THE OKA PRINCIPLE FOR TAME FAMILIES OF STEIN MANIFOLDS

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ABSTRACT. Let X be a smooth open manifold of even dimension, T be a topological space, and $\mathscr{J} = \{J_t\}_{t\in T}$ be a continuous family of smooth integrable Stein structures on X. Under suitable additional assumptions on T and $\mathscr{J}$, we prove an Oka principle for continuous families of maps from the family of Stein manifolds (X, J_t), $t \in T$, to any Oka manifold, showing that every family of continuous maps is homotopic to a family of J_t-holomorphic maps depending continuously on t. We also prove the Oka–Weil theorem for sections of $\mathscr{J}$-holomorphic vector bundles on $Z = T \times X$ and the Oka principle for isomorphism classes of such bundles. The assumption on the family $\mathscr{J}$ is that the J_t-convex hulls on any compact set in X are upper semicontinuous with respect to $t \in T$; such a family is said to be tame. For suitable parameter spaces T, we characterise tameness by the existence of a continuous family $\rho_t : X \to \mathbb{R}_+ = [0, +\infty)$, $t \in T$, of strongly J_t-plurisubharmonic exhaustion functions on X. Every family of complex structures on an open orientable surface is tame. We give an example of a nontame smooth family of Stein structures J_t on $\mathbb{R}^{2n}$ $(t \in \mathbb{R},\ n > 1)$ such that $(\mathbb{R}^{2n}, J_t)$ is biholomorphic to $\mathbb{C}^n$ for every $t \in \mathbb{R}$. We show that the Oka principle fails on any nontame family.

CONTENTS

1. INTRODUCTION

Let X be a smooth manifold of dimension $2n \geq 2$. An almost complex structure J on X is an endomorphism $J : TX \to TX$ of its tangent bundle satisfying $J^2 = -\mathrm{Id}$. When $n = 1$, i.e., X is a smooth surface, every such J of local Hölder class $\mathscr{C}^\alpha$, $0 < \alpha < 1$, determines on X the structure of a Riemann surface [2, Theorem 5.3.4]; if X is an open surface then (X, J) is a Stein manifold according to Behnke and Stein [3]. In [14] the first named author showed that, under suitable

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regularity assumptions on the parameter space T and on a family $\mathscr{J} = \{J_t\}_{t\in T}$ of complex structures on a smooth open surface X, the Oka principle holds for families of J_t-holomorphic maps from X to any Oka manifold Y, with continuous or smooth dependence on $t \in T$ and with approximation on suitable families of Runge subsets of X. The notion of an Oka manifold (see [11], [12, Sect. 5.4], and [13]) developed from the classical Oka–Grauert–Gromov principle [46, 22, 25].

In this paper we study the mapping problem for families of Stein structures on smooth manifolds of dimension $2n \geq 4$. Integrability is then a nontrivial condition; see Section 2. However, this is not the only new issue. The construction in [14] strongly uses the fact that the holomorphic hull of a compact set in a smooth surface X is independent of the choice of the complex structure on X. This is no longer the case on higher dimensional manifolds. In Theorem 4.1 we give an example of a smooth family of integrable Stein structures $\{J_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}^{2n}$ for any $n > 1$ such that $(\mathbb{R}^{2n}, J_t)$ is biholomorphic to $\mathbb{C}^n$ for every $t \in \mathbb{R}$ but the J_t-convex hulls of the closed ball explode when $t \in \mathbb{R} \setminus \{0\}$ approaches 0. This phenomenon excludes the possibility of any reasonable analysis of global analytic problems. Motivated by this example, we introduce a tameness condition on a family of Stein structures $\{J_t\}_{t\in T}$ on a smooth manifold X which excludes this type of pathology. Such a family is said to be tame if the J_t-convex hulls of any compact set in X are upper semicontinuous with respect to $t \in T$; see Definition 5.1. Tameness is characterised by the existence of a continuous family of strongly J_t-plurisubharmonic exhaustion functions $\rho_t : X \to \mathbb{R}_+$; see Theorem 5.5. Every family of Riemann surface structures is tame. We give several examples of tame families of Stein structures on higher dimensional manifolds.

The following Oka principle is a special case of our main result, Theorem 6.1.

