RUNGE AND MERGELYAN THEOREMS ON FAMILIES OF OPEN RIEMANN SURFACES

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ABSTRACT. In this paper, we develop the Oka theory for maps from families of complex structures on smooth open surfaces to any Oka manifold. Along the way, we prove Runge and Mergelyan approximation theorems and Weierstrass interpolation theorem on such families, with continuous or smooth dependence of the data and the approximating functions on the complex structure. This implies global solvability of the $\overline{\partial}$-equation on such families. We also obtain an Oka principle for complex line bundles on families of open Riemann surfaces, and we show that the canonical bundles of open Riemann surfaces are holomorphically trivial in families. We include applications to families of directed holomorphic immersions and conformal minimal immersions.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

The Runge approximation theorem is one of the cornerstones of complex analysis. Its basic version, proved by Runge [73] in 1885, says that for a compact set K in the complex plane $\mathbb{C}$ with connected complement $\mathbb{C} \setminus K$, every holomorphic function on an open neighbourhood of K is a uniform limit on K of holomorphic polynomials. Runge's theorem was extended by Behnke and Stein [16] (1949) to any compact set K in an open Riemann surface X such that $X \setminus K$ has no relatively compact connected components. A set K with this property is said to be Runge in X, a topological condition. The analogue of Runge's theorem in Stein manifolds of arbitrary dimension is the Oka–Weil theorem; see [67, 82] and [32, Theorem 2.8.4]. A major generalization concerns holomorphic maps from Stein manifolds to Oka manifolds, the subject of modern Oka theory [32].

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In geometric applications, it is often necessary to approximate families of functions or maps which depend continuously or smoothly on parameters. For maps from a fixed Stein manifold X, see [32, Theorem 2.8.4] for the parametric Cartan–Oka–Weil theorem and [32, Theorem 5.4.4] for the parametric Oka principle for maps to any Oka manifold. In those results, the parameter set is a compact Hausdorff space, the maps depend continuously on the parameter, and they can be left unchanged for parameter values in a closed subset for which they are already holomorphic on X.

In the present paper, we consider the more general situation where not only the maps, but also the complex structures on the source manifold depend on a parameter. In this first work on the subject, we limit ourselves to the case of a smooth open oriented surface endowed with a family of complex structures. Our main result is an Oka principle, saying that every family of continuous maps from such a family of Riemann surfaces to an Oka manifold is homotopic to a family of holomorphic maps, with approximation on a family of compact Runge subsets where the given maps are already holomorphic; see Theorem 1.4. Our method also applies to products of variable open Riemann surfaces and a fixed Stein manifold; see Theorem 7.4. We also prove Mergelyan-type theorems for maps to Oka manifolds; see Theorems 7.5 and 7.7. These results open a new direction in Oka theory. On the way, we prove special cases which concern approximation of functions (i.e., maps to the complex number field $\mathbb{C}$) on families of open Riemann surfaces, the Runge Theorem 1.1 and the Mergelyan Theorem 1.2. Their proofs are simpler since we can use partitions of unity instead of dealing with homotopies, and the parameter spaces can be more general. Our results, combined with the techniques from Gromov's convex integration theory, enable the construction of families of holomorphic curves of prescribed (variable) conformal types having additional properties such as being immersed, directed by a conical subvariety of $\mathbb{C}^n$, etc., and the construction of families of minimal surfaces of prescribed conformal types in Euclidean spaces. In Section 10 we give a sample of such applications and indicate further problems which can possibly be treated by our methods.

We now introduce the setup. By X we always denote a smooth orientable surface without boundary, which will be an open surface in most of our results. A complex structure on X is given by an endomorphism J of its tangent bundle TX satisfying $J^2 = -\mathrm{Id}$. Thus, J is a section of the vector bundle $T^*X \otimes TX \to X$ whose fibre over any point $x \in X$ is the space $\mathrm{Hom}(T_xX, T_xX)$ of linear maps $T_xX \mapsto T_xX$. Such an operator J is usually called an almost complex structure on X, but due to integrability we shall not distinguish between these two notions. A differentiable function $f : X \to \mathbb{C}$ is said to be J-holomorphic if the Cauchy–Riemann equation $df_x \circ J_x = \mathrm{i}\, df_x$ holds for every $x \in X$, where $\mathrm{i} = \sqrt{-1}$. Assuming that J is of Hölder class $\mathscr{C}^{(k,\alpha)}$ for some $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $0 < \alpha < 1$, there is an atlas $\{(U_i, \phi_i)\}$ of open sets $U_i \subset X$ with $\bigcup_i U_i = X$ and J-holomorphic charts $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}$ of class $\mathscr{C}^{(k+1,\alpha)}(U_i)$ (see Theorem 2.2). Since the transition maps $\phi_i \circ \phi_j^{-1}$ are biholomorphic in the standard structure J_{st} on $\mathbb{C}$, J determines on X the structure of a Riemann surface, denoted (X, J), which is $\mathscr{C}^{k+1}$ compatible with the smooth structure on X, that is, the identity map on X is of class $\mathscr{C}^{k+1}$ as a map between these two structures.

We shall consider families of complex structures $\{J_b\}_{b\in B}$ on a smooth surface X which depend continuously or smoothly on a parameter b in a topological space B as a map $B \ni b \mapsto J_b \in \Gamma^{(k,\alpha)}(X, T^*X \otimes TX)$ into the vector space of sections of Hölder class $\mathscr{C}^{(k,\alpha)}$ of the smooth vector bundle $T^*X \otimes TX \to X$. If B is a manifold of class $\mathscr{C}^l$ for some $l \in \mathbb{N} = \{1, 2, \ldots\}$, or a topological space if $l = 0$, we say that $\{J_b\}_{b\in B}$ is of class $\mathscr{C}^{l,(k,\alpha)}(B \times X)$ if the derivatives of J_b up to order l with respect to $b \in B$ are continuous on $B \times X$ and of class $\mathscr{C}^{(k,\alpha)}(X)$ for any fixed value of $b \in B$. The analogous definition applies to maps $B \times X \to Y$ to a smooth manifold Y. Such a family $\{J_b\}_{b\in B}$ can also be given by a family $\{\mu_b\}_{b\in B}$ of maps from X to the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ of the same smoothness class $\mathscr{C}^{l,(k,\alpha)}(B \times X)$; see Remark 4.5 (C).

Consider the projection $\pi : B \times X \to B$. We endow the fibre $X_b = \pi^{-1}(b) \cong X$ with the structure of the Riemann surface (X, J_b) determined by J_b. A continuous map $f : B \times X \to Y$ to a complex manifold Y is said to be X*-holomorphic* if the restriction $f(b, \cdot) : X_b \to Y$ is J_b-holomorphic for every $b \in B$. Assuming that the family $\{J_b\}_{b \in B}$ is of class $\mathscr{C}^{l,(k,\alpha)}$, the space $B \times X$ admits fibre preserving X-holomorphic charts of class $\mathscr{C}^{l,(k+1,\alpha)}$ with values in $B \times \mathbb{C}$ (see Theorem 4.1).

Recall that a topological space is said to be paracompact if every open cover has an open locally finite refinement. A Hausdorff space is paracompact if and only if it admits a continuous partition of unity subordinate to any given open cover.

Our first result extends the classical Runge–Behnke–Stein approximation theorem on open Riemann surfaces [73, 16], combined with the Weierstrass–Florack interpolation theorem [81, 27], to families of complex structures on a smooth open surface.

Theorem 1.1. *Assume that $l \in \mathbb{Z}_+$, B is a locally compact and paracompact Hausdorff space if $l = 0$ and a manifold of class $\mathscr{C}^l$ if $l > 0$, X is a smooth open oriented surface, $\{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathscr{C}^{l,(k,\alpha)}$ ($k \in \mathbb{Z}_+$, $l \le k+1$, $0 < \alpha < 1$), K is a compact Runge subset of X, A is a closed discrete subset of X, $U \subset B \times X$ is an open set containing $B \times (K \cup A)$, $f : U \to \mathbb{C}$ is a function of class $\mathscr{C}^{l,0}(U)$ such that $f_b = f(b, \cdot)$ is J_b-holomorphic on $U_b = \{x \in X : (b, x) \in U\}$ for every $b \in B$, and $r \in \{0, 1, \ldots, k+1\}$. Then, $f \in \mathscr{C}^{l,k+1}(U)$ and there is a function $F \in \mathscr{C}^{l,k+1}(B \times X)$ satisfying the following conditions.*

(a) *The function $F_b = F(b, \cdot) : X \to \mathbb{C}$ is J_b-holomorphic for every $b \in B$.*
(b) *F approximates f as closely as desired in the fine $\mathscr{C}^{l,k+1}$-topology on $B \times K$.*
(c) *$F_b - f_b$ vanishes to order r at every point $a \in A$ for every $b \in B$.*

Theorem 1.1 is proved in Section 5. The reason for assuming that $l \le k+1$ will become evident in the proof of Lemma 5.4. The complex structures J_b in the theorem are compatible with one another only to order $k+1$, which necessitates the assumption $r \le k+1$ in condition (c).

A more sophisticated approximation theorem was proved by Mergelyan in 1951. Its original version [63] says that a continuous function on a compact Runge set $K \subset \mathbb{C}$, which is holomorphic in the interior of K, is a uniform limit on K of holomorphic polynomials. This was generalized to open Riemann surfaces by Bishop [17], with different proofs and generalizations given by Sakai [74], Scheinberg [75], Gauthier [40], and others. Results on approximation by rational function on the plane with poles in $\mathbb{C} \setminus K$ were obtained by Vitushkin [79, 80]. See also the surveys [28, 38]. In Section 6 we prove the following Mergelyan theorem for families of complex structures on a smooth surface.

Theorem 1.2. *Assume that X is a smooth oriented surface without boundary, B is a locally compact and paracompact Hausdorff space, $\{J_b\}_{b \in B}$ is a continuous family of complex structures of some Hölder class $\mathscr{C}^\alpha$ ($0 < \alpha < 1$) on X, K is compact set in X such that, for some $c > 0$ and a Riemannian distance function on X, each relatively compact connected component of $X \setminus K$ has diameter at least c, A is a finite subset of the interior $\mathring{K}$ of K, and $f : B \times K \to \mathbb{C}$ is a continuous function such that $f_b = f(b, \cdot) : K \to \mathbb{C}$ is J_b-holomorphic on $\mathring{K}$ for every $b \in B$. Given a continuous function $\epsilon : B \to (0, +\infty)$, there is a continuous function F on a neighbourhood $U \subset B \times X$ of $B \times K$ such that for every $b \in B$ the function $F_b = F(b, \cdot) : U_b = \{x \in X : (b, x) \in U\} \to \mathbb{C}$ is J_b-holomorphic, $\sup_{x \in K} |F_b(x) - f_b(x)| < \epsilon(b)$, and $F_b - f_b$ vanishes to order 1 in every point $a \in A$.*

If in addition the surface X is open and the set K is Runge in X, then there is a continuous function $F : B \times X \to \mathbb{C}$ such that $F_b = F(b, \cdot) : X \to \mathbb{C}$ is J_b-holomorphic for every $b \in B$ and F satisfies the above approximation and interpolation conditions.

The last part of the theorem clearly follows from the first part and Theorem 1.1. The condition on the set K in Theorem 1.2 is not optimal, but it more than suffices for the intended applications. We also obtain Mergelyan approximation for families of manifold-valued maps (see Theorem 7.5), and Mergelyan approximation in the $\mathscr{C}^l$ topology on admissible sets (see Theorem 7.7).

We now introduce the parameter spaces that will be used in our main result, Theorem 1.4.

Definition 1.3. In the following, all topological spaces are assumed to be metrizable.

(i) A space B is an *absolute neighbourhood retract* (ANR) if, whenever B is a closed subset of a space B', then B is a retract of a neighbourhood of B in B' (a neighbourhood retract).
(ii) A space B is a *Euclidean neighbourhood retract* (ENR) if it admits a topological embedding $\iota : B \hookrightarrow \mathbb{R}^n$ for some n whose image $\iota(B) \subset \mathbb{R}^n$ is a neighbourhood retract.
(iii) A space B is a *local ENR* if every point of B has an ENR neighbourhood.

We refer to Mardešić [60] for the theory of ANRs and ENRs, and to Hatcher [50] and May [62] for CW complexes. Note that CW complexes generalize both manifolds and simplicial complexes, and they have particular significance for algebraic topology. A CW complex is locally compact if and only if its collection of closed cells is locally finite, if and only if it is metrizable (see Fritsch and Piccinini [37, Theorem B]). If a CW complex B can be embedded in a Euclidean space $\mathbb{R}^m$, then B has at most countably many cells, it is locally compact and has dimension $\leq m$; see [37, Theorem D]. Conversely, every countable locally compact CW-complex B of finite dimension m is an ENR. Indeed, by [37, Theorem A] such B admits a closed embedding in $\mathbb{R}^{2m+1}$. Since every metrizable CW-complex is an ANR (see Dugundji [25]), the image of the embedding is a neighbourhood retract, so B is an ENR. In particular, every finite CW complex is an ENR (see [50, Corollary A.10]).

Our main result is the following Oka principle with approximation for maps from families of open Riemann surfaces to an Oka manifold (see [32, Sect. 5.4] for this notion). It is proved in Section 7.

Theorem 1.4. *Assume the following:*

(a) *B is a paracompact Hausdorff space which is a local ENR (see Definition 1.3).*
(b) *X is a smooth open surface and $\pi : B \times X \to B$ is the projection.*
(c) *$\{J_b\}_{b\in B}$ is a continuous family of complex structures on X of Hölder class $\mathscr{C}^\alpha$, $0 < \alpha < 1$.*
(d) *$K \subset B \times X$ is a closed subset such that $\pi|_K : K \to B$ is proper, and for every $b \in B$ the fibre $K_b = \{x \in X : (b, x) \in K\}$ is a compact Runge set in X, possibly empty.*
(e) *Y is an Oka manifold endowed with a distance function dist_Y inducing the manifold topology.*
(f) *$f : B \times X \to Y$ is a continuous map, and there is an open set $U \subset B \times X$ containing K such that $f_b = f(b, \cdot\,) : X \to Y$ is J_b-holomorphic on $U_b = \{x \in X : (b, x) \in U\}$ for every $b \in B$.*

Given a continuous function $\epsilon : B \to (0, +\infty)$, there are a neighbourhood $U' \subset U$ of K and a homotopy $f_t : B \times X \to Y$ $(t \in I = [0, 1])$ satisfying the following conditions.

(i) $f_0 = f$.
(ii) *The map $f_{t,b} = f_t(b, \cdot\,) : X \to Y$ is J_b-holomorphic on $U'_b \supset K_b$ for every $b \in B$.*
(iii) $\sup_{x\in K_b} \mathrm{dist}_Y(f_t(b, x), f(b, x)) < \epsilon(b)$ *for every $b \in B$ and $t \in I$.*
(iv) *The map $F = f_1$ is such that $F_b = F(b, \cdot\,) : X \to Y$ is J_b-holomorphic for every $b \in B$.*
(v) *If Q is a closed subset of B and $U_b = X$ for all $b \in Q$, then the homotopy $f_{t,b}$ can be chosen to be fixed for every $b \in Q$, and in particular $F = f$ on $Q \times X$.*

If B is a manifold of class $\mathscr{C}^l$ $(l \in \mathbb{N})$, the set Q in (v) is a closed $\mathscr{C}^l$ submanifold of B, the family $\{J_b\}_{b\in B}$ is of class $\mathscr{C}^{l,(k,\alpha)}$ where $l \leq k + 1$, and the map $f : B \times X \to Y$ is X-holomorphic on a neighbourhood U of K and $f|_U \in \mathscr{C}^{l,0}(U, Y)$, then $f|_U \in \mathscr{C}^{l,k+1}(U, Y)$ and there is a homotopy $f_t : B \times X \to Y$ $(t \in I)$ which is of class $\mathscr{C}^{l,k+1}$ on a neighbourhood of K, it satisfies conditions (i)–(v), and every f_t approximates f in the fine $\mathscr{C}^{l,k+1}$-topology on K to a desired precision uniformly in $t \in I$ (i.e., condition (iii) is upgraded to fine $\mathscr{C}^{l,k+1}$ approximation on K).

Remark 1.5. (A) In analogy to Theorem 1.1, one can also include (jet) interpolation in Theorem 1.4. Assuming that $x_1, \ldots, x_m : B \to X$ are maps of class $\mathscr{C}^l$ with pairwise disjoint graphs contained in K and $r \in \{0, 1, \ldots, k+1\}$, the homotopy $\{f_t\}_{t \in I}$ in the theorem can be chosen such that the r-jet of the map $f_t(b, \cdot) : X \to Y$ in the point $x_j(b) \in K_b$ agrees with the corresponding r-jet of $f(b, \cdot)$ for every $b \in B$, $j = 1, \ldots, m$ and $t \in I$. We invite the reader to supply the details. It is a general fact of Oka theory that interpolation follows from approximation; see [32, Proposition 5.15.1] and [56].

(B) Condition (v) can be established under the weaker assumption that the map $f_b : X \to Y$ is J_b-holomorphic for every $b \in Q$. See [32, Theorem 5.4.4] where a result of this kind is proved for a family of maps from a fixed Stein manifold.

Oka manifolds are complex manifolds for which the h-principle (also called the *Oka principle* in this context) holds for maps from any Stein manifold, and in particular from any open Riemann surface; see [32, Sect. 5.4]. The appear naturally in many existence results in complex geometry. Model Oka manifolds are the complex Euclidean spaces $\mathbb{C}^n$, in which case Theorem 1.4 generalizes the approximation statement in Theorem 1.1 to a variable family of compact Runge sets. Every complex homogeneous manifold and, more generally, every Gromov elliptic manifold is an Oka manifold (see Grauert [44] and Gromov [45]). Many further examples can be found in the surveys [32, Chapter 7] and [30]. In [31] it is shown that in semipositive ample line bundles on certain projective Oka manifolds, the unit disc bundle is an Oka manifold. Oka theory has recently been developed for maps from open Riemann surfaces to the bigger class of Oka-1 manifolds which properly contains the class of Oka manifolds; see [9, 35]. However, it does not seem possible to use this bigger class in Theorem 1.4 since its proof relies on the Oka principle for maps from higher dimensional Stein manifolds.

Theorem 1.4 seems to be the first result of its kind in Oka theory. Its proof in Section 7 also applies to products $X \times Z$, where X is a smooth open surface endowed with a family $\{J_b\}_{b \in B}$ of complex structures as in Theorem 1.4 and Z is a Stein manifold with a fixed complex structure; see Theorem 7.4. Essentially the same proof, combined with Theorem 1.2, yields Mergelyan approximation for maps from families of open Riemann surfaces to an Oka manifold; see Theorems 7.5 and 7.7.

Our results apply in particular if B is the Teichmüller space $\mathcal{T}(g, k)$ of marked complex structures on the k-punctured surface $X = \overline{X} \setminus \{p_1, \ldots, p_k\}$, where $\overline{X}$ is an oriented smooth compact surface of genus g and $k \geq 1$. The spaces $\mathcal{T}(g, k)$ and $\mathcal{T}(g, k) \times X$ carry natural complex structures such that the projection $\pi : \mathcal{T}(g, k) \times X \to \mathcal{T}(g, k)$ is a holomorphic submersion and the complex structure on each fibre $X_b = \pi^{-1}(b)$ is the one determined by $b \in \mathcal{T}(g, k)$. See Nag [65, Chapter 3] or Imayoshi and Taniguchi [53] for a precise description, and also the surveys [26, 61] for infinite dimensional Teichmüller spaces of hyperbolic complex structures. However, we do not see how to use this theory in order to obtain or simplify the proofs of our main results for a general parameter space B.

The results and methods developed in the paper can be used in a range of constructions of families of holomorphic curves with special properties, and of related objects such as conformal minimal surfaces. To illustrate the point, we give two such applications in Section 10. The first one in Theorem 10.2 gives families of J_b-holomorphic immersions $X \to \mathbb{C}^n$ directed by an irreducible conical complex subvariety $\overline{A} = A \cup \{0\} \subset \mathbb{C}^n$ such that $A = \overline{A} \setminus \{0\}$ is an Oka manifold. By taking $A = \mathbb{C}^n_*$ we obtain families of ordinary immersions $X \to \mathbb{C}^n$. For $n = 1$ this gives an extension of the Gunning–Narasimhan theorem [46] to families of J_b-holomorphic immersions $X \to \mathbb{C}$; see Corollary 10.3. Another application pertains to the null cone $\mathbf{A} \subset \mathbb{C}^n$ for $n \geq 3$, see (10.3). Holomorphic immersions directed by this cone are the $(1, 0)$-derivatives of minimal surfaces, and we obtain continuous or smooth families of conformally immersed minimal surfaces $X \to \mathbb{R}^n$ $(n \geq 3)$ for any continuous or smooth family of complex structures J_b on X (see Corollary 10.6). Several other possible applications are indicated in Problem 10.7. A common feature of these examples is that their construction combines

Oka theory with methods from Gromov's convex integration theory to ensure the period vanishing conditions of the derivative maps on a basis of the first homology group $H_1(X,\mathbb{Z})$.

The paper is organised as follows. In Section 2 we recall the connection between Riemannian metrics, conformal structures, and the Beltrami equation. Section 3 contains preparatory results on the Cauchy and Beurling transforms. In Section 4 we develop results on deformations of complex structures which are used in the proofs. Theorem 4.1 gives a solution of the Beltrami equation on any smoothly bounded relatively compact domain Ω in an open Riemann surface X for Beltrami coefficients μ with small Hölder $\mathscr{C}^{(k,\alpha)}$ norm ($k \in \mathbb{Z}_+,\ 0<\alpha<1$) and with smooth dependence on μ. Using also a theorem of Gunning and Narasimhan [46], it is shown in Theorem 4.3 that any small $\mathscr{C}^{(k,\alpha)}$ perturbation of the complex structure on Ω can be realised by a small $\mathscr{C}^{(k+1,\alpha)}$ perturbation of Ω in X, with smooth dependence of the map on the complex structure. This extends the Ahlfors–Bers theory [3] of quasiconformal maps of the plane. With these tools in hand, Theorem 1.1 is proved in Section 5 and Theorem 1.2 is proved in Section 6. In Section 7 we prove our main result, Theorem 1.4, and obtain further Runge and Mergelyan type approximation results for families of manifold-valued maps; see Theorems 7.4, 7.5, and 7.7. In Section 8 we show that for a family of complex structures on a smooth open surface, the family of their holomorphic cotangent (canonical) bundles admits a family of holomorphic trivializations (see Theorem 8.1), which can be given by a family of holomorphic immersions to $\mathbb{C}$ (see Corollary 10.3). In Section 9 we obtain a global solution of the $\overline{\partial}$-equation on families of open Riemann surfaces (see Theorem 9.1) and the Oka principle for continuous families of complex line bundles (see Theorem 9.3). In Section 10 we apply our results to the construction of families of directed holomorphic immersions and conformal minimal immersions.

Open Riemann surfaces are Stein manifolds of complex dimension one. It would be of interest to develop the Oka theory for families of integrable Stein structures on manifolds of higher dimension. To this end, one would need an analogue of Theorem 4.3 for such families on strongly pseudoconvex domains. A special case of results of Hamilton [48, 49] is that a sufficiently small smooth integrable deformation of a complex structure on a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold can be realised by a small deformation of the domain. Recently, Gong and Shi [43] obtained Hamilton's results under substantially weaker regularity assumptions. However, smooth dependence of the deformation map on the complex structure is not clear from these results. Another complication in the higher dimensional case is caused by the behaviour of hulls of compact sets in families of Stein structures. For these reason, we defer this project to a future work.

Notation and terminology. We denote by $\mathscr{C}(X)$ and $\mathscr{O}(X)$ the Fréchet algebras of continuous and holomorphic functions on a complex manifold X, respectively, endowed with the compact-open topology. Given a compact set K in a complex manifold X, we denote by $\mathscr{C}(K)$ the Banach algebra of continuous complex valued functions on K with the supremum norm, by $\mathscr{O}(K)$ the space of functions that are holomorphic in a neighbourhood of K (depending on the function), and by $\overline{\mathscr{O}}(K)$ the uniform closure of $\{f|_K : f \in \mathscr{O}(K)\}$ in $\mathscr{C}(K)$. By $\mathscr{A}(K)$ we denote the closed subspace of $\mathscr{C}(K)$ consisting of functions $K \to \mathbb{C}$ which are holomorphic in the interior $\mathring{K}$ of K. For $r \in \mathbb{Z}_+ \cup \{\infty\}$ we denote by $\mathscr{C}^r(K)$ the space of all functions on K which extend to r-times continuously differentiable functions on X, and $\mathscr{A}^r(K) = \{f \in \mathscr{C}^r(K) : f|_{\mathring{K}} \in \mathscr{O}(\mathring{K})\}$. The analogous notation is used for manifold-valued maps. A compact set K in a complex manifold X is said to be $\mathscr{O}(X)$-*convex*, or holomorphically convex in X, if K equals its holomorphically convex hull:

$$K = \widehat{K}_{\mathscr{O}(X)} = \big\{p \in X : |f(p)| \le \max_{x\in K} |f(x)| \ \text{ for all } f \in \mathscr{O}(X)\big\}.$$

If X is a Riemann surface then K is $\mathscr{O}(X)$-convex if and only if it is Runge in X. A compact $\mathscr{O}(\mathbb{C}^n)$-convex set $K \subset \mathbb{C}^n$ is said to be polynomially convex. A compact set K in a complex manifold X is said to be a *Stein compact* if it admits a basis of open Stein neighbourhoods in X.

2. RIEMANNIAN METRICS, COMPLEX STRUCTURES, AND THE BELTRAMI EQUATION

In this section, we recall the relevant background on the topics mentioned in the title. The details can be found in standard texts on quasiconformal mappings and Teichmüller spaces; see e.g. Ahlfors [2], Lehto and Virtanen [57], Nag [65], and Imayoshi and Taniguchi [53].

