APPROXIMATION OF BIHOLOMORPHIC MAPS BETWEEN RUNGE DOMAINS BY HOLOMORPHIC AUTOMORPHISMS

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ABSTRACT. We show that biholomorphic maps between certain pairs of Runge domains in the complex affine space $\mathbb{C}^n$, $n > 1$, are limits of holomorphic automorphisms of $\mathbb{C}^n$. A similar result holds for volume preserving maps and also in Stein manifolds with the density property. This generalises several results in the literature with considerably simpler proofs.

1. INTRODUCTION

A holomorphic vector field V on the complex Euclidean space $\mathbb{C}^n$ is said to be complete if its flow $\phi_t(z)$, solving the initial value problem

$$\frac{d}{dt}\phi_t(z) = V(\phi_t(z)), \quad \phi_0(z) = z \in \mathbb{C}^n,$$

exists for every $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$. Such a vector field V is also complete in complex time $t \in \mathbb{C}$ (see [9, Corollary 2.2]), and $\{\phi_t\}_{t\in\mathbb{C}}$ is a complex 1-parameter subgroup of the holomorphic automorphism group $\mathrm{Aut}(\mathbb{C}^n)$ of $\mathbb{C}^n$. The same conclusion holds if V is assumed to be complete in positive real time; see Ahern, Flores, and Rosay [2].

Let $\mathbb{B}(0,\epsilon)$ denote the ball of radius ϵ around the origin $0 \in \mathbb{C}^n$. We say that 0 is a *globally attracting fixed point* of V if $V(0) = 0$ and the following two conditions hold:

(1) $\lim_{t\to+\infty} \phi_t(z) = 0$ holds for all $z \in \mathbb{C}^n$.
(2) For every $\epsilon > 0$ there exists a $\delta > 0$ such that $\phi_t(z) \in \mathbb{B}(0,\epsilon)$ for every $z \in \mathbb{B}(0,\delta)$ and $t \geq 0$.

A domain $\Omega \subset \mathbb{C}^n$ is said to be invariant under the positive time flow of V if $\phi_t(z) \in \Omega$ for every $z \in \Omega$ and $t \geq 0$. Such a domain is sometimes called *spirallike* for V (see [12]). It was shown by Chatterjee and Gorai [6, Theorem 1.1] (see also El Kasimi [7] for starshaped domains and Hamada [13, Theorem 3.1] for linear vector fields) that a spirallike domain Ω containing the origin is Runge in $\mathbb{C}^n$, that is, the restrictions of holomorphic polynomials on $\mathbb{C}^n$ to Ω form a dense subset of the space $\mathscr{O}(\Omega)$ of holomorphic functions on Ω.

In this note we prove the following result.

Theorem 1.1. *Assume that V is a complete holomorphic vector field on $\mathbb{C}^n$, $n > 1$, with a globally attracting fixed point $0 \in \mathbb{C}^n$ and the domain $0 \in \Omega \subset \mathbb{C}^n$ is invariant under the positive time flow of V. Then, every biholomorphic map from Ω onto a Runge domain in $\mathbb{C}^n$ can be approximated uniformly on compacts in Ω by holomorphic automorphisms of $\mathbb{C}^n$.*

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By Andersén and Lempert [4], it follows that a biholomorphic map $F : \Omega \to F(\Omega) \subset \mathbb{C}^n$ in Theorem 1.1 can be approximated uniformly on compacts by compositions of holomorphic shears and generalized shears. We refer to [10, Chapter 4] and [8] for surveys of this theory.

For a starshaped domain $\Omega \subset \mathbb{C}^n$, Theorem 1.1 coincides with [4, Theorem 2.1] due to Andersén and Lempert. Theorem 1.1 generalizes results of Hamada [13, Theorem 4.2] (in which the vector field V is linear) and Chatterjee and Gorai [6, V5, Theorem 1.5]. Their results give the same conclusion under additional conditions on the flow of V, and their proofs (especially the one in [6]) are fairly involved. The papers [13, 6] include applications to the theory of Loewner partial differential equation; see Arosio, Bracci and Wold [5] for the latter.

Here we show that Theorem 1.1 is an elementary corollary to [11, Theorem 1.1] and no additional conditions on the vector field V are necessary.

