THE NONHOMOGENEOUS CAUCHY–RIEMANN EQUATION ON FAMILIES OF OPEN RIEMANN SURFACES

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ABSTRACT. In this paper we give an optimal solution to the $\overline\partial$-equation for continuous or smooth families of complex structures and $(0,1)$-forms of a Hölder class on a smooth open orientable surface. As an application, we obtain the Oka–Grauert principle for complex line bundles on such families.

1. THE $\overline\partial$-EQUATION ON A FAMILY OF OPEN RIEMANN SURFACES

In this paper, we show that the nonhomogeneous Cauchy–Riemann equation, or the $\overline\partial$-equation for short, can be solved for very general families of complex structures and $(0,1)$-forms of a Hölder class on a smooth open orientable surface, with the usual gain of one derivative in the space variable and without loss of regularity in the parameter. (The same approach can be carried out in Sobolev spaces.) Our main results are Theorem 1.1 and its global version, Corollary 1.2. The proof uses nonhomogeneous Beltrami equation on smoothly bounded relatively compact domains in open Riemann surfaces, together with the Runge approximation theorem for families of holomorphic functions on families of open Riemann surfaces (see [9, Theorem 1.1]). An application is a Dolbeault cohomology vanishing theorem (see Proposition 1.5) and the classification of holomorphic line bundles on such families (see Theorem 2.1).

We begin by introducing the setup, referring to [9] for more details. Let X be a smooth orientable surface. A complex structure on X is an endomorphism J of its tangent bundle TX satisfying $J^2=-\mathrm{Id}$. Thus, J is a section of the smooth vector bundle $T^*X\otimes TX\to X$ whose fibre over $x\in X$ is the space $\mathrm{Hom}(T_xX,T_xX)$ of linear maps $T_xX\mapsto T_xX$. We endow the plane $\mathbb{R}^2\cong\mathbb{C}$ with the standard complex structure $J_{\rm st}$ given in standard basis by the matrix $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and in complex notation by multiplication by $\mathfrak{i}=\sqrt{-1}$. A differentiable function $f:U\to\mathbb{C}$ on an open set $U\subset X$ is said to be *J-holomorphic* if the Cauchy–Riemann equation $df_x\circ J_x=\mathfrak{i}\,df_x$ holds for every $x\in U$. We say that J is of (local) Hölder class $\mathscr{C}^{(k,\alpha)}$ for some $k\in\mathbb{Z}_+=\{0,1,2,\ldots\}$ and $0<\alpha<1$ if for any relatively compact domain $\Omega\Subset X$, the restriction $J|_\Omega\in\Gamma^{(k,\alpha)}(\Omega,T^*\Omega\otimes T\Omega)$ is a section of $T^*\Omega\otimes T\Omega$ of class $\mathscr{C}^{(k,\alpha)}(\Omega)$. (Hölder norms are defined in the usual way with respect to a smooth Riemannian metric on X; see [10, Sect. 4.1]. If Ω has $\mathscr{C}^1$ boundary, as will be the case in our results, then a function $f\in\mathscr{C}^{(k,\alpha)}(\Omega)$ extends to a unique function in $\mathscr{C}^{(k,\alpha)}(\overline\Omega)$. The same holds for complex structures.) For such J, there is an atlas $\{(U_i,\phi_i)\}_i$ of open sets $U_i\subset X$ with $\bigcup_i U_i=X$ and J-holomorphic charts $\phi_i:U_i\to\phi_i(U_i)\subset\mathbb{C}$ of class $\mathscr{C}^{(k+1,\alpha)}(U_i)$; see [2, Theorem 5.3.4]. Since the transition maps $\phi_i\circ\phi_j^{-1}$ are $J_{\rm st}$-biholomorphic, J determines the structure of a Riemann surface on X, and every J-holomorphic function is of local class $\mathscr{C}^{(k+1,\alpha)}$ in the given smooth structure on X. Since the inverse of a diffeomorphism of local class $\mathscr{C}^{(k+1,\alpha)}$ is again of the same class (see Norton [25] and Bojarski et al. [4, Theorem 2.1]), the smooth structure on X determined by a complex structure J of class $\mathscr{C}^{(k,\alpha)}$ is $\mathscr{C}^{(k+1,\alpha)}$ compatible with the given smooth structure.

Date: 14 July 2025.

2020 *Mathematics Subject Classification.* Primary 32W05; secondary 32Q56, 32L20.

Key words and phrases. Riemann surface, Cauchy–Riemann equation, Oka manifold.