Let $z = x + \mathrm{i}y$ be the complex coordinate on $\mathbb{C}$. Set $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, $dz = dx + \mathrm{i}dy$, $d\bar{z} = dx - \mathrm{i}dy$,

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\partial_x - \mathrm{i}\partial_y\right), \qquad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\partial_x + \mathrm{i}\partial_y\right).$$

For a differentiable function f we shall write $f_z = \partial_z f$ and $f_{\bar{z}} = \partial_{\bar{z}} f$. Note that f is holomorphic if and only if $f_{\bar{z}} = 0$. The exterior differential on functions splits in the sum of its $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts: $d = \partial + \overline{\partial} = \partial_z dz + \partial_{\bar{z}} d\bar{z}$.

A Riemannian metric on a smooth surface X is given in any local coordinates (x, y) by

$$g = Edx \otimes dx + F(dx \otimes dy + dy \otimes dx) + Gdy \otimes dy = Edx^2 + 2Fdxdy + Gdy^2, \tag{2.1}$$

where E, F, G are real functions satisfying $EG - F^2 > 0$. The area form determined by the metric g is $\sqrt{EG - F^2}\, dx \wedge dy$. The Euclidean metric and the area form on $\mathbb{R}^2 \cong \mathbb{C}$ with the coordinate $z = x + \mathrm{i}y$ are given by $g_{\mathrm{st}} = dx^2 + dy^2 = |dz|^2$ and $dx \wedge dy = \frac{\mathrm{i}}{2} dz \wedge d\bar{z}$. On every tangent space T_pX, a Riemannian metric g defines a scalar product having the matrix $\left(\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right)$ in the basis $\partial_x,\ \partial_y$. Hence, g determines a unique conformal structure on X, and two Riemannian metrics g_1, g_2 determine the same conformal structure if and only if $g_2 = \lambda g_1$ for a positive function λ. A pair of nonzero tangent vectors $\xi, \eta \in T_pX$ is said to be a conformal frame if ξ and η have the same g-length and are g-orthogonal to each other. If X is oriented, there is a unique endomorphism $J : TX \to TX$ on the tangent bundle of X such that for any tangent vector $0 \neq v \in T_pX$, (v, Jv) is a positively oriented g-conformal frame. Note that $J^2 = -\mathrm{Id}$; an endomorphism of TX satisfying this condition is called an *almost complex structure* on X. We have the following local expression for the matrix of J (in the standard oriented basis $\partial_x,\ \partial_y$) in terms of the metric g (2.1):

$$[J] = \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix} = \begin{pmatrix} -b & -c \\ (b^2+1)/c & b \end{pmatrix} \tag{2.2}$$

where $\delta = EG - F^2 > 0$, $b = F/\sqrt{\delta}$, and $c = G/\sqrt{\delta} > 0$. Every almost complex structure J is of this form for some Riemannian metric g, which is unique up to conformal equivalence. The standard almost complex structure J_{st} on $\mathbb{C}$, defined by the Euclidean metric g_{st}, has the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In complex notation, J_{st} amounts to multiplication by i. A differentiable function $f : U \to \mathbb{C}$ on a domain $U \subset X$ is said to be J-holomorphic (more precisely, (J, J_{st})-holomorphic) if it satisfies the Cauchy–Riemann equation $df_p \circ J_p = J_{\mathrm{st}} \circ df_p$ at all points $p \in U$, where J_p denotes the restriction of J to T_pX. At a point where $df_p \neq 0$, such f is an orientation preserving conformal map from the conformal structure on X determined by $J = J_g$ to the standard conformal structure on $\mathbb{C}$.

Assume that the metric g is given in local coordinates (x, y) on an open set $U \subset X$ by (2.1). Taking $z = x + \mathrm{i}y$ as a complex coordinate on U, we can write g in the complex form as

$$g = \lambda |dz + \mu d\bar{z}|^2 \tag{2.3}$$

for a positive function $\lambda > 0$ and the complex function

$$\mu = \frac{1 - c + \mathrm{i}b}{1 + c + \mathrm{i}b} : X \to \mathbb{D} \tag{2.4}$$

with values in the unit disc, where the numbers b and c are as in (2.2); see [11, p. 51]. A diffeomorphism $f : U \to f(U) \subset \mathbb{C}$ is conformal from the g-structure on X to the standard conformal structure on

$\mathbb{C}$ if and only if $g = h|df|^2$ for a positive function $h > 0$. A chart f with this property is said to be *isothermal* for g. Assume that f is orientation preserving, which amounts to $|f_z| > |f_{\bar z}|$. Then

$$|df|^2 = |f_z dz + f_{\bar z} d\bar z|^2 = |f_z|^2 \cdot \left| dz + \frac{f_{\bar z}}{f_z} d\bar z \right|^2,$$

and comparison with (2.3) shows that f is isothermal if and only if it satisfies the Beltrami equation

$$f_{\bar z} = \mu f_z \tag{2.5}$$

with the Beltrami coefficient μ given by (2.4). We shall say that f is μ-conformal if (2.5) holds. Equivalently, f is a biholomorphic map from (U, J) to $(f(U), J_{\rm st})$ where J is the complex structure on X determined by g (or by μ). The transition map between any pair of μ-conformal charts is a conformal diffeomorphism between domains in the plane $\mathbb{C}$ endowed with the Euclidean metric.

One can also consider quasiconformal maps $f : X \to Y$ between a pair of Riemann surfaces. The quantity $\mu_f(z) = f_{\bar z}/f_z$, defined in a local holomorphic coordinate z on X, is independent of the choice of the local holomorphic coordinates on Y, and $\mu_f(z) d\bar z/dz$ is a section of the bundle $K_X^{-1} \otimes \overline{K}_X \to X$ where $K_X = T^*X$ is the canonical bundle of X (see [65, p. 46]).

Remark 2.1. The formulas (2.2)–(2.4) show that the conformal class of a Riemannian metric g, the associated complex structure J, and the Beltrami coefficient μ are of the same smoothness class.

The situation is especially simple if we fix a reference complex structure on X, so it is an open Riemann surface. By a theorem of Gunning and Narasimhan [46], such a surface admits a holomorphic immersion $z = u + \mathrm{i}v : X \to \mathbb{C}$. Its differential $dz = du + \mathrm{i}dv$ is a nowhere vanishing holomorphic 1-form on X trivializing the canonical bundle $T^*X = K_X$, $|dz|^2 = du^2 + dv^2$ is a Riemannian metric on X determining the given complex structure, $\frac{\mathrm{i}}{2} dz \wedge d\bar z = du \wedge dv$ is the associated area form, and $d\sigma = du\, dv$ is the surface measure on X. The function z is a local holomorphic coordinate on X at every point. Given a differentiable function $f : X \to \mathbb{C}$, its partial derivatives

$$f_z = \partial_z f = \partial f/dz, \qquad f_{\bar z} = \partial_{\bar z} f = \overline{\partial} f/d\bar z \tag{2.6}$$

are globally defined functions on X. Any Riemannian metric g on X is globally of the form (2.1) for some real functions E, F, G on X. (However, these coefficients are not functions of z unless z is injective.) We can write g in the form (2.3) where the function $\mu : X \to \mathbb{D}$ is given by (2.4). Conversely, any such function μ determines a Riemannian metric by (2.3), and hence a complex structure J_μ by (2.2). Note that $\mu = 0$ corresponds to the given reference complex structure on X. This global viewpoint will be important in the sequel.

The existence of isothermal charts on a Riemannian surface is a classical subject going back to Lagrange and Gauss. For a Hölder continuous μ see Korn [54], Lichtenstein [58], and Chern [20]. The existence of global quasiconformal homeomorphisms $\mathbb{C} \to \mathbb{C}$ follow from the local theorem by use of the uniformization theorem, and direct proofs were given by Ahlfors [1] and Vekua [78]. For a measurable function μ satisfying $\|\mu\|_\infty \le k < 1$ (where $\|\mu\|_\infty$ denotes the essential supremum), see Morrey [64] and Bojarski [18]. In this case, solutions of (2.5) are k-quasiconformal homeomorphisms having distributional derivatives in L^p for some $p \ge 1$. More precise results in $L^p(\mathbb{C})$ spaces, with continuous or smooth dependence of solutions of the Beltrami equation (2.5) on the Beltrami coefficient μ, are due to Ahlfors and Bers [3]; see also Ahlfors [2, Chapter V] and Astala et al. [14]. The result that we use in this paper is the following; see [14, Theorem 5.3.4].

Theorem 2.2. *An almost complex structure J of Hölder class $\mathscr{C}^{(k,\alpha)}$ $(k \in \mathbb{Z}_+,\ 0 < \alpha < 1)$ on a smooth Riemannian surface X admits a J-holomorphic chart of class $\mathscr{C}^{(k+1,\alpha)}$ at any point of X.*

A semiglobal version of this result with parameters is given by Theorem 4.1.

3. THE CAUCHY AND BEURLING TRANSFORMS ON DOMAINS IN OPEN RIEMANN SURFACES

In this section, we consider regularity properties of the Cauchy and Beurling transforms on smoothly bounded relatively compact domains in open Riemann surfaces. Theorem 3.2 is an important analytic ingredient for solving the Beltrami equation on such domains; see Theorems 4.1 and 4.3.

Let X be an open Riemann surface. Fix a holomorphic immersion $z = u + \mathrm{i}v : X \to \mathbb{C}$ (see [46]) and let $d\sigma = du\, dv$ denote the associated area measure on X. Given a differentiable function $f : U \to \mathbb{C}$ on a domain $U \subset X$, its derivatives f_z and $f_{\bar z}$ given by (2.6) are well-defined functions on U. The pullback of the Cauchy kernel $C(\zeta, z) = \frac{dz}{z-\zeta}$ on $\mathbb{C}$ by the immersion $z : X \to \mathbb{C}$ is a Cauchy-type kernel on X with the correct behaviour near the diagonal $D_X = \{(x, x) : x \in X\}$ (see (3.2)), but with additional poles if z is not injective. Since D_X has a basis of Stein neighbourhoods in $X \times X$ and $X \times X \setminus D_X$ is also Stein, one can remove the extra poles by solving a Cousin problem (see Scheinberg [75, Lemma 2.1]). This gives a meromorphic 1-form on $X \times X$ of the form

$$\omega(q, x) = \xi(q, x)dz(x) \quad \text{for } q, x \in X, \tag{3.1}$$

where $dz(x)$ denotes the restriction of dz to T_xX, ξ is a meromorphic function on $X \times X$ which is holomorphic on $X \times X \setminus D_X$, and the 1-form $\omega(q, \cdot\,)$ has a simple pole at $q \in X$ with residue 1. In a neighbourhood $U \subset X \times X$ of D_X the coefficient ξ of ω is of the form

$$\xi(q, x) = \frac{1}{z(x) - z(q)} + h(q, x), \tag{3.2}$$

where h is a holomorphic function on U. Such Cauchy kernels were constructed by Scheinberg [75] and Gauthier [40], following the work by Behnke and Stein [16, Theorem 3]. (See also Behnke and Sommer [15, p. 584] and [28, Remark 1, p. 141] for additional references.) Given a relatively compact smoothly bounded domain $\Omega \Subset X$, the usual argument using Stokes formula and the residue calculation gives the following Cauchy–Green formula for any $f \in \mathscr{C}^1(\overline{\Omega})$ and $q \in \Omega$:

$$\begin{aligned} f(q) &= \frac{1}{2\pi\mathrm{i}} \int_{x \in b\Omega} f(x)\, \omega(q, x) - \frac{1}{2\pi\mathrm{i}} \int_{x\in\Omega} \overline{\partial} f(x) \wedge \omega(q, x) \\ &= \frac{1}{2\pi\mathrm{i}} \int_{x \in b\Omega} f(x)\, \xi(q, x) dz(x) - \frac{1}{\pi} \int_{x\in\Omega} f_{\bar z}(x) \xi(q, x) d\sigma(x). \end{aligned}$$

If f is holomorphic in Ω, we obtain the Cauchy representation formula

$$f(q) = \frac{1}{2\pi\mathrm{i}} \int_{x\in b\Omega} f(x)\, \omega(q, x), \quad q \in \Omega.$$

On the other hand, for a function $f \in \mathscr{C}^1_0(X)$ with compact support we have

$$f(q) = -\frac{1}{\pi} \int_{x\in X} f_{\bar z}(x) \xi(q, x) d\sigma(x), \quad q \in X. \tag{3.3}$$

To the Cauchy kernel ω we associate two transforms, defined for $\phi \in \mathscr{C}_0(X)$ and $q \in X$ by

$$P(\phi)(q) = -\frac{1}{\pi} \int_X \phi(x) \xi(q, x) d\sigma(x), \tag{3.4}$$

$$S(\phi)(q) = \partial_z P(\phi)(q) = -\frac{1}{\pi} \int_X \phi(x) \partial_{z(q)} \xi(q, x) d\sigma(x). \tag{3.5}$$

Here, $\partial_{z(q)}\xi(q, x)$ denotes the ∂_z derivative (2.6) of the function $\xi(\cdot\,, x)$ at the point $q \in X$.

The operator P is called the Cauchy–Green transform, or simply the Cauchy operator associated to the Cauchy kernel (3.1). The integral converges absolutely, and we have that

$$\partial_{\bar z} \circ P = \mathrm{Id} = P \circ \partial_{\bar z} \quad \text{on } \mathscr{C}^1_0(X).$$

The second identity follows from (3.3). The first identity holds in a more precise form: for every relatively compact domain $\Omega \subset X$ with piecewise $\mathscr{C}^1$ boundary,

$$\partial_{\bar z} P(\phi) = \phi \ \text{ holds on } \Omega \text{ for every } \phi \in \mathscr{C}^1(\overline{\Omega}), \tag{3.6}$$

where the integral defining P is applied only over Ω. (Equivalently, we apply (3.4) to the function which equals ϕ on $\overline{\Omega}$ and equals 0 on $X \setminus \overline{\Omega}$.) The equation (3.6) holds in the distributional sense for every integrable ϕ. It is obtained by following the proof in the case when $\xi(q,x) = \frac{1}{x-q}$ on $X = \mathbb{C}$, when P equals the standard Cauchy–Green operator on $\mathbb{C}$:

$$\mathcal{C}(\phi)(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\zeta)}{\zeta - z} d\sigma(\zeta), \quad z \in \mathbb{C}. \tag{3.7}$$

The operator S (3.5) is an analogue of the Beurling transform $\mathcal{B}$ in the plane [2, 14, 57]:

$$\mathcal{B}(\phi)(z) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|z-\zeta|>\epsilon} \frac{\phi(\zeta)}{(z-\zeta)^2} d\sigma(\zeta), \quad z \in \mathbb{C}. \tag{3.8}$$

This is a singular convolution operator of Calderón–Zygmund type with nonintegrable kernel $-1/\pi z^2$, hence one must use the Cauchy principal value. Its main property is that $\mathcal{B} \circ \partial_{\bar z} = \partial_z$ on $\mathscr{C}^1_0(\mathbb{C})$, so $\mathcal{B}$ interchanges the operators $\partial_{\bar z}$ and ∂_z. Likewise, it follows from (3.3)–(3.5) that

$$S(\phi_{\bar z}) = \partial_z P(\phi_{\bar z}) = \phi_z \ \text{ for every } \phi \in \mathscr{C}^1_0(\Omega). \tag{3.9}$$

In order to understand the local regularity properties of P and S, we look more closely at their kernel functions $\xi(q,x)$ and $\partial_{z(q)}\xi(q,x)$. We consider the latter one, which is more involved; the analogous analysis applies to the former. Let $U \subset X \times X$ be an open neighbourhood of the diagonal D_X on which (3.2) holds. On U we have

$$\partial_{z(q)}\xi(q,x) = \partial_{z(q)} \frac{1}{z(x)-z(q)} + \partial_{z(q)} h(q,x) = \frac{1}{(z(x)-z(q))^2} + \partial_{z(q)} h(q,x),$$

and the function $\partial_{z(q)} h(q,x)$ is holomorphic on U. Fix a point $q_0 \in X$ and choose a neighbourhood $V \subset X$ of q_0 such that $V \times V \subset U$ and the immersion $z : X \to \mathbb{C}$ is injective on V. Pick a smooth cut-off function $\chi : X \to [0,1]$ with $\mathrm{supp}\chi \subset V$ such that $\chi = 1$ on a smaller neighbourhood $V' \Subset V$ of q_0. For $q \in V$ we have

$$\begin{aligned} S(\phi)(q) &= -\frac{1}{\pi} \int_X \chi(x)\phi(x)\partial_{z(q)}\xi(q,x) d\sigma(x) + \frac{1}{\pi} \int_X (\chi(x)-1)\phi(x)\partial_{z(q)}\xi(q,x) d\sigma(x) \\ &= S_1(\phi)(q) + S_2(\phi)(q), \end{aligned}$$

where the operators S_1 and S_2 are given by

$$\begin{aligned} S_1(\phi)(q) &= -\frac{1}{\pi} \int_X \frac{\chi(x)\phi(x)}{(z(x)-z(q))^2} d\sigma(x), \\ S_2(\phi)(q) &= -\frac{1}{\pi} \int_X \chi(x)\phi(x)\partial_{z(q)} h(q,x) d\sigma(x) \\ &\qquad + \frac{1}{\pi} \int_X (\chi(x)-1)\phi(x)\partial_{z(q)}\xi(q,x) d\sigma(x). \end{aligned}$$

In the complex coordinate z on V, $S_1(\phi) = \mathcal{B}(\chi\phi)$ is the Beurling operator applied to $\chi\phi$, while S_2 has smooth kernel. The same construction can be carried out with the operator P.

The conclusion is that the operators P and S have the same local regularity properties as their classical models $\mathcal{C}$ (3.7) and $\mathcal{B}$ (3.8), respectively.

Let Ω be a relatively compact, smoothly bounded domain in X. One may consider the truncated operators P and S, defined by integration over Ω (that is, extending the function ϕ under the integral to X by setting $\phi = 0$ on $X \setminus \overline{\Omega}$). While P has the expected regularity on Hölder spaces, the regularity of S fails at the boundary points of Ω since the effect of averaging in (3.7) is lost. To circumvent this problem, we shall use a bounded linear extension operator, which we now describe.

Let dist denote the distance function on a surface X induced by a smooth Riemannian metric. We recall the basics concerning Hölder spaces; see [41, Sect. 4.1] for more information. Let Ω be a domain in X. For $\alpha \in (0,1)$, the Hölder $\mathscr{C}^\alpha(\Omega)$ norm of a function $f : \Omega \to \mathbb{C}$ is given by

$$\|f\|_\alpha = \sup_{x\in\Omega} |f(x)| + \sup\{|f(x) - f(y)|/\mathrm{dist}(x,y)^\alpha : x, y \in \Omega,\ x \neq y\}, \tag{3.10}$$

and the associated Hölder space is $\mathscr{C}^\alpha(\Omega) = \{f : \Omega \to \mathbb{C} : \|f\|_\alpha < \infty\}$. Similarly we define the norm $\|f\|_{k,\alpha}$ for $k > 0$, and the corresponding Hölder space $\mathscr{C}^{(k,\alpha)}(\Omega)$, by adding to $\|f\|_\alpha$ in (3.10) the $\mathscr{C}^\alpha(\Omega)$ norms of the partial derivatives of f of highest order k. In particular, $\mathscr{C}^\alpha(\Omega) = \mathscr{C}^{(0,\alpha)}(\Omega)$. These spaces are Banach algebras with the pointwise product of functions. Every function in $\mathscr{C}^{(k,\alpha)}(\Omega)$ has a unique extension to a function in $\mathscr{C}^{(k,\alpha)}(\overline{\Omega})$. We shall need the following lemma.

Lemma 3.1. *Given a smoothly bounded relatively compact domain $\Omega \Subset X$ in a smooth open Riemannian surface X and a domain $\Omega' \subset X$ containing $\overline{\Omega}$, there is for every $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$ a continuous linear extension operator $E : \mathscr{C}^{(k,\alpha)}(\Omega) \to \mathscr{C}_0^{(k,\alpha)}(\Omega')$ with range in the space of compactly supported functions in $\mathscr{C}^{(k,\alpha)}(\Omega')$.*

Proof. For domains in Euclidean spaces and $k \geq 1$, this is [41, Lemma 6.37]; it is clear from the construction that one obtains a linear extension operator. (The cited lemma is stated for any relatively compact domain $\Omega \subset \mathbb{R}^n$ with $\mathscr{C}^{(k,\alpha)}$ boundary, but we believe that a smoothness of class at least $\mathscr{C}^{k+1}$ is needed since the composition of a $\mathscr{C}^{(k,\alpha)}$ function and a $\mathscr{C}^{(k,\alpha)}$ diffeomorphism, which is used in the proof to locally straighten the boundary $b\Omega$, is only of class $\mathscr{C}^{(k,\alpha^2)}$.) We can reduce to this case by noting that every component S of $b\Omega$ has a neighbourhood $U \subset \Omega'$ smoothly diffeomorphic to an annulus in $\mathbb{R}^2$, with S corresponding to the unit circle. Assume now that $k = 0$. Using the above notation, let $\tau : U \to S$ denote the (smooth) radial projection of the annulus onto the circle S. Set $U_+ = U \setminus \Omega$, and let $\chi : \overline{\Omega} \cup U \to [0,1]$ be a smooth function which equals 1 on $\overline{\Omega}$ and the restriction $\chi|_{U_+}$ has compact support. Given $f \in \mathscr{C}^\alpha(\overline{\Omega})$, we let $E(f) : \overline{\Omega} \cup U \to \mathbb{C}$ be defined by $E(f)(x) = f(x)$ for $x \in \overline{\Omega}$ and $E(f)(x) = \chi(x) f(\tau(x))$ for $x \in U_+$. We perform the same construction on each of the finitely many boundary components of Ω. It is easily verified that this extension has the required properties. □

With the notation of Lemma 3.1, define the operators P_Ω and S_Ω on $\phi \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega})$ and $q \in \overline{\Omega}$ by

$$P_\Omega(\phi)(q) \quad = \quad -\frac{1}{\pi} \int_{x\in\Omega'} E(\phi)(x)\xi(q,x)d\sigma(x), \tag{3.11}$$

$$S_\Omega(\phi)(q) \quad = \quad -\frac{1}{\pi} \int_{x\in\Omega'} E(\phi)(x)\partial_{z(q)}\xi(q,x)d\sigma(x). \tag{3.12}$$

Theorem 3.2. *Let X be an open Riemann surface with a Cauchy kernel* (3.1)*,* (3.2)*.*

(a) *$P_\Omega : \mathscr{C}^{(k,\alpha)}(\overline{\Omega}) \to \mathscr{C}^{(k+1,\alpha)}(\overline{\Omega})$ is a bounded linear operator for every $0 < \alpha < 1$ and $k \in \mathbb{Z}_+$, and it satisfies $\partial_{\bar z} P_\Omega(\phi) = \phi$ on $\overline{\Omega}$ for every $\phi \in \mathscr{C}^\alpha(\overline{\Omega})$.*
(b) *$S_\Omega : \mathscr{C}^{(k,\alpha)}(\overline{\Omega}) \to \mathscr{C}^{(k,\alpha)}(\overline{\Omega})$ is a bounded linear operator for every $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, and it satisfies $S_\Omega(\phi) = \partial_z P_\Omega(\phi)$ for every $\phi \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega})$.*

Proof. We have seen above that the operators P and S have the same local regularity properties as their classical models $\mathcal{C}$ (3.7) and $\mathcal{B}$ (3.8), respectively. Part (a) then follows from [14, Theorem 4.7.2] and (3.3), and part (b) follows from [14, Theorem 4.7.1] and (3.12). (The analogous properties hold on Sobolev spaces, but we shall not use them.) □

In this section, $z : X \to \mathbb{C}$ denotes a holomorphic immersion from an open Riemann surface X (see [46]). We shall call the pair (X, z) a *Riemann domain over* $\mathbb{C}$. Let Ω be a domain in X. Recall that for any $\mathscr{C}^1$ function $f : \Omega \to \mathbb{C}$, the derivatives $f_z = \partial f/dz$ and $f_{\bar z} = \overline{\partial} f/d\bar z$ (2.6) are well-defined functions on Ω. The following result gives a solution of the Beltrami equation on any smoothly bounded relatively compact domain for Beltrami coefficients with sufficiently small Hölder norm.

Theorem 4.1. *Let Ω be a smoothly bounded relatively compact domain in a Riemann domain (X, z). For any $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$ there is a constant $c = c(k, \alpha) > 0$ such that for every $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ with $\|\mu\|_{k,\alpha} < c$ there is function $f = f(\mu) \in \mathscr{C}^{(k+1,\alpha)}(\Omega)$ solving the Beltrami equation $f_{\bar z} = \mu f_z$, with $f(\mu)$ depending smoothly on μ and satisfying $f(0) = z|_\Omega$.*

Interpreting the function $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ as a complex structure J_μ on Ω (see (2.2)–(2.4)), with J_0 coinciding with the initial complex structure, the function $f(\mu) : \Omega \to \mathbb{C}$ is J_μ-holomorphic, and it is an immersion for μ close to 0 since $f(\mu)$ is then close to $f(0) = z|_\Omega$ in $\mathscr{C}^{(k+1,\alpha)}(\Omega)$. Thus, $(\Omega, J_\mu, f(\mu))$ is a family of Riemann domains over $\mathbb{C}$ depending smoothly on μ.

Proof of Theorem 4.1. The idea is inspired by the proof of the corresponding result for $\mu \in L^p(\mathbb{C})$ $(p > 2)$, due to Ahlfors and Bers [3, Theorem 4]. For simplicity of exposition we shall consider the case $k = 0$, noting that the same arguments apply for any $k \in \mathbb{Z}_+$.

