EVERY PROJECTIVE OKA MANIFOLD IS SUBELLIPTIC

FRANC FORSTNERIČ AND FINNUR LÁRUSSON

ABSTRACT. We show that every projective Oka manifold is subelliptic. This solves a long-standing open problem. We present further results concerning the relationship between the Oka property, ellipticity, subellipticity, and a new property that we call weak ellipticity.

1. INTRODUCTION

A complex manifold Y is said to be an Oka manifold if it satisfies all forms of the h-principle (also called the Oka principle) for holomorphic maps $X \to Y$ from any Stein manifold and, more generally, from reduced Stein spaces X. In Gromov's terminology [18, 3.1, p. 878], Oka manifolds are called Ell_∞ manifolds. Two simple characterisations of the class of Oka manifolds are the convex approximation property introduced by the first named author in [9] and the convex Ell_1 property (see Kusakabe [21, Theorem 1.3]).

A complex manifold Y is said to be elliptic in the sense of Gromov if it admits a dominating holomorphic spray $s : E \to Y$ defined on the total space of a holomorphic vector bundle $\pi : E \to X$ (see Gromov [18, 0.5, p. 855]). This means that s restricts to the identity map on the zero section $E_0 \cong Y$ of E, and for every $y \in Y$ the differential ds_{0_y} at the origin $0_y \in E_y = \pi^{-1}(y)$ maps E_y onto T_yY. An ostensibly weaker condition, subellipticity, was introduced by the first named author in [7, Definition 2]. It asks for the existence of finitely many holomorphic sprays (E_j, π_j, s_j) on Y, $j = 1, \dots, m$, satisfying

$$(ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) + \cdots + (ds_m)_{0_y}(E_{m,y}) = T_yY \quad \text{for all } y \in Y. \tag{1.1}$$

One of the main results of Oka theory is that every elliptic manifold is an Oka manifold (see [18, 0.6, p. 855] and [14]), and every subelliptic manifold is an Oka manifold [7, Theorem 1.1]. See also the survey in [11, Chap. 5]. Examples of elliptic and subelliptic manifolds can be found in [11, Sect. 6.4] and in the surveys [13, 10, 6]. In particular, every complex homogeneous manifold is elliptic but the converse fails in general.

In this paper we prove the following main result.

Theorem 1.1. *Every projective Oka manifold is subelliptic.*

Theorem 1.1 solves a long-standing open problem, originating in Gromov's seminal 1989 paper [18, 3.2.A" Question], whether every Oka manifold is elliptic or subelliptic; see also [11, Problem 6.4.21]. The first counterexamples for noncompact manifolds were found only very recently. In 2024, Kusakabe showed that the complement $\mathbb{C}^n \setminus K$ of any compact

Date: 25 February 2025. Minor edits on 10 March 2025.

2010 *Mathematics Subject Classification.* Primary 32Q56; secondary 32E10.

Key words and phrases. Projective manifold, Oka manifold, elliptic manifold, subelliptic manifold, Stein manifold.

polynomially convex set $K \subset \mathbb{C}^n$ for $n > 1$ is an Oka manifold [22, Theorem 1.6]. A few years earlier it was shown by Andrist, Shcherbina, and Wold [2] that if K is a compact set with nonempty interior in $\mathbb{C}^n$ for $n \geq 3$, then $\mathbb{C}^n \setminus K$ fails to be subelliptic. Taking K to be polynomially convex, $\mathbb{C}^n \setminus K$ is Oka but not subelliptic. These examples are non-Stein, and every Stein Oka manifold is elliptic [18, 3.2.A, p. 879]. However, there seems to be no example in the literature of a compact Oka manifold that fails to be elliptic or subelliptic. In light of Theorem 1.1, the remaining open questions on this topic are the following.

Problem 1.2. (a) Is every projective Oka manifold elliptic?
(b) Is there a compact non-projective Oka manifold that fails to be subelliptic?