Let B be a topological space whose nature will depend on the integer $l \geq 0$ to be specified. When $l = 0$, we assume that B is a paracompact Hausdorff space, and if $l > 0$ then B will be a (paracompact) manifold of class $\mathscr{C}^l$. A family $\{J_b\}_{b\in B}$ of complex structures on X is said to be of class $\mathscr{C}^{l,(k,\alpha)}$ if for any relatively compact domain $\Omega \Subset X$, the map $B \ni b \mapsto J_b|_\Omega \in \Gamma^{(k,\alpha)}(\Omega, T^*\Omega \otimes T\Omega)$ is of class $\mathscr{C}^l$ as a map to the Hölder space $\Gamma^{(k,\alpha)}(\Omega, T^*\Omega \otimes T\Omega)$. Following Kodaira and Spencer [21] and Kirillov [20], the collection $\{(X, J_b)\}_{b\in B}$ is called a *family of Riemann surfaces* of class $\mathscr{C}^{l,(k,\alpha)}$. Such a family $\{J_b\}_{b\in B}$ can equivalently be given by a family $\{\mu_b\}_{b\in B}$ of maps from X to the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ of the same smoothness class $\mathscr{C}^{l,(k,\alpha)}$; see [9, Sect. 2]. A continuous map $f : B \times X \to Y$ to a complex manifold Y is said to be X*-holomorphic* if the map $f_b = f(b, \cdot) : X \to Y$ is J_b-holomorphic for every $b \in B$. Assuming that $\{J_b\}_{b\in B}$ is of class $\mathscr{C}^{l,(k,\alpha)}$, the space $Z = B \times X$ endowed with the complex structure J_b on the fibre $\{b\} \times X$ admits fibre preserving X-holomorphic charts of class $\mathscr{C}^{l,(k+1,\alpha)}$ with values in $B \times \mathbb{C}$ which are local in $b \in B$ and semiglobal in the space variable $x \in X$ (see [9, Theorem 4.1]). Every X-holomorphic function $f : B \times X \to \mathbb{C}$ of class $\mathscr{C}^{l,0}$ is of local class $\mathscr{C}^{l,(k+1,\alpha)}$ (see [9, Lemma 5.6]).

Assume now that X is a smooth *open* surface. Fix a complex structure J of class $\mathscr{C}^{(k,\alpha)}$ $(k \in \mathbb{Z}_+,\ 0 < \alpha < 1)$ on X. By Gunning and Narasimhan [15], there is a J-holomorphic immersion $z : X \to \mathbb{C}$. By what was said above, z is of class $\mathscr{C}^{(k+1,\alpha)}$ in the given smooth structure on X. Its differentials dz and $d\bar z$ trivialise the respective cotangent bundles $T_J^{*(1,0)}X$ and $T_J^{*(0,1)}X$, and they are of the same class $\mathscr{C}^{(k,\alpha)}$ as J. We have $\mathbb{C} \otimes T^*X = T_J^{*(1,0)}X \oplus T_J^{*(0,1)}X$, and every 1-form β on X can be uniquely written as $\beta = Adz + Bd\bar z$ for a pair of functions $A, B : X \to \mathbb{C}$. Note that β is of class $\mathscr{C}^{(k,\alpha)}$ if and only if the functions A, B are of class $\mathscr{C}^{(k,\alpha)}$. Given a differentiable function $f : X \to \mathbb{C}$, its differential equals

$$df = \partial_J f + \overline{\partial}_J f = f_z \cdot dz + f_{\bar z} \cdot d\bar z, \tag{1.1}$$

where the partial derivatives f_z and $f_{\bar z}$ with respect to z and $\bar z$ are defined by the above equation.

Given a $(0,1)$-form $\beta = u\, d\bar z$ on a domain $\Omega \subset X$, the $\overline{\partial}_J$-equation asks for a solution $f : \Omega \to \mathbb{C}$ of $\overline{\partial}_J f = \beta$; equivalently, $f_{\bar z} = u$. If Ω is relatively compact and has sufficiently regular boundary then this elliptic equation is solvable in many function spaces with a gain of one derivative; see e.g. Ahlfors [1] and Astala et al. [2].

We shall prove the following result, which gives families of solutions for families of complex structures and $(0,1)$-forms on domains in a smooth open surface.