Recall that the algebra $\mathrm{Lin}(E)$ of all bounded linear operators on a Banach space E, with functional composition as multiplication and the operator norm, is a unital Banach algebra (see Conway [22]). In our case, E will be the Banach space $\mathscr{C}^\alpha(\Omega)$.

We look for a solution of the Beltrami equation $f_{\bar z} = \mu f_z$ on Ω in the form

$$f = f(\mu) = z|_\Omega + P(\phi), \quad \phi \in \mathscr{C}^\alpha(\Omega). \tag{4.1}$$

Here, $P = P_\Omega : \mathscr{C}^\alpha(\Omega) \to \mathscr{C}^{1,\alpha}(\Omega)$ is the Cauchy–Green operator (3.11). Thus, $\phi = 0$ corresponds to $f = z|_\Omega$. By Theorem 3.2 (a), P is a continuous linear operator. We have

$$f_{\bar z} = \partial_{\bar z} P(\phi) = \phi, \qquad f_z = 1 + \partial_z P(\phi) = 1 + S(\phi),$$

where $S = S_\Omega \in \mathrm{Lin}(\mathscr{C}^\alpha(\Omega))$ is the (continuous, linear) Beltrami operator (3.12). The first identity follows from Theorem 3.2 (a) and the second one from the definition (3.12) of S. Inserting the above expressions in the Beltrami equation $f_{\bar z} = \mu f_z$ gives the following equation for ϕ:

$$\phi = \mu(S(\phi) + 1) = \mu S(\phi) + \mu.$$

Let $c_0 = \|S\|_\alpha > 0$ denote the operator norm of S on $\mathscr{C}^\alpha(\Omega)$ (see Theorem 3.2 (b)), so

$$\|S(\phi)\|_\alpha \le c_0 \|\phi\|_\alpha \quad \text{for all } \phi \in \mathscr{C}^\alpha(\Omega). \tag{4.2}$$

Note that the multiplication by $\mu \in \mathscr{C}^\alpha(\Omega)$ on the space $\mathscr{C}^\alpha(\Omega)$ is a linear operator of norm $\|\mu\|_\alpha$. Let us rewrite the above equation for ϕ in the form

$$(I - \mu S)\phi = \mu, \tag{4.3}$$

where I denotes the identity map on $\mathscr{C}^\alpha(\Omega)$. We now assume that

$$\|\mu\|_\alpha < c := 1/c_0 = 1/\|S\|_\alpha. \tag{4.4}$$

Then, $\|\mu S\|_\alpha \le \|\mu\|_\alpha c_0 < 1$, so the operator $I - \mu S$ is invertible with the bounded inverse

$$\Theta(\mu) = (I - \mu S)^{-1} = \sum_{j=0}^{\infty} (\mu S)^j \in \mathrm{Lin}(\mathscr{C}^\alpha(\Omega)). \tag{4.5}$$

The equation (4.3) then has a unique solution $\phi = \phi(\mu)$ given by

$$\phi = \Theta(\mu)\mu = \sum_{j=0}^{\infty} (\mu S)^j \mu.$$

Inserting into (4.1) gives a solution to the Beltrami equation $f_{\bar z} = \mu f_z$ on Ω:

$$f(\mu) = z|_\Omega + P(\Theta(\mu)\mu) = z|_\Omega + P((I - \mu S)^{-1}\mu) \in \mathscr{C}^{(1,\alpha)}(\Omega).$$

We claim that the map $\mu \mapsto f(\mu)$ is smooth on $\{\mu \in \mathscr{C}^\alpha(\Omega) : \|\mu\|_\alpha < c\}$. Since the map $P : \mathscr{C}^\alpha(\Omega) \to \mathscr{C}^{(1,\alpha)}(\Omega)$ is continuous linear and the evaluation map $\mathrm{Lin}(\mathscr{C}^\alpha(\Omega)) \times \mathscr{C}^\alpha(\Omega) \ni (A, \mu) \mapsto A\mu \in \mathscr{C}^\alpha(\Omega)$ is continuous bilinear, it suffices to see that the map $\mu \mapsto \Theta(\mu) = (I - \mu S)^{-1} \in \mathrm{Lin}(\mathscr{C}^\alpha(\Omega))$ is smooth on $\{\mu \in \mathscr{C}^\alpha(\Omega) : \|\mu\|_\alpha < c\}$. This follows from the fact that the inversion $A \mapsto A^{-1}$ is an analytic map on a neighbourhood of the identity on $\mathrm{Lin}(\mathscr{C}^\alpha(\Omega))$, as is seen from the expansion (4.5). Let us give a more explicit argument. For $\nu \in \mathscr{C}^\alpha(\Omega)$ we have

$$I - (\mu + \nu)S = (I - \mu S) - \nu S = (I - \mu S)(I - \Theta(\mu)\nu S).$$

If $\|\nu\|_\alpha$ is small enough then $\|\Theta(\mu)\nu S\|_\alpha < 1$, and taking inverses gives

$$\Theta(\mu + \nu) = (I - \Theta(\mu)\nu S)^{-1}\Theta(\mu). \tag{4.6}$$

As $\|\nu\|_\alpha \to 0$, the operator $\Theta(\mu)\nu S$ converges to zero in the operator norm, and hence the first term on the right hand side converges to the identity. This establishes continuity. Next, we show that $\Theta(\mu)$ is continuously differentiable with respect to μ on $\|\mu\|_\alpha < c$. Fix a constant $c_1 \in (0, c)$. Assume for a moment that $\|\mu\|_\alpha \le c_1$, and hence $\|\mu S\|_\alpha \le c_1 c_0 < c c_0 = 1$; see (4.4). From (4.5) we obtain

$$\|\Theta(\mu)\|_\alpha \le \sum_{j=0}^{\infty} (c_0 c_1)^j = \frac{1}{1 - c_0 c_1} =: C. \tag{4.7}$$

Pick a constant $c_2 \in (0, 1)$ and consider $\nu \in \mathscr{C}^\alpha(\Omega)$ such that $\|\nu\|_\alpha \le c_2/(C c_0)$. Then,

$$\|\Theta(\mu)\nu S\|_\alpha \le \|\Theta(\mu)\|_\alpha \|\nu\|_\alpha \|S\|_\alpha \le C\,\|\nu\|_\alpha c_0 \le c_2 < 1. \tag{4.8}$$

For such ν we consider the series expansion

$$(I - \Theta(\mu)\nu S)^{-1} = I + \Theta(\mu)\nu S + \sum_{j=2}^{\infty} (\Theta(\mu)\nu S)^j.$$

Note that $\nu \mapsto \Theta(\mu)\nu S$ is a linear map $\mathscr{C}^\alpha(\Omega) \mapsto \mathrm{Lin}(\mathscr{C}^\alpha(\Omega))$. In view of (4.8) we have

$$\begin{aligned}
\Big\| \sum_{j=2}^{\infty} (\Theta(\mu)\nu S)^j \Big\|_\alpha &\le \sum_{j=2}^{\infty} \|\Theta(\mu)\|_\alpha^j \, \|\nu\|_\alpha^j \, \|S\|_\alpha^j \\
&\le \|\nu\|_\alpha^2 \sum_{j=2}^{\infty} C^j \Big(\frac{c_2}{c_0 C}\Big)^{j-2} c_0^j = \frac{c_0^2 C^2}{1 - c_2} \|\nu\|_\alpha^2.
\end{aligned}$$

This shows that $\nu \mapsto (I - \Theta(\mu)\nu S)^{-1}$ is differentiable at $\nu = 0$ and its differential equals $\nu \mapsto \Theta(\mu)\nu S$. From this and (4.6) it follows that $\mu \mapsto \Theta(\mu)$ is differentiable at every μ with $\|\mu\|_\alpha < c = 1/\|S\|_\alpha$, and its differential equals

$$D_\mu \Theta(\mu)(\nu) = \Theta(\mu)\, \nu S\, \Theta(\mu), \quad \nu \in \mathscr{C}^\alpha(\Omega). \tag{4.9}$$

Since $\Theta(\mu)$ is continuous in μ, we see that $D_\mu\Theta(\mu)$ is continuous, so $\Theta(\mu)$ is continuously differentiable. In particular, $\Theta(\mu)$ is strongly differentiable in the sense of Nijenhuis [66]. Since the composition of continuously differentiable families of bounded linear operators is continuously differentiable, we see from (4.9) that the operator $D_\mu\Theta(\mu)$ is continuously differentiable on μ, so $\Theta(\mu)$ is of class $\mathscr{C}^2$ in μ. We can continue inductively to conclude that $\Theta(\mu)$ is smooth in μ. □

Remark 4.2. The fact that $\mu \to f(\mu)$ is smooth for $\|\mu\|_{k,\alpha} < c$ implies that if $\mu(t)$ is a $\mathscr{C}^l$ function of parameters $t = (t_1, \ldots, t_m) \in U \subset \mathbb{R}^m$ with $\|\mu(t)\|_{k,\alpha} < c$ for all $t \in U$, then the map $U \ni t \mapsto f(\mu(t)) \in \mathscr{C}^{k+1,\alpha}(\Omega)$ is also of class $\mathscr{C}^l(U)$. The analogous statement was proved by Ahlfors and Bers [3, Theorem 2] for solutions of the Beltrami equation with $\mu \in L^p(\mathbb{C})$ for $p > 2$. They also showed that if $\mu(t)$ is a holomorphic function of $t \in U \subset \mathbb{C}^m$ then the solution $f(\mu(t))$ is holomorphic in t as well. Holomorphic dependence of $f(\mu(t))$ on t also holds in the context of Theorem 4.1 if the map $t \mapsto \mu(t)$ is holomorphic.

Given an open Riemann surface (X, J) and a domain $\Omega \subset X$, a family of smooth diffeomorphisms $\Phi_b : \Omega \to \Phi_b(\Omega) \subset X$ $(b \in B)$ induces a family of complex structures $J_b = \Phi_b^* J$ on Ω. The following result shows that a converse also holds on any smoothly bounded domain $\Omega \Subset X$ for sufficiently small variations of the complex structure. This provides an essential tool used in the proofs of our main results. Recall that $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.

Theorem 4.3. *Assume that (X, z) is a Riemann domain over $\mathbb{C}$, Ω is a relatively compact smoothly bounded domain in X, and $a_1, \ldots, a_m \in \Omega$ are distinct points. For any $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$ there is a constant $c = c(k, \alpha) > 0$ such that for every function $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ with $\|\mu\|_{k,\alpha} < c$ there is a μ-conformal diffeomorphism $\Phi_\mu : \Omega \to \Phi_\mu(\Omega) \subset X$ of class $\mathscr{C}^{(k+1,\alpha)}$, depending smoothly on μ, such that $\Phi_0 = \mathrm{Id}_\Omega$ and $\Phi_\mu(a_j) = a_j$ for all such μ and $j = 1, \ldots, m$.*

Denoting by J_μ the complex structure on Ω determined by the function $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ (see Section 2), Theorem 4.3 gives a family $\Phi_\mu : \Omega \to \Phi_\mu(\Omega) \subset X$ of (J_μ, J_0)-biholomorphic maps onto their images in X with its given complex structure $J_0 = J$, with Φ_μ depending smoothly on μ in a neighbourhood of $\mu_0 = 0$ in $\mathscr{C}^{(k,\alpha)}(\Omega)$.

Remark 4.4. Note that every diffeomorphism $\Phi_\mu : \Omega \to \Phi_\mu(\Omega) \subset X$ in Theorem 4.3 is homotopic to the identity map on Ω by the homotopy $[0, 1] \ni t \mapsto \Phi_{t\mu}$. In particular, if the domain Ω is Runge in X then so is $\Phi_\mu(\Omega)$. In fact, a domain Ω in a Riemann surface X is Runge if and only if the inclusion-induced homomorphism $H_1(\Omega, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ of the first homology groups is injective, and this condition is clearly invariant under homotopies.

Proof of Theorem 4.3. If $c > 0$ is small enough then for every $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{k,\alpha} < c$ the function $f(\mu) : \Omega \to \mathbb{C}$, furnished by Theorem 4.1, is so close to the holomorphic immersion $f(0) = z|_\Omega : \Omega \to \mathbb{C}$ in the $\mathscr{C}^{(k+1,\alpha)}(\Omega)$ norm that it is an immersion. In the course of the proof, we shall decrease the constant $c > 0$ whenever needed.

If $f(\mu)$ is sufficiently close to $f(0) = z|_\Omega$, we can lift it with respect to the holomorphic immersion $z : X \to \mathbb{C}$ to a unique diffeomorphism $\Phi_\mu : \Omega \to \Phi_\mu(\Omega) \subset X$ of class $\mathscr{C}^{(k+1,\alpha)}(\Omega)$, close to $\Phi_0 = \mathrm{Id}_\Omega$, such that

$$z \circ \Phi_\mu = f(\mu) \quad \text{holds on } \Omega. \tag{4.10}$$

To see this, pick $r > 0$ such that for any $q \in \overline{\Omega}$ the immersion $z : X \to \mathbb{C}$ is injective on the disc $U_r(q) \subset X$ of radius r around q in the metric $|dz|^2$. If $f(\mu)(q) \in \mathbb{C}$ is close enough to $z(q) \in \mathbb{C}$, which holds if μ is close to 0, there is a unique point $p \in U_r(q)$ such that $z(p) = f(\mu)(q)$, and we set $\Phi_\mu(q) = p$. Thus, $\Phi_\mu(q)$ is the unique closest point to q among the points in the closed discrete set $z^{-1}(f(\mu)(q)) \cap \subset X$, so Φ_μ is well-defined on $\overline{\Omega}$. This implies $z(\Phi_\mu(q)) = z(p) = f(\mu)(q)$, so (4.10) holds. Since Φ_μ is locally obtained by postcomposing the immersion $f(\mu) : \overline{\Omega} \to \mathbb{C}$ with a local inverse of the J-holomorphic immersion $z : X \to \mathbb{C}$, Φ_μ is an immersion, its Beltrami coefficient is the same as that of $f(\mu)$ (which is μ), and the regularity properties remain unchanged. It is easily seen that Φ_μ is injective if $f(\mu)$ is close enough to z, which holds if $c > 0$ is small enough.

This shows that $\Phi_\mu : \Omega \to \Phi_\mu(\Omega) \subset X$ is a family of μ-conformal diffeomorphisms with the stated properties depending smoothly on μ, except that they need not satisfy the interpolation conditions $\Phi_\mu(a_j) = a_j$ for $j = 1, \ldots, m$. These are achieved as follows. For every $j = 1, \ldots, m$ we choose a holomorphic vector field v_j on X which is nonzero at the point a_j and it vanishes at all points a_i for $i \in \{1, \ldots, m\} \setminus \{j\}$. Let $t \to \psi_{j,t}$ denote the (local) flow of v_j for complex time t. If $c > 0$ is small enough, there is an open relatively compact domain $\Omega' \Subset X$ such that $\Phi_\mu(\Omega) \subset \Omega'$ holds for all $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{k,\alpha} < c$. Choose a bigger domain $\Omega'' \Subset X$ such that $\overline{\Omega'} \subset \Omega''$. Since $\overline{\Omega''}$ is compact, there is a $t_0 > 0$ such that the holomorphic map $\Psi_t := \psi_{1,t_1} \circ \cdots \circ \psi_{m,t_m} : \Omega'' \to X$ is well-defined for all $t = (t_1, \ldots, t_m) \in \mathbb{C}^m$ in the polydisc $\Delta^m_{t_0} = \{|t_j| < t_0,\ j = 1, \ldots, m\}$. Note that for every $t \in \Delta^m_{t_0}$ the map Ψ_t, being a composition of flows of holomorphic vector fields, is biholomorphic onto its image $\Psi_t(\Omega'') \subset X$. The choice of the vector fields v_j ensures, by the inverse function theorem, that for every m-tuple of points $a' = \{a'_1, \ldots, a'_m\} \subset \Omega$ such that a'_j is close enough to a_j for $j = 1, \ldots, m$ there is a unique $t = t(a') \in \Delta^m_{t_0}$ close to the origin such that $\Psi_t(a_j) = a'_j$ for $j = 1, \ldots, m$, and the map $a' \mapsto t(a')$ is holomorphic. Let $a'(\mu) = (\Phi_\mu(a_1), \ldots, \Phi_\mu(a_m))$. Then, for all $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{k,\alpha} < c$ for a small enough $c > 0$, the injective holomorphic map $\Psi^{-1}_{t(a'(\mu))} : \Omega' \to X$ sends the point $\Phi_\mu(a_j)$ back to a_j for $j = 1, \ldots, m$. Hence, replacing Φ_μ by $\Psi^{-1}_{t(a'(\mu))} \circ \Phi_\mu$ we also satisfy the interpolation condition at the points $a_1, \ldots, a_m$. □

Remark 4.5. (A) If ω is a Cauchy kernel on an open Riemann surface X (see Section 3) and $\Phi : Y \to X$ is an injective holomorphic map from another open Riemann surface Y, then the pullback $\Phi^*\omega$ is a Cauchy kernel on Y. Applying this observation to the smooth family of conformal diffeomorphisms $\Phi_\mu : \Omega \to \Phi_\mu(\Omega) \subset X$, furnished by Theorem 4.3, gives a family of Cauchy kernels $\omega_\mu = \Phi^*_\mu\omega$ on (Ω, J_μ) of the form (3.1), (3.2), with $\omega_0 = \omega$, whose entry functions $z_\mu = z \circ \Phi_\mu$ and h_μ depend smoothly on $\mu \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega}, \mathbb{D})$ in a neighbourhood of $\mu = 0$. In turn, we obtain a family of Cauchy–Green operators P_μ (3.4) on (Ω, J_μ) solving the $\overline{\partial}$-equation

$$\overline{\partial}_\mu P_\mu(\phi) = \phi \cdot d\bar{z}_\mu \quad \text{for } \phi \in \mathscr{C}^1(\overline{\Omega}).$$

For a fixed μ, the operator P_μ has the regularity properties given by Theorem 3.2. However, the joint regularity of the map $(\mu, \phi) \to P_\mu(\phi)$ seems less well understood. See Theorem 9.1 for a partial results in this direction, and [42, Theorem 4.5] due to Gong and Kim for regularity of the Cauchy operators on a 1-parameter family of domains in $\mathbb{C}$.

(B) Given $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ close to 0, the induced complex structures J_μ on Ω are $\mathscr{C}^{k+1}$-compatible with one another according to Theorem 2.2. It follows that for any smoothly bounded relatively compact domain D in Ω, the spaces $\mathscr{C}^{(s,\alpha)}(D, J_\mu)$ are independent of μ for $s + \alpha \leq k + 1$ and the norms are comparable. However, this no longer holds if $s + \alpha > k + 1$.

(C) Assume that B is a connected parameter space and $\{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathscr{C}^{l,(k,\alpha)}$ as in Theorem 1.1 or 1.4. Fix $b_0 \in B$ and let $z : X \to \mathbb{C}$ be a J_{b_0}-holomorphic immersion [46]. We obtain a family of Beltrami multipliers $\mu_b : X \to \mathbb{D}$ ($b \in B$) of the same class $\mathscr{C}^{l,(k,\alpha)}$, with $\mu_{b_0} = 0$, such that μ_b represents J_b as explained in Section 2. In particular, solutions of the Beltrami equation $f_{\bar{z}} = \mu f_z$ on a domain $\Omega \subset X$ are J_b-holomorphic functions on Ω. Here is a direct way to see this correspondence. Pick a smooth nowhere vanishing vector field v on X; such exists since the tangent bundle TX is trivial. Then, $V_b = v - \mathrm{i}J_b v$ for $b \in B$ is a family of nowhere vanishing $(1,0)$-vector fields on the Riemann surfaces (X, J_b), of class $\mathscr{C}^{l,(k,\alpha)}(B \times X)$. By duality, the family of complex $(1,0)$-forms θ_b on (X, J_b), determined by $\theta_b(V_b) = 1$, provides a simultaneous trivialization of the canonical bundles (T^*X, J_b). Then, $g_b = |\theta_b|^2$ is a Riemannian metric on X determining the complex structure J_b, and μ_b is the associated family of Beltrami multipliers of class $\mathscr{C}^{l,(k,\alpha)}$ given by (2.3) and (2.4). Theorems 4.1 and 4.3 will often be used in this way.

5. Runge theorem on families of open Riemann surfaces

In this section we prove Theorem 1.1. We first consider the basic case $l = k = 0$. Given a continuous function $\epsilon : B \to (0, +\infty)$, it suffices to prove that for any compact Runge set $L \subset X$ containing K in its interior there exist an open set $\Omega \subset X$ containing L and a continuous function $F \in \mathscr{C}(B \times \Omega)$ satisfying the following conditions for every $b \in B$.

(a) The function $F_b = F(b, \cdot\,) : \Omega \to \mathbb{C}$ is J_b-holomorphic.
(b) $\sup_{x \in K} |F_b(x) - f_b(x)| < \epsilon(b)$.
(c) $F_b - f_b$ vanishes in the points of the finite set $A' = A \cap L = \{a_1, \ldots, a_m\}$.

By slightly increasing K and adding to it small pairwise disjoint discs around the finitely many points in $A \cap (L \setminus K)$, we may assume that $A \cap L$ is contained in the interior of K. A function $B \times X \to \mathbb{C}$ satisfying Theorem 1.1 for $l = k = 0$ is then obtained by an obvious induction with respect to an exhaustion of X by an increasing family of compact Runge sets.

Secondly, it suffices to prove the result locally in the parameter. More precisely, given a point $b_0 \in B$, it suffices to find an open neighbourhood $B_0 \subset B$ of b_0 and a function $F : B_0 \times \Omega \to \mathbb{C}$ satisfying conditions (a)–(c) above for all $b \in B_0$. This gives a locally finite cover of B by open sets B_j and functions $F_j : B_j \times \Omega \to \mathbb{C}$ satisfying conditions (a)–(c) for $b \in B_j$. Choose a partition of unity $1 = \sum_j \chi_j$ with $\operatorname{supp} \chi_j \subset B_j$ for every j. The function $F : B \times \Omega \to \mathbb{C}$ defined by

$$F(b, x) = \sum_j \chi_j(b) F_j(b, x) \quad \text{for } b \in B \text{ and } x \in \Omega \tag{5.1}$$

then clearly satisfies conditions (a)–(c).

With these reductions in mind, we now consider the problem near a parameter value $b_0 \in B$. In the following exposition, we assume that $\{J_b\}_{b \in B}$ is of class $\mathscr{C}^{0,(k+1,\alpha)}$ for an arbitrary $k \in \mathbb{Z}_+$. We endow X with the Riemann surface structure determined by J_{b_0}. By Theorem 2.2, this structure is $\mathscr{C}^{k+1}$-compatible with the given smooth structure on X. Choose a J_{b_0}-holomorphic immersion $z : X \to \mathbb{C}$ (see [46]). Let μ_b denote the Beltrami coefficient corresponding to the complex structure J_b (see (2.2) and (2.4)), with $\mu_{b_0} = 0$. Choose a relatively compact, smoothly bounded domain $\Omega \subset X$ with $L \subset \Omega$. Note that $\mu_b \in \mathscr{C}^{(k,\alpha)}(\overline{\Omega}, \mathbb{D})$ depends continuously on $b \in B$ (see Remark 2.1). Let $c > 0$ be chosen such that Theorem 4.3 applies to all $\mu \in \mathscr{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{k,\alpha} < c$. By continuity of the map $b \mapsto \mu_b$ there is a neighbourhood $B_0' \subset B$ of b_0 such that $\|\mu_b\|_{k,\alpha} < c$ for all $b \in B_0'$. Let $\Phi_b : \Omega \to \Phi_b(\Omega) \subset X$ for $b \in B_0'$ be a continuous family of μ_b-conformal diffeomorphisms, furnished by Theorem 4.3. Thus, Φ_b is a biholomorphic map from the domain (Ω, J_b) onto its image $\Phi_b(\Omega) \subset (X, J_{b_0})$ such that $\Phi_b(a) = a$ for all $a \in A$ and $b \in B_0'$, and $\Phi_{b_0} = \mathrm{Id}_\Omega$. Choose a compact Runge set K' in X containing K in its interior such that f_{b_0} is holomorphic on a neighbourhood of K', and then pick a neighbourhood $B_0 \subset B_0'$ of b_0 such that $\Phi_b(K) \subset \mathring{K}'$ holds for all $b \in B_0$. By Runge theorem in open Riemann surfaces (see Behnke and Stein [16]), we can approximate f_{b_0} as closely as desired uniformly on K' by a J_{b_0}-holomorphic function $F_{b_0} : X \to \mathbb{C}$. The function $F_b = F_{b_0} \circ \Phi_b : \Omega \to \mathbb{C}$ for $b \in B_0$ is then J_b-holomorphic, it depends continuously on $b \in B_0$, and F_b is as close as desired to f_b uniformly on K provided that b is close enough to b_0 and F_{b_0} is close enough to f_{b_0} on K'. Indeed, for $x \in K$ we have

$$\begin{aligned} |F_b(x) - f_b(x)| \;\; &\le \;\; |F_{b_0} \circ \Phi_b(x) - f_{b_0} \circ \Phi_b(x)| \\ &\quad + |f_{b_0} \circ \Phi_b(x) - f_{b_0}(x)| + |f_{b_0}(x) - f_b(x)|, \end{aligned}$$

and each term on the right hand side is as small as desired if b is close enough to b_0 and F_{b_0} is close enough to f_{b_0} on K'. Hence, shrinking the neighbourhood B_0 around b_0 if necessary, the family $\{F_b\}_{b \in B_0}$ satisfies conditions (a) and (b). The interpolation condition (c) will be dealt with later.

Next, we consider fine approximation in the $\mathscr{C}^{l,k+1}$-topology for any pair of integers $0 \le l \le k+1$. As before, locally in the parameter we can use Theorem 4.3 in order to reduce the approximation problem for a variable family of complex structures to the case of a moving family of compact Runge sets in a fixed complex structure. With future applications in mind, we consider a more general situation for a family of compact holomorphically convex sets in a Stein manifold of arbitrary dimension.