Theorem 1.1. *Assume that T is a finite CW complex, X is a smooth manifold, $\mathscr{J} = \{J_t\}_{t\in T}$ is a tame family of smooth Stein structures on X depending continuously on t, and Y is an Oka manifold. Then, every continuous map $f : Z = T \times X \to Y$ is homotopic to a $\mathscr{J}$-holomorphic map $F : Z \to Y$, i.e. such that $F(t, \cdot) : X \to Y$ is J_t-holomorphic for every $t \in T$ and continuous in t. If f is $\mathscr{J}$-holomorphic on a neighbourhood of a closed subset $K \subset Z$ with proper projection $K \to T$ and J_t-convex fibres K_t $(t \in T)$, then F can be chosen to approximate f in the fine topology on K.*

The special case when Y is the complex number field $\mathbb{C}$ is the Oka–Weil theorem for such families; see Theorem 6.3. We show in Corollary 6.4 that the Oka principle fails on any nontame family, so tameness is a necessary and sufficient condition for the Oka principle. The Oka–Weil theorem is also proved for sections of fibrewise holomorphic vector bundles on tame families of Stein structures; see Theorem 7.2. This is used to obtain global solutions of the $\overline{\partial}$-equation for fibrewise smooth (p, q)-forms in all bidigrees, see Theorem 8.1. We also prove the Oka principle for the classification of complex vector bundles on such families, extending the classical results of Oka [46] and Grauert [22]; see Theorems 9.1 and 9.2 . Our results open a new direction in modern Oka theory.

An important ingredient in the proofs is a theorem of Hamilton [27], also called the global Newlander–Nirenberg theorem, on representing small integrable deformations of the complex structure on the closure of a smoothly bounded, relatively compact, strongly pseudoconvex domain Ω in a Stein manifold X by small deformations of Ω in X. We need a version with continuous dependence on parameters; see Theorem 3.1, which is obtained from the proof of Hamilton's theorem by Greene and Krantz [24, Theorem 1.13]. Unlike the original proof and its improvements [16, 21], which use the Nash–Moser technique, the proof in [24] is based on stability of the canonical (Kohn) solution of the $\overline{\partial}$-equation with respect to perturbations of the complex structure, obtained in [24, Theorem 3.10] by following the pioneering work of Kohn [32, 33] on the $\overline{\partial}$-Neumann problem. A special case of Hamilton's theorem with parameters for smoothly bounded domains in Riemann surfaces, and under considerably lower regularity assumptions on the family of complex structures, was obtained by the first named author in [14, Theorem 4.3] using the Beltrami equation.

In this section we recall the relevant background concerning almost complex structures.

Let X be a smooth manifold of real dimension $2n$. An almost complex structure J on X is an endomorphism $J : TX \to TX$ of its tangent bundle satisfying $J^2 = -\mathrm{Id}$. Every point $x_0 \in X$ has an open coordinate neighbourhood $U \subset X$ such that $TX|_U \cong U \times \mathbb{R}^{2n}$ and $J : TX|_U \to TX|_U$ is given by $(x, \xi) \mapsto (x, A(x)\xi)$, where the matrix $A(x) \in GL_{2n}(\mathbb{R})$ satisfies $A(x)^2 = -I$ with $I \in GL_{2n}(\mathbb{R})$ the identity matrix. We say that J is of class $\mathscr{C}^k$ if its matrix $A(x)$ in any smooth local coordinate on $U \subset X$ is a $\mathscr{C}^k$ map $U \to GL_{2n}(\mathbb{R})$. Similarly one defines (local) Hölder classes $\mathscr{C}^{(k,\alpha)}$ with $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$; see [17, Sect. 4.1]. An almost complex structure J extends to an endomorphism of the complexified tangent bundle $\mathbb{C}TX = TX \otimes_{\mathbb{R}} \mathbb{C}$. Since $J_x^2 = -\mathrm{Id}$ holds for every $x \in X$, J induces a decomposition $\mathbb{C}TX = H \oplus \overline{H}$ into a direct sum of complex subbundles of rank n whose fibres H_x and $\overline{H}_x$ over $x \in X$ are, respectively, the $+\mathfrak{i} = \sqrt{-1}$ and $-\mathfrak{i}$ eigenspaces of J_x on $\mathbb{C}T_xX$. This gives complex vector bundle projections $\pi_{1,0} : \mathbb{C}TX \to H$ and $\pi_{0,1} : \mathbb{C}TX \to \overline{H}$ satisfying

$$\pi_{1,0} = \bar{\pi}_{0,1}, \quad \pi_{1,0} + \pi_{0,1} = \mathrm{Id}, \quad \pi_{1,0} \circ \pi_{0,1} = 0 = \pi_{0,1} \circ \pi_{1,0}. \tag{2.1}$$