Theorem 1.1. *Assume that X is a smooth open orientable surface, $\Omega \Subset X$ is a relatively compact domain with $\mathscr{C}^{(k+1,\alpha)}$ boundary for some $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, $l \in \mathbb{Z}_+$, B is a paracompact Hausdorff space if $l = 0$ and a $\mathscr{C}^l$ manifold if $l > 0$, and $\{J_b\}_{b\in B}$ is a family of complex structures of class $\mathscr{C}^{l,(k,\alpha)}(B \times \overline{\Omega})$ on $\overline{\Omega}$. Given a family of $(0,1)$-forms $\{\beta_b\}_{b\in B}$ on $\overline{\Omega}$ of class $\mathscr{C}^{l,(k,\alpha)}(B \times \overline{\Omega})$, there is a function $f \in \mathscr{C}^{l,(k+1,\alpha)}(B \times \overline{\Omega})$ satisfying*

$$\overline{\partial}_{J_b} f(b, \cdot) = \beta_b \ \text{ on } \Omega \text{ for every } b \in B. \tag{1.2}$$

Theorem 1.1, together with [9, Theorem 1.1], implies the following corollary concerning solutions of families of global $\overline{\partial}$-equations.

Corollary 1.2. *Assume that $\{J_b\}_{b\in B}$ is a family of complex structures of local class $\mathscr{C}^{l,(k,\alpha)}$ on a smooth open orientable surface X, where $l, k \in \mathbb{Z}_+$, $l \leq k + 1$, $0 < \alpha < 1$ and B is as in Theorem 1.1. Given $(0,1)$-forms $\beta_b \in \Gamma(X, T_{J_b}^{*(0,1)}X)$ such that the family $\{\beta_b\}_{b\in B}$ is of local class $\mathscr{C}^{l,(k,\alpha)}$, there is a function $f : B \times X \to \mathbb{C}$ of local class $\mathscr{C}^{l,(k+1,\alpha)}$ satisfying*

$$\overline{\partial}_{J_b} f(b, \cdot) = \beta_b \ \text{ on } X \text{ for every } b \in B. \tag{1.3}$$

The condition $l \le k+1$ is due to the use of the Runge approximation theorem on families of open Riemann surfaces [9, Theorem 1.1]. See also Remark 1.4 concerning the $\mathscr{C}^\infty$ case.

These results are optimal since we have the expected gain of one derivative in the space variable and no loss of regularity in the parameter $b \in B$. Note that the 1-forms β_b and the function f in (1.2) are expressed in terms of the smooth coordinates on X. Expressing them in terms of local J_b-holomorphic charts leads to a loss of derivatives in the space variable if $l > 0$; see [9, V3, Theorem 9.1].

Proof of Theorem 1.1. We shall use the connection between complex structures and Beltrami multipliers; see [9, Sect. 2] or any standard text on quasiconformal maps. Choose a smooth complex structure J on X and a J-holomorphic immersion $z : X \to \mathbb{C}$. Every function $\mu : \overline{\Omega} \to \mathbb{D}$ of class $\mathscr{C}^{(k,\alpha)}(\overline{\Omega})$ with values in the unit disc determines a complex structure J_μ on $\overline{\Omega}$ of the same class, with $\mu = 0$ corresponding to $J|_{\overline{\Omega}}$. A function $f : \overline{\Omega} \to \mathbb{C}$ satisfies the $\overline{\partial}_{J_\mu}$-equation (and hence is holomorphic on Ω) if and only if it satisfies the Beltrami equation $f_{\bar z} = \mu f_z$, with the partial derivatives f_z and $f_{\bar z}$ defined by (1.1). Conversely, every complex structure J' on $\overline{\Omega}$ of class $\mathscr{C}^{(k,\alpha)}$ in the same orientation class as J equals J_μ for a unique $\mu \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega}, \mathbb{D})$.

To prove the theorem, it suffices to show that for every point $b_0 \in B$ there is a neighbourhood $B_0 \subset B$ of b_0 such that equation (1.2) is solvable on $B_0 \times \Omega$ with $f \in \mathscr{C}^{l,(k+1,\alpha)}(B_0 \times \overline{\Omega})$. By using $\mathscr{C}^l$ partitions of unity on B we then obtain a solution on $B \times \Omega$, thereby proving the theorem.

Note that the complex structure J_{b_0} on $\overline{\Omega}$ extends to a complex structure on X of local class $\mathscr{C}^{(k,\alpha)}$. To see this, we represent J_{b_0} in terms of J by a Beltrami multiplier $\mu \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega}, \mathbb{D})$. Since $b\Omega$ is of class $\mathscr{C}^{(k+1,\alpha)}$, μ extends from $\overline{\Omega}$ to a function $\mu' : X \to \mathbb{D}$ of class $\mathscr{C}^{(k,\alpha)}$ with compact support (see Gilbarg and Trudinger [10, Lemma 6.37]). The associated complex structure $J_{\mu'}$ on X is of local class $\mathscr{C}^{(k,\alpha)}$ and it coincides with J_μ on $\overline{\Omega}$. For $k \ge 1$ the same conclusion holds if $b\Omega$ is of class $\mathscr{C}^{(k,\alpha)}$.