Lemma 5.1. *Assume that B is a locally compact and paracompact Hausdorff space if $l = 0$, and a manifold of class $\mathscr{C}^l$ if $l > 0$. Let X be a Stein manifold, $\pi : B \times X \to B$ be the projection, and K be a closed subset of $B \times X$ such that the restricted projection $\pi|_K : K \to B$ is proper and for every $b \in B$ the fibre $K_b = \{x \in X : (b, x) \in K\}$ is $\mathscr{O}(X)$-convex. Assume that $U \subset B \times X$ is an open set containing K and $f : U \to \mathbb{C}$ is a function of class $\mathscr{C}^{l,0}(U)$ such that for every $b \in B$ the function $f_b = f(b, \cdot) : U_b = \{x \in X : (b, x) \in U\} \to \mathbb{C}$ is holomorphic. Then, $f \in \mathscr{C}^{l,\infty}(U)$ and for any $s \in \mathbb{Z}_+$, f can be approximated in the fine $\mathscr{C}^{l,s}$ topology on K by $\mathscr{C}^{l,\infty}$ functions $F : B \times X \to \mathbb{C}$ such that $F_b = F(b, \cdot) \in \mathscr{O}(X)$ for every $b \in B$. If B is a topologically closed $\mathscr{C}^l$ submanifold of $\mathbb{R}^n \subset \mathbb{C}^n$ (possibly with boundary), or a closed subset of $\mathbb{R}^n$ when $l = 0$, then f can be approximated in the fine $\mathscr{C}^{l,s}$ topology on K by holomorphic functions $F : \mathbb{C}^n \times X \to \mathbb{C}$.*

Remark 5.2. (a) Given K as in the lemma, for every $b_0 \in B$ and neighbourhood $W \supset K_{b_0}$ we have

$$K_b \subset W \ \text{ for all } b \in B \text{ sufficiently close to } b_0. \tag{5.2}$$

Otherwise, since $\pi : K \to B$ is proper, there would exist a sequence $(b_j, x_j) \in K$ with $\lim_{j\to\infty} b_j = b_0$ such that the sequence x_j has an accumulation point in $X \setminus K_{b_0}$, a contradiction since K is closed. Given an open neighbourhood $U \subset B \times X$ of K, it follows that there are a compact neighbourhood $B_0 \subset B$ of b_0 and an open set $U_0 \subset X$ such that $K \cap (B_0 \times X) \subset B_0 \times U_0 \subset U$.

(b) If B is a subset of $\mathbb{R}^n$, then a compact set $K \subset B \times X$ with $\mathscr{O}(X)$-convex fibres K_b is $\mathscr{O}(\mathbb{C}^n \times X)$-convex (see Remark 1.3 and Proposition 1.4 in [36]).

Proof of Lemma 5.1. The assumptions on the function f in the theorem clearly imply that it is of class $\mathscr{C}^{l,\infty}(U)$, so we may talk of $\mathscr{C}^{l,s}$ approximation for any $s \in \mathbb{Z}_+$.

We first consider the special case when B is compact and $U = B \times U'$, where $U' \subset X$ is an open set. Pick a compact $\mathscr{O}(X)$-convex set $K' \subset X$ such that $K' \subset U'$ and $K \subset B \times \mathring{K}'$. Choose a smoothly bounded strongly pseudoconvex domain D in X with $K' \subset D \Subset U'$. On D, there is a Henkin–Ramirez type kernel $\omega(x, \zeta)$, which is holomorphic in $x \in D$ for every $\zeta \in bD$, such that every $f \in \mathscr{O}(\overline{D})$ can be represented on D by the integral $f(x) = \int_{\zeta \in bD} f(\zeta)\omega(x, \zeta)$, $x \in D$. (See Henkin and Leiterer [51] or Lieb and Michel [59]. When X is an open Riemann surface, we can use a Cauchy kernel (3.1) on X.) In the case at hand, we have that $f(b, x) = \int_{\zeta \in bD} f(b, \zeta)\omega(x, \zeta)$ for all $x \in D$ and $b \in B$. Approximating the integral by Riemann sums gives a uniform approximation of f on $B \times K'$ by finite sums $\sum_i f(b, \zeta_i)g_i(x)$ $(b \in B,\ x \in D)$ for suitably chosen points $\zeta_i \in bD$ and functions $g_i \in \mathscr{O}(D)$ which come from the kernel $\omega(x, \zeta_i)$. If $l > 0$ then for any linear differential operator L of order $\le l$ in the variable $b \in B$ we have that $Lf(b, x) = \int_{\zeta \in bD} Lf(b, \zeta)\omega(x, \zeta)$. By adding more points $\zeta_i \in bD$ to the Riemann sum if necessary, we approximate $Lf(b, x)$ uniformly on $B \times K'$ by functions $\sum_i Lf(b, \zeta_i)g_i(x) = L \sum_i f(b, \zeta_i)g_i(x)$. This shows that f can be approximated in $\mathscr{C}^{l,0}(B \times K')$ by finite sums $\sum_i f(b, \zeta_i)g_i(x)$. By the Oka–Weil theorem, we can approximate g_i uniformly on K' by entire functions $g_i : X \to \mathbb{C}$. This gives approximation of f in $\mathscr{C}^{l,0}(B \times K')$ by functions $F : B \times X \to \mathbb{C}$ which are holomorphic on the fibres $\{b\} \times X$ and of class $\mathscr{C}^l$ in $b \in B$. Since $K \subset B \times \mathring{K}'$, it follows that $F - f$ can be arbitrarily small in $\mathscr{C}^{l,s}(K)$.

In the general case, Remark 5.2 (a) shows that we can find a locally finite cover of K by open sets of the form $B_j \times U_j \subset B \times X$ such that, for every j, the set $\overline{B}_j$ is compact and

$$K_j := K \cap (\overline{B}_j \times X) \subset \overline{B}_j \times U_j \subset U. \tag{5.3}$$

Fix a $\mathscr{C}^l$ partition of unity $\{\chi_j\}_j$ on a neighbourhood of $\pi(K)$ in B, subordinate to the cover $\{B_j\}_j$. The argument given in the special case shows that for every j we can approximate $f|_{K_j}$ as closely as desired in $\mathscr{C}^{l,s}(K_j)$ by functions $F_j : \overline{B}_j \times X \to \mathbb{C}$ of the form

$$F_j(b,x) = \sum_i h_{j,i}(b) g_{j,i}(x), \quad b \in \overline{B}_j,\ x \in X,$$

where $h_{j,i} \in \mathscr{C}^l(\overline{B}_j)$ and $g_{j,i} \in \mathscr{O}(X)$. (The partition of unity χ_j is kept fixed while performing the approximation on K_j.) We define the function $F : B \times X \to \mathbb{C}$ by

$$F(b,x) = \sum_j \chi_j(b) F_j(b,x) = \sum_{j,i} \chi_j(b) h_{j,i}(b) g_{j,i}(x) \quad \text{for } b \in B \text{ and } x \in X. \tag{5.4}$$

Clearly, F approximates f to a given precision in the fine $\mathscr{C}^{l,s}$ topology on K provided that $F_j|_{K_j}$ is sufficiently close to $f|_{K_j}$ in $\mathscr{C}^{l,s}(K_j)$ for every j.

Finally, if B is a closed submanifold of $\mathbb{R}^n \subset \mathbb{C}^n$ of class $\mathscr{C}^l$, possibly with boundary, we can apply [71, Theorem 1] by Range and Siu to approximate each function $\chi_j h_{j,i} \in \mathscr{C}^l(B)$ in (5.4) (which has compact support contained in B_j) in the fine $\mathscr{C}^l(B)$ topology by an entire function $\tilde h_{j,i} \in \mathscr{O}(\mathbb{C}^n)$. (Another argument is to extend $\chi_j h_{j,i}$ from the submanifold $B \subset \mathbb{R}^n$ to a $\mathscr{C}^l$ function on $\mathbb{R}^n$ and then approximate it in the fine $\mathscr{C}^l(\mathbb{R}^n)$ topology by entire functions using Carleman approximation theorem [19].) The function $\widetilde F = \sum_{j,i} \tilde h_{j,i} g_{j,i}$ is then holomorphic on $\mathbb{C}^n \times X$ and it approximates f in the fine $\mathscr{C}^{l,s}$ topology on K. For $l = 0$, the same holds if B is any closed subset of $\mathbb{R}^n$, which is seen by combining Tietze's extension theorem with Carleman approximation theorem. □

We now prove Theorem 1.1 for arbitrary integers $k \geq 0$ and $0 \leq l \leq k+1$. Let $K \subset X$ be as in the theorem, and pick a compact Runge set $L \subset X$ containing K in its interior. By the argument in the beginning of the proof, we may assume that K contains the finite set $A \cap L$ in its interior. Choose a smoothly bounded Runge domain $\Omega \Subset X$ such that $L \subset \Omega$. Fix a point $b_0 \in B$. By Theorem 4.3 and Remark 4.5 (C) there are a compact neighbourhood $B_0 \subset B$ of b_0 and a diffeomorphism

$$\Phi : B_0 \times \Omega \xrightarrow{\cong} \Phi(B_0 \times \Omega) \subset B_0 \times X, \quad \Phi(b,x) = (b, \phi(b,x)) = (b, \phi_b(x)) \tag{5.5}$$

of class $\mathscr{C}^{l,(k+1,\alpha)}$ (hence, of class $\mathscr{C}^l$ jointly in both variables (b,x)) such that for every $b \in B_0$, the map $\phi_b : \Omega \to \phi_b(\Omega) \subset X$ is a biholomorphism from (Ω, J_b) onto $(\phi_b(\Omega), J_{b_0})$ satisfying

$$\phi_b(a) = a \ \text{ for all } a \in A \cap L, \text{ and } \phi_{b_0} = \mathrm{Id}_\Omega. \tag{5.6}$$

Clearly, Φ has a continuous inverse $\Phi^{-1}(b,z) = (b, \psi(b,z))$, and if $l > 0$ then Φ^{-1} and hence ψ are of class $\mathscr{C}^l$ by the inverse function theorem. Furthermore, the following observations are simple consequences of the chain rule, and we leave the proof to the reader.

Lemma 5.3. (a) *If Φ as above is of class $\mathscr{C}^{l,k+1}$ and $l \leq k+1$, then Φ^{-1} is of class $\mathscr{C}^{l,k+1-l}$.*
(b) *If $f(b,x)$ is of class $\mathscr{C}^{l,k}$ and $g(b,z)$ is of class $\mathscr{C}^{l,l+k}$, then $g(b,f(b,x))$ is of class $\mathscr{C}^{l,k}$.*

Lemma 5.4. *Assume that $0 \leq l \leq k+1$, $f \in \mathscr{C}^{l,0}(B_0 \times \Omega)$, and $f(b,\cdot) : \Omega \to \mathbb{C}$ is J_b-holomorphic for every $b \in B_0$. Then, the function $F = f \circ \Phi^{-1} : \Phi(B_0 \times \Omega) \to \mathbb{C}$ is of class $\mathscr{C}^{l,\infty}(B_0 \times X)$ when X is endowed with the complex structure J_{b_0}, $F(b,\cdot) : \phi_b(\Omega) \to \mathbb{C}$ is J_{b_0}-holomorphic for every $b \in B_0$, and $f \in \mathscr{C}^{l,(k+1,\alpha)}(B_0 \times \Omega)$ in the complex structure J_{b_0}. (Hence, $f \in \mathscr{C}^{l,k+1}(B_0 \times \Omega)$ in the original smooth structure on Ω.) The analogous result holds for maps to any complex manifold.*

Proof. Clearly, F is continuous. Since $F(b,\cdot) = f(b, \psi(b,\cdot))$ is a composition of the (J_{b_0}, J_b)-holomorphic map $\psi(b,\cdot)$ and the J_b-holomorphic function $f(b,\cdot)$, $F(b,\cdot)$ is J_{b_0}-holomorphic for every $b \in B_0$. It follows that $F \in \mathscr{C}^{0,\infty}$ in the complex structure J_{b_0} on X. Since $f = F \circ \Phi$ and Φ is of class $\mathscr{C}^{l,(k+1,\alpha)}$, we infer that f is of class $\mathscr{C}^{0,(k+1,\alpha)}$ in the J_{b_0} structure. This proves the lemma for $l = 0$. Suppose now that $l > 0$. Then, ψ is of class $\mathscr{C}^l$. We shall prove that F is of class $\mathscr{C}^l$ in the variable $b \in B_0$, and hence of class $\mathscr{C}^{l,\infty}$ (since it is J_{b_0}-holomorphic in the space variable). We make the calculation in a local coordinate b of class $\mathscr{C}^l$ on B_0, and we assume for simplicity of exposition that $B_0 = [0,1] \subset \mathbb{R}$. On X, we use a J_{b_0}-holomorphic coordinate z. Differentiating the equation $F(b,z) = f(b,\psi(b,z))$ on b and denoting the partial derivatives by the lower case indices gives

$$F_b(b,z) = f_b(b,\psi(b,z)) + f_x(b,\psi(b,z))\psi_b(b,z). \tag{5.7}$$

Here, f_x denotes the total derivative of f with respect to a smooth local coordinate $x = (u,v)$ on X. This shows that $F_b(b,z)$ exists and is continuous in (b,z). Since F is also holomorphic in z, it follows that $F \in \mathscr{C}^{1,\infty}$ and therefore $f = F \circ \Phi \in \mathscr{C}^{1,(k+1,\alpha)}$ in the complex structure J_{b_0} on X. (Hence, $f \in \mathscr{C}^{1,k+1}$ in the original smooth structure on X.) Suppose now that $l \geq 2$, so $k+1 \geq l \geq 2$, $\psi \in \mathscr{C}^2$, and $f \in \mathscr{C}^{2,0} \cap \mathscr{C}^{1,2}$. Differentiating the equation (5.7) on b gives

$$F_{bb} = f_{bb} + 2f_{bx}\psi_b + f_{xx}(\psi_b)^2 + f_x\psi_{bb},$$

and F_{bb} is continuous in (b,z). Since it is holomorphic in z, it follows that $F \in \mathscr{C}^{2,\infty}$ and therefore $f \in \mathscr{C}^{2,(k+1,\alpha)}$. This process can be continued up to $l = k+1$ but not beyond. □

We continue with the proof of Theorem 1.1. The compact set $\widetilde{K} = \Phi(B_0 \times K) \subset B_0 \times X$ clearly satisfies the conditions of Lemma 5.1. In particular, its fibre $\widetilde{K}_b = \Phi_b(K)$ is Runge in $\Phi_b(\Omega)$ for every $b \in B_0$, and since $\Phi_b(\Omega)$ is Runge in X (see Remark 4.4), $\widetilde{K}_b$ is Runge in X as well. Recall that U is an open neighbourhood of $B \times K$ and $f : U \to \mathbb{C}$ is an X-holomorphic function of class $\mathscr{C}^{l,0}$. Pick an open set $V \subset B_0 \times X$ such that $\overline{V} \subset U \cap (B_0 \times \Omega)$. Let $\widetilde{V} = \Phi(V) \subset B_0 \times X$. We have $f = \tilde{f} \circ \Phi$ where by Lemma 5.4 the function $\tilde{f} = f \circ \Phi^{-1} : \widetilde{V} \to \mathbb{C}$ is X-holomorphic and of class $\mathscr{C}^{l,\infty}$ with respect to the complex structure J_{b_0} on X. By Lemma 5.1, for any given $s \in \mathbb{Z}_+$ we can approximate $\tilde{f}$ in $\mathscr{C}^{l,s}(\widetilde{K})$ by an X-holomorphic function $\widetilde{F} : B_0 \times X \to \mathbb{C}$ of class $\mathscr{C}^{l,\infty}$. If s is chosen big enough and the approximation is close enough, then the function $F = \widetilde{F} \circ \Phi : B_0 \times \Omega \to \mathbb{C}$ is of class $\mathscr{C}^{l,k+1}$, it is X-holomorphic, and it approximates f in the $\mathscr{C}^{l,k+1}$ topology on $B_0 \times K$.

This gives a locally finite open cover B_j of B such that f can be approximated as closely as desired in the $\mathscr{C}^{l,k+1}$ topology on $B_j \times K$ by X-holomorphic functions $F_j : \overline{B}_j \times \Omega \to \mathbb{C}$ of class $\mathscr{C}^{l,k+1}$. Choose a $\mathscr{C}^l$ partition of unity $1 = \sum_j \chi_j$ on B with $\mathrm{supp}\,\chi_j \subset B_j$ for every j. Assuming that each F_j is close enough to f in $\mathscr{C}^{l,k+1}(B_j \times K)$ for every j, the X-holomorphic function $F : B \times \Omega \to \mathbb{C}$ defined by (5.1) is of class $\mathscr{C}^{l,k+1}$ and it satisfies the required approximation condition.

It remains to obtain the interpolation conditions (c) at the points of $A' = A \cap L = \{a_1, \ldots, a_m\} \subset \mathring{K}$. It suffices to explain this in the local situation given by Lemma 5.1; the subsequent steps in the proof preserve this condition up to order $k+1$ (the degree of smoothness of $F(b,x)$ in x). In view of (5.5) and (5.6), the points of A' are fixed under the maps $\phi(b,\cdot)$. Choose $r \in \mathbb{Z}_+$ and set $n = m(r+1)$; this is the complex dimension of the space of complex r-jets (including the values) of holomorphic functions on X at the points of A'. By the classical function theory on open Riemann surfaces, we can find a family of J_{b_0}-holomorphic functions $\xi_t : X \to \mathbb{C}$, depending holomorphically on $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$, such that $\xi_0 = 0$ and for every collection of r-jets at the points of A' there is precisely one member ξ_t of this family which assumes these r-jets at the given points. Let F be a function in (5.4) which approximates f to a given precision in the fine $\mathscr{C}^{l,k+1}$ topology on $B \times K$. Hence, the r-jets of the function $F_b = F(b,\cdot)$ at the points of A' are close to the respective r-jets of $f_b = f(b,\cdot)$ for any $b \in B$. We subtract from each F_b the appropriate uniquely determined member of the family ξ_t so that the r-jets of the new function at the points of A' agree with those of f_b (i.e., the interpolation condition (c) holds.) This does not affect the approximation condition (b) very much since the jets of ξ_t in question are close to those of the zero function, and hence ξ_t is uniformly close to zero on K. This completes the proof of Theorem 1.1.

6. MERGELYAN THEOREM ON FAMILIES OF OPEN RIEMANN SURFACES

In this section, we prove Theorem 1.2. See also Theorems 7.5 and 7.7 for manifold-valued maps.

We begin with the basic case when K is a compact Runge set in $X = \mathbb{C}$. Thus, let $\{J_b\}_{b\in B}$ be a continuous family of complex structures of class $\mathscr{C}^\alpha$ on $\mathbb{C}$ and $f : B \times K \to \mathbb{C}$ be a continuous function such that $f_b = f(b, \cdot\,)$ is J_b-holomorphic in $\mathring{K}$ for every $b \in B$. Choose an open disc $D \subset \mathbb{C}$ with $K \subset D$. Given a point $b_0 \in B$ and a number $\epsilon > 0$, we shall find an open neighbourhood $B_0 \subset B$ of b_0 and a continuous function $F : B_0 \times D \to \mathbb{C}$ such that for every $b \in B_0$, the function $F_b = F(b, \cdot\,) : D \to \mathbb{C}$ is J_b-holomorphic and satisfies $\sup_{z\in K} |F_b(z) - f_b(z)| < \epsilon$ and the interpolation conditions in the points of the finite set $A \subset \mathring{K}$. The proof is then concluded by using a partition of unity on B together with Theorem 1.1.

By the Riemann mapping theorem, J_{b_0} is either the standard complex structure $J_{\rm st}$ on $\mathbb{C}$ or the standard complex structure on a disc containing $\overline{D}$. The proof is the same in both cases, so let us assume the former case. Choose a pair of discs $D' \Subset \Omega \Subset \mathbb{C}$ such that $\overline{D} \subset D'$. By Theorem 4.1 and Remark 4.5 (C) there are a compact neighbourhood $B_1 \subset B$ of b_0 and a family of diffeomorphisms $\Phi_b : \Omega \to \Phi_b(\Omega) =: \Omega_b \subset \mathbb{C}$ of class $\mathscr{C}^{1,\alpha}(\Omega)$, depending continuously on $b \in B_1$, such that $\Phi_{b_0} = \mathrm{Id}_\Omega$ and Φ_b is biholomorphic from (Ω, J_b) onto $(\Omega_b, J_{\rm st})$. For every $b \in B_1$ set

$$K_b = \Phi_b(K), \quad D_b = \Phi_b(D), \quad \tilde f_b = f_b \circ \Phi_b^{-1} : K_b \to \mathbb{C}. \tag{6.1}$$

The function $\tilde f_b$ is continuous on K_b, $J_{\rm st}$-holomorphic in the interior $\mathring{K}_b$, and it depends continuously on $b \in B_1$. After shrinking B_1 around b_0 if necessary, we may assume that

$$K_b \subset D_b \subset D' \subset \Omega_b \ \text{ for all } b \in B_1. \tag{6.2}$$

It now suffices to find a neighbourhood $B_0 \subset B_1$ of b_0 and a continuous function $\widetilde F : B_0 \times D' \to \mathbb{C}$ such that for every $b \in B_0$, the function $\widetilde F_b = \widetilde F(b, \cdot\,) : D' \to \mathbb{C}$ is holomorphic in the standard structure $J_{\rm st}$ on $\mathbb{C}$ and it satisfies $\sup_{z\in K_b} |\widetilde F_b(z) - \tilde f_b(z)| < \epsilon$. Indeed, the function $F_b = \widetilde F_b \circ \Phi_b$ is then J_b-holomorphic on D and satisfies $\sup_{z\in K} |F_b(z) - f_b(z)| < \epsilon$ for every $b \in B_0$. In summary, locally in the parameter we changed the Mergelyan approximation problem with respect to a family of complex structures to a similar problem on a moving family of compact sets in $\mathbb{C}$ with respect to the standard complex structure. We now show that this task can be realised by inspecting the proof of the classical Mergelyan's theorem [63]; see also Gaier [38, p. 97], Gamelin [39], and Rudin [72].

To simplify the notation, we drop the tildes and consider $f_b : K_b \to \mathbb{C}$ as a family of continuous functions that are holomorphic on $\mathring{K}_b$ and depend continuously on $b \in B_1$. Since the compact sets K_b in (6.1) vary continuously with b, the set $\bigcup_{b\in B_1} K_b$ is compact, and by (6.2) it is contained in D'. By Tietze's extension theorem, there is a continuous function $f : B_1 \times \mathbb{C} \to \mathbb{C}$ with compact support contained in $B_1 \times D'$ such that $f(b, \cdot\,)$ agrees with the given function f_b on K_b for every $b \in B_1$. For $\delta > 0$ we denote by $\omega(\delta)$ the modulus of continuity of f with respect to the second variable:

$$\omega(\delta) = \max\{|f(b,z) - f(b,w)| : b \in B_1,\ z, w \in \mathbb{C},\ |z - w| \le \delta\}.$$

Note that $\lim_{\delta\to 0} \omega(\delta) = 0$, and we choose δ small enough such that

$$6000\, \omega(\delta) < \epsilon. \tag{6.3}$$

Define the function $A_\delta : \mathbb{C} \to \mathbb{R}_+$ by setting $A_\delta(z) = 0$ for $|z| > \delta$ and

$$A_\delta(z) = \frac{3}{\pi\delta^2}\left(1 - \frac{|z|^2}{\delta^2}\right)^2, \qquad 0 \le |z| \le \delta.$$

The convolution of f and A_δ with respect to the z variable is a continuous function f^δ on $B_1 \times \mathbb{C}$ with compact support whose differential with respect to the z-variable exists and is also continuous in both variables. Furthermore, for every $b \in B_1$ and $z \in \mathbb{C}$ we have that

$$|f(b,z) - f^\delta(b,z)| < \omega(\delta) \ \text{ and } \ \Big|\frac{\partial f^\delta}{\partial \bar z}(b,z)\Big| < \frac{2\omega(\delta)}{\delta},$$

and

$$f^\delta_b := f^\delta(b,\cdot\,) = f_b \ \text{ on } \ K^\delta_b := \{z \in K_b : \mathrm{dist}(z, \mathbb{C}\setminus K_b) > \delta\}.$$

These properties follow directly from the case for a single b, which is proved in the cited papers. Decreasing $\delta > 0$ if necessary, we may assume that $\mathrm{supp} f^\delta_b \subset D'$ for all $b \in B_1$. Note that K^δ_b is an open set contained in $\mathring{K}_b$. Since $f^\delta_b = f_b$ on K^δ_b, it follows that f^δ_b is holomorphic on K^δ_b and hence

$$\mathrm{supp}(\overline\partial f^\delta_b) \subset L_b := \overline{D}' \setminus K^\delta_b \quad \text{for all } b \in B_1. \tag{6.4}$$

Recall that $K_{b_0} = K$. We cover the compact set $L_{b_0} = \overline{D}' \setminus K^\delta_{b_0}$ by finitely many open discs $\Delta_j = \mathbb{D}(z_j, 2\delta)$ $(j = 1,\ldots,n)$ of radius 2δ with centres $z_j \in \mathbb{C}\setminus K$. Since $\mathbb{C}\setminus K$ is connected, each disc $\overline\Delta_j$ contains a compact Jordan arc E_j connecting its centre z_j to a boundary point of Δ_j such that $E_j \cap K = \varnothing$. Set $E = \bigcup_{j=1}^n E_j$. Since the compact sets K_b in (6.1) depend continuously on b, there is a neighbourhood $B_0 \subset B_1$ of b_0 such that for all $b \in B_0$ and $j = 1,\ldots,n$ we have

$$L_b \subset \bigcup_{i=1}^n \Delta_i \quad \text{and} \quad E \cap K_b = \varnothing.$$

In particular, the open set $\mathbb{C}\setminus E$ contains $\bigcup_{b\in B_0} K_b$. For $b \in B_0$ consider the sets

$$L_{b,1} = L_b \cap \overline\Delta_1, \quad L_{b,j} = L_b \cap \overline\Delta_j \setminus (L_{b,1}\cup\ldots\cup L_{b,j-1}) \ \text{ for } \ j = 2,\ldots,n.$$

These sets depend continuously on b. Write $\zeta = u + \mathrm{i}v$. By the Cauchy–Green formula (3.3) we have

$$f^\delta_b(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f^\delta_b(\zeta)}{\partial\bar\zeta}\frac{du\,dv}{z-\zeta} = \sum_{j=1}^n \frac{1}{\pi}\int_{L_{b,j}} \frac{\partial f^\delta_b(\zeta)}{\partial\bar\zeta}\frac{du\,dv}{z-\zeta} \quad \text{for } z \in \mathbb{C} \text{ and } b \in B_1. \tag{6.5}$$

The main point now is to approximate the Cauchy kernel $\frac{1}{z-\zeta}$ for $z \in \mathbb{C}\setminus E_j$ and $\zeta \in \Delta_j$ sufficiently well by a function of the form

$$P_j(z,\zeta) = g_j(z) + (\zeta - c_j) g_j(z)^2,$$

where $g_j \in \mathscr{O}(\mathbb{C}\setminus E_j)$ and $c_j \in \mathbb{C}$. This is accomplished by Mergelyan's lemma [63], which says that g_j and c_j can be chosen such that for all $z \in \mathbb{C}\setminus E_j$ and $\zeta \in \Delta_j$ we have

$$|P_j(z,\zeta)| < \frac{50}{\delta} \quad \text{and} \quad \left|P_j(z,\zeta) - \frac{1}{z-\zeta}\right| < \frac{4000\,\delta^2}{|z-\zeta|^3}. \tag{6.6}$$

(See [38, p. 101] or [72, Lemma 20.2].) The function

$$F^\delta_b(z) = \sum_{j=1}^n \frac{1}{\pi}\int_{L_{b,j}} \frac{\partial f^\delta_b(\zeta)}{\partial\bar\zeta} P_j(z,\zeta)\,du\,dv \tag{6.7}$$

depends continuously on $b \in B_0$, for every b it is holomorphic in the open set $\mathbb{C}\setminus E \supset K_b$ (since $P_j(\cdot\,,\zeta)$ is holomorphic on $\mathbb{C}\setminus E_j$ for every j), and it follows from (6.3) and (6.5)–(6.7) that

$$|F^\delta_b(z) - f^\delta_b(z)| < 6000\,\omega(\delta) < \epsilon \quad \text{for all } z \in \mathbb{C}\setminus E \text{ and } b \in B_0.$$

The proof of the first estimate can be found in the cited works. Finally, the interpolation in finitely many interior points of K can be handled as in the proof of Theorem 1.1.