In [18, 3.2.A', p. 879], Gromov proposed a way to see that every projective Oka manifold is elliptic, although this was not formally stated and no details were provided. We were unable to verify step 2 in his outline (see Remark 3.1), while step 3 follows from a later result by Prezelj [27, 28]; see Lemma 2.2.

A much studied property of algebraic manifolds is the algebraic version of ellipticity. A complex algebraic manifold Y is said to be algebraically elliptic if it admits an algebraic dominating spray $s : E \to Y$ defined on the total space of an algebraic vector bundle $\pi : E \to Y$; see [11, Definition 5.6.13 (e)]. Similarly, Y is algebraically subelliptic if it admits finitely many algebraic sprays (E_j, π_j, s_j) satisfying (1.1). It was recently shown by Kaliman and Zaidenberg [20] that every algebraically subelliptic manifold is algebraically elliptic; the converse is a tautology. Algebraic ellipticity is a Zariski local condition as shown by Gromov [18, 3.5.B, 3.5.C]; see also [11, Proposition 6.4.2]. No such results are known in the holomorphic category. Every algebraically elliptic manifold Y satisfies the algebraic homotopy approximation theorem for maps $X \to Y$ from affine manifolds X, showing in particular that every holomorphic map which is homotopic to an algebraic map is a limit of algebraic maps in the compact-open topology; see [8, Theorem 3.1], [11, Theorem 6.15.1], and the recent generalisations in [1, Sect. 2]. As shown by Lárusson and Truong [25], this is the closest analogue of the Oka principle in the algebraic category. The optimal known geometric sufficient condition for a compact algebraic manifold to be algebraically elliptic is uniform rationality; see Arzhantsev, Kaliman, and Zaidenberg [3, Theorem 1.3]. However, there are examples of projective Oka manifolds that fail to be algebraically elliptic, for example, abelian varieties. Hence, the algebraic counterpart to Theorem 1.1 is not true, and the GAGA principle of Serre [29] fails for subellipticity of projective manifolds.

2. Proof of Theorem 1.1

Let $Y \subset \mathbb{CP}^n$ be a projective manifold. Denote by $\mathbb{U} \to \mathbb{CP}^n$ the universal line bundle. We shall need the following lemma.

Lemma 2.1. *Given a point $y_0 \in Y$ and a tangent vector $0 \neq v_0 \in T_{y_0}Y$, there are an integer $k > 0$ and an algebraic vector field V on the total space of the line bundle $\pi : L = \mathbb{U}^k \to \mathbb{CP}^n$ (the k-th tensor power of $\mathbb{U}$) with the following properties.*

(a) *V vanishes on the zero section $L_0 \cong \mathbb{CP}^n$ of L.*
(b) *For every $e \in L|Y = \pi^{-1}(Y)$, we have that $d\pi_e V(e) \in T_{\pi(e)}Y$.*

(c) *There are an affine chart $U_0 \subset \mathbb{CP}^n$ with $y_0 \in U_0$, isomorphic to $\mathbb{C}^n$, with coordinates $x = (x_1, \ldots, x_n)$, and an algebraic line bundle chart $L|U_0 \cong U_0 \times \mathbb{C}$ in which*

$$V(x,t) = \sum_{i=1}^{n} t\, V_i(x) \partial_{x_i}, \tag{2.1}$$

where $\partial_{x_i} = \partial/\partial x_i$, $V_i(x)$ are polynomials, and $t \in \mathbb{C}$ is the fibre coordinate.

(d) *We have $d\pi_{e_0} V(e_0) = v_0$ for some $e_0 \in L_{y_0}$.*

Proof. Let $z = [z_0 : z_1 : \cdots : z_n]$ be homogeneous coordinates on $\mathbb{CP}^n$. Set $\Lambda_i = \{z_i = 0\}$ for $i = 0, 1, \ldots, n$, and let $U_i = \mathbb{CP}^n \setminus \Lambda_i \cong \mathbb{C}^n$ be the affine chart with coordinates $(z_0/z_i, \ldots, z_n/z_i)$ where the term $z_i/z_i = 1$ is omitted. We may assume that $y_0 \in U_0$. Denote the affine coordinates on U_0 by $x = (x_1, \ldots, x_n)$, with $x_i = z_i/z_0$.