This reduces the proof of the theorem to the following proposition. In this result, the smooth structure on X is the one defined by the complex structure J, and the Hölder norms are with respect to a fixed smooth Riemannian metric on X (whose precise choice is unimportant).

Proposition 1.3. *Let (X, J) be an open Riemann surface, and let $\Omega \Subset X$, k, α be as in Theorem 1.1. For $\mu \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega}, \mathbb{D})$ let J_μ denote the associated complex structure on $\overline{\Omega}$, with $J_0 = J|_{\overline{\Omega}}$. For $c > 0$ set $B_c = \{\mu \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega}) : \|\mu\|_{k,\alpha} < c\}$. There exists $c > 0$ such that for any map $B_c \ni \mu \mapsto \beta_\mu \in \Gamma^{(k,\alpha)}(\overline{\Omega}, T^{*(0,1)}_{J_\mu}\overline{\Omega})$ of class $\mathscr{C}^l$, $l \in \{0, 1, \ldots, \infty, \omega\}$, there is a function $f \in \mathscr{C}^{l,(k+1,\alpha)}(B_c \times \overline{\Omega})$ such that for every $\mu \in B_c$ the function $f_\mu = f(\mu, \cdot\,) : \overline{\Omega} \to \mathbb{C}$ satisfies*

$$\overline{\partial}_{J_\mu} f_\mu = \beta_\mu. \tag{1.4}$$

Proof. We begin by recalling some technical tools from [9, Secs. 3-4]. Choose a J-holomorphic immersion $z : X \to \mathbb{C}$. There is a Cauchy kernel on (X, J) which determines on any smoothly bounded domain $\Omega \Subset X$ a pair of bounded linear operators $P : \mathscr{C}^{(k,\alpha)}(\overline{\Omega}) \to \mathscr{C}^{(k+1,\alpha)}(\overline{\Omega})$ and $S : \mathscr{C}^{(k,\alpha)}(\overline{\Omega}) \to \mathscr{C}^{(k,\alpha)}(\overline{\Omega})$ ($k \in \mathbb{Z}_+$, $0 < \alpha < 1$) such that for any $\phi \in \mathscr{C}^{(k,\alpha)}(\Omega)$, the Cauchy operator P solves the equation $P(\phi)_{\bar z} = \phi$ on Ω, while the Beurling operator S is given by $S(\phi) = P(\phi)_z$. Their properties are summarised in [9, Theorem 3.2]. Although the cited result is stated for domains Ω with $\mathbb{C}^\infty$ boundaries, it is clear from the proof and [10, Lemma 6.37] that it holds if $b\Omega$ is of class $\mathscr{C}^{(k,\alpha)}$ if $k \ge 1$ and of class $\mathscr{C}^{(1,\alpha)}$ if $k \ge 0$.

By [9, Theorem 4.1] there are a constant $c > 0$ and a function $h : B_c \times \overline{\Omega} \to \mathbb{C}$ such that for every $\mu \in B_c$, $h_\mu = h(\mu, \cdot\,) : \Omega \to \mathbb{C}$ is a J_μ-holomorphic immersion of class $\mathscr{C}^{(k+1,\alpha)}(\overline{\Omega})$ depending analytically on μ. We recall the proof since we shall use a similar idea in the sequel. The function $f = h_\mu$ must solve the Beltrami equation $f_{\bar z} = \mu f_z$. We look for a solution in the form

$f = z|_{\overline{\Omega}} + P(\phi)$ with $\phi \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega})$. Note that $\phi = 0$ corresponds to $f = z|_{\overline{\Omega}}$. We have that

$$f_{\bar z} = P(\phi)_{\bar z} = \phi, \quad f_z = 1 + P(\phi)_z = 1 + S(\phi).$$

Inserting in the Beltrami equation $f_{\bar z} = \mu f_z$ gives $(I - \mu S)\phi = \mu$, where I denotes the identity map on $\mathscr{C}^{(k,\alpha)}(\overline{\Omega})$. For $\|\mu\|_{k,\alpha}$ small enough we have that $\|\mu S\|_{k,\alpha} < 1$, so the operator $I - \mu S$ is invertible and its bounded inverse depends analytically on μ:

$$\Theta(\mu) = (I - \mu S)^{-1} = \sum_{j=0}^{\infty} (\mu S)^j \in \mathrm{Lin}(\mathscr{C}^{k,\alpha}(\overline{\Omega})).$$