We wish to point out that very little seems to be known about globally attracting complete nonlinear holomorphic vector fields on $\mathbb{C}^n$ for $n > 1$. It was proved by Rebelo [14] that a complete holomorphic vector field on $\mathbb{C}^2$ has a nonvanishing two-jet at each fixed point. It seems unknown whether such a vector field can have more than one fixed point.

Proof of Theorem 1.1. Let V and $\Omega \subset \mathbb{C}^n$ be as in the theorem. By [6, Theorem 1.1], Ω is Runge in $\mathbb{C}^n$. We shall prove that every biholomorphic map $F : \Omega \to \Omega'$ onto a Runge domain $\Omega' \subset \mathbb{C}^n$ is a limit of holomorphic automorphisms of $\mathbb{C}^n$, uniformly on compacts in Ω.

We may assume that $F(0) = 0$ and the derivative $DF(0)$ is the identity map. Thus, F is a small perturbation of the identity near the origin. In particular, choosing $\epsilon > 0$ small enough, we have that $\mathbb{B}(0, \epsilon) \subset \Omega$ and the image $F(\mathbb{B}(0, \epsilon))$ is convex. Hence, the restricted map $F : \mathbb{B}(0, \epsilon) \to F(\mathbb{B}(0, \epsilon))$ is a limit of holomorphic automorphisms by [4, Theorem 2.1]. Fix such an ϵ. Note that for each $t \geq 0$ the domain $\Omega_t := \phi_t(\Omega) \subset \Omega$ is Runge in $\mathbb{C}^n$ (and hence in Ω) since $\phi_t \in \mathrm{Aut}(\mathbb{C}^n)$.

Assume first that $\overline{\Omega}$ is compact. Conditions (1) and (2) on the vector field V imply that there is a $t_0 > 0$ such that $\phi_t(\overline{\Omega}) \subset \mathbb{B}(0, \epsilon)$ for all $t \geq t_0$. Indeed, given a point $p \in \overline{\Omega}$, condition (1) gives a number $t(p) > 0$ such that $\phi_{t(p)}(p) \in \mathbb{B}(0, \delta)$. By continuity, there is a neighbourhood $U_p \subset \mathbb{C}^n$ of p such that $\phi_{t(p)}(U_p) \subset \mathbb{B}(0, \delta)$. This gives a finite open cover $U_1, \ldots, U_m$ of $\overline{\Omega}$ and numbers $t_1 > 0, \ldots, t_m > 0$ such that

$$\phi_{t_j}(U_j) \subset \mathbb{B}(0, \delta) \ \text{ holds for } j = 1, \ldots, m. \tag{1.1}$$

Set $t_0 = \max\{t_1, \ldots, t_m\}$. By property (2) of the flow and (1.1) we have that $\phi_t(\overline{\Omega}) \subset \mathbb{B}(0, \epsilon)$ for all $t \geq t_0$, which proves the claim.

Recall that $\Omega' = F(\Omega)$. Let $\psi_t : \Omega' \to \Omega'$ for $t \geq 0$ be the unique holomorphic map which is F-conjugate to $\phi_t : \Omega \to \Omega$, defined by the condition

$$F \circ \phi_t = \psi_t \circ F \ \text{ for all } t \geq 0.$$

Thus, ψ_t maps Ω' biholomorphically onto the domain $\Omega'_t := \psi_t(\Omega') = F(\Omega_t) \subset \Omega'$ for every $t \geq 0$, and ψ_0 is the identity on Ω'. Since Ω_t is Runge in Ω for every $t \geq 0$ and the map $F : \Omega \to \Omega'$ is biholomorphic, we infer that Ω'_t is Runge in Ω', and hence also in $\mathbb{C}^n$ (since Ω' is Runge in $\mathbb{C}^n$). Consider the family of maps

$$F_t = F \circ \phi_t : \Omega \xrightarrow{\cong} \Omega'_t, \quad t \geq 0. \tag{1.2}$$