Since $Y \cap U_0$ is an algebraic submanifold of $U_0 \cong \mathbb{C}^n$, there is a polynomial vector field $W(x) = \sum_{i=1}^n V_i(x) \partial_{x_i}$ on $\mathbb{C}^n$ whose restriction to Y is tangential to Y and satisfies $W(y_0) = v_0$. We associate to W the vector field $V(x,t) = \sum_{i=1}^n t\, V_i(x) \partial_{x_i}$ on the trivial line bundle $U_0 \times \mathbb{C} \cong \mathbb{C}^{n+1}$, where $t \in \mathbb{C}$ is the fibre coordinate. Thus, V is a horizontal vector field depending linearly on t. It clearly satisfies conditions (c) and (d) in the lemma, and it satisfies conditions (a) and (b) on $U_0 \times \mathbb{C}$.

We now show that, for a sufficiently large $k > 0$, V extends to an algebraic vector field on $L = \mathbb{U}^k$ satisfying conditions (a) and (b).

For every $i = 0, 1, \ldots, n$, we have a line bundle trivialisation $\theta_i : L|U_i \xrightarrow{\cong} U_i \times \mathbb{C}$ with transition maps $\theta_{i,j} = \theta_i \circ \theta_j^{-1}$ on $(U_i \cap U_j) \times \mathbb{C}$ given by

$$\theta_{i,j}([z], t) = \big([z], (z_i/z_j)^k t\big), \quad 0 \le i, j \le n.$$

In particular, $\theta_{i,0}([z], t) = \big([z], (z_i/z_0)^k t\big)$. We shall analyse the behaviour of V near the hyperplane $\Lambda_0 \setminus \Lambda_i$ for all $i = 1, \ldots, n$. It suffices to consider the case $i = 1$ since the same argument will apply to every i. Replacing the first coordinate $x_1 = z_1/z_0$ by $1/x_1 = z_0/z_1$, the vector field V has the same form (2.1), where the coefficient functions $V_j(x)$ are rational with poles along the hyperplane $\{x_1 = 0\} = \{z_0 = 0\}$. (The component V_1 gets changed, but this will not affect the subsequent argument.) In these coordinates, the transition map $\theta_{1,0}$ is given by $\theta_{1,0}(x,t) = \big(x, x_1^{-k} t\big)$. Its differential has the block form

$$D\theta_{1,0}(x,t) = \begin{pmatrix} I_n & 0 \\ b & x_1^{-k} \end{pmatrix}$$

where I_n is the identity $n \times n$ matrix and $b = (-k x_1^{-k-1} t, 0, \ldots, 0)$. Hence, the image vector field $V' = (\theta_{1,0})_* V$ on the chart $L|U_1$ for $x \in U_0 \cap U_1$ equals

$$V'(x,t) = D\theta_{1,0}(x,t) V(x,t) = \sum_{i=1}^{n} t V_i(x) \partial_{x_i} + (-k) t^2 x_1^{-k-1} V_1(x) \partial_{t'}.$$

In terms of the new fibre variable $t' = x_1^{-k} t$ (so $t = x_1^k t'$) we have

$$V'(x,t') = \sum_{i=1}^{n} t' x_1^k V_i(x) \partial_{x_i} - k (t')^2 x_1^{k-1} V_1(x) \partial_{t'}.$$

By choosing $k > 0$ big enough, V' extends to the points of L over the hyperplane $\Lambda_0 \setminus \Lambda_1 = \{x_1 = 0\}$ and it vanishes there. Applying this argument for every $i = 1, \ldots, n$, we see that for $k>0$ big enough the vector field V extends to the line bundle $L=\mathbb{U}^k$ and it vanishes on $L_0\cup(L|\Lambda_0)$. □