The gives the following solution $h_\mu = f$ to the Beltrami equation $f_{\bar z} = \mu f_z$ on Ω:

$$h_\mu = z|_\Omega + P(\Theta(\mu)\mu) = z|_\Omega + P((I - \mu S)^{-1}\mu) \in \mathscr{C}^{(k+1,\alpha)}(\overline{\Omega}).$$

Note that h_μ depend analytically on μ. For $\|\mu\|_{k,\alpha}$ small enough, h_μ is so close to $h_0 = z|_{\overline{\Omega}}$ in $\mathscr{C}^{(k+1,\alpha)}(\overline{\Omega})$ that it is an immersion. Hence, for $c > 0$ small enough, $\{\theta_\mu = dh_\mu\}_{\mu \in B_c}$ is a family of nowhere vanishing holomorphic 1-forms on (Ω, J_b) of class $\mathscr{C}^{(k,\alpha)}(\overline{\Omega})$ with analytic dependence on μ. The conjugate $\bar\theta_\mu = d\bar h_\mu$ is a nowhere vanishing antiholomorphic $(0,1)$-form with respect to J_μ for every $\mu \in B_c$. Thus, every family $\{\beta_\mu\}_{\mu \in B_c}$ on $\overline{\Omega}$, where β_μ is a $(0,1)$-form with respect to J_μ, is of the form $\beta_\mu = u_\mu \, d\bar h_\mu$ for a family of functions $u_\mu : \overline{\Omega} \to \mathbb{C}$, $\mu \in B_c$. If the family $\{\beta_\mu\}_{\mu \in B_c}$ is of class $\mathscr{C}^{l,(k,\alpha)}(B_c \times \overline{\Omega})$ then $\{u_\mu\}_{\mu \in B_c}$ is of the same class, and vice versa.

We shall express the $\overline\partial_{J_\mu}$-equation (1.4) as a nonhomogeneous Beltrami equation with respect to the immersion z. For $\mu \in B_c$ we can uniquely express any complex 1-form β on $\overline{\Omega}$ as

$$\beta = A dz + B d\bar z = A_\mu dh_\mu + B_\mu d\bar h_\mu.$$

Note that β is of class $\mathscr{C}^{(k,\alpha)}(\overline{\Omega})$ if and only if the coefficients $A, B, A_\mu, B_\mu : \overline{\Omega} \to \mathbb{C}$ are of this class. We shall now express A_μ and B_μ in terms of the functions A, B, μ, and

$$g_\mu := (h_\mu)_z \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega}).$$

We have

$$dh_\mu = (h_\mu)_z \, dz + (h_\mu)_{\bar z} \, d\bar z = g_\mu \, dz + \mu g_\mu \, d\bar z$$

where the second identity follows from the Beltrami equation $(h_\mu)_{\bar z} = \mu (h_\mu)_z$. It follows that

$$\begin{aligned} A dz + B d\bar z &= A_\mu (g_\mu dz + \mu g_\mu d\bar z) + B_\mu (\overline{\mu g_\mu} \, dz + \overline{g_\mu} \, d\bar z) \\ &= (A_\mu g_\mu + B_\mu \overline{\mu g_\mu}) dz + (A_\mu \mu g_\mu + B_\mu \overline{g_\mu}) d\bar z \end{aligned}$$

and hence

$$A = A_\mu g_\mu + B_\mu \overline{\mu g_\mu}, \quad B = A_\mu \mu g_\mu + B_\mu \overline{g_\mu}.$$

Solving these equations on A_μ and B_μ gives

$$A_\mu = \frac{1}{(1 - |\mu|^2) g_\mu} (A - \bar\mu B), \quad B_\mu = \frac{1}{(1 - |\mu|^2) \overline{g_\mu}} (B - \mu A).$$