This is an isotopy of biholomorphic maps from the Runge domain Ω onto the family of Runge domains $\Omega'_t \subset \mathbb{C}^n$, with F_t depending smoothly on t. Since $\overline{\Omega}_{t_0} = \phi_{t_0}(\overline{\Omega}) \subset \mathbb{B}(0,\epsilon)$, the restricted map $F : \mathbb{B}(0,\epsilon) \to \mathbb{C}^n$ is a limit of automorphisms of $\mathbb{C}^n$ and $\phi_{t_0} \in \mathrm{Aut}(\mathbb{C}^n)$, the map $F_{t_0} = F \circ \phi_{t_0}$ is also a limit of automorphisms of $\mathbb{C}^n$. By [11, Theorem 1.1] it follows that every map F_t in the isotopy (1.2) is a limit of automorphisms of $\mathbb{C}^n$. In particular, this holds for the map $F_0 = F : \Omega \to \Omega'$.

This proves the theorem in the case when $\overline{\Omega}$ is compact. The general case follows by observing that Ω is exhausted by relatively compact domains $\Omega_0 \Subset \Omega$ containing the origin which are invariant under the positive time flow of V. To see this, choose an open relatively compact subset W of Ω and set $\Omega_0 = \bigcup_{t\ge 0} \phi_t(W) \subset \Omega$. Obviously, Ω_0 is open and positive time invariant. Pick $\epsilon > 0$ such that $\overline{\mathbb{B}(0,\epsilon)} \subset \Omega$. We see as before that there is a number $t_0 > 0$ such that $\phi_t(\overline{W}) \subset \mathbb{B}(0,\epsilon)$ for all $t \ge t_0$. It follows that

$$\Omega_0 \subset \bigcup_{0\le t\le t_0} \phi_t(\overline{W}) \cup \overline{\mathbb{B}(0,\epsilon)}.$$

Since the first set on the right hand side is compact and contained in Ω, we see that $\overline{\Omega}_0 \subset \Omega$. By [6, Theorem 1.1], Ω_0 is Runge in $\mathbb{C}^n$, and hence in Ω. If follows that $F(\Omega_0) = \Omega'_0$ is Runge in $\Omega' = F(\Omega)$, and hence also in $\mathbb{C}^n$ (since Ω' is Runge in $\mathbb{C}^n$). The above argument in the special case then shows that $F : \Omega_0 \to \Omega'_0$ is a limit of holomorphic automorphisms of $\mathbb{C}^n$ uniformly on compacts in Ω. By the construction, Ω_0 can be chosen to contain any given compact subset of Ω, which proves the theorem. □

The Runge domain Ω in Theorem 1.1 need not be pseudoconvex. Replacing $\mathbb{C}^n$ by a Stein manifold X with the density property (see Varolin [15, 16] and [10, Section 4.10]) and assuming that Ω is a pseudoconvex Runge domain in X which is positive time invariant for a complete holomorphic vector field V on X with a globally attracting fixed point in Ω, the conclusion of Theorem 1.1 still holds, with the same proof. The relevant version of the result on approximation of isotopies of biholomorphic maps between pseudoconvex Runge domains in X by holomorphic automorphisms of X is given by [10, Theorem 4.10.5]. (A recent survey on Stein manifolds with the density property can be found in [8, Sect. 2].) However, we do not know any example of a Stein manifold with the density property and with a globally attracting complete holomorphic vector field, other than the Euclidean spaces $\mathbb{C}^n$, $n > 1$.

A version of Theorem 1.1 also holds for biholomorphic maps $F : \Omega \to \Omega'$ between certain Runge domains in $\mathbb{C}^n$ with coordinates $z_1, \dots, z_n$ preserving the holomorphic volume form

$$\omega = dz_1 \wedge \cdots \wedge dz_n, \tag{1.3}$$

in the sense that $F^*\omega = \omega$. Note that $F^*\omega = (JF)\,\omega$ where JF denotes the complex Jacobian determinant of F. Recall that the *divergence* of a holomorphic vector field V with respect to ω is the holomorphic function $\mathrm{div}_\omega V$ satisfying the equation

$$L_V\omega = d(V\rfloor\omega) + V\rfloor d\omega = d(V\rfloor\omega) = \mathrm{div}_\omega V \cdot \omega, \tag{1.4}$$

where $L_V\omega$ is the Lie derivative of ω and $V\rfloor\omega$ is the inner product of V and ω. The first equality is Cartan's formula (see [1, Theorem 6.4.8]), and we used that $d\omega = 0$. Let ϕ_t denote the flow of V. From (1.4) we obtain Liouville's formula

$$\frac{d}{dt}\,\phi_t^*\omega = \phi_t^*(L_V\omega) = \phi_t^*(\mathrm{div}_\omega V\cdot\omega). \tag{1.5}$$