Proof of Theorem 1.1. Given a point $y_0\in Y$ and a vector $v_0\in T_{y_0}Y$, let V be a vector field on $L=\mathbb{U}^k$ given by Lemma 2.1. Since V vanishes on the zero section L_0 of L, there is a neighbourhood $\Omega\subset L$ of L_0 such that the flow $\phi_\tau(e)$ of V, starting at $\tau=0$ at any point $e\in\Omega$, exists for all $\tau\in[0,1]$. We may assume that Ω has convex fibres. The map

$$s_0=\pi\circ\phi_1:\Omega\to\mathbb{CP}^n$$

is a local holomorphic spray on $\mathbb{CP}^n$. Set $L|Y=\pi^{-1}(Y)$. Condition (b) in Lemma 2.1 implies that $\pi\circ\phi_\tau$ maps the domain $\Omega\cap L|Y$ to Y for every $\tau\in[0,1]$, so it is a family of local holomorphic sprays on Y. On the zero section L_0 we have a natural direct sum splitting $TL|L_0=L\oplus T\mathbb{CP}^n$. Identifying a vector $e\in L_y=\pi^{-1}(y)$ with $e\in T_{0_y}L_y$, we let

$$(Vds_0)_y(e)=(ds_0)_{0_y}(e)\in T_y\mathbb{CP}^n$$

denote the vertical derivative of s_0 at y applied to the vector e. We claim that conditions (c) and (d) in Lemma 2.1 imply

$$(Vds_0)_{y_0}(e_0)=v_0. \tag{2.2}$$

To see this, choose $\delta_0\in(0,1]$ such that $\delta_0e_0\in\Omega$. We make the calculation in the line bundle chart on $L|U_0$ over the affine chart $e_0\in U_0\subset\mathbb{CP}^n$ on which V is of the form (2.1). It follows that $\pi\circ\phi_\tau(\delta e)=\pi\circ\phi_{\delta\tau}(e)$ holds for every $e\in L|U_0$, $0\le\delta\le1$, and all τ for which the flow exists. Taking $\delta\in[0,\delta_0]$ gives $\pi\circ\phi_\tau(\delta e_0)=\pi\circ\phi_{\delta\tau}(e_0)$ for $0\le\tau\le1$. At $\tau=1$ we obtain

$$s_0(\delta e_0)=\pi\circ\phi_1(\delta e_0)=\pi\circ\phi_\delta(e_0),\quad 0\le\delta\le\delta_0.$$

Differentiating with respect to δ at $\delta=0$ and noting that $\left.\frac{d}{d\delta}\right|_{\delta=0}\phi_\delta(e_0)=V(e_0)$ and $d\pi_{e_0}V(e_0)=v_0$ (see condition (d)) gives (2.2).

So far, we have not used the hypothesis that Y is an Oka manifold. At this point, we replace L by $L|Y$ and Ω by $\Omega\cap L|Y$. Since the line bundle $L\to Y$ is negative, its total space L is a 1-convex manifold with the exceptional subset $L_0\cong Y$ (see Grauert [15, Satz 1, p. 341]). Hence, there is a plurisubharmonic exhaustion function $\rho:L\to[0,\infty)$ with $\rho^{-1}(0)=L_0$ which is strongly plurisubharmonic on $L\setminus L_0$. (In fact, the squared norm of a negatively curved hermitian metric on the line bundle L has this property.) In particular, L_0 admits a basis of strongly pseudoconvex neighbourhoods $\Omega\subset L$ with convex fibres. Assuming that Y is an Oka manifold, the results of Prezelj [27, 28] give a global holomorphic map $s:L\to Y$ which agrees with $s_0:\Omega\to Y$ to the second order along the zero section L_0 of L. Hence, s satisfies (2.2). Since the point $y_0\in Y$ and the vector $v_0\in T_{y_0}Y$ were arbitrary and Y is compact, finitely many sprays of this type dominate Y, so Y is subelliptic.