Taking $\beta = df$, we have $A = f_z$, $B = f_{\bar z}$, $A_\mu = f_{h_\mu}$, $B_\mu = f_{\bar h_\mu}$. This shows that the equation

$$\overline\partial_{J_\mu} f = \beta_\mu = u_\mu d\bar h_\mu \iff f_{\bar h_\mu} = \frac{f_{\bar z} - \mu f_z}{(1 - |\mu|^2) \overline{g_\mu}} = u_\mu$$

is equivalent to nonhomogeneous Beltrami equation

$$f_{\bar z} - \mu f_z = (1 - |\mu|^2) \overline{g_\mu} \, u_\mu. \tag{1.5}$$

We look for solution in the form $f = f(\mu) = P(\phi)$ with $\phi \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega})$. Inserting $f_{\bar z} = P(\phi)_{\bar z} = \phi$ and $f_z = P(\phi)_z = S(\phi)$ into the above equation gives

$$f_{\bar z} - \mu f_z = (I - \mu S)\phi = (1 - |\mu|^2) \overline{g_\mu} \, u_\mu.$$

For $\|\mu\|_{k,\alpha}$ small enough the operator $I-\mu S$ is invertible and we obtain

$$\phi = \phi_\mu = (I-\mu S)^{-1}\left((1-|\mu|^2)\overline{g_\mu}\, u_\mu\right).$$

Since the bounded linear operator $(I-\mu S)^{-1} \in \mathrm{Lin}(\mathscr{C}^{k,\alpha}(\overline{\Omega}))$ is analytic in μ and $(1-|\mu|^2)\overline{g_\mu}\, u_\mu \in \mathscr{C}^{l,(k,\alpha)}(B_c \times \overline{\Omega})$, the map $(\mu, x) \mapsto \phi_\mu(x)$ also belongs to $\mathscr{C}^{l,(k,\alpha)}(B_c \times \overline{\Omega})$. Finally, the solution of (1.5) is $f_\mu = P(\phi_\mu)$, and the map $(\mu, x) \to f_\mu(x)$ belongs to $\mathscr{C}^{l,(k+1,\alpha)}(B_c \times \overline{\Omega})$. □

By what has been said before, this complete the proof of Theorem 1.1. □

Proof of Corollary 1.2. Global solvability of (1.3) is obtained by exhausting X by an increasing family of relatively compact, smoothly Runge domains, solving the equation (1.3) on each of them by using Theorem 1.1, and applying the Runge approximation theorem on families of open Riemann surfaces [9, Theorem 1.1] at every step of the induction to ensure convergence of solutions. One follows the standard scheme in the proof of Cartan's Theorem B, see e.g. [16, Section VIII.14]. □

Remark 1.4. If B is a manifold of class $\mathscr{C}^l$ with $0 < l \le k+1$, then the space $Z = B \times X$ in Theorem 1.1, endowed with an atlas of class $\mathscr{C}^{l,(k+1,\alpha)}$ given by [9, Theorem 4.1], is a *mixed manifold* of class $\mathscr{C}^l$ in the sense of Jurchescu [17, 18], and a Levi-flat CR manifold of CR-dimension one in the sense of the Cauchy–Riemann geometry; see [3]. In Jurchescu's papers, maps which are holomorphic on complex leaves of a mixed manifold are called *morphic*, while in CR geometry they are called CR maps. The Runge approximation theorem [9, Theorem 1.1] shows that Z is also a *Cartan manifold* of class $\mathscr{C}^l$ in the sense of [18, Sect. 6]. Cartan manifolds are analogues of Stein manifolds in the category of mixed manifolds. Solvability of the tangential $\overline{\partial}$-complex on $\mathscr{C}^\infty$ Cartan manifolds was shown by Jurchescu in [19, Sect. 3] by using sheaf-theoretic approach, similar to the one in the classical theory of Stein manifolds.

There are results in the literature concerning the $\overline{\partial}$-equation on a moving family of domains, also in higher dimensional manifolds; see Diederich and Ohsawa [7], Cho and Choi [6], Gong and Kim [11, Theorem 4.5], Simon [27], Kruse [22], among others. Pulling back a complex structure by a family of diffeomorphisms, the case of moving domains is related to the variation of the complex structure on a fixed domain. In [14], Greene and Krantz studied stability of the $\overline{\partial}$-Neumann operator and the Kohn solution of the $\overline{\partial}$-equation under small integrable variations of a complex structure J on a compact strongly J-pseudoconvex domain $\overline{M}$. Assuming that the boundary bM is of class $2s+5$ for some $s \ge 1$, they obtained continuous dependence of the Neumann operator N_J in the Sobolev L^2-space W^s under small variations of J of class $\mathscr{C}^{2s+5}$, and hence continuous dependence of solutions of the $\overline{\partial}_J$-equation in W^{s-1} [14, Theorems 3.9, and 3.10]. We could not find results in the literature with smooth (better than continuous) dependence of solutions on the complex structure.