This proves Theorem 1.2 when K is a compact Runge set in $X = \mathbb{C}$. Let us now consider the general case when X is an arbitrary open Riemann surface. We shall adjust Bishop's localization theorem [17] to the variable complex structure setting, using solutions of the $\overline\partial$-equation as in Sakai's paper [74]. See also the exposition in [28, pp. 142–143].

Assume that $K \subset X$ and $f : B \times K \to \mathbb{C}$ are as in the statement of Theorem 1.2. In particular, there is a number $c > 0$ such that each relatively compact connected component of $X \setminus K$ has diameter at least c in some fixed Riemannian metric on X. By deleting a point from X if necessary, we may assume that X is an open surface. Let $\widehat{K}$ denote the smallest compact Runge set in X containing K. (In any complex structure on X, $\widehat{K}$ is the holomorphically convex hull of K.) Choose a smoothly bounded relatively compact Runge domain $\Omega \Subset X$ with $\widehat{K} \subset \Omega$. Let $B_0 \subset B$ be a compact neighbourhood of b_0 such that Theorem 4.1 applies on Ω for all $b \in B_0$. In particular, we have continuous families of J_b-holomorphic immersions $z_b : \Omega \to \mathbb{C}$, Cauchy kernels $\omega_b = \Phi_b^* \omega$ where Φ_b are given by Theorem 4.3, and Cauchy operators P_b on (Ω, J_b) for $b \in B_0$ (see Remark 4.5 (A)). We cover K by finitely many open coordinate discs $U_1, \ldots, U_m \subset \Omega$ of diameter at most c. Choose closed discs $D_j \subset U_j$ for $j = 1, \ldots, m$ whose interiors still cover K. Note that $U_j \setminus (K \cap D_j)$ is connected for every j. (Indeed, a relatively compact component of $U_j \setminus (K \cap D_j)$ is also a connected component of $X \setminus K$ which is contained in $\mathring{D}_j$, so it has diameter $< c$, contradicting the assumption.) Let χ_j be a smooth partition of unity on a neighbourhood of K with respect to the cover $\{U_j\}$. Since every U_j is a planar domain, the special case of the theorem proved above shows that for any $\epsilon > 0$ there are continuous functions $f_j : B_0 \times U_j \to \mathbb{C}$ such that $f_j(b, \cdot\,) : U_j \to \mathbb{C}$ is J_b-holomorphic for every $b \in B_0$ and satisfies

$$\max_{x \in K \cap D_j} |f_j(b,x) - f(b,x)| < \epsilon \ \text{ for every } b \in B_0 \text{ and } j = 1, \ldots, m.$$

Set $g = \sum_{j=1}^m \chi_j f_j$ and $g_b = g(b, \cdot\,)$ for $b \in B_0$. Let $\overline{\partial}_{J_b}$ denote the $(0,1)$-derivative with respect to the complex structure J_b and the immersion $z_b : \Omega \to \mathbb{C}$ (see (2.6)). On some open neighbourhood $U = U_\epsilon \subset \Omega$ of K we then have $\|g - f\|_{\mathscr{C}(B_0 \times U)} = O(\epsilon)$ and

$$\overline{\partial}_b\, g = \sum_{j=1}^m \overline{\partial}_b \chi_j \cdot f_j = \sum_{j=1}^m \overline{\partial}_b \chi_j \cdot (f_j - f), \qquad \|\overline{\partial}_b\, g\|_{\mathscr{C}(U)} = O(\epsilon),$$

where the bound in $O(\epsilon)$ is uniform for $b \in B_0$. (In the second equality we used that $\sum_{j=1}^m \overline{\partial}_b \chi_j = 0$ near K.) Let $\chi \in \mathscr{C}_0^\infty(U)$ be a cut-off function with $0 \le \chi \le 1$ and $\chi \equiv 1$ near K. Then we have $\|\chi \cdot \overline{\partial}_b g\|_{\mathscr{C}(U)} = O(\epsilon)$, and hence $\|P_b(\chi \cdot \overline{\partial} g)\|_{\mathscr{C}(U)} = O(\epsilon)$. For every $b \in B_0$ the function

$$F(b, \cdot\,) = g_b - P_b(\chi \cdot \overline{\partial}_b\, g_b)$$

is then J_b-holomorphic on a neighbourhood of K, which can be chosen independent of $b \in B_0$, it depends continuously on $b \in B_0$ and satisfies $\|F - f\|_{\mathscr{C}(B_0 \times K)} = O(\epsilon)$. Continuity in b is seen by noting that the $(0,1)$-forms $\xi_b = \chi \cdot \overline{\partial}_b\, g_b$ have compact support contained in the interior of Ω, and solving $\overline{\partial} u_b = \xi_b$ reduces to solving the same problems for the family of $(0,1)$-forms $(\Phi_b^{-1})^*(\xi_b)$ in the fixed complex structure J_0 on X. (See Theorem 9.1 for the case $l = 0$.)

This proves the approximation result for $b \in B_0$. Interpolation in finitely many points of $\mathring{K}$ can be obtained by the same argument as in the proof of Theorem 1.1. The proof is concluded by finding a locally finite cover $\{B_j\}_j$ of B by sets on which the above conditions hold and combining the approximants F_j by a partition of unity on B subordinate to this cover; see (5.1).

7. The Oka principle for maps from families of open Riemann surfaces to Oka manifolds

In this section we prove Theorem 1.4. The same proof also gives the generalization stated in Theorem 7.4. We then prove a couple of Mergelyan-type approximation theorems for manifold-valued maps from families of open Riemann surfaces; see Theorems 7.5 and 7.7. With future applications in mind, the technical results in Lemmas 7.2 and 7.3 are obtained in the bigger generality when X is a Stein manifold of arbitrary dimension.

Recall that a compact set K in a complex manifold X is said to be a *Stein compact* if it admits a basis of open Stein neighbourhoods. Every compact $\mathscr{O}(X)$-convex set in a Stein manifold X is a Stein compact (see [52, Theorem 5.1.6]). Given a complex manifold Y, we denote by

$$\overline{\mathscr{O}}(K,Y)$$

the space of continuous maps $K \to Y$ which are uniform limits of holomorphic maps on open neighbourhoods of K in X. Furthermore, we denote by

$$\overline{\mathscr{O}}_{\rm loc}(K,Y)$$

the space of continuous maps $f : K \to Y$ with the property that every point $x \in K$ has an open neighbourhood $U \subset X$ such that $f|_{K\cap\overline{U}} \in \overline{\mathscr{O}}(K\cap\overline{U})$. Clearly, we have the inclusions

$$\{f|_K : f \in \mathscr{O}(K,Y)\} \subset \overline{\mathscr{O}}(K,Y) \subset \overline{\mathscr{O}}_{\rm loc}(K,Y) \subset \mathscr{A}(K,Y).$$

The importance of the space $\overline{\mathscr{O}}_{\rm loc}(K,Y)$ lies in the following result of Poletsky [70, Theorem 3.1].

Theorem 7.1 (Poletsky [70])**.** *If K is a Stein compact in a complex manifold X, Y is a complex manifold and $f \in \overline{\mathscr{O}}_{\rm loc}(K,Y)$, then the graph of f on K is a Stein compact in $X \times Y$.*

We consider $\mathbb{R}^n$ as the standard real subspace of $\mathbb{C}^n$. By using Theorem 7.1 we prove the following.

Lemma 7.2. *Let X be a Stein manifold and $\pi : \mathbb{C}^n \times X \to \mathbb{C}^n$ be the projection. Assume that $K \subset \mathbb{C}^n \times X$ is a compact set such that $B := \pi(K) \subset \mathbb{R}^n$ and $K_b = \{x \in X : (b,x) \in K\}$ is $\mathscr{O}(X)$-convex for every $b \in B$. Let U be an open neighbourhood of K in $B \times X$, Y be a complex manifold, and $f : U \to Y$ be a continuous map such that for every $b \in B$ the map $f_b = f(b,\cdot\,) : U_b = \{x \in X : (b,x) \in U\} \to Y$ is holomorphic. Then, the graph*

$$G_f = \{(b,x,f(b,x)) : (b,x) \in K\} \subset \mathbb{C}^n \times X \times Y \tag{7.1}$$

of f on K is a Stein compact in $\mathbb{C}^n \times X \times Y$, and $f \in \overline{\mathscr{O}}(K,Y)$. Furthermore, given $\epsilon > 0$ there are a neighbourhood V of K in $\mathbb{C}^n \times X$, a neighbourhood $\widetilde{U} \subset U \cap V$ of K in $B \times X$, a holomorphic map $\tilde{f} : V \to Y$, and a homotopy $g_t : \widetilde{U} \to Y$ $(t \in I = [0,1])$ satisfying the following conditions.

(a) $g_0 = f|_{\widetilde{U}}$ *and* $g_1 = \tilde{f}|_{\widetilde{U}}$.
(b) $g_t(b,\cdot\,) : \widetilde{U}_b \to Y$ *is holomorphic for every* $b \in B$ *and* $t \in I$.
(c) $\sup_K \mathrm{dist}_Y(g_t,f) < \epsilon$ *for all* $t \in I$.

If in addition B is a $\mathscr{C}^l$ submanifold of $\mathbb{R}^n$ (possibly with boundary) for some $l \in \mathbb{N}$ and $f \in \mathscr{C}^{l,0}(U,Y)$, then for any $s \in \mathbb{Z}_+$ and after shrinking $U \supset K$, the homotopy f_t can be chosen such that, in addition to the above, $f_t \in \mathscr{C}^{l,s}(U,Y)$ for all $t \in I$ and the approximation in (c) holds in $\mathscr{C}^{l,s}(K)$.

Proof. By Remark 5.2 (b), the set K is $\mathscr{O}(\mathbb{C}^n \times X)$-convex, whence a Stein compact. We shall now verify that the map f satisfies the conditions in Theorem 7.1, and hence G_f (7.1) is a Stein compact.

Fix a point $b_0 \in B$. Since the map $f_{b_0} : U_{b_0} \to Y$ is holomorphic and K_{b_0} is a Stein compact, the graph $G_{b_0} = \{(b_0,x,f_{b_0}(x)) : x \in K_{b_0}\}$ has an open Stein neighbourhood $\Gamma \subset \mathbb{C}^n \times X \times Y$ by Siu's theorem [76] (see also Colţoiu [21], Demailly [23, Theorem 1], and [32, Theorem 3.1.1]). Choose a holomorphic embedding $\Theta : \Gamma \hookrightarrow \mathbb{C}^N$. By a theorem of Docquier and Grauert [24] (see also [32, Theorem 3.3.3]) there are a neighbourhood $O \subset \mathbb{C}^N$ of $\Theta(\Gamma)$ and a holomorphic retraction $\rho : O \to \Theta(\Gamma)$. By continuity of $f(b,\cdot\,)$ with respect to $b \in B$ and in view of (5.2) there is a compact neighbourhood $B_0 \subset B$ of b_0 such that, setting

$$S := \{(b,x) : b \in B_0,\ x \in K_b\} \subset U,$$

we have that $\widetilde{S} := \{(b,x,f(b,x)) : (b,x) \in S\} \subset \Gamma$. Hence, the map

$$h(b,x) := \Theta(b,x,f(b,x)) \in O \subset \mathbb{C}^N \tag{7.2}$$

is well-defined on a neighbourhood of S in $B \times X$, and $h(b,\cdot)$ is holomorphic on a neighbourhood of K_b in X for every $b \in B_0$. By Lemma 5.1 we can approximate h as closely as desired uniformly on S by a holomorphic map $\tilde{h} : W \to \mathbb{C}^N$ from a neighbourhood $W \subset \mathbb{C}^n \times X$ of S. Assuming that the approximation is close enough and the neighbourhood $W \supset S$ is small enough, we have that $\tilde{h}(W) \subset O$. Let $\tau : \mathbb{C}^n \times X \times Y \to Y$ denote the projection. The map

$$\tilde{f} := \tau \circ \Theta^{-1} \circ \rho \circ \tilde{h} : W \to Y \tag{7.3}$$

is then well defined and holomorphic, and it approximates f uniformly on S. Since this holds for every $b_0 \in B$, we see that $f \in \overline{\mathscr{O}}_{\rm loc}(K, Y)$. Hence, Theorem 7.1 implies that G_f (7.1) is a Stein compact.

To prove that $f \in \overline{\mathscr{O}}(K, Y)$ and the last statement in the lemma, we apply the same argument with the entire parameter space B. Choose a Stein neighbourhood $\Gamma \subset \mathbb{C}^n \times X \times Y$ of G_f (7.1), a holomorphic embedding $\Theta : \Gamma \hookrightarrow \mathbb{C}^N$, and a holomorphic retraction $\rho : O \to \Theta(\Gamma)$ from a neighbourhood $O \subset \mathbb{C}^N$ of $\Theta(\Gamma)$. The map h given by (7.2) is now defined on a neighbourhood $\widetilde{U} \subset (B \times X) \cap U$ of K, and the map $h(b,\cdot)$ is holomorphic on $\widetilde{U}_b$ for every $b \in B$. By Lemma 5.1 we can approximate h as closely as desired uniformly on K by a holomorphic map $\tilde{h} : V \to \mathbb{C}^N$ from a neighbourhood $V \subset \mathbb{C}^n \times X$ of K. As before, we may assume that $\tilde{h}(V) \subset O$. The map $\tilde{f} : V \to Y$ given by the formula (7.3) is then holomorphic and it approximates f on K as closely as desired. Furthermore, if $\tilde{h}$ is close enough to h on K and after shrinking the neighbourhood $\widetilde{U} \supset K$ if necessary, the family of convex combinations

$$h_t = (1-t)h + t\tilde{h} : \widetilde{U} \to \mathbb{C}^N, \quad t \in I \tag{7.4}$$

assumes values in O. The family of maps

$$g_t = \tau \circ \Theta^{-1} \circ \rho \circ h_t : \widetilde{U} \to Y, \quad t \in I \tag{7.5}$$

is then a homotopy from $g_0 = f|_{\widetilde{U}}$ to $g_1 = \tilde{f}|_{\widetilde{U}}$ with the stated properties.

The last statement of the lemma follows by the same argument, using Lemma 5.1 with approximation in $\mathscr{C}^{l,s}(K)$. ☐

The next result is a version of Lemma 5.1 for maps with values in an Oka manifold and with homotopies added to the picture. This is the main technical ingredient in the proof of Theorem 1.4.

Lemma 7.3. *Assume that $B'' \subset \mathbb{R}^n$ is a neighbourhood retract and $B_0 \subset B_1 \subset B \subset B'$ are compact subsets of B'', each of them contained in the interior of the next one. Let X be a Stein manifold, $\pi : \mathbb{C}^n \times X \to \mathbb{C}^n$ be the projection, and $K \subset \mathbb{C}^n \times X$ be a compact subset such that $\pi(K) \subset B$ and the fibre $K_b = \{x \in X : (b,x) \in K\}$ is $\mathscr{O}(X)$-convex for every $b \in B$. Assume that U is an open neighbourhood of K in $B' \times X$, Y is an Oka manifold, and $f : B' \times X \to Y$ is a continuous map such that for every $b \in B$ the map $f_b = f(b,\cdot) : X \to Y$ is holomorphic on $U_b = \{x \in X : (b,x) \in U\}$. Fix $\epsilon > 0$ and $s \in \mathbb{Z}_+$. After shrinking the open set $U \supset K$, there is a homotopy $f_t : B \times X \to Y$ $(t \in I = [0,1])$ with the following properties.*

(a) $f_0 = f|_{B \times X}$.
(b) *$f_t(b,\cdot) : X \to Y$ is holomorphic on U_b for every $b \in B$ and $t \in I$.*
(c) *f_t approximates f in $\mathscr{C}^{0,s}(K)$ to precision ϵ.*
(d) *$f_t(b,\cdot) = f(b,\cdot)$ for all $b \in B \setminus B_1$ and $t \in I$.*
(e) *The map $f_1(b,\cdot) : X \to Y$ is holomorphic for every b in a neighbourhood of B_0.*

If in addition B is a $\mathscr{C}^l$ submanifold of $\mathbb{R}^n$ (possibly with boundary) for some $l \in \mathbb{N}$ and $f \in \mathscr{C}^{l,0}(B \times X, Y)$, then for any $s \in \mathbb{Z}_+$ the homotopy f_t can be chosen such that, in addition to the above, $f_t \in \mathscr{C}^{l,s}(B \times X, Y)$ for all $t \in I$ and the approximation in (c) holds in $\mathscr{C}^{l,s}(K)$.

Proof. We focus on the case $l = 0,\ s = 0$. It will be clear that the same proof gives the corresponding results in the general case by using the corresponding versions of Lemmas 5.1 and 7.2.

By the assumption, there are a neighbourhood $V'' \subset \mathbb{C}^n$ of B'' and a retraction $\rho : V'' \to B''$ onto B''. The conditions imply that B' is a neighbourhood of B in B''. Since $\rho|_{B'}$ is the identity map, it follows that there is an open neighbourhood $V \subset \mathbb{C}^n$ of B such that $V \subset V''$ and $\rho(V) \subset B'$. Replacing $f(b,x)$ by $f(\rho(b),x)$ extends f to a continuous map $V \times X \to Y$, still denoted f. Since every compact subset B of $\mathbb{R}^n$ is polynomially convex in $\mathbb{C}^n$ [77, p. 3], V may be chosen Stein. If B is a $\mathscr{C}^l$ submanifold of $\mathbb{R}^n$ then the retraction ρ as above always exists and can be chosen of class $\mathscr{C}^l$.

We claim that there are an open neighbourhood $W \subset V \times X$ of K in $\mathbb{C}^n \times X$ and a homotopy $g_t : W \to Y$ $(t \in I = [0,1])$ connecting $g_0 = f$ to a holomorphic map $g_1 : W \to Y$ such that

$$\sup_{(b,x)\in K} \mathrm{dist}_Y(g_t(b,x), f(b,x)) < \epsilon/2 \ \text{ holds for all } t \in I$$

and the map $g_t(b,\cdot\,) : W_b \to Y$ is holomorphic for every $b \in B$ and $t \in I$. Note that Lemma 7.2 furnishes a homotopy g_t with the desired properties on a neighbourhood of K in $B \times X$; see (7.2), (7.4), and (7.5). In the present situation, all maps in the construction of g_t are defined on a neighbourhood of K in $\mathbb{C}^n \times X$. Hence, the same argument, using convex combinations as in (7.4) and defining g_t by (7.5), yields a desired homotopy on a neighbourhood $W \subset V \times X$ of K in $\mathbb{C}^n \times X$.

Pick a smooth function $\chi : \mathbb{C}^n \times X \to [0,1]$ with support in W such that $\chi = 1$ on a smaller neighbourhood $W' \Subset W$ of K. Consider the map $h_0 : V \times X \to Y$ given by

$$h_0(z,x) = g_{\chi(z,x)}(z,x) \quad \text{for } z \in V \text{ and } x \in X. \tag{7.6}$$

For $(z,x) \in W'$ we have $\chi = 1$ and hence $h_0|_{W'} = g_1|_{W'}$, which is a holomorphic map. On $(V \times X) \setminus W$ we have $\chi = 0$ and hence $h_0 = g_0 = f$. Furthermore, h_0 is homotopic to f by the homotopy $I \ni t \mapsto g_{t\chi}$, and every map in this homotopy has the same properties as f.

Since V is Stein, the set $K \subset V \times X$ is $\mathscr{O}(\mathbb{C}^n \times X)$-convex (see Remark 5.2 (b)), and Y is an Oka manifold, the main result of Oka theory (see [32, Theorem 5.4.4]) furnishes a homotopy $h_t : V \times X \to Y$ $(t \in I)$ from h_0 to a holomorphic map $h_1 : V \times X \to Y$ such that the homotopy $f_t : V \times X \to Y$ $(t \in I)$ given by

$$f_t = \begin{cases} g_{2t\chi}, & 0 \le t \le 1/2, \\ h_{2t-1}, & 1/2 \le t \le 1 \end{cases}$$

satisfies conditions (a)–(c) (where we take $s = 0$ in (c)), and it satisfies condition (e) for all $b \in B$ since $f_1 = h_1$. To obtain (d), choose a smooth function $\xi : \mathbb{R}^n \to [0,1]$ which equals 1 on a neighbourhood of B_0 and vanishes on $B \setminus B_1$, and replace $f_t(b,\cdot\,)$ by $f_{t\xi(b)}(b,\cdot\,)$ for $b \in B$ and $t \in I$.

This proves the lemma for $l = s = 0$. The same arguments apply when $s > 0$, and also for $l > 0$ when B is a $\mathscr{C}^l$ submanifold of $\mathbb{R}^n$, noting that the approximation by holomorphic functions in $\mathscr{C}^{l,s}(K)$ is furnished by Lemma 5.1 and the existence of a homotopy $\{g_t\}_{t\in I}$ with approximation in $\mathscr{C}^{l,s}(K)$ is given by Lemma 7.2. □

Proof of Theorem 1.4. We consider the case $l = k = 0$. The arguments in the general case are similar by using the corresponding version of Lemma 7.3.

Let $\{J_b\}_{b\in B}$ be a family of complex structures on X as in the theorem. We shall say that a closed subset $K \subset B \times X$ is Runge in $B \times X$ if it satisfies the assumptions of the theorem, that is, the projection $\pi|_K : K \to B$ is proper and every fibre $K_b = \{x \in X : (b,x) \in K\}$ $(b \in B)$ is Runge in X or empty. Recall that a continuous map $f : B \times X \to Y$ is said to be X-holomorphic on an open set $U \subset B \times X$ if $f_b = f(b,\cdot\,)$ is J_b-holomorphic on $U_b = \{x \in X : (b,x) \in U\}$ for every $b \in B$.

We first explain the proof in the case when the parameter space B is compact. Let $K^0 = K \subset B \times X$ and $f^0 = f : B \times X \to Y$ be as in the theorem, so f^0 is X-holomorphic on a neighbourhood of K^0. Choose an increasing sequence of compact Runge sets $K'_1 \subset K'_2 \subset \cdots \subset \bigcup_{j=1}^\infty K'_j = X$ such that every set is contained in the interior of the next one, and let $K^j = B \times K'_j \subset B \times X$ for $j = 1, 2, \ldots$. We choose K'_1 big enough such that $K^0 \subset K^1$. Given a decreasing sequence $\epsilon_j > 0$, we shall find a sequence of maps $f^j : B \times X \to Y$ and homotopies $f^j_t : B \times X \to Y$ $(t \in I = [0, 1])$ satisfying the following conditions for every $j = 1, 2, \ldots$.

(i) f^j is X-holomorphic on a neighbourhood of K^j. (By the assumption, this also holds for $j = 0$.)
(ii) $f^j_0 = f^{j-1}$ and $f^j_1 = f^j$.
(iii) f^j_t is X-holomorphic on a neighbourhood of K^{j-1} for all $t \in I$.
(iv) $\max_{K^{j-1}} \mathrm{dist}_Y(f^{j-1}, f^j_t) < \epsilon_j$ for all $t \in I$.
(v) The homotopy $f^j_t(b, \cdot\,)$ is fixed for all b in a neighbourhood of Q in B.