Conversely, a pair of such projections determines an almost complex structure J on X. Note that $\pi_{1,0}$ and $\pi_{0,1}$ are as smooth as J. An almost complex structure J of class $\mathscr{C}^1$ on X is said to be (formally) *integrable* if the subbundle $H = \pi_{1,0}(\mathbb{C}TX)$ satisfies the commutator condition $[H, H] \subset H$ for its sections, which are called vector fields of type $(1, 0)$. Every such vector field is of the form $v - \mathfrak{i}Jv$ where v is a real vector field on X. If $n = 1$ then the commutator condition is void, but integrability is a nontrivial condition when $n \geq 2$. For later reference, we state the following precise version of the Newlander–Nirenberg integrability theorem.

Theorem 2.1. *If X is a smooth manifold of dimension $2n$ and J is an integrable almost complex structure on X of local Hölder class $\mathscr{C}^{(k,\alpha)}$, with $k \geq 1$ an integer (or $k \geq 0$ when X is a surface) and $0 < \alpha < 1$, then every point $x_0 \in X$ has a neighbourhood $U \subset X$ with a J-holomorphic coordinate map $z : U \to \mathbb{C}^n$ of class $\mathscr{C}^{(k+1,\alpha)}$. Thus, (X, J) is a complex manifold, and the smooth structure on X determined by J is $\mathscr{C}^{(k+1,\alpha)}$ compatible with the given smooth structure.*

This result has a complex genesis. For surfaces ($n = 1$), see Korn [34], Lichtenstein [37], Chern [6], and Astala et al. [2, Theorem 5.3.4]. For $n > 1$ the result is due to Newlander and Nirenberg [42] under stronger regularity assumptions. Improvements were given by Nijenhuis and Woolf [43], Kohn [32, Theorem 12.1], Malgrange [38], Webster [50, Theorem 3.1], Treves [49], and possibly others. (See also Nirenberg [44] and Hörmander [29, Sect. 5.7].) The last statement concerning the compatibility of smooth structures follows from the fact that the inverse of a diffeomorphism of local Hölder class $\mathscr{C}^{(k,\alpha)}$ with $k \geq 1$ is of the same class; see Norton [45] and Bojarski et al. [5, Theorem 2.1]. A 1-parametric version of the Newlander–Nirenberg theorem was proved by Gong [20].

Denote by $\Lambda^l(\mathbb{C}T^*X)$ the l-th exterior power of the complexified cotangent bundle $\mathbb{C}T^*X$. Its sections are complex differential l-forms on X. The projections $\pi_{1,0}$ and $\pi_{0,1}$ in (2.1) give rise to projections $\pi_{p,q} : \Lambda^l(\mathbb{C}T^*X) \to \Lambda^{p,q}(\mathbb{C}T^*X)$ onto complex vector subbundles of $\Lambda^l(\mathbb{C}T^*X)$ for $0 \leq p, q \leq n$, with $p + q = l \in \{1, \ldots, 2n\}$, such that $\oplus_{p+q=l}\Lambda^{p,q}(\mathbb{C}T^*X) = \Lambda^l(\mathbb{C}T^*X)$. Sections of $\Lambda^{p,q}(\mathbb{C}T^*X)$ are differential forms of bidegree (p, q) with respect to the complex structure J on X. Assuming that J is of class $j \in \{0, 1, \ldots, \infty\}$, these subbundles are also of class $\mathscr{C}^j$. Let $\mathscr{D}_j^{p,q}(X)$ denote the space of (p, q)-forms of class $\mathscr{C}^j$ on X, and let d be the exterior differential on X. We have the operators $\partial = \partial_J$ and $\overline{\partial} = \overline{\partial}_J$ defined by

$$\partial = \pi_{p+1,q} \circ d : \mathscr{D}_j^{p,q}(X) \to \mathscr{D}_{j-1}^{p+1,q}(X), \quad \overline{\partial} = \pi_{p,q+1} \circ d : \mathscr{D}_j^{p,q}(X) \to \mathscr{D}_{j-1}^{p,q+1}(X).$$

