 \subset \dots$ of M by smoothly bounded compact Runge sets, each contained in the interior of the next one, such that $P \cap K_0 = \emptyset$. We shall proceed inductively, using the parameter interval $[i, i+1] \subset \mathbb{R}$ in the i -th step of the induction and finally reparametrising $[0, +\infty]$ to $[0, 1]$ as in the proof of Proposition 3.2.

We begin by explaining the initial step of the construction with $i = 0$; every subsequent step will be of the same kind. Choose real analytic paths $p_j(t) \in M \setminus K_0$, $t \in [0, 1]$, with pairwise disjoint graphs in $[0, 1] \times M$ such that $p_j(0) = p_j$ and $p_j(1) \in M \setminus K_1$ for all $j = 1, 2, \dots$. The path $p_j(t)$ is chosen to be independent of t if $p_j \in M \setminus K_1$, which holds for all but finitely many j . Let $\gamma_j(t) = (t, p_j(t))$ for $t \in [0, 1]$. There is a convex neighbourhood $D \subset \mathbb{C}$ of $[0, 1]$ such that every p_j extends from $[0, 1]$ to a holomorphic map $p_j : D \rightarrow M$, and $\Gamma_j = \gamma_j(D)$ is a family of pairwise disjoint closed complex curves (graphs of p_j over D) in the Stein surface $D \times M$. Let V_j denote the holomorphic vector field on $D \times M$ given by

$$V_j = \frac{\partial}{\partial t} + \dot{p}_j(t) \frac{\partial}{\partial z} \Big|_x,$$

where t is the coordinate on $\mathbb{C}$ and $z : M \rightarrow \mathbb{C}$ is the holomorphic immersion. Note that V_j is tangent to the complex curve Γ_j . Let $\phi_{j,t}(x) = (t, \varphi_{j,t}(x))$ denote the holomorphic flow of V_j satisfying the initial condition $\varphi_{j,0}(x) = x$. Shrinking D around $[0, 1]$ if necessary, the flow is defined for all $t \in D$ and all x in a disc neighbourhood $U_j \subset M$ of the point $p_j = p_j(0)$ for every j , and the map

$$\varphi_{j,t} : U_j \rightarrow U_{j,t} := \varphi_{j,t}(U_j) \subset M$$

is biholomorphic and satisfies $\varphi_{j,t}(p_j) = p_j(t)$ for every $t \in D$ and every j . Note that for all but finitely many j we have $V_j = \frac{\partial}{\partial t}$ and hence $\varphi_{j,t}$ is the identity on U_j for all $t \in D$. Choosing the discs U_j small enough, we may assume that the sets $\{U_{j,t}\}_j$ are pairwise disjoint for all $t \in D$, and they are also disjoint from a neighbourhood $W_0 \subset M$ of K_0 . Write $2\partial u_0 = f_0\theta$. Consider the path of abelian differentials ω_t on the domains $\tilde{U}_t := W_0 \cup \bigcup_j U_{j,t} \subset M$, $t \in D$, defined by

$$(4.1) \quad \omega_t = \begin{cases} (\varphi_{j,t}^{-1})^*(f_0\theta) & \text{on } U_{j,t}, \\ f_0\theta & \text{on } W_0. \end{cases}$$

Note that ω_t is full, it assumes values in the null quadric $\mathbf{A}_*$ (1.2) for every $t \in D$, it has a pole of order $n_j \geq 2$ at the point $p_j(t)$ for every j (with n_j independent of t), and it has no other zeros or poles. Furthermore, the residue of ω_t at $p_j(t)$ is independent of $t \in D$, so it has vanishing real part (as this holds for $\omega_0 = 2\partial u_0$). Write $\omega_t = f_t\theta$ for $t \in D$. Then, $f_t : \tilde{U}_t \cup W_0 \rightarrow Y = \mathbf{A}_* \cup Q$ is a holomorphic

map depending holomorphically on $t \in D$, and $f_0 = 2\partial u_0/\theta$ is holomorphic on all of M . (Recall that $Q \subset \mathbb{CP}^{n-1}$ is the hyperquadric (1.4).) Writing $f(t, x) = f_t(x)$, the map f with values in Y is holomorphic on the open set

$$O := (D \times W_0) \cup \bigcup_{t \in D} (\{t\} \times \tilde{U}_t) \subset D \times M$$

and on the complex submanifold $\{0\} \times M$. Note that $f^{-1}(Q) = \Gamma := \bigcup_j \Gamma_j$ is a complex submanifold contained in O . It is obvious that f extends from a somewhat smaller open set containing $(D \times K_0) \cup \Gamma$ to a continuous map $f : D \times M \rightarrow Y$ which agrees with f_0 on $\{0\} \times M$ and maps $(D \times M) \setminus \Gamma$ to $\mathbf{A}_*$.