Let B, X, and $\{J_b\}_{b\in B}$ be as in Theorem 1.1, where the family J_b is of class $\mathscr{C}^{l,(k,\alpha)}$ for some $0 \le l \le k+1$ and $0 < \alpha < 1$. Denote by $\mathscr{O}$ the sheaf of germs of functions f of class $\mathscr{C}^l$ on $Z = B \times X$ such that $f_b = f(b,\cdot)$ is J_b-holomorphic for each $b \in B$. By [9, Lemma 5.6], $\mathscr{O}$ is a subsheaf of the sheaf $\mathscr{C}^{l,(k+1,\alpha)}$ of functions of the indicated class. These are sheaves of unital abelian rings; in particular, of abelian groups. We have the following corollary to Theorem 1.1.

Proposition 1.5. *(Assumptions as above.)* $H^q(Z, \mathscr{O}) = 0$ *for all* $q = 1, 2, \ldots$.

Proof. Consider the sequence of homomorphisms of sheaves of abelian groups

$$0 \longrightarrow \mathscr{O} \hookrightarrow \mathscr{C}^{l,(k+1,\alpha)} \xrightarrow{\overline{\partial}} \mathscr{C}^{l,(k,\alpha)}_{(0,1)} \longrightarrow 0, \tag{1.6}$$

where $\mathscr{C}^{l,(k,\alpha)}_{(0,1)}$ is the sheaf of germs of $(0,1)$-forms of class $\mathscr{C}^{l,(k,\alpha)}$ on the fibres $Z_b = (X, J_b)$ and $\overline{\partial}$ is the operator which equals $\overline{\partial}_{J_b}$ on Z_b for every $b \in B$. By Theorem 1.1 the sequence (1.6) is exact.

The second and the third sheaf in (1.6) are fine sheaves (as they admit partitions of unity), so they are acyclic, i.e., their cohomology groups of order ≥ 1 vanish. It follows that

$$H^1(Z,\mathscr{O}) = \Gamma\big(Z,\mathscr{C}^{l,(k,\alpha)}_{(0,1)}\big)/\overline{\partial}\,\Gamma\big(Z,\mathscr{C}^{l,(k+1,\alpha)}\big)$$

and $H^q(Z,\mathscr{O}) = 0$ for $q \geq 2$ (see [16, Chapter VI]). Here, Γ denotes the space of global sections of a sheaf. The quotient group on the right hand side above vanishes by Theorem 1.1. □

2. The Oka principle for line bundles on families of open Riemann surfaces

Every holomorphic vector bundle on an open Riemann surface is holomorphically trivial by the Oka–Grauert principle; see Oka [26], Grauert [13], and [8, Theorem 5.3.1]. We now show that Proposition 1.5 implies the Oka principle for isomorphism classes of families of holomorphic line bundles on families of open Riemann surfaces.

Let B be a paracompact Hausdorff space, X be a smooth open surface, and $\{J_b\}_{b\in B}$ be a continuous family of complex structures on X of class $\mathscr{C}^\alpha$ for some $0<\alpha<1$. Let $\mathscr{C}$ denote the sheaf of germs of continuous functions on $Z = B\times X$, and let $\mathscr{O}$ denote the subsheaf of $\mathscr{C}$ consisting of germs of X-holomorphic functions. These are sheaves of unital abelian rings. Furthermore, let $\mathscr{O}^* \subset \mathscr{O}$ and $\mathscr{C}^* \subset \mathscr{C}$ denote the subsheaves consisting of germs with nonzero values; these are sheaves of (multiplicative) abelian groups. A topological complex line bundle $E\to Z = B\times X$ is said to be X-holomorphic (or fibrewise holomorphic) if it admits a transition cocycle consisting of sections of the sheaf $\mathscr{O}^*$. The restriction of such a line bundle to any fibre $Z_b = (X,J_b)$ is a holomorphic line bundle on the Riemann surface (X,J_b). We denote by $\mathrm{Pic}(Z)\cong H^1(Z,\mathscr{O}^*)$ the set of isomorphism classes of X-holomorphic line bundles on $Z = B\times X$.

We have the following Oka principle for families of complex line bundles.

Theorem 2.1. *Every topological complex line bundle on $Z = B\times X$ is isomorphic to an X-holomorphic line bundle, and any two X-holomorphic line bundles on Z which are topologically isomorphic are also isomorphic as X-holomorphic line bundles. Furthermore,* $\mathrm{Pic}(Z)\cong H^2(Z,\mathbb{Z})$.