Let us provide the details for the last step of the proof. We recall the special case of Prezelj's result which will be used. Assume that X is a 1-convex manifold with the maximal compact nowhere-discrete complex submanifold Y (called the exceptional submanifold of X), $h:Z\to X$ is a holomorphic fibre bundle with an Oka fibre, K is a compact holomorphically convex subset of X containing Y, and $a:X\to Z$ is a continuous section of $h:Z\to X$ which is holomorphic on a neighbourhood of K. The main result of [27] (whose proof is completed in [28]) gives a homotopy of continuous sections $a_t:X\to Z$, $t\in[0,1]$, such that a_t agrees with a to any given finite order along Y, the sections a_t are holomorphic on a neighbourhood of K, they approximate a as closely as desired uniformly on K and uniformly in $t \in [0,1]$, and the section a_1 is holomorphic on X.

In the case at hand, it suffices to apply Prezelj's result to the 1-convex manifold $X = L$ with the exceptional submanifold $L_0 \cong Y$, letting $h : Z = X \times Y \to X$ be the trivial projection with the Oka fibre Y and $a : X \to Z$ be the graph of a continuous extension $a_0 : X \to Y$ of the holomorphic spray $s_0 : \Omega \to Y$ constructed above. Such an extension clearly exists if we choose Ω to be strongly pseudoconvex and with convex fibres. Let $a_1 : X \to Z$ be a holomorphic section furnished by Prezelj's theorem which agrees with a_0 to the second order along Y. The map $s = h \circ a_1 : L \to Y$ is then a holomorphic spray on Y with the required property. This completes the proof of Theorem 1.1. $\square$

For later reference we formulate the last part of the proof of Theorem 1.1 as a lemma. A holomorphic vector bundle $\pi : E \to Y$ is said to be negative if it admits a hermitian metric h that is negatively curved in the sense of Griffiths [16, 17]. For such h, the square norm function $\phi : E \to \mathbb{R}_+$, $\phi(e) = |e|_h^2$, $e \in E$, is plurisubharmonic on E and strongly plurisubharmonic on the complement $E \setminus E_0$ of the zero section E_0. If Y is compact, then the zero section $E_0 \cong Y$ is the exceptional submanifold of E (see [15, Satz 1, p. 341]).

Lemma 2.2. *Assume that $\pi : E \to Y$ is a Griffiths negative holomorphic vector bundle on a compact complex manifold Y and $s : \Omega \to Y$ is a local holomorphic spray defined on a neighbourhood $\Omega \subset E$ of the zero section E_0. If Y is an Oka manifold, then there exists a global holomorphic spray $s : E \to Y$ which agrees with s_0 to any given finite order along E_0. In particular, if s_0 is dominating, then s can be chosen to be dominating.*

3. Further results on ellipticity and subellipticity

In this section we collect some further results, remarks, and open problems concerning the relationship between the Oka property, ellipticity, and subellipticity of a complex manifold. We also introduce a new property that we call weak ellipticity (see Definition 3.6), which implies the Oka property for all complex manifolds and characterises the Oka property in the class of projective manifolds; see Theorem 3.7.

We begin with the following remark concerning [18, 3.2.A', p. 879].

Remark 3.1. If $L \to Y$ is a negative holomorphic line bundle on a compact (hence projective) manifold Y, then for a sufficiently large $k > 0$ the vector bundle $\mathrm{Hom}(L^k, TY) \cong L^{-k} \otimes TY$ on Y is generated by finitely many global holomorphic sections $h_1, \ldots, h_N$ (theorem of Hartshorne; see Lazarsfeld [26, Theorem 6.1.10]). Let $E = NL^k$ denote the direct sum of N copies of L^k. Considering h_i as a homomorphism $h_i : L^k \to TY$, it follows that the holomorphic vector bundle map $h = \oplus_{i=1}^N h_i : E \to TY$ is an epimorphism. More precisely, h is defined by $h(e_1, \ldots, e_N) = \sum_{i=1}^N h_i(e_i)$, where $e_1, \ldots, e_N \in L_y^k$ for some $y \in Y$ and the sum takes place in T_yY. Gromov proposed [18, 3.2.A', Step 2, p. 879] that such h is the vertical derivative of a local (dominating) holomorphic spray $s_0 : U \to Y$ from an open neighbourhood $U \subset E$ of its zero section $E_0 \cong Y$. If this holds true and Y is an Oka manifold, then Lemma 2.2 gives a dominating spray on Y, so Y is elliptic. However, we do not know how to prove Gromov's claim.