Integrability of J is equivalent to each of the conditions $\partial^2 = 0$, $\overline{\partial}^2 = 0$, and $d = \partial + \overline{\partial}$ (see [10, Proposition 1.2.1]). If J is integrable then the kernel of the operator $\overline{\partial}$ (resp. ∂) on functions are precisely the J-holomorphic (resp. the J-antiholomorphic) functions. We also have the conjugate differential $d^c = d^c_J = \mathfrak{i}(\overline{\partial} - \partial)$ and the operator $dd^c = 2\mathfrak{i}\partial\overline{\partial}$. For a $\mathscr{C}^2$ function $\rho : X \to \mathbb{R}$, $dd^c\rho$ is a $(1,1)$-form called the *Levi form* of ρ. A function ρ is said to be (strongly) *J-plurisubharmonic* if $dd^c\rho \geq 0$ (resp. $dd^c\rho > 0$), in the sense that for any $x \in X$ and $0 \neq v \in T_xX$ we have $\langle dd^c\rho(x), v \wedge Jv\rangle \geq 0$ (resp. > 0); see [12, Eq. (1.39), p. 30]. A complex manifold (X,J) is a Stein manifold if and only if it admits a strongly J-plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}_+$ (see Grauert [23]). A necessary and sufficient topological condition for the existence of an integrable Stein structure on a smooth manifold of dimension $2n \geq 6$ was given by Eliashberg [9, 7]. The situation is more complicated on manifolds of dimension 4; see Gompf [18, 19] and [12, Chap. 10].

A domain $D \Subset X$ with $\mathscr{C}^2$ boundary is said to be *strongly pseudoconvex* (or *strongly J-pseudoconvex* if we wish to emphasise the choice of the complex structure J) if it admits a defining function $\rho : U \to \mathbb{R}$ on a neighbourhood U of $\bar{D}$ such that $D = \{\rho < 0\}$, $d\rho \neq 0$ on $bD = \{\rho = 0\}$, and $dd^c\rho(x) > 0$ for every $x \in bD$. See Krantz [35] for the basic theory of such domains.

Fix a smooth Riemannian metric g on X. Such g extends to a field of $\mathbb{C}$-bilinear forms on the complexified tangent spaces $\mathbb{C}T_xX$, $x \in X$. Given an almost complex structure J on X determined by the projections (2.1), write a vector $u \in \mathbb{C}TX$ in the form $u = u_{1,0} + u_{0,1}$ where $u_{1,0} = \pi_{1,0}(u)$ and $u_{0,1} = \pi_{0,1}(u)$. Then, g and J determine a field of inner products on the fibres of $\mathbb{C}TX$ by

$$\langle u, v\rangle_J = g(u_{1,0}, \overline{v_{1,0}}) + g(u_{0,1}, \overline{v_{0,1}}), \quad u, v \in \mathbb{C}T_xX,\ x \in X \tag{2.2}$$

(cf. [10, p. 8]). This inner product is J-hermitian on the subbundle $H \subset \mathbb{C}TX$ on which $J = \mathfrak{i}$, J-antihermitian on the conjugate subbundle $\overline{H} \subset \mathbb{C}TX$ on which $J = -\mathfrak{i}$, the subbundles H and $\overline{H}$ are $\langle\cdot\,,\cdot\rangle_J$-orthogonal, and for every $u \in \mathbb{C}TX$ we have

$$\|u\|^2 = \langle u, u\rangle_J = \|\Re u_{1,0}\|_g^2 + \|\Im u_{1,0}\|_g^2 + \|\Re u_{0,1}\|_g^2 + \|\Im u_{0,1}\|_g^2.$$