Let $\{C_l : l \in I \subset \mathbb{N}\}$ be a collection of smooth oriented Jordan curves in M determining a homology basis of M such that $\bigcup_{l \in J} C_l$ is a Runge compact set in M for every finite set $J \subset I$, and each curve C_l contains a nontrivial arc $\tilde{C}_l$ that is disjoint from C_i for all $i \in I \setminus \{l\}$. In addition, we choose these curves such that $C \cap p_j([0, 1]) = \emptyset$ for every $j = 1, 2, \dots$ where $C = \bigcup_{l \in I} C_l$, and $C_1, \dots, C_\ell$ determine a homology basis of K_0 . We assume that $\ell > 0$ since the proof is much simpler otherwise. Let $\mathcal{P}^t = (\mathcal{P}_1^t, \dots, \mathcal{P}_\ell^t) : \mathcal{C}(C^0) \rightarrow (\mathbb{C}^n)^\ell$ be the period map whose l -th component is given by

$$\mathcal{P}_l^t(g) = \int_{C_l} g \omega_t, \quad g \in \mathcal{C}(C_l).$$

Up to shrinking the domain $D \supset [0, 1]$, Lemma 2.3 furnishes a nowhere vanishing holomorphic function $\Xi : D \times \mathbb{C}^N \times M \rightarrow \mathbb{C}^*$ satisfying the following conditions:

- (I) $\Xi(t, \zeta, \cdot) - 1$ vanishes to order n_j at $p_j(t)$ for every $t \in [0, 1]$, $\zeta \in \mathbb{C}^N$, and $j = 1, 2, \dots$
- (II) The map $\mathbb{C}^N \ni \zeta \mapsto \mathcal{P}^t(\Xi(t, \zeta, \cdot)) \in (\mathbb{C}^n)^\ell$ has maximal rank at $\zeta = 0$ for each $t \in [0, 1]$.
- (III) $\Xi(\cdot, 0, \cdot) \equiv 1$.

Recall that the manifolds $Q \subset \mathbb{CP}^{n-1}$ (1.4), $Y = \mathbf{A}_* \cup Q \subset \mathbb{CP}^n$ (1.6), and $\mathbf{A}_* = Y \setminus Q$ are Oka manifolds. Indeed, $\mathbf{A}_*$ is a homogeneous space of the complex Lie group $O(n, \mathbb{C})$, and hence an Oka manifold by Grauert's theorem [14]. (See also [11, Proposition 5.6.1 and Example 5.6.2].) The projection $\pi : \mathbf{A}_* \rightarrow Q$, $\pi(z_1, \dots, z_n) = [z_1 : \dots : z_n]$, is a holomorphic fibre bundle with Oka fibre $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, so Q is Oka by [11, Theorem 5.6.5]. Finally, $\pi : Y \rightarrow Q$ is a holomorphic line bundle, so Y is Oka by the same theorem.

Recall that $n_j \geq 2$ denotes the order of the pole of $f_0 = 2\partial u_0/\theta$ at the point p_j for $j = 1, 2, \dots$. By the Oka principle (see [11, Theorem 5.4.4]), there is a holomorphic map $F : D \times M \rightarrow Y$ which agrees with f to order n_j along Γ_j for every j , it agrees with $f(0, \cdot)$ on $\{0\} \times M$, and it approximates f as closely as desired uniformly on $[0, 1] \times K_0$. Moreover, after shrinking D around $[0, 1]$ we can ensure that $F^{-1}(Q) = f^{-1}(Q) = \Gamma$, that is, F maps $(D \times M) \setminus \Gamma$ to the Oka domain $\mathbf{A}_* = Y \setminus Q \subset Y$. This can be obtained by inductively using [10, Theorem 1.3]. Then, $\theta_t = F(t, \cdot)\theta$ ($t \in [0, 1]$) is an analytic path of abelian differentials on M with values in $\mathbf{A}_*$ satisfying the following conditions.

- (i) $\theta_0 = \omega_0 = 2\partial u_0$.
- (ii) $\theta_t - \omega_t$ is holomorphic near $p_j(t)$ for every $t \in [0, 1]$ and $j = 1, 2, \dots$
- (iii) θ_t approximates ω_t uniformly on K_0 and uniformly in $t \in [0, 1]$.
- (iv) θ_t has no zeros on M for any $t \in [0, 1]$ and its polar locus is $P_t = \{p_j(t)\}_j$.