Problem 3.2. Which holomorphic vector bundles $\pi : E \to Y$ of rank $\geq \dim Y$ admit a local dominating spray $s : U \to Y$ from a neighbourhood $U \subset E$ of the zero section E_0 of E?

The discussion in Remark 3.1, together with Lemma 2.2, imply the following.

Corollary 3.3. *A projective Oka manifold Y which admits a local dominating spray is elliptic.*

Proof. Assume that $E \to Y$ is a holomorphic vector bundle and $s_0 : U \to Y$ is a local dominating holomorphic spray from a neighbourhood $U \subset E$ of the zero section E_0. By the argument in Remark 3.1, there exists a Griffiths negative holomorphic vector bundle $\widetilde{E} \to Y$ with a vector bundle epimorphism $h : \widetilde{E} \to E$ over Y. Then, $\widetilde{E}$ is a 1-convex manifold whose exceptional variety is the zero section $\widetilde{E}_0$ [15, Satz 1, p. 341]. Choose a neighbourhood $V \subset \widetilde{E}$ of $\widetilde{E}_0$ such that $h(V) \subset U$. The composition $s = s_0 \circ h : V \to Y$ is then a local dominating spray. Since Y is Oka, Lemma 2.2 implies that Y is elliptic. □

The following observation generalises [12, Proposition 6.2]. Recall that every complex homogeneous manifold is elliptic [11, Proposition 5.6.1], and hence an Oka manifold.

Proposition 3.4. *Assume that a compact complex manifold Y admits a local dominating holomorphic spray (E, π, s). If the bundle $\pi : E \to Y$ is generated by global holomorphic sections, then Y is a complex homogeneous manifold.*

The condition on E to be globally generated holds for a trivial bundle and for any sufficiently Griffiths positive bundle, but it fails for negative bundles.

Proof. Let $s : U \to Y$ be a local dominating spray defined on a neighbourhood $U \subset E$ of the zero section E_0. The vertical derivative $Vds|E_0 : VT(E)|E_0 = E \to TY$ is a vector bundle epimorphism. Given a holomorphic section $\xi : Y \to E$, the map

$$Y \ni y \to V_\xi(y) := Vds(y)(\xi(y)) \in T_yY$$

is a holomorphic vector field on Y. (We are using the natural identification of the vertical tangent bundle $VT(E)|E_0$ on the zero section E_0 with the bundle E itself.) Applying this argument to sections $\xi_1, \ldots, \xi_m : Y \to E$ generating E gives holomorphic vector fields $V_1, \ldots, V_m$ on Y spanning the tangent bundle TY (since Vds is surjective). Thus, the manifold Y is holomorphically flexible. Since Y is compact, these vector fields are complete, so their flows are complex 1-parameter subgroups of the holomorphic automorphism group $\mathrm{Aut}(Y)$. The spanning property implies that $\mathrm{Aut}(Y)$ acts transitively on Y. Since the holomorphic automorphism group of a compact complex manifold is a finite dimensional complex Lie group [5], it follows that Y is a homogeneous space of the complex Lie group $\mathrm{Aut}(Y)$. □

There are projective Oka manifolds that are not homogeneous, for instance, blowups of certain projective manifolds such as projective spaces, Grassmannians, etc.; see [11, Propositions 6.4.5 and 6.4.6], the papers [19, 24], and the survey [6, Subsect. 6.3]. Many of these manifolds are algebraically elliptic. Another class of non-homogeneous projective surfaces which are algebraically elliptic are the Hirzebruch surfaces H_l for $l = 1, 2, \ldots$; see [4, p. 191] and [11, Proposition 6.4.5]. In view of Proposition 3.4, such manifolds do not admit a local dominating spray from any globally generated holomorphic vector bundle.