Assuming that the sequence $\epsilon_j > 0$ is chosen to converge to 0 sufficiently fast, the limit map $F = \lim_{j\to\infty} f^j : B \times X \to Y$ exists and is X-holomorphic, it approximates f as closely as desired uniformly on K, and $F(b, \cdot\,) = f(b, \cdot\,)$ holds for all $b \in Q$. Furthermore, the homotopies f^j_t $(j \in \mathbb{N},\ t \in I)$ can be assembled into a single homotopy $f_t : B \times X \to Y$ $(t \in I)$ from $f_0 = f$ to $f_1 = F$ such that f_t is X-holomorphic on a neighbourhood of K, it approximates f on K for every $t \in I$, and it is fixed for all $b \in Q$.

Every step in the induction is of the same kind, so it suffices to explain the initial step, that is, the construction of a homotopy $f^1_t : B \times X \to Y$ $(t \in I)$ which is X-holomorphic on a neighbourhood of K^0, it approximates $f = f^0$ on K^0, it is fixed for b in a neighbourhood of the subset $Q \subset B$ (see condition (v) in the theorem), and such that the map $f^1_1 = f^1$ is X-holomorphic on a neighbourhood of K^1. This is accomplished by a finite induction with respect to an increasing family of compact subsets of the parameter space B, which we now explain.

Recall that $K^0 \subset K^1 = B \times L$, where L is a compact Runge set in X. If the subset $Q \subset B$ in condition (v) is nonempty, it has a compact neighbourhood $\widetilde{Q} \subset B$ such that the map f^0_b is holomorphic on a neighbourhood of L for every $b \in \widetilde{Q}$. Let $\widetilde{K}^0 \subset B \times X$ denote the compact set with fibres

$$\widetilde{K}^0_b = \begin{cases} L, & b \in \widetilde{Q}; \\ K^0_b, & b \in B \setminus \widetilde{Q}. \end{cases} \tag{7.7}$$

Clearly, $K^0 \subset \widetilde{K}^0 \subset K^1$ and $\widetilde{K}^0$ is Runge in $B \times X$. If $Q = \varnothing$, we take $\widetilde{Q} = \varnothing$ and hence $K^0 = \widetilde{K}^0$. Pick a smoothly bounded domain $\Omega \Subset X$ and domains $V, V' \subset X$ such that

$$L \subset V \subset V' \subset \Omega \tag{7.8}$$

and the closure of each of these sets is contained in the interior of the next one. Fix a point $b_0 \in B$. The conditions on B imply that there is a neighbourhood $P'' \subset B$ of b_0 which is an ENR (see Definition 1.3). We may therefore consider P'' as a neighbourhood retract in some $\mathbb{R}^n \subset \mathbb{C}^n$. By Theorem 4.3 and Remark 4.5 (C), there are a compact neighbourhood P' of b_0, contained in the interior of P'', and a continuous family of biholomorphic maps $\Phi_b : (\Omega, J_b) \to (\Phi_b(\Omega), J_{b_0})$ $(b \in P')$ such that

$$\Phi_b(V) \subset V' \subset \Phi_b(\Omega) \ \text{ holds for every } b \in P'. \tag{7.9}$$

Pick a compact neighbourhood $P \subset B$ of b_0 contained in the interior of P'. Let $K' \subset L'$ be compact subsets of $P \times X$ whose fibres over any point $b \in P$ are given by

$$K'_b = \Phi_b(\widetilde{K}^0_b), \qquad L'_b = \Phi_b(L).$$

By (7.7)–(7.9) we have that

$$K'_b \subset L'_b \subset \Phi_b(V) \subset V' \quad \text{for all } b \in P.$$

Consider the maps

$$f'_b = f_b \circ \Phi_b^{-1} : \Phi_b(\Omega) \to Y, \quad b \in P.$$

Since f_b is J_b-holomorphic on a neighbourhood of $\widetilde{K}_b^0$ and the map $\Phi_b : (\Omega, J_b) \to (\Phi_b(\Omega), J_{b_0})$ is biholomorphic, f'_b is J_{b_0}-holomorphic on a neighbourhood of K'_b for every $b \in P$. Pick a pair of smaller neighbourhoods $P_0 \subset P_1 \subset P$ of b_0, each of them contained in the interior of the next one. Lemma 7.3, applied with X replaced by $V' \subset X$, furnishes a homotopy of maps

$$f'_{t,b} : V' \to Y \quad \text{for } b \in P \text{ and } t \in I$$

satisfying conditions (a)–(e) in the lemma with the sets $B_0 \subset B_1 \subset B$ replaced by $P_0 \subset P_1 \subset P$. In particular, $f'_{t,b} = f'_{0,b} = f'_b$ holds for $b \in P \setminus P_1$, $f'_{t,b}$ approximates f'_b on K'_b for $b \in P$, and the map $f'_{1,b}$ is J_{b_0}-holomorphic on V' for b in a neighbourhood of P_0. By (7.9) we have $\Phi_b(V) \subset V'$. Hence,

$$f_{t,b} := f'_{t,b} \circ \Phi_b : V \to Y \quad \text{for } b \in P \text{ and } t \in I \tag{7.10}$$

is a homotopy of maps which are J_b-holomorphic on a neighbourhood of $\widetilde{K}_b^0$, they approximate f_b uniformly on $\widetilde{K}_b^0$, they agree with $f_{0,b} = f_b$ for $b \in P \setminus P_1$ (so we can extend the family to all $b \in B$), and the map $f_b^1 := f_{1,b} : V \to Y$ is J_b-holomorphic for all b in a neighbourhood of P_0. By using a cut-off function in the parameter of the homotopy, we can extend the maps $f_{t,b}$ to X without changing their values on a neighbourhood of L (compare with (7.6)).

If the sets $Q \subset \widetilde{Q}$ are nonempty, we make another modification to the above homotopy to ensure condition (v) in the theorem. Choose a function $\chi : B \to [0,1]$ such that $\chi = 1$ on $B \setminus \widetilde{Q}$ and $\chi = 0$ on a neighbourhood of Q. With $f_{t,b}$ as in (7.10), we set

$$\tilde{f}_{t,b} = f_{t\chi(b),b} \quad \text{for } b \in B \text{ and } t \in I.$$

For $b \in B$ in a neighbourhood of Q we then have $\tilde{f}_{t,b} = f_{0,b} = f_b$ as desired, and the other required properties still hold.

What was just explained serves as a step in a finite induction which we now describe.

The assumptions imply that there is a finite family of triples $P_0^j \subset P_1^j \subset P^j$ $(j = 1, 2, \ldots, m)$ of compact sets in B such that $\bigcup_{j=1}^m P_0^j = B$ and the above construction can be performed on each of these triples with the same sets in (7.8). The induction proceeds as follows.

In the first step, we perform the procedure explained above on the first triple (P_0^1, P_1^1, P^1) with the set K^0 and the map $g^0 := f^0 = f$. The resulting map $g^1 : B \times X \to Y$ is X-holomorphic on a neighbourhood of the compact set

$$S^1 := \left[(P_0^1 \times X) \cap K^1\right] \cup \left[((B \setminus P_0^1) \times X) \cap K^0\right] \subset B \times X, \tag{7.11}$$

and $g_b^1 = f_b^0$ holds for all b in a neighbourhood of Q. Note that the fibre S_b^1 of S^1 over any point $b \in B$ is Runge in X. Indeed, we have $S_b^1 = L$ for $b \in P_0^1$ and $S_b^1 = K_b^0$ for $b \in B \setminus P_0^1$. Since $K_b^0 \subset L$ for every $b \in B$, the set S^1 is compact and Runge in $B \times X$. Furthermore, we obtain a homotopy from $f^0 = g^0$ to g^1 such that every map in the homotopy is X-holomorphic on a neighbourhood of K^0 and it approximates f^0 there, and the homotopy is fixed for b in a neighbourhood of $(B \setminus P_1^1) \cup Q$.

In the second step, the same argument is applied to the map g^1 on the triple (P_0^2, P_1^2, P^2) with respect to the set S^1 in (7.11). The resulting map $g^2 : B \times X \to Y$ is X-holomorphic on a neighbourhood of the compact Runge set

$$S^2 = \left[((P_0^1 \cup P_0^2) \times X) \cap K^1\right] \cup \left[((B \setminus (P_0^1 \cup P_0^2) \times X) \cap K^0\right] \subset B \times X. \tag{7.12}$$

Note that $S_b^2 = L$ for $b \in P_0^1 \cup P_0^2$ and $S_b^2 = S_b^1 = K_b^0$ for $b \in B \setminus (P_0^1 \cup P_0^2)$. We also obtain a homotopy from g^1 to g^2 consisting of maps which are X-holomorphic on a neighbourhood of S^1, they approximate g^1 there, and the homotopy is fixed for b in a neighbourhood of $(B \setminus P_1^2) \cup Q$.

Proceeding inductively, we obtain after m steps a map $g^m : B \times X \to Y$ which is X-holomorphic on a neighbourhood of $S^m = K^1 = B \times L$. We define $f^1 := g^m$. Furthermore, the individual homotopies between the subsequent maps g^j and g^{j+1} for $j = 0, 1, \ldots, m-1$ can be assembled into a homotopy f^1_t $(t \in I)$ from $f^1_0 = f^0 = g^0$ to $f^1_1 = f^1 = g^m$ such that f^1_t is X-holomorphic on a neighbourhood of K^0 for all $t \in I$ and the homotopy is fixed for $b \in B$ in a neighbourhood of Q. This completes the proof of the theorem if the parameter space B is compact.

In the general case when B is paracompact and locally compact, we choose a normal exhaustion $B_1 \subset B_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} B_j = B$ by compact sets and a normal exhaustion $L_1 \subset L_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} L_j = X$ by compact Runge subsets of X such that

$$(B_j \times X) \cap K \subset B_j \times L_j \quad \text{holds for all } j = 1, 2, \ldots.$$

Define the increasing sequence of subsets $K = K^0 \subset K^1 \subset \cdots \subset \bigcup_{j=0}^{\infty} K^j = B \times X$ by

$$K^j = (B_j \times L_j) \cup \big[((B \setminus B_j) \times X) \cap K\big], \quad j = 1, 2, \ldots.$$

Note that K^j is closed, the projection $\pi : K^j \to B$ is proper, and K^j is Runge in $B \times X$. (Indeed, $K^j_b = L_j$ if $b \in B_j$ and $K^j_b = K_b$ if $b \in B \setminus B_j$.) Applying the special case proved above gives a sequence of maps $f^j : B \times X \to Y$ $(j = 0, 1, \ldots)$ with $f^0 = f$ such that for every $j = 1, 2, \ldots$ the map f^j is X-holomorphic on a neighbourhood of K^j, it approximates f^{j-1} in the fine topology on K^{j-1}, it is homotopic to f^{j-1} by a homotopy of maps which are X-holomorphic on a neighbourhood of K^{j-1} and approximate f^{j-1} on K^{j-1}, and the homotopy is fixed for b in a neighbourhood of Q. (Indeed, by the construction the map f^j approximates f^{j-1} on $K^{j-1} \cap (B_{j-1} \times X)$ and it agrees with f^{j-1} outside a small neighbourhood of $B_{j-1} \times X$.) Assuming that the approximation is close enough at every step, we obtain a limit map $F = \lim_{j \to \infty} f^j : B \times X \to Y$ which is X-holomorphic, it approximates the initial map f as closely as desired in the fine topology on K, it agrees with f on $Q \times X$, and it is homotopic to f by maps having the same properties. □

The following result generalizes Theorem 1.4. The proof is essentially the same and is omitted.

Theorem 7.4. *Let B, X, $\{J_b\}_{b \in B}$, $K \subset B \times X$, and Y be as in Theorem 1.4. Assume that Z is a Stein manifold and L is a compact $\mathscr{O}(Z)$-convex set in Z. For every $b \in B$ let $\widetilde{J}_b$ be the almost complex structure on $X \times Z$ which equals J_b on TX and equals the given almost complex structure on TZ. Assume that $f : B \times X \times Z \to Y$ is a continuous map, and there is an open set $U \subset B \times X \times Z$ containing $K \times L$ such that $f_b = f(b, \cdot, \cdot) : X \times Z \to Y$ is $\widetilde{J}_b$-holomorphic on $U_b = \{(x, z) \in X \times Z : (b, x, z) \in U\}$ for every $b \in B$. Given a continuous function $\epsilon : B \to (0, +\infty)$, there is a homotopy $f_t : B \times X \times Z \to Y$ $(t \in I = [0, 1])$ satisfying the following conditions:*

(i) $f_0 = f$.
(ii) *The map $f_{t,b} = f_t(b, \cdot, \cdot) : X \times Z \to Y$ is $\widetilde{J}_b$-holomorphic near $K_b \times L$ for every $b \in B$.*
(iii) $\sup_{(x,z) \in K_b \times L} \mathrm{dist}_Y(f_{t,b}(x, z), f_b(x, z)) < \epsilon(b)$ *for every $b \in B$ and $t \in I$.*
(iv) *The map $F = f_1$ is such that $F_b = F(b, \cdot, \cdot) : X \times Z \to Y$ is $\widetilde{J}_b$-holomorphic for every $b \in B$.*

If in addition $0 \le l \le k+1$, B is a $\mathscr{C}^l$ manifold if $l > 0$, the family $\{J_b\}_{b \in B}$ is of class $\mathscr{C}^{l,(k,\alpha)}(B \times X)$ $(0 < \alpha < 1)$, and f is of class $\mathscr{C}^{l,0}(U)$, then $f \in \mathscr{C}^{l,k+1}(U)$ and the homotopy $\{f_t\}_{t \in I}$ can be chosen to be of class $\mathscr{C}^{l,k+1}(B \times X \times Z)$ and to approximate f in the fine $\mathscr{C}^{l,k+1}$-topology on $K \times L$.

By using the techniques in the proof of Theorem 1.4, we can also extend Mergelyan approximation in Theorem 1.2 to manifold-valued maps as in the following theorem. For a similar result in the nonparametric case, see [28, Corollary 5, p. 176].

Theorem 7.5. *Let X be a smooth open surface, B be a paracompact Hausdorff space which is a local ENR (see Definition 1.3), $\{J_b\}_{b\in B}$ be a family of complex structures on X of class $\mathscr{C}^\alpha$ $(0<\alpha<1)$, $K\subset X$ be a compact Runge set, and $A\subset \mathring{K}$ be a finite set. Assume that Y is a complex manifold and $f: B\times K\to Y$ is a continuous map such that for every $b\in B$ the map $f_b=f(b,\cdot\,):K\to Y$ is J_b-holomorphic on $\mathring{K}$. Given a continuous function $\epsilon: B\to(0,+\infty)$, there are a neighbourhood $U\subset B\times X$ of $B\times K$ and a continuous map $F:U\to Y$ such that for every $b\in B$ the map $F_b:U_b\to Y$ is J_b-holomorphic, $\sup_{x\in K}\mathrm{dist}_Y(F_b(x),f_b(x))<\epsilon(b)$, and F_b agrees with f_b to order 1 in every point $a\in A$.*

Proof. We have seen in the proof of Theorem 1.2 that there are arbitrarily small open coordinate discs $U_1,\ldots,U_N\subset X$ and compact discs $D_j\subset U_j$ for $j=1,\ldots,N$ such that $K\subset\bigcup_{j=1}^N \mathring{D}_j$ and $U_j\setminus(K\cap D_j)$ is connected for every j. Fix a parameter value $b_0\in B$. We may assume that the discs U_j are chosen small enough so that $f_{b_0}(K\cap D_j)\subset Y$ is contained in a coordinate chart of Y for each j. Hence, by Theorem 1.2 we can approximate f_{b_0} as closely as desired uniformly on $K\cap D_j$ by holomorphic maps from open neighbourhoods of $K\cap D_j$ to Y for $j=1,\ldots,N$. This shows that the hypotheses of Theorem 7.1 hold, so the graph of f on K is a Stein compact in $X\times Y$. By the argument in the proof of Lemma 7.2 (choosing a Stein neighbourhood $\Gamma\subset X\times Y$ of the graph of f on K, embedding it in a Euclidean space, and using a holomorphic retraction onto the embedded submanifold), we reduce the Mergelyan approximation problem for maps $f_b=f(b,\cdot\,):K\to Y$, with $b\in B$ close enough to b_0, to the scalar-valued case furnished by Theorem 1.2. The local J_b-holomorphic approximants of f_b can be glued together by finding homotopies as in the proof of Lemma 7.2 (see (7.4) and (7.5)) and using cut-off functions in the parameter of the homotopy. The inductive procedure is similar to the one in the proof of Theorem 1.4 and will not be repeated. □

Before stating our next result, we recall the following notion; see [11, p. 69].

Definition 7.6. Let X be a smooth surface. An *admissible set* in X is a compact set of the form $S=K\cup E$, where K is a (possibly empty) finite union of pairwise disjoint compact domains with smooth boundaries in X and $E=S\setminus\mathring{K}$ is a union of finitely many pairwise disjoint Jordan arcs and closed smooth Jordan curves meeting K only at their endpoints (if at all) such that their intersections with the boundary bK of K are transverse.

Admissible sets arise in handlebody decompositions of surfaces; see [11, Sect. 1.4]. For this reason, approximation on such sets plays a major role in constructions of directed holomorphic maps, minimal surfaces and related objects, as is evident from the results in [11]. The basic case for continuous functions follows from Theorem 1.2. In Section 10 we shall also use the following version.

Theorem 7.7. *Assume that X is a smooth open surface, $1\le l\le k+1$ are integers, B is a manifold of class $\mathscr{C}^l$, $\{J_b\}_{b\in B}$ is a family of complex structures on X of class $\mathscr{C}^{l,(k,\alpha)}(B\times X)$ for some $0<\alpha<1$, $S=K\cup E$ is a Runge admissible set in X, $U\subset X$ is an open set containing K, and $f:B\times(U\cup E)\to\mathbb{C}$ is a function of class $\mathscr{C}^l$ such that for every $b\in B$, the function $f_b=f(b,\cdot\,)$ is J_b-holomorphic on U. Then, f can be approximated in the fine $\mathscr{C}^l$ topology on $B\times S$ by functions $F:B\times X\to\mathbb{C}$ of class $\mathscr{C}^{l,k+1}$ such that $F_b=F(b,\cdot\,):X\to\mathbb{C}$ is J_b-holomorphic for every $b\in B$. The analogous result holds for maps to any complex manifold Y, where the approximating maps F are defined on small open neighbourhoods of $B\times S$ in $B\times X$. If Y is an Oka manifold then there are maps $F:B\times X\to Y$ satisfying the same conclusion.*

Proof. It suffices to prove the result locally in the parameter. Thus, fix a point $b_0\in B$, a smoothly bounded domain $\Omega\Subset X$ containing S, and a compact neighbourhood $B_0\subset B$ of b_0 for which Theorem 4.3 applies and gives a family of (J_b,J_{b_0})-biholomorphic maps $\Phi_b:\Omega\to\Phi_b(\Omega)\subset X$

$(b \in B_0)$ of class $\mathscr{C}^{l,k+1}$. We may assume that B_0 is a $\mathscr{C}^l$ submanifold of $\mathbb{R}^n \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$. Recall that the function f_b in the theorem is J_b-holomorphic on a neighbourhood $U \subset X$ of K for every $b \in B_0$. (Since B_0 is compact, we may choose U independent of $b \in B_0$.) As in the proof of Theorem 1.4, this reduces the approximation problem for $b \in B_0$ to the situation in Lemma 7.3 where the compact sets $\widetilde{K}, \widetilde{E}, \widetilde{S}$ in $B_0 \times X \subset \mathbb{C}^n \times X$ have fibres $K_b = \Phi_b(K)$, $E_b = \Phi_b(E)$, and $S_b = \Phi_b(S) = K_b \cup E_b$, respectively. Note that $\widetilde{K}$ and $\widetilde{S}$ are holomorphically convex in $\mathbb{C}^n \times X$ (see Remark 5.2 (b)). Let $\widetilde{U} \subset B_0 \times X$ be the set with fibres $U_b = \Phi_b(U)$. The function $\tilde{f}_b : U_b \cup E_b \to \mathbb{C}$, defined by $\tilde{f}_b \circ \Phi_b = f_b$ on $U \cup E$ $(b \in B_0)$, is J_{b_0}-holomorphic on U_b for every $b \in B_0$. Let $\tilde{f} : \widetilde{U} \cup \widetilde{E} \to \mathbb{C}$ be given by $\tilde{f}(b, \cdot\,) = \tilde{f}_b$ for $b \in B_0$. Note that $\tilde{f}$ is of class $\mathscr{C}^l$. Choose a compact $\mathscr{O}(\mathbb{C}^n \times X)$-convex set $L \subset \widetilde{U}$ containing $\widetilde{K}$ in its relative interior. By Lemma 5.1 we can approximate $\tilde{f}$ as closely as desired in $\mathscr{C}^l(L)$ by a function $h : V \to \mathbb{C}$ on an open neighbourhood $V \subset \mathbb{C}^n \times X$ of L which is holomorphic with respect to the standard complex structure on $\mathbb{C}^n$ and the complex structure J_{b_0} on X. By smoothly gluing h with $\tilde{f} : \widetilde{E} \to \mathbb{C}$ on the set $L \setminus \widetilde{K}$, we may assume that h is unchanged (and hence holomorphic) in a neighbourhood $\widetilde{V} \subset V$ of $\widetilde{K}$ in $\mathbb{C}^n \times X$, it is of class $\mathscr{C}^l$ on $\widetilde{E}$, it agrees with $\tilde{f}$ on $\widetilde{E} \setminus L$, and it approximates $\tilde{f}$ to a desired precision in $\mathscr{C}^l(\widetilde{E})$. Note that $\widetilde{E}$ is a totally real submanifold of class $\mathscr{C}^l$ in $\mathbb{C}^n \times X$, and the set $\widetilde{S} = \widetilde{K} \cup \widetilde{E}$ is admissible in the sense of [28, Definition 5 (a), p. 156]. Hence, by [28, Theorem 20, p. 161] we can approximate h in $\mathscr{C}^l(\widetilde{S})$ by entire functions on $\mathbb{C}^n \times X$. By the argument in the proof of Lemma 5.1 this gives functions F, defined on open neighbourhoods of $B_0 \times S$ in $B_0 \times X$, which approximate f in $\mathscr{C}^l(B_0 \times S)$ and such that $F(b, \cdot\,)$ is holomorphic for every $b \in B_0$. The proof is completed by using $\mathscr{C}^l$ partitions of unity on B and Theorem 1.1. For maps to manifolds, we follow the argument in the proof Lemma 7.3, and the statement for maps to an Oka manifold Y follows from Theorem 1.4. $\square$

8. Trivialization of canonical bundles of families of open Riemann surfaces

Every open Riemann surface X has trivial holomorphic cotangent bundle $K_X = T^*X$, trivialized by a nowhere vanishing holomorphic 1-form θ on X. (In fact, every holomorphic vector bundle on an open Riemann surface is holomorphically trivial by the Oka–Grauert principle; see Oka [68], Grauert [44], and [32, Theorem 5.3.1].) We prove the following generalization to families of complex structures. See also Corollary 10.3, which extends the theorem of Gunning and Narasimhan [46].

Theorem 8.1. *Given a smooth open surface X and a family $\{J_b\}_{b\in B}$ of complex structures of class $\mathscr{C}^{l,(k,\alpha)}(B \times X)$ on X as in Theorem 1.4 (with $l \le k+1$), there exists a family $\{\theta_b\}_{b\in B}$ of nowhere vanishing holomorphic 1-forms on (X, J_b) of class $\mathscr{C}^{l,k+1}(B \times X)$.*

Note that a family of holomorphic 1-forms $\{\theta_b\}_{b\in B}$ as in the theorem, which is of class $\mathscr{C}^l$ in $b \in B$, is necessarily of class $\mathscr{C}^{l,k+1}(B \times X)$ by Lemma 5.4. Theorem 8.1 is used in Section 10.

Proof. Write $X_b = (X, J_b)$ for $b \in B$. By Remark 4.5 (C), there is a family of nowhere vanishing complex $(1,0)$-forms θ_b on X_b of class $\mathscr{C}^{l,(k,\alpha)}$. We will show that the family $\{\theta_b\}_{b\in B}$ can be deformed to a family of nowhere vanishing J_b-holomorphic 1-forms $\{\tilde{\theta}_b\}_{b\in B}$ of class $\mathscr{C}^l$ in the parameter $b \in B$.

Note that $\theta = \{\theta_b\}_{b\in B}$ is a section of the complex line bundle $E \to B \times X$ whose restriction to the fibre X_b over $b \in B$ equals T^*X_b. We shall inductively deform θ so as to make it X-holomorphic (that is, holomorphic on fibres of the projection $\pi : B \times X \to B$) on larger and larger subsets of $B \times X$. We follow the scheme in the proof of Theorem 1.4. Assuming that $K \subset L$ are compact Runge sets in X and θ is X-holomorphic on a neighbourhood of $B \times K$, we find a multiplier $f : B \times X \to \mathbb{C}^*$ which is homotopic to the constant function 1, and is X-holomorphic and close to 1 on a neighbourhood of $B \times K$ (thereby ensuring that $f\theta$ is X-holomorphic and close to θ on $B \times K$), such that $f\theta$ is X-holomorphic on a neighbourhood of $B \times L$. We then conclude the proof by an induction on a normal exhaustion of X by an increasing family of compact Runge sets.