Assume now that $\mathrm{div}_\omega V = c \in \mathbb{C}$ is constant. This holds in particular for every linear holomorphic vector field on $\mathbb{C}^n$ as is seen from the formula

$$\mathrm{div}_\omega\left(\sum_{j=1}^n a_j(z)\frac{\partial}{\partial z_j}\right) = \sum_{j=1}^n \frac{\partial a_j}{\partial z_j}(z). \tag{1.6}$$

In this case, (1.5) reads $\frac{d}{dt}\,\phi_t^*\omega = c\,\phi_t^*\omega$. Since $\phi_0 = \mathrm{Id}$, it follows that

$$\phi_t^*\omega = e^{ct}\omega \ \text{ for all } t. \tag{1.7}$$

In particular, if V is globally contracting then $\Re c < 0$. The case $c = 0$ corresponds to ω-preserving vector fields whose flow maps have Jacobian 1. The following result should be compared with [6, V5, Theorem 1.10 (i)]. As before, ω is given by (1.3).

Theorem 1.2. *Let V be a complete holomorphic vector field on $\mathbb{C}^n$, $n > 1$, with a globally attracting fixed point $0 \in \mathbb{C}^n$, whose divergence $\mathrm{div}_\omega V = c$ is constant. Assume that the domain $0 \in \Omega \subset \mathbb{C}^n$ is pseudoconvex, invariant under the positive time flow $\{\phi_t\}_{t\geq 0}$ of V, it satisfies $H^{n-1}(\Omega,\mathbb{C}) = 0$, and $\phi_{t_0}(\Omega) \Subset \Omega$ holds for some $t_0 > 0$. Then, every volume preserving biholomorphic map of Ω onto a Runge domain $\Omega' \subset \mathbb{C}^n$ can be approximated uniformly on compacts in Ω by volume preserving automorphisms of $\mathbb{C}^n$.*

By Andersén [3], every volume preserving holomorphic automorphism of $\mathbb{C}^n$ is a locally uniform limit of compositions of shears.

Proof. Since V is globally attracting, we have that $\Re c < 0$. Let $W = \frac{-c}{n}\sum_{j=1}^n z_j\frac{\partial}{\partial z_j}$. From (1.6) we see that $\mathrm{div}_\omega W = -c$. The flow ψ_t of W is complete on $\mathbb{C}^n$ and satisfies

$$\psi_t^*\omega = e^{-ct}\omega \ \text{ for all } t \in \mathbb{C}. \tag{1.8}$$

(Compare with (1.7).) Consider the family of injective holomorphic maps

$$F_t := \psi_t \circ F \circ \phi_t : \Omega \to F_t(\Omega) \subset \mathbb{C}^n, \quad t \geq 0.$$

Note that $F_0 = F : \Omega \to \Omega'$. Since $JF = 1$, it follows from (1.7) and (1.8) that $JF_t = 1$ for all $t \geq 0$. The conclusion now follows by the same argument as in the proof of Theorem 1.1, using the second part of [11, Theorem 1.1] on approximation of isotopies of volume preserving biholomorphic maps by volume preserving automorphisms of $\mathbb{C}^n$. (See the Erratum to [11] concerning the condition $H^{n-1}(\Omega,\mathbb{C}) = 0$.) $\square$

Remark 1.3. (A) Theorem 1.2 can be generalized to Stein manifolds (X,ω) having the volume density property; see [16], [8], and [10, Sect. 4.10] for this topic.

(B) Chatterjee and Gorai stated an analogue of [6, V5, Theorem 1.5] for holomorphic vector fields on $\mathbb{C}^{2n}$ with coordinates $(z_1,\ldots,z_n,w_1,\ldots,w_n)$ preserving the holomorphic symplectic form $\omega = \sum_{j=1}^n dz_j \wedge dw_j$ [6, V5, Theorem 1.10 (ii)]. Note however that such a vector field also preserves the volume form ω^n, so it does not have any attracting fixed points.

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