Remark 3.5. Let $\mathscr{S}$ be the largest class of complex manifolds for which the Oka property implies subellipticity, that is, the class of manifolds that are either subelliptic or not Oka. As remarked above, it is long known that every Stein manifold belongs to $\mathscr{S}$. By Theorem 1.1, so does every projective manifold. We know of two ways to produce new members of $\mathscr{S}$ from old. If $Y \to X$ is a covering map and X is subelliptic, so is Y. Also, X is Oka if and only if Y is. Hence, a covering space of a manifold in $\mathscr{S}$ is in $\mathscr{S}$. Also, it is easily seen that a product of manifolds in $\mathscr{S}$ is in $\mathscr{S}$.

Every projective manifold Y carries an affine bundle $\pi : A \to Y$, whose total space is Stein. (This is the so-called Jouanolou trick; its relevance in Oka theory was noted in [23].) Suppose that Y is Oka. Then A is Oka and, being Stein, therefore elliptic. Let $p : E \to A$ be a vector bundle with a dominating spray $s : E \to A$. Let E_0 be the zero section of E. Since A is Stein, E can be taken to be trivial; it is then clear that the composition $\pi \circ p : E \to A \to Y$ is an affine bundle on Y with E_0 as an affine subbundle. The holomorphic map $\pi \circ s : E \to Y$ resembles a dominating spray over Y in that for every $y \in Y$, its restriction to the fibre $(\pi \circ p)^{-1}(y)$ is a submersion at each point of $E_0 \cap (\pi \circ p)^{-1}(y)$ and maps each such point to y.

We turn this setting into a new definition as follows.

Definition 3.6. A complex manifold Y is *weakly elliptic* if there is an affine bundle $\pi : A \to Y$ with an affine subbundle B and a holomorphic map $s : A \to Y$, such that for every $y \in Y$, the restriction of s to the fibre $\pi^{-1}(y)$ is a submersion at each point of $B \cap \pi^{-1}(y)$ and maps each such point to y.

Note that ellipticity is precisely the special case of B having rank zero: then A has a compatible vector bundle structure with B as its zero section.

Theorem 3.7. (a) *Every weakly elliptic manifold is Oka.*
(b) *A projective manifold is Oka if and only if it is weakly elliptic.*

Proof. (a) Let Y be as in the definition of weak ellipticity and let $f : X \to Y$ be a holomorphic map from a Stein manifold X. The pullback $f^*B \to X$, being a fibre bundle with a contractible Oka fibre over a Stein base, has a holomorphic section, which may serve as the zero section in a compatible vector bundle structure on the pullback bundle $f^*A \to X$. The composition of the bundle morphism $f^*A \to A$ over f followed by the map $s : A \to Y$ is then a dominating relative spray on Y with core f. This shows that Y is relatively elliptic (or Ell_1 in Gromov's terminology) and hence Oka by [21, Theorem 1.3].

(b) is clear from (a) and the discussion preceding the theorem. □

Acknowledgements. Forstnerič is supported by the European Union (ERC Advanced grant HPDR, 101053085) and grants P1-0291 and N1-0237 from ARIS, Republic of Slovenia. The work on this paper was done during his visit to the University of Adelaide in February 2025, and he wishes to thank the institution for hospitality.

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Franc Forstnerič, Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

Email address: franc.forstneric@fmf.uni-lj.si

Finnur Lárusson, Discipline of Mathematical Sciences, University of Adelaide, Adelaide SA 5005, Australia

Email address: finnur.larusson@adelaide.edu.au