It remains to explain the basic case described above. As in the proof of Theorem 1.4, we proceed by induction with respect to a normal exhaustion of B by compact subsets. To explain the inductive step, assume that L is a compact Runge set in X and $K^0 \subset B \times L$ is a closed subset whose compact fibres K^0_b, $b \in B$, are Runge in X. We allow for the possibility that some fibres are empty. (At every inductive step, the fibre K^0_b will equal the given Runge set K for some parameter values and will equal L for the other values.) Assume that θ as above is X-holomorphic on a neighbourhood $U \subset B \times X$ of K^0, that is, θ_b is J_b-holomorphic on the neighbourhood U_b of K^0_b for every $b \in B$. Pick a smoothly bounded domain $\Omega \Subset X$ with $L \subset \Omega$. Fix a point $b_0 \in B$. Theorem 4.3 furnishes a compact neighbourhood $P \subset B$ of b_0 and a family of biholomorphic maps $\Phi_b : (\Omega, J_b) \to (\Phi_b(\Omega), J_{b_0})$ $(b \in P)$ of class $\mathscr{C}^{l,k+1}$. By the Oka–Grauert principle there is a function $g : X \to \mathbb{C}^*$, homotopic to the constant $X \to 1$ through functions $X \to \mathbb{C}^*$, such that the 1-form $g\theta_{b_0}$ is J_{b_0}-holomorphic on X. Hence,

$$\phi_b := \Phi_b^*(g\theta_{b_0}) = f_b\theta_b, \quad b \in P$$

is a family of nowhere vanishing J_b-holomorphic 1-forms on Ω. Since $g\theta_{b_0}$ is independent of $b \in P$, the family $\{\phi_b\}_{b\in P}$ is of class $\mathscr{C}^l(P)$. Hence, the same holds for the family of functions $f_b = \phi_b/\theta_b : \Omega \to \mathbb{C}^*$. By shrinking $U \supset K^0$ if necessary we may assume that $U \subset B \times \Omega$. Since θ_b is J_b-holomorphic on $U_b \supset K^0_b$ for every $b \in P$, the function $f_b = \phi_b/\theta_b$ is also J_b-holomorphic on U_b for every $b \in P$. Theorem 1.4, applied with the Oka manifold $Y = \mathbb{C}^*$, furnishes a homotopy of functions $f_{t,b} : \Omega \to \mathbb{C}^*$ $(b \in P,\ t \in I)$ of class $\mathscr{C}^{l,k}$ satisfying the following conditions:

(i) $f_{0,b} = f_b$ for all $b \in P$,
(ii) $f_{1,b}$ is J_b-holomorphic on Ω for all $b \in P$ and of class $\mathscr{C}^l$ in b, and
(iii) $f_{t,b}$ is J_b-holomorphic on a neighbourhood of K^0_b and it approximates f_b on K_b as closely as desired for all $b \in P$ and $t \in I$. (In fact, the approximation is in the $\mathscr{C}^{l,k}$ topology.)

The homotopy of 1-forms $\theta'_{t,b} = \phi_b/f_{t,b}$ $(b \in P,\ t \in I)$ on Ω is of class $\mathscr{C}^{l,k}$ and satisfies

(i') $\theta'_{0,b} = \phi_b/f_{0,b} = \phi_b/f_b = \theta_b$ for all $b \in P$,
(ii') $\theta'_{1,b} = \phi_b/f_{1,b}$ is J_b-holomorphic on Ω for every $b \in P$, and
(iii') $\theta'_{t,b}$ is J_b-holomorphic on a neighbourhood of K^0_b and it approximates θ_b on K^0_b for all $b \in P$. (The approximation is in the $\mathscr{C}^{l,k}$ topology.)

Pick a pair of neighbourhoods $P_0 \subset P_1 \subset P$ of b_0, each contained in the interior of the next one, and a function $\xi : B \to [0,1]$ of class $\mathscr{C}^l$ which equals 1 on a neighbourhood of P_0 and vanishes on $B \setminus P_1$. We define a new homotopy of 1-forms on Ω of class $\mathscr{C}^{l,k}(B \times \Omega)$ by

$$\theta_{t,b} = \theta'_{t\xi(b),b} \ \text{ for every } b \in B \text{ and } t \in I.$$

Then, $\theta_{t,b} = \theta'_{t,b}$ holds for b in a neighbourhood of P_0 (where $\xi = 1$), and $\theta_{t,b} = \theta_{0,b} = \theta_b$ holds for all $b \in B \setminus P_1$ (where $\xi = 0$) and $t \in I$. It follows that

(i”) $\theta_{0,b} = \theta'_{0,b} = \theta_b$ for all $b \in B$,
(ii”) $\theta_{t,b}$ is J_b-holomorphic on a neighbourhood of K^0_b and it approximates θ_b on K^0_b for all $b \in B$ and $t \in I$ (the approximation is in the fine $\mathscr{C}^{l,k}$ topology), and
(iii”) $\theta_{1,b} = \theta'_{1,b}$ is J_b-holomorphic on Ω for all $b \in P_0$.

By using another cut-off function in the parameter of the homotopy, we can extend $\theta_{t,b}$ for $p \in P$ to all of X_b withouth changing its values on a neighbourhood of $(\mathring{P} \times L) \cup ((B \setminus \mathring{P}) \times X)$.

Using this device inductively with respect to an exhaustion of B as in the proof of Theorem 1.4, we can approximate θ in the fine $\mathscr{C}^{l,k}$ topology on K^0 by a family of nowhere vanishing 1-forms $\{\tilde\theta_b\}_{b\in X}$ of class $\mathscr{C}^{l,k}(B \times X)$ which are X-holomorphic on a neighbourhood of $B \times L$. Theorem 8.1 then follows by an obvious induction with respect to a normal exhaustion of X by compact Runge sets. □

A standard application of Runge's theorem on open Riemann surfaces (and, more generally, of the Oka–Weil theorem on Stein manifolds) is the global solvability of the $\overline{\partial}$-equation. By using Theorems 1.1, 3.2, and 4.3 we will now show that the same holds on families of open Riemann surfaces.

Assume that X is a smooth open oriented surface, B is a parameter space as in Theorem 1.4 (its precise type depending on the value of the integer $l \in \mathbb{Z}_+$), and $\{J_b\}_{b\in B}$ is a family of complex structures of Hölder class $\mathscr{C}^{l,(k,\alpha)}$ with $k \geq 1$, $0 \leq l \leq k+1$, and $0 < \alpha < 1$. Let $\tau : Z = B\times X \to B$ denote the projection onto the parameter space, and endow the fibre $Z_b = \tau^{-1}(b) \cong X$ ($b \in B$) with the complex structure J_b. For a fixed $b \in B$ we denote by $\overline{\partial}_{J_b}$ the $(0,1)$-differential in the complex structure J_b in X. By Theorem 8.1 there exists a family $\{\theta_b\}_{b\in B}$ of nowhere vanishing holomorphic 1-forms on (X, J_b) of class $\mathscr{C}^{l,k+1}$. Given an integer $s \in \{0,\dots,k-1\}$, a number $0 < a < 1$, and a function $f : Z \to \mathbb{C}$ of class $\mathscr{C}^{l,(s+1,a)}$, we have that

$$\overline{\partial}_{J_b} f_b = \beta_b = g_b\bar{\theta}_b, \quad b \in B, \tag{9.1}$$

where the function $g : Z \to \mathbb{C}$ determined by $g_b = g(b,\cdot\,)$ ($b \in B$) is of class $\mathscr{C}^{l,(s,a)}$. Note that the family of $(0,1)$-forms $\beta_b = g_b\bar{\theta}_b$ is of class $\mathscr{C}^{l,(s,a)}$ if and only if g is such. The following result gives a global solution of the equation (9.1) for suitable values of the integers k, l, and s.

Theorem 9.1. *Let k, l, s be integers satisfying $0 \leq 2l \leq s < k$, and let $0 < a, \alpha < 1$. Assume that X is a smooth open surface, $\{J_b\}_{b\in B}$ is a family of complex structures of class $\mathscr{C}^{l,(k,\alpha)}$ on X as in Theorem 1.4, and $\{\theta_b\}_{b\in B}$ is a family of J_b-holomorphic 1-forms of class $\mathscr{C}^{l,k+1}$ furnished by Theorem 8.1. Given a family of $(0,1)$-forms $\beta_b = g_b\bar{\theta}_b$ on X ($b \in B$) of class $\mathscr{C}^{l,(s,a)}$, there is a function $f : Z = B\times X \to \mathbb{C}$ of class $\mathscr{C}^{l,(s+1-2l,a)}$ such that $f_b = f(b,\cdot\,) : X \to \mathbb{C}$ satisfies* (9.1).

Compared with the standard result in a fixed complex structure, where the solution of the $\overline{\partial}$ equation gains one derivative on Hölder spaces, we have a loss of $2l$ derivatives in Theorem 9.1. This may be due to the method of proof and is caused by the fact that compositions and inverses of $\mathscr{C}^{l,k}$ maps need not be of the same class; see Lemma 5.3. Note however that there is no loss when $l = 0$, i.e., for continuous dependence of the data β_b and the solutions f_b on the parameter $b \in B$.

Proof. It suffices to show that for every point $b_0 \in B$ and smoothly bounded relatively compact domain $\Omega \Subset X$ there is a neighbourhood $B_0 \subset B$ of b_0 such that the equation (9.1) is solvable on $B_0 \times \Omega$. By using partitions of unity on B we then obtain solvability on $B \times \Omega$. The proof is completed by an exhaustion of X by an increasing family of compact Runge sets, using Theorem 1.1 at every step of the induction to ensure convergence of solutions. One follows the standard scheme in the proof of Cartan's Theorem B, see e.g. [47, Section VIII.14]. Furthermore, it suffices to consider the case when the forms β_b have compact support contained in slightly smaller domain $\Omega' \Subset \Omega$ for every $b \in B_0$.

Fix a point $b_0 \in B$ and an open relatively compact neighbourhood $B_0 \subset B$ of b_0 such that Theorem 4.3 furnishes a family of (J_b, J_{b_0})-biholomorphic maps $\Phi_b : \Omega \to \Phi_b(\Omega) \subset X$ ($b \in \overline{B}_0$) of class $\mathscr{C}^{l,k+1}$. Let $h : X \to \mathbb{C}$ be a J_{b_0}-holomorphic immersion (see [46]). Then,

$$h_b = h \circ \Phi_b : \Omega \to \mathbb{C}, \quad b \in B_0$$

is a family of J_b-holomorphic immersions of class $\mathscr{C}^{l,k+1}$, and we may take $\bar{\theta}_b = d\bar{h}_b$ as the family of $(0,1)$-forms in (9.1). Note that $\bar{\theta}_b$ is antiholomorphic for every $b \in B_0$. Thus, $\beta_b = g_b\, d\bar{h}_b$ where the function $g : B_0 \times \Omega \to \mathbb{C}$ with $g_b = g(b,\cdot\,)$ is of class $\mathscr{C}^{l,(s,a)}$. Since the family of inverse maps $\Psi_b = \Phi_b^{-1}$ is of class $\mathscr{C}^{l,k+1-l}$ by Lemma 5.3, the family of composition $\tilde{g}_b = g_b \circ \Psi_b$ is of class $\mathscr{C}^{l,(s-l,a)}$. To see this, note that by the chain rule each partial derivative of $\tilde{g}_b$ of biorder $(l, s-l)$ (in any system of local coordinates on $B \times X$) is a sum of terms obtained by precomposing a derivative of biorder (l, s) of g with a derivative of biorder $(l, s-l)$ of Ψ_b. (This is the argument behind Lemma

5.3 (b).) Since $0 \leq s-l < k+1-l$ (where the latter number is the order of smoothness of Ψ_b in the space variable) and each derivative of biorder (l,s) of g is of Hölder class $\mathscr{C}^a$ in the space variable, the derivatives of $\tilde{g}_b$ of biorder $(l,s-l)$ are also of class $\mathscr{C}^a$ in the space variable, so $\tilde{g}_b$ is of class $\mathscr{C}^{l,(s-l,a)}$ as claimed. Note that $h_b \circ \Psi_b = h$ and hence $\Psi_b^* d\bar{h}_b = d\bar{h}$ holds for all $b \in B_0$. Thus, the family of $(0,1)$-forms with compact supports in the complex structure J_{b_0} on X, given by

$$\Psi_b^*\beta_b = (g_b \circ \Psi_b)\Psi_b^* d\bar{h}_b = \tilde{g}_b\, d\bar{h}, \quad b \in B_0,$$

is of class $\mathscr{C}^{l,(s-l,a)}$. Let ω be a Cauchy kernel on (X, J_{b_0}) associated to the holomorphic immersion $h : X \to \mathbb{C}$ (see Sect. 3), and let P be the associated Cauchy operator (3.4). Our assumption implies that the union of supports of the functions $\tilde{g}_b$ $(b \in B_0)$ lies in a compact subset of X. Hence, by Theorem 3.2 (a) the family of functions $\tilde{f}_b = P(\tilde{g}_b)$ $(b \in B_0)$ is of class $\mathscr{C}^{l,(s-l+1,a)}$ and

$$\overline{\partial}_{J_{b_0}} \tilde{f}_b = \tilde{g}_b\, d\bar{h} \quad \text{holds for every } b \in B_0.$$

Regularity in the parameter b follows from the fact that P is a linear operator independent of b.

Set $f_b = \tilde{f}_b \circ \Phi_b : \Omega \to \mathbb{C}$ for $b \in B_0$. Recall that $\tilde{g}_b \circ \Phi_b = g_b$ and $h \circ \Phi_b = h_b$ for $b \in B_0$. Since Φ_b is (J_b, J_{b_0})-holomorphic, it follows that

$$\overline{\partial}_{J_b} f_b = \overline{\partial}_{J_b}(\Phi_b^* \tilde{f}_b) = \Phi_b^*(\overline{\partial}_{J_{b_0}} \tilde{f}_b) = \Phi_b^*(\tilde{g}_b\, d\bar{h}) = g_b\, d\bar{h}_b$$

for every $b \in B_0$, so (9.1) holds. As $\{\tilde{f}_b\}_{b\in B_0}$ is of class $\mathscr{C}^{l,(s+1-l,a)}$, $\{\Phi_b\}_{b\in B_0}$ is of class $\mathscr{C}^{l,k+1}$, and $s < k$, the family $f_b := \tilde{f}_b \circ \Phi_b$ $(b \in B_0)$ is of class $\mathscr{C}^{l,(s+1-2l,a)}$ by the same argument as above. □

We now consider the Cousin-I problem on a family of open Riemann surfaces. Let B, X, and $\{J_b\}_{b\in B}$ be as in Theorem 1.4, where the family J_b is of class $\mathscr{C}^{0,(k,\alpha)}$ for some $k \geq 1$ and $0 < \alpha < 1$ (so $l = 0$ and J_b is continuous in $b \in B$). Let $\mathscr{O}$ denote the sheaf of germs of continuous functions f on $Z = B \times X$ such that $f_b = f(b,\cdot)$ is J_b-holomorphic for each b. By Lemma 5.4, $\mathscr{O}$ is a subsheaf $\mathscr{C}^{0,k+1}$, the sheaf of germs of continuous functions on Z which are of class $\mathscr{C}^{k+1}$ in the space variable.

Theorem 9.2. *(Assumptions as above.) We have that $H^q(Z,\mathscr{O}) = 0$ for all $q = 1, 2, \ldots$.*

Proof. Pick an integer s with $0 \leq s < k$ and a number $0 < a < 1$. Then, $\mathscr{C}^{0,k+1}$ is a subsheaf of the sheaf $\mathscr{C}^{0,(s+1,a)}$, and we have the following sequence of sheaf homomorphisms

$$0 \longrightarrow \mathscr{O} \hookrightarrow \mathscr{C}^{0,(s+1,a)} \xrightarrow{\overline{\partial}} \mathscr{C}^{0,(s,a)}_{(0,1)} \longrightarrow 0, \tag{9.2}$$

where $\mathscr{C}^{l,(s,a)}_{(0,1)}$ is the sheaf of germs of $(0,1)$-forms of class $\mathscr{C}^{0,(s,a)}$ on the fibres $Z_b = (X, J_b)$ and $\overline{\partial}$ is the operator which equals $\overline{\partial}_{J_b}$ on Z_b for every $b \in B$. These are sheaves of unital abelian rings. By Theorem 9.1 applied with $l = 0$, the resolution (9.2) is exact. Since the second and the third sheaf in (9.2) are fine sheaves (as they admit partitions of unity), they are acyclic and we conclude that

$$H^1(Z,\mathscr{O}) = \Gamma(Z, \mathscr{C}^{0,(s,a)}_{(0,1)})/\overline{\partial}(\mathscr{C}^{0,(s+1,a)}(Z))$$

and $H^q(Z,\mathscr{O}) = 0$ for $q \geq 2$ (see [47, Chapter VI]). Here, Γ denotes the space of global sections. The quotient group on the right hand side above vanishes by Theorem 9.1. □

Recall that every holomorphic vector bundle on an open Riemann surface is holomorphically trivial by the Oka–Grauert principle; see Oka [68], Grauert [44], and [32, Theorem 5.3.1]. We can use Theorem 9.2 to obtain an Oka principle for isomorphism classes of families of holomorphic line bundles on families of open Riemann surfaces. Using the above notation, let $\mathscr{O}^* \subset \mathscr{O}$ and $\mathscr{C}^* \subset \mathscr{C}$ denote the subsheaves of the respective sheaves consisting of functions with nonzero values; these are sheaves of abelian groups. A topological complex line bundle $E \to Z$ is said to be X-holomorphic if it admits a transition cocycle consisting of sections of the sheaf $\mathscr{O}$. The restriction of such a line bundle to any fibre $Z_b = (X, J_b)$ is a holomorphic line bundle on the Riemann surface (X, J_b). We denote by

$\mathrm{Pic}(Z)$ the set of isomorphism classes of topological X-holomorphic line bundles on $Z = B \times X$. By the standard argument, $\mathrm{Pic}(Z) \cong H^1(Z, \mathscr{O}^*)$. We have the following Oka principle.

Theorem 9.3. *(Assumptions as above and $k \geq 1$.) Every topological complex line bundle on $Z = B \times X$ is isomorphic to an X-holomorphic topological line bundle, and any two X-holomorphic topological line bundles on Z which are topologically isomorphic are also isomorphic as X-holomorphic topological line bundles. Furthermore, $\mathrm{Pic}(Z) \cong H^2(Z, \mathbb{Z})$. In particular, if B is contractible then every X-holomorphic line bundle on $B \times X$ is trivial.*

Proof. The proof follows the standard argument for complex line bundles on a complex manifold; see [32, Theorem 5.2.2]. Let $\sigma(f) = \mathrm{e}^{2\pi \mathrm{i} f}$. Consider the following commutative diagram whose rows are exponential sheaf sequences and the vertical arrows are the natural inclusions:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \hookrightarrow & \mathscr{O} & \xrightarrow{\sigma} & \mathscr{O}^* & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \hookrightarrow & \mathscr{C} & \xrightarrow{\sigma} & \mathscr{C}^* & \longrightarrow & 1
\end{array}
\tag{9.3}
$$

Note that $\mathscr{C}$ is a fine sheaf, and hence $H^q(Z, \mathscr{C}) = 0$ for all $q \in \mathbb{N}$. By Theorem 9.2 we also have $H^q(Z, \mathscr{O}) = 0$ for all $q \in \mathbb{N}$. Hence, the relevant part of the long exact sequence of cohomology groups associated to the diagram (9.3) gives

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^1(Z, \mathscr{O}^*) & \longrightarrow & H^2(Z; \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \\
0 & \longrightarrow & H^1(Z, \mathscr{C}^*) & \longrightarrow & H^2(Z; \mathbb{Z}) & \longrightarrow & 0
\end{array}
$$

Thus, all arrows in the central square are isomorphisms. Since $\mathrm{Pic}(Z) \cong H^1(Z, \mathscr{O}^*)$ and $H^1(Z, \mathscr{C}^*)$ is the set of isomorphisms classes of topological line bundles on Z, the theorem follows. $\square$

Remark 9.4. The following observation was communicated to me by Finnur Lárusson. Theorem 1.4 allows us to extend Theorem 9.3 to vector bundles of arbitrary rank by using an approach from the classical Oka–Grauert theory. A topological vector bundle on $B \times X$ is the pullback $f^*\mathbb{U}$ by a continuous map f from $B \times X$ to a suitable Grassmannian G of the universal bundle $\mathbb{U} \to G$. Since G is complex homogeneous and hence an Oka manifold, Theorem 1.4 allows us to deform f to a map $F : B \times X \to G$ such that $F(b, \cdot) : X \to G$ is J_b-holomorphic for every $b \in B$. The pullback $F^*\mathbb{U}$ is then an X-holomorphic vector bundle isomorphic to $f^*\mathbb{U}$. The second part of Theorem 9.3 is obtained similarly with the pair of parameter spaces $B \times \{0, 1\} \subset B \times [0, 1]$. This argument generalizes to other types of bundles for which we have an Oka classifying space with a universal bundle. We postpone a more complete treatment of this subject to a subsequent publication.

10. Families of directed holomorphic immersions and of conformal minimal immersions

In this section, we illustrate how the results of this paper can be used to construct families of directed holomorphic immersions and conformal minimal immersions from a family of open Riemann surfaces. There are many further problems of this kind which may possibly or even likely be treated by these new methods, and we indicate a few of them in Problem 10.7.

A connected compact complex submanifold Y of the complex projective space $\mathbb{CP}^{n-1}$, $n \in \mathbb{N}$, determines the punctured complex cone

$$
A = \{(z_1, \ldots, z_n) \in \mathbb{C}^n_* : [z_1 : \cdots : z_n] \in Y\}. \tag{10.1}
$$

Note that A is smooth and connected, and its closure $\overline{A} = A \cup \{0\} \subset \mathbb{C}^n$ is an algebraic subvariety of $\mathbb{C}^n$ by Chow's theorem. By [32, Theorem 5.6.5], A is an Oka manifold if and only if Y is an

Oka manifold. By [11, Lemma 3.5.1], the convex hull of A is the smallest complex subspace of $\mathbb{C}^n$ containing A, and we shall assume without loss of generality that this hull is all of $\mathbb{C}^n$.

Let X be a connected open Riemann surface and θ be a nowhere vanishing holomorphic 1-form on X. A holomorphic immersion $h : X \to \mathbb{C}^n$ is said to be *directed by* A, or an A*-immersion*, if its complex derivative with respect to any local holomorphic coordinate on X takes its values in A. Clearly, this holds if and only if the holomorphic map $f = dh/\theta : X \to \mathbb{C}^n$ assume values in A. Conversely, a holomorphic map $f : X \to A$ satisfying the period vanishing conditions

$$\int_C f\theta = 0 \quad \text{for all closed curves } C \subset X \tag{10.2}$$

integrates to a holomorphic A-immersion $h : X \to \mathbb{C}^n$ by setting

$$h(x) = v + \int_{x_0}^{x} f\theta, \quad x \in X$$

for any $x_0 \in X$ and $v \in \mathbb{C}^n$. Since $f\theta$ is a holomorphic 1-form, it suffices to verify conditions (10.2) on a basis of the homology group $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^r$, a free abelian group of some rank $r \in \mathbb{Z}_+ \cup \{\infty\}$.

Note that a map directed by the cone $A = \mathbb{C}^n_*$ is simply an immersion. Another case of major interest is the *null quadric*

$$\mathbf{A} = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n_* = \mathbb{C}^n \setminus \{0\} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\big\}, \quad n \geq 3. \tag{10.3}$$

Holomorphic immersions directed by $\mathbf{A}$ are called *holomorphic null curves* in $\mathbb{C}^n$. The real and the imaginary part of a holomorphic null immersion $X \to \mathbb{C}^n$ are conformal harmonic immersions $X \to \mathbb{R}^n$. Such immersions parameterize minimal surfaces, hence are called conformal minimal immersions. Conversely, every conformal minimal immersion $X \to \mathbb{R}^n$ is locally (on any simply connected domain) the real part of a holomorphic null curve. See [11, 69] for more information.

Directed holomorphic immersions were studied by Alarcón and Forstnerič in [8]. Under the assumption that A is an Oka manifold, they proved an Oka principle with Runge and Mergelyan approximation for holomorphic A-immersions [8, Theorems 2.6 and 7.2]. They also showed that every holomorphic A-immersion can be approximated by holomorphic A-embeddings when $n \geq 3$, and by proper holomorphic A-embeddings under an additional assumption on the cone [8, Theorem 8.1]. Alarcón and Castro-Infantes [6] added interpolation to the picture. A parametric Oka principle for A-immersions was proved in [34, Theorem 5.3]. Algebraic A-immersions from affine Riemann surfaces were studied in [12] under the assumption that A is algebraically elliptic in the sense of Gromov [45] (see also [32, Definition 5.6.13]). Several cones arising in geometric applications, in particular the null quadric $\mathbf{A}$ (10.3), are algebraically elliptic. More recently, Alarcón et al. [7] obtained h-principles for algebraic immersions directed by cones which are flexible in the sense of Arzhantsev et al. [13]. Minor variations of these results for the null cone (10.3) yield similar results for conformal minimal immersions of open Riemann surfaces in Euclidean spaces; see the monograph [11].

The main advantage of the techniques in the mentioned papers, when compared to the previously known results, is that they allow a complete control of the conformal structure of the resulting directed curves or minimal surfaces. By using the approximation results developed in the present paper, one can go substantially further and construct families of such objects with a control of the conformal structure of every member of the family, which may depend continuously or smoothly on a parameter. We now present a few specific results in this direction, which are only the tip of an iceberg of possibilities.

In the following, X is a connected, smooth, open oriented surface, $\{J_b\}_{b\in B}$ is a family of complex structures on X as in Theorem 1.4, and $\{\theta_b\}_{b\in B}$ is a family of nowhere vanishing J_b-holomorphic 1-forms on X, furnished by Theorem 8.1. Recall that a continuous map $f : B \times X \to Y$ is said to be

X-holomorphic if the map $f(b,\cdot\,) : X \to Y$ is J_b-holomorphic for every $b \in B$. The first two items in the following definition come from [8, Definition 2.2].