It follows in particular that if B is contractible then $\mathrm{Pic}(Z) = 0$.

Proof. The proof follows the standard argument for complex line bundles on a complex manifold, due to Oka [26]; see [8, Theorem 5.2.2]. Let $\sigma(f) = \mathrm{e}^{2\pi \mathrm{i} f}$. Consider the following commutative diagram whose rows are exponential sheaf sequences and whose vertical arrows are the natural inclusions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \hookrightarrow & \mathscr{O} & \xrightarrow{\sigma} & \mathscr{O}^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \hookrightarrow & \mathscr{C} & \xrightarrow{\sigma} & \mathscr{C}^* & \longrightarrow & 1 \end{array} \tag{2.1}$$

Note that $\mathscr{C}$ is a fine sheaf, and hence $H^q(Z,\mathscr{C}) = 0$ for all $q\in\mathbb{N}$. By Proposition 1.5 we also have $H^q(Z,\mathscr{O}) = 0$ for all $q\in\mathbb{N}$. Hence, the relevant part of the long exact sequence of cohomology groups associated to the diagram (2.1) gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(Z,\mathscr{O}^*) & \longrightarrow & H^2(Z;\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^1(Z,\mathscr{C}^*) & \longrightarrow & H^2(Z;\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Thus, all arrows in the central square are isomorphisms. Since $\mathrm{Pic}(Z)\cong H^1(Z,\mathscr{O}^*)$ and $H^1(Z,\mathscr{C}^*)$ is the set of isomorphisms classes of topological line bundles on Z, the theorem follows. □

The Oka principle for maps from families of open Riemann surfaces to Oka manifolds (see [9, Theorem 1.6]) allows us to extend the first part of Theorem 2.1 to vector bundles of arbitrary rank by using the approach from the classical Oka–Grauert theory. However, the assumptions on the parameter space B must be more restrictive for the cited result to imply. We state the following special case when B is a CW complex and refer to the discussion preceding [9, Theorem 1.6] for more information.

Theorem 2.2. *Assume that B is a finite CW complex or a countable locally compact CW-complex of finite dimension, X is a smooth open surface, and $\{J_b\}_{b\in B}$ is a continuous family of complex structures on X of local Hölder class $\mathscr{C}^\alpha$ for some $0<\alpha<1$. Then, every topological vector bundle on $B\times X$ is isomorphic to a fibrewise holomorphic vector bundle.*

Proof. A topological vector bundle E on $B\times X$ is the pullback $f^*\mathbb{U}$ by a continuous map f from $B\times X$ to a suitable Grassmannian $G=G(r,N)$ (consisting of complex r-planes in $\mathbb{C}^N$) of the universal bundle $\mathbb{U}\to G$. (We take N big enough such that E embeds as a topological vector subbundle of the trivial bundle $(B\times X)\times\mathbb{C}^N$; this is possible since $B\times X$ is paracompact.) Since G is a complex homogeneous manifold, and hence an Oka manifold by Grauert [12], the Oka principle in [9, Theorem 1.6] shows that f is homotopic to a map $F:B\times X\to G$ such that $F(b,\cdot):X\to G$ is J_b-holomorphic for every $b\in B$. The pullback $F^*\mathbb{U}\to B\times X$ is then a fibrewise holomorphic vector bundle topologically isomorphic to $f^*\mathbb{U}$. □

Remark 2.3. Under the assumptions in Theorem 2.2, it is also possible to show that any two fibrewise holomorphic vector bundles on $B\times X$ that are topologically isomorphic are also isomorphic as fibrewise holomorphic vector bundles. One may follow the classical case for a single complex structure on a Stein manifold X, due to Grauert [13]; see also the expositions by Cartan [5], Leiterer [23], and [8, Theorem 5.3.1]. However, to complete the proof, we need need an Oka principle for sections of fibrewise holomorphic principal bundles on $B\times X$, thereby extending the Oka principle for maps from $B\times X$ to Oka manifolds in [9, Theorem 1.6]. We shall treat this in a subsequent publication.

Note that Mongodi and Tomassini [24] obtained the Oka principle for more general CR vector bundles on certain real analytic Levi-flat submanifolds of complex Euclidean spaces by reducing the problem to the Oka–Grauert theorem [13]. Our techniques use considerably less regularity.

Acknowledgements. The author is supported by the European Union (ERC Advanced grant HPDR, 101053085) and grants P1-0291 and N1-0237 from ARIS, Republic of Slovenia. I wish to thank Finnur Lárusson for helpful conversations concerning Remark 2.3.

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