Assuming that the approximation of f by F is close enough, every θ_t is full, and the implicit function theorem provides in view of conditions (II), (III), and (4.1) a path $\beta : [0, 1] \rightarrow \mathbb{C}^N$ such that $\beta(0) = 0$, $g_t := \Xi(t, \beta(t), \cdot)$ is uniformly close to 1 on K_0 for all $t \in [0, 1]$, and the continuous family of abelian differential $g_t \theta_t$ ($t \in [0, 1]$) satisfies

$$\Re \int_{C_l} g_t \theta_t = \Re(\mathcal{P}_l^t(1)) = \Re \int_{C_l} \omega_t = \Re \int_{C_l} f_0 \theta = 0, \quad l = 1, \dots, \ell.$$

(Cf. [6, proof of Proposition 3.1]. Note that fullness of ω_t ($t \in [0, 1]$) has been used here in an important way to kill the real periods, which is seen in condition (II) above.) Thus, taking into account conditions (I) and (III) and replacing θ_t by $g_t\theta_t$, we may assume in addition to conditions (i)–(iv) above that

$$(4.2) \quad \Re \int_{C_l} \theta_t = 0 \quad \text{for every } l \in \{1, \dots, \ell\} \text{ and } t \in [0, 1].$$

(Note that $g_0 \equiv 1$.) Proposition 2.2 then furnishes a path of holomorphic functions $h_t : M \rightarrow \mathbb{C}^*$ ($t \in [0, 1]$) such that $h_0 \equiv 1$ and the following conditions hold for every $t \in [0, 1]$.

- (a) $\Re \int_{C_l} h_t \theta_t = 0$ for every $l \in I$; take into account (4.2) and that $\Re \int_{C_l} \theta_0 = 2\Re \int_{C_l} \partial u_0 = 0$ for all $l \in I$.
- (b) $h_t - 1$ vanishes to order n_j at the point $p_j(t)$ for every $j = 1, 2, \dots$
- (c) h_t is uniformly close to 1 on K_0 .

Condition (b) implies that the full abelian differential $h_t\theta_t$ has the same residue as θ_t at the point $p_j(t)$ for every j and t . By the construction, this agrees with the residue of $\theta_0 = 2\partial u_0$ at $p_j = p_j(0)$, so its real part vanishes. Hence, condition (a) implies that $\Re(h_t\theta_t)$ is exact on M . Therefore, taking also (c) into account and replacing θ_t by $h_t\theta_t$, we may assume in addition to conditions (i)–(iv) above that $\Re(\theta_t)$ is exact on M for every $t \in [0, 1]$. It follows that the path of abelian differentials θ_t integrates by the Weierstrass formula (1.3) to a path of maps $u_t \in \text{CMI}_s^f(M, \mathbb{R}^n)$ with $2\partial u_t = \theta_t$ for every $t \in [0, 1]$, so $u_0 = u$ (see (i)), $P(u_t) = P_t$ (see (iv)), and u_t approximates u_0 on K_0 (see (iii) and (4.1)) for every $t \in [0, 1]$. In particular, the minimal surface $u_1 \in \text{CMI}_s^f(M, \mathbb{R}^n)$ has no singularities on K_1 . This completes the first step of the construction.

In the second step, we apply the same procedure, starting with u_1 and finding a path $\{u_t\}_{t \in [1, 2]} \subset \text{CMI}_s^f(M, \mathbb{R}^n)$ such that for all $t \in [1, 2]$ we have $P(u_t) \subset M \setminus K_1$, u_t approximates u_1 on K_1 , and u_2 is nonsingular on K_2 . Clearly, the induction may be continued so that the limit $u_\infty = \lim_{t \rightarrow \infty} u_t$ exists uniformly on compacts in M and $u_\infty \in \text{CMI}(M, \mathbb{R}^n)$. Since all maps $u_t \in \text{CMI}_s^f(M, \mathbb{R}^n)$ in the construction approximate u_0 on K_0 , which is full, we can ensure that they all are full as well.

This completes the proof of Theorem 1.2.

The following extension of this result, concerning moving complete ends of finite total curvature within the surface, follows by a straightforward modification of the proof. Recall the notation in (1.7).

Corollary 4.1. *If M , n , and u are as in Theorem 1.2 and $C \subset M$ is a (possibly empty) closed discrete subset that is in bijection with a subset of $\mathcal{E}_u^\infty$, then there is an isotopy $u_t \in \text{CMI}_s^f(M, \mathbb{R}^n)$, $t \in [0, 1]$, such that $u_0 = u$ and $\mathcal{E}_{u_1}^\infty = C$.*

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