Definition 10.1. Let $A \subset \mathbb{C}^n_*$ be a punctured complex cone of the form (10.1).

(i) A holomorphic map $f : X \to A$ is nondegenerate if the tangent spaces $T_{f(x)}A \subset T_{f(x)}\mathbb{C}^n \cong \mathbb{C}^n$ over all points $x \in X$ span $\mathbb{C}^n$.
(ii) A holomorphic A-immersion $h : X \to \mathbb{C}^n$ is nondegenerate if the map $f = dh/\theta : X \to A$ is such, where θ is any nowhere vanishing holomorphic 1-form on X.
(iii) An X-holomorphic map $f : B \times X \to A$ is nondegenerate if $f_b = f(b,\cdot\,) : X \to A$ is nondegenerate for every $b \in B$.
(iv) A map $h : B \times X \to \mathbb{C}^n$ is an A-immersion if $h_b = h(b,\cdot\,) : X \to \mathbb{C}^n$ is a J_b-holomorphic A-immersion for every $b \in B$, and is nondegenerate if $dh_b/\theta_b : X \to A$ is such for every $b \in B$.

If X is disconnected, the maps f and h as above are called nondegenerate if the respective conditions hold on each connected component of X.

By [11, Lemma 3.1.1], an immersion $h : X \to \mathbb{C}^n$ directed by the null cone (10.3) is nondegenerate if and only if it is nonflat, meaning that its image $h(X)$ is not contained in an affine complex line of $\mathbb{C}^n$. Equivalently, the range of the map $f = dh/\theta : X \to \mathbf{A}$ is not contained in a ray of $\mathbf{A}$.

We shall prove the following h-principle for families of directed holomorphic immersions from a family of open Riemann surfaces. Compare with the h-principles for maps a single open Riemann surface in [8, Theorem 2.6] and [34, Theorem 5.3].

Theorem 10.2. *Assume that $A \subset \mathbb{C}^n$ is a smooth Oka cone* (10.1) *which is not contained in any hyperplane, X is a smooth open surface, B is a manifold of class $\mathscr{C}^l$ if $l \in \mathbb{N}$ or a paracompact Hausdorff space which is a local ENR (see Definition 1.3) if $l = 0$, $\{J_b\}_{b\in B}$ is a family of complex structures on X of class $\mathscr{C}^{l,(k,\alpha)}$ ($k \ge 1$, $l \le k+1$, $0 < \alpha < 1$), and $\{\theta_b\}_{b\in B}$ is a family of nowhere vanishing J_b-holomorphic 1-forms on X furnished by Theorem 8.1. Given a continuous map $f_0 : B \times X \to A$, there is a nondegenerate A-immersion $h : B \times X \to \mathbb{C}^n$ of class $\mathscr{C}^{l,k+1}(B \times X)$ such that the map $f : B \times X \to A$ defined by $f(b,\cdot\,) = dh_b/\theta_b$ for all $b \in B$ is homotopic to f_0.*

One can also add approximation and interpolation conditions as in Theorem 1.4 and Remark 1.5.

By taking $A = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we obtain the following corollary to Theorem 10.2, which extends the Gunning–Narasimhan theorem [46] to families of complex structures on a smooth open surface.

Corollary 10.3. *Given a smooth open surface X and a family $\{J_b\}_{b\in B}$ of complex structures on X as in Theorem 10.2, there is a function $h : B \times X \to \mathbb{C}$ of class $\mathscr{C}^{l,k+1}$ such that $h(b,\cdot\,) : X \to \mathbb{C}$ is a J_b-holomorphic immersion for every $b \in B$.*

If h is as in the corollary then $|h(b,\cdot\,)|^2$ is a smooth strongly subharmonic function on the Riemann surface (X, J_b) for every $b \in B$. By using Theorem 1.1 it is easy to find a function $\rho : B \times X \to \mathbb{R}_+$ of the form $\rho = \sum_i |f_i|^2$, where each $f_i : B \times X \to \mathbb{C}$ is X-holomorphic, satisfying following.

Corollary 10.4. *Given a smooth open oriented surface X and a family $\{J_b\}_{b\in B}$ of complex structures on X as in Theorem 10.2, there is a function $\rho : B \times X \to \mathbb{R}_+$ of class $\mathscr{C}^{l,k+1}$ such that $\rho(b,\cdot\,) : X \to \mathbb{R}_+$ is a smooth strongly subharmonic exhaustion function on (X, J_b) for every $b \in B$.*

Proof of Theorem 10.2. We shall adapt the proof of the parametric h-principle for directed holomorphic immersions from an open Riemann surface in [34, Section 5]. For the nonparametric case, see [8, Theorem 2.6] and [11, Theorem 3.6.1] where the reader can find further details.

For simplicity of exposition, we shall assume that the parameter space B is compact. The general case requires an additional induction with respect to a normal exhaustion of B, which proceeds as in the proof of Theorem 1.4 and will not be repeated.

By Theorem 1.4 we can deform f_0 to an X-holomorphic map $f_1 : B \times X \to A$ of class $\mathscr{C}^{l,k+1}$. The following lemma shows that we can choose f_1 to be nondegenerate in the sense of Definition 10.1.

Lemma 10.5. *(Assumptions as in Theorem 10.2.) Every X-holomorphic map $f : B \times X \to A$ of class $\mathscr{C}^{l,k+1}$ can be approximated in the $\mathscr{C}^{l,k+1}$ topology on compacts by nondegenerate X-holomorphic maps of class $\mathscr{C}^{l,k+1}$ homotopic to f.*

Proof. Since the cone A is not contained in any hyperplane of $\mathbb{C}^n$, there is an integer $q \in \mathbb{N}$ such that for a generic q-tuple of points $z_1, \ldots, z_q$ in A we have $\sum_{i=1}^{q} T_{z_i} A = \mathbb{C}^n$. (We consider the tangent space $T_z A$ as a subspace of $\mathbb{C}^n$.) Hence, to deform a map $f : X \to A$ to a nondegenerate one, it suffices to push a set of q points of $f(X)$ to such a generic position. An explicit procedure for a single map is given in [8, Theorem 3.2 (a)]. For a family of maps from a single Riemann surface, see [34, Theorem 5.4]. We adapt the argument in the latter proof to our situation.

Recall that if V is a holomorphic vector field on a domain $D \subset \mathbb{C}^n$ then its flow $\phi_\zeta(z)$ in complex time ζ is holomorphic in (ζ, z) on its fundamental domain in $\mathbb{C} \times D$. If U is a domain in $B \times X$ and $f : U \to D$ and $h : U \to \mathbb{C}$ are maps of class $\mathscr{C}^{l,k+1}$, then the map $(\zeta, b, x) \mapsto \phi_{\zeta h(b,x)}(f(b,x)) \in D$ is defined in a neighbourhood of $\{0\} \times U$ in $\mathbb{C} \times B \times X$, it is holomorphic in ζ, J_b-holomorphic in x for a fixed b and ζ, and of class $\mathscr{C}^{l,k+1}$ for every fixed ζ.

Consider first the case when A is flexible, in the sense that there exist finitely many $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_m$ tangent to A which span the tangent space of A at every point. (This holds in particular if A is the null quadric $\mathbf{A}$, cf. [8, Example 4.4]. A partial list of references to examples of flexible varieties can be found in [30, p. 394] preceding Example 6.3 ibid.) Consider a map $\Psi : \mathbb{C}^N \times B \times X \to A$ of the form

$$\Psi(\zeta, b, x) = \phi^1_{\zeta_1 h_1(b,x)} \circ \cdots \circ \phi^N_{\zeta_N h_N(b,x)}(f(b,x)) \tag{10.4}$$

where $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N$, $(b, x) \in B \times X$, every ϕ^j is the flow of one of the vector fields $V_1, \ldots, V_m$ (possibly with repetitions), and the functions $h_j \in \mathscr{C}^{l,k+1}(B \times X)$ are X-holomorphic. By what was said above, the map Ψ is holomorphic in ζ, of class $\mathscr{C}^{l,k+1}$ in (b, x), and J_b-holomorphic in x for every fixed $b \in B$ and ζ. A suitable choice of the functions h_j ensures that for a generic choice of $\zeta \in \mathbb{C}^N$ the homotopy $f^t := \Psi(t\zeta, \cdot\,, \cdot\,) : B \times X \to A$ $(t \in [0, 1])$ satisfies the lemma. Approximation is achieved by choosing ζ close to 0.

If A is not necessarily flexible, we can still find finitely many holomorphic vector fields $V_1, \ldots, V_m$ tangent to A and spanning the tangent space of A at every point (see the extension of Cartan's Theorem A due to Forster [29, Corollary 4.4] and Kripke [55]). Pick a compact Runge set $K \subset X$ with nonempty interior. Let $\mathbb{B}^N$ denote the unit ball of $\mathbb{C}^N$. The map Ψ in (10.4) is now defined on $r\mathbb{B}^N \times B \times K$ for some $r > 0$ and satisfies $\Psi(0, \cdot\,, \cdot\,) = f$. Pick a number $r_0 \in (0, r)$. By Theorem 7.4 we can approximate Ψ on $r_0\mathbb{B}^N \times B \times K$ by a map $\widetilde\Psi : r_0\mathbb{B}^N \times B \times X \to A$ of the same smoothness class such that $\widetilde\Psi(\cdot\,, b, \cdot\,) : r\mathbb{B}^N \times X \to A$ is holomorphic in the complex structure J_b on X and the standard structure on $\mathbb{C}^N$, and $\widetilde\Psi(\zeta, \cdot\,, \cdot\,)$ is homotopic to f for every ζ. For a generic choice of $\zeta \in r_0\mathbb{B}^N$ close to 0 this map satisfies the lemma. □

Fix a complex structure J on X and a strongly J-subharmonic Morse exhaustion function $\rho : X \to \mathbb{R}_+$. There is an exhaustion $\varnothing = K_0 \subset K_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} K_i = X$ by smoothly bounded compact Runge sets $K_i = \{x \in X : \rho(x) \le c_i\}$ for a sequence of regular values $0 < c_1 < c_2 \cdots$ of ρ, with $\lim_i c_i = +\infty$, such that $K_1 \ne \varnothing$ and for every $i \in \mathbb{Z}_+$ the domain $D_i = \mathring{K}_{i+1} \setminus K_i$ contains at most one critical point of ρ. (See [11, Sect. 1.4].) Recall that $\{\theta_b\}_{b\in B}$ is a family of nowhere vanishing J_b-holomorphic 1-forms on X furnished by Theorem 8.1. We shall inductively construct a sequence of open neighbourhoods $U_i \subset B \times X$ of $B \times K_i$, maps $f_i : U_i \to A$ of class $\mathscr{C}^{l,k+1}$, and numbers $\epsilon_i > 0$ such that the following conditions hold for every $i = 1, 2, \ldots$.

(a) The map $f_{i,b} : U_{i,b} \to A$ is J_b-holomorphic and nondegenerate for every $b \in B$.
(b) $\int_C f_{i,b}\theta_b = 0$ for every closed curve $C \subset K_i$.
(c) f_i is homotopic to $f_1|_{U_i}$ through maps $U_i \to A$.
(d) $\|f_{i+1} - f_i\|_{\mathscr{C}^l(B\times K_i)} < \epsilon_i$.
(e) $0 < \epsilon_{i+1} < \epsilon_i/2$, and if $f : B \times X \to A \cup \{0\}$ is an X-holomorphic map of class $\mathscr{C}^l$ such that $\|f - f_i\|_{\mathscr{C}^l(B\times K_i)} < 2\epsilon_i$ then f is nondegenerate and $f(B \times K_{i-1}) \subset A$.

Under these conditions, the limit map $f = \lim_{i\to\infty} f_i : B \times X \to A$ exists and is of class $\mathscr{C}^l(B \times X)$, it is X-holomorphic (hence of class $\mathscr{C}^{l,k+1}(B \times X)$ by Lemma 5.4), nondegenerate, homotopic to f_0, and $\int_C f_b\theta_b = 0$ holds for every closed curve $C \subset X$ and $b \in B$. Fixing a point $x_0 \in X$, the map $h : B \times X \to \mathbb{C}^n$ given by

$$h(b, x) = \int_{x_0}^{x} f(b, \cdot\,)\theta_b, \qquad b \in B,\ x \in X$$

is a well-defined nondegenerate A-immersion, and $f_b = dh_b/\theta_b$ holds for every $b \in B$.

We now explain the induction. The assumptions imply that K_1 is a smoothly bounded compact disc. Let U_1 be an open disc containing K_1, and let f_1 be the restriction to $B \times U_1$ of the initial nondegenerate map $f_1 : B \times X \to A$. Assume inductively that $i \in \mathbb{N}$ and we have already found maps f_j with the required properties for $j = 1, \ldots, i$, and let us explain how to obtain the next map f_{i+1}. We distinguish two cases.

The noncritical case: The domain $D_i = \mathring{K}_{i+1} \setminus K_i$ does not contain any critical point of ρ.

The critical case: D_i contains a unique (Morse) critical point of ρ.

We begin with the noncritical case. Then, K_i is a strong deformation retract of K_{i+1} and D_i is a finite union of annuli. In particular, the inclusion $K_i \hookrightarrow K_{i+1}$ induces an isomorphism of their homology groups $H_1(K_i, \mathbb{Z}) \cong H_1(K_{i+1}, \mathbb{Z})$. Assume that K_i is connected; the procedure that we shall explain can be performed independently on every connected component. Fix a point $x_0 \in \mathring{K}_i$. There are finitely many smooth Jordan curves $C_1, \ldots, C_m \subset K_i$ such that any two of them only intersect at x_0, they form a basis of the homology group $H_1(K_i, \mathbb{Z})$, and their union $C = \bigcup_{j=1}^m C_j$ is Runge in X. The same curves then form a basis of $H_1(K_{i+1}, \mathbb{Z})$. Consider the period map $\mathcal{P} : B \times \mathscr{C}(B \times C, A) \to (\mathbb{C}^n)^m$ given for any $b \in B$ and $f \in \mathscr{C}(B \times C, A)$ by

$$\mathcal{P}(b, f) = \left(\oint_{C_j} f(b, \cdot\,)\theta_b\right)_{j=1,\ldots,m} \in (\mathbb{C}^n)^m. \tag{10.5}$$

By condition (b) we have that $\mathcal{P}(b, f_i) = 0$ for all $b \in B$. Since the map $f_i : B \times K_i \to A$ is nondegenerate, we can apply [8, Lemma 5.1] (see also [11, Lemma 3.2.1]) to find a *period dominating spray* of J_b-holomorphic maps

$$F_i(\zeta, b, \cdot\,) : K_i \to A \quad \text{for } b \in B,$$

of class $\mathscr{C}^l(B \times K_i)$, depending holomorphically on $\zeta = (\zeta_1, \ldots, \zeta_N)$ in a ball $\mathbb{B} \subset \mathbb{C}^N$, such that $F_i(0, \cdot\,, \cdot\,) = f_i$. (Recall that a map is called holomorphic on a compact set if it is holomorphic in an open neighbourhood of the said set.) The period domination property means that the map

$$\mathbb{B} \ni \zeta \longmapsto \mathcal{P}(b, F_i(\zeta, b, \cdot\,)) = \left(\oint_{C_j} F_i(\zeta, b, \cdot\,)\theta_b\right)_{j=1,\ldots,m} \in (\mathbb{C}^n)^m \tag{10.6}$$

is submersive at $\zeta = 0$, i.e., its differential at $\zeta = 0$ is surjective for every $b \in B$. Such a map F_i can be chosen of the same form as Ψ in (10.4), and it has the same regularity properties as that map. We begin by choosing functions $h_j : B \times C \to \mathbb{C}$ of class $\mathscr{C}^l$ as in (10.4) to ensure that the map Ψ is period dominating on the curves in C; see [11, proof of Lemma 3.2.1] for the details. By the parametric Mergelyan theorem (see Theorem 1.2 when $l = 0$ and Theorem 7.7 when $l > 0$) we can approximate the functions h_j in $\mathscr{C}^l(B \times C)$ by X-holomorphic functions of class $\mathscr{C}^l(B \times K_i)$ depending holomorphically on ζ. By Lemma 5.4 these approximants are then of class $\mathscr{C}^{l,k+1}(B \times K_i)$. If the approximation is close enough then the resulting map F_i has the stated properties.

For each $b \in B$ let $V_b \subset \mathbb{C}^N$ denote the kernel of the differential of the period map (10.6) at $\zeta = 0$. This is a complex vector subspace of $\mathbb{C}^N$ with $\dim V_b = N - mn$ which is of class $\mathscr{C}^l$ in $b \in B$. Let $W_b \subset \mathbb{C}^N$ denote the orthogonal complement of V_b. Fix a number $0 < r < 1$. Since A is an Oka manifold and K_i is a strong deformation retract of K_{i+1}, Theorem 7.4 allows us to approximate F_i in the $\mathscr{C}^l$ topology on $r\mathbb{B} \times B \times K_i$ by a family of J_b-holomorphic maps $g(\zeta, b, \cdot\,) : K_{i+1} \to A$ $(\zeta \in r\mathbb{B},\ b \in B)$ which are holomorphic in ζ and of the same regularity class as F_i. If the approximation is sufficiently close, the implicit function theorem gives a map $\zeta : B \to \mathbb{B}$ of class $\mathscr{C}^l(B)$, close to the zero map, such that $\zeta(b) \in W_b$ for all $b \in B$ and the J_b-holomorphic map

$$f_{i+1}(b, \cdot\,) := g(\zeta(b), b, \cdot\,) : K_{i+1} \to A$$

satisfies the period vanishing conditions $\mathcal{P}(b, f_{i+1}) = 0$ (10.6) for every $b \in B$. If the approximations were close enough then the map f_{i+1} is nondegenerate. To complete the induction step, we choose a number ϵ_{i+1} satisfying condition (e).

Next, we consider the critical case. Let $x_i \in D_i$ be the unique critical point of ρ in D_i. Since ρ is strongly subharmonic, its Morse index is either 0 or 1. If the Morse index is 0, the point x_i is a local minimum of ρ, and hence a new connected component of the sublevel set $\{\rho \le t\}$ appears when t passes the value $\rho(x_i)$. On this new component of K_{i+1} we can take f_{i+1} to be any nondegenerate X-holomorphic map to A. On the remaining connected components of K_{i+1} we proceed as in the noncritical case explained above.

If the critical point $x_i \in D_i$ of ρ has Morse index 1, there is a smooth embedded arc $x_i \in E_i \subset D_i \cup bK_i$, which is transversely attached with both endpoints to bK_i and is otherwise disjoint from K_i, such that $S_i = K_i \cup E_i$ is a Runge admissible set in X (see Definition 7.6), and S_i is a strong deformation retract of K_{i+1}. (See [11, pp. 21–22] for the details.) We assume that the Runge admissible set $S_i = K_i \cup E_i$ is connected, since on the remaining components of K_i we are faced with the noncritical case described above. We extend f_i from a small open neighbourhood U_i of $B \times K_i$ to a map $U_i \cup (B \times S_i) \to A$ of class $\mathscr{C}^l$ such that the extended map is homotopic to f_1 through a homotopy that is fixed on U_i, and for every $b \in B$ the map $f_i(b, \cdot\,) : E_i \to A$ is nondegenerate (see Definition 10.1). Nondegeneracy of f_i on E_i can be ensured as in the proof of Lemma 10.5.

If the arc E_i connects two different connected components of K_i then the homology basis of $H_1(S_i, \mathbb{Z})$ is the union of homology bases of these two components, and there is no further condition on the extended map f_i on $B \times E_i$. If on the other hand the endpoints of E_i are attached to the same connected component of K_i, then the arc E_i closes in K_i to a Jordan curve C, which is an additional element of the homology basis of S_i. In this case, we choose the extension of f_i to E_i so that $\int_C f_i(b, \cdot\,)\theta_b = 0$ holds for all $b \in B$. This can be done by [34, Lemma 3.1 and Claim, p. 26].

We can now proceed as in the noncritical case. Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a homology basis of S_i such that $C = \bigcup_{j=1}^m C_j$ is a Runge set (see [11, Lemma 1.12.10]). As in the noncritical case, we find a period dominating spray $F_i : \mathbb{B} \times B \times S_i \to A$ of the form (10.4), where $\mathbb{B} \subset \mathbb{C}^N$ is a ball, such that $F_i(0, \cdot\,, \cdot\,) = f_i$ and the map $F_i(\cdot\,, b, \cdot\,)$ is holomorphic in the complex structure $J_{\rm st} \times J_b$ on $\mathbb{B} \times K_i$ for every $b \in B$. By Theorem 7.5 (if $l = 0$) or Theorem 7.7 (if $l > 0$) we can approximate f_i in

$\mathscr{C}^l(B\times S_i)$ by X-holomorphic maps $f_i' : B\times V_i \to A$, where V_i is a neighbourhood of S_i. Likewise, we can approximate the X-holomorphic functions h_j in the expression (10.4) in $\mathscr{C}^l(B \times S_i)$ by functions h_j' which are X-holomorphic on $B \times V_i$. Pick a number $0 < r < 1$. Inserting these approximants in the expression (10.4) for F_i and shrinking the neighbourhood V_i around S_i if necessary gives a map $g_i : r\mathbb{B} \times B \times V_i \to A$ approximating F_i in $\mathscr{C}^l(r\mathbb{B} \times B \times S_i)$ such that $g_i(\cdot\,, b, \cdot\,)$ is holomorphic in the complex structure $J_{\rm st} \times J_b$ on $r\mathbb{B} \times X$ for every $b \in B$.

The final step is exactly as in the noncritical case, and we obtain a nondegenerate X-holomorphic map $f_{i+1} : B\times V_i \to A$ approximating f_i in $\mathscr{C}^l(B\times K_i)$ such that for every $b \in B$, the J_b-holomorphic map $f_{i+1}(b, \cdot\,) : V_i \to A$ satisfies the period vanishing conditions $\mathcal{P}(b, f_{i+1}) = 0$ (see (10.5)) for the curves in the homology basis of $H_1(S_i, \mathbb{Z})$. Next, we extend f_{i+1} by approximation in $\mathscr{C}^l(B \times S_i)$ to an X-holomorphic map $f_{i+1} : B \times K_{i+1} \to A$, keeping the period vanishing conditions. This is accomplished by the noncritical case since S_i has a compact neighbourhood $S_i' \subset V_i$ such that $K_{i+1} \setminus S_i'$ is an annulus. We conclude the induction step by choosing ϵ_{i+1} satisfying condition (e). □

The following h-principle for families of conformal minimal immersions is an immediate corollary to Theorem 10.2 applied with the null quadric $\mathbf{A}$ (10.3).

Corollary 10.6. *Let X, $\{J_b\}_{b\in B}$, and $\{\theta_b\}_{b\in B}$ be as in Theorem 10.2. Given a continuous map $f_0 : B \times X \to \mathbf{A}$ to the punctured null quadric $\mathbf{A} \subset \mathbb{C}^n$ (10.3) for some $n \geq 3$, there is a map $u : B \times X \to \mathbb{R}^n$ of class $\mathscr{C}^{l,k+1}$ such that $u_b = u(b, \cdot\,) : (X, J_b) \to \mathbb{R}^n$ is a nonflat conformal minimal immersion for every $b \in B$, and the X-holomorphic map $f : B \times X \to \mathbf{A}$, defined by $f(b, \cdot\,) = \partial_{J_b} u_b/\theta_b$ for all $b \in B$, is homotopic to f_0.*

This result can also be improved by adding approximation and interpolation. We invite the reader to supply the precise statement and proof of this generalization.

By adding various global conditions on the maps in Theorem 10.2 and Corollary 10.6 such as properness, embeddedness, or completeness, the construction methods become more intricate, and we do not know whether they can be made in families. We pose the following problems.

Problem 10.7. Let $\{(X, J_b)\}_{b\in B}$ be a family of open Riemann surfaces as in Theorem 1.4.

(a) Is there a continuous or a smooth family of proper J_b-holomorphic immersions $X \to \mathbb{C}^2$ and embeddings $X \hookrightarrow \mathbb{C}^3$? (The basic case is classical, see [32, Theorem 2.4.1] and the references therein. Without the properness condition, the affirmative result is given by Theorem 10.2.)
(b) Assuming that $A \subset \mathbb{C}^n$ is an Oka cone (10.1), is there a continuous or a smooth family of proper J_b-holomorphic A-immersions or A-embeddings $X \to \mathbb{C}^n$? (For the basic case, see [8]. Without the properness or embeddednes condition, the affirmative answer is given by Theorem 10.2.)
(c) Is there a continuous or a smooth family of proper conformal harmonic immersions $(X, J_b) \to \mathbb{R}^n$ for $n \geq 3$? (For the basic case, see [11, Theorem 3.6.1] and the references therein.)
(d) Assume that X is a bordered Riemann surface. Is there a family of complete conformal minimal immersions $(X, J_b) \to \mathbb{R}^n$ ($n \geq 3$) with bounded images, i.e., does the Calabi–Yau phenomenon for minimal surfaces hold in families? (For the nonparametric case, see [11, Chapter 7] and [4].) The analogous question can be asked for holomorphic (directed) immersions $(X, J_b) \to \mathbb{C}^n$, $n \geq 2$ in the context of the problem asked by Paul Yang [83]; see the survey by Alarcón [5].
(e) Let $\eta = dz + \sum_{j=1}^n x_j dy_j$ be the standard complex contact form on $\mathbb{C}^{2n+1}$, $n \geq 1$. Is there a continuous or a smooth family of proper J_b-holomorphic Legendrian immersions $f_b : X \to \mathbb{C}^{2n+1}$, that is, such that $f_b^*\eta = 0$ holds for all $b \in B$? (For the basic case, see [10]. For the parametric case for maps from a single Riemann surface and without the properness condition, see [33].)

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