Here, $\Re$ and $\Im$ denote the real and imaginary part. By duality and multilinear algebra, the field of inner products $\langle\cdot\,,\cdot\rangle_J$ in (2.2) extends to the bundles $\Lambda^{p,q}(\mathbb{C}T^*X)$. If J is of class $\mathscr{C}^j$ then so is $\langle\cdot\,,\cdot\rangle_J$, and if an almost complex structure J' on X is $\mathscr{C}^j$ close to J then $\langle\cdot\,,\cdot\rangle_{J'}$ is $\mathscr{C}^j$ close to $\langle\cdot\,,\cdot\rangle_J$. Given a domain $D \Subset X$ with $\mathscr{C}^1$ boundary, we have an inner product of forms $\phi, \psi \in \mathscr{D}^{p,q}_j(\bar{D})$ given by

$$(\phi, \psi)_J = \int_D \langle\phi, \psi\rangle_J \, dV \tag{2.3}$$

where dV is the volume form on X determined by g. If $\{J_t\}_{t\in T}$ is a continuous family of almost complex structures on X then the inner products $(\cdot\,,\cdot)_{J_t}$ also vary continuously, and the L^2 norms $\|\phi\|^2_{J_t} = (\phi,\phi)_{J_t}$ are comparable for t in any compact subset of T.

We shall be dealing with families $\mathscr{J} = \{J_t\}_{t\in T}$ of integrable complex structures on given smooth manifold X, where T is a topological space whose precise properties will be specified in the individual results. A continuous map $f : Z = T \times X \to Y$ to a complex manifold Y is said to be *$\mathscr{J}$-holomorphic* if the map $f(t,\cdot) : X \to Y$ is J_t-holomorphic for every $t \in T$. Such a family $\mathscr{J}$ is said to be of class $\mathscr{C}^{0,k}$, where $k \in \{0, 1, \ldots, \infty\}$, if J_t admits partial derivatives of order up to k in the space variable $x \in X$ and these derivative depend continuously on $t \in T$. If $k \in \mathbb{R}_+$ is fractional, $k = [k] + \alpha$ for $0 < \alpha < 1$, we ask that J_t is of local Hölder class $\mathscr{C}^{([k],\alpha)}$ on X and it depends continuously on t. More precisely, for every smoothly bounded relatively compact domain $\Omega \Subset X$, $J_t|_{T\Omega} \in \mathrm{Hom}^{([k],\alpha)}(T\Omega, T\Omega)$ depends continuously on $t \in T$ as an element of this space. The analogous definition applies to functions or maps on $T \times X$. If T is a $\mathscr{C}^l$ manifold then a function is of class $\mathscr{C}^{l,k}(T \times X)$ if it has l derivatives in $t \in T$ followed by k derivatives in $x \in X$, and these derivatives are continuous. Similarly one defines the Hölder classes $\mathscr{C}^{l,(k,\alpha)}$.

3. A THEOREM OF HAMILTON FOR FAMILIES OF COMPLEX STRUCTURES

The main result of this section, Theorem 3.1, is a version of Hamilton's theorem [27] (also called the global Newlander–Nirenberg theorem) for a family of smooth integrable complex structures on a compact strongly pseudoconvex domain in a Stein manifold. It is used in the proof of all main results in the paper. Its proof uses stability of Kohn's solution of the $\overline{\partial}$-equation on such domains, obtained by Greene and Krantz [24] and based on the work of Kohn [32, 33]; see Theorem 3.2.

Assume that (X, J) is a Stein manifold and $D \Subset X$ is a relatively compact, smoothly bounded, strongly J-pseudoconvex domain. A theorem of Hamilton [27] says that for every sufficiently small smooth integrable deformation J' of the complex structure J on $\bar{D}$ there is a smooth diffeomorphism $F : \bar{D} \to F(\bar{D}) \subset X$, close to the identity map on $\bar{D}$, such that $J' = F^*J$ is the pullback of J by F. Equivalently, the map $F : D \to F(D)$ is biholomorphic from (D, J') onto $(F(D), J)$. (Hamilton's result applies to a wider class of domains but we shall restrict the attention to this case.) The proof in [27] is nonlinear in nature and uses the Nash–Moser technique. Improvements in terms of the required regularity of the almost complex structure and of the boundary of the domain were obtained by Gan and Gong [16], Shi [47] (for strongly pseudoconvex domains in $\mathbb{C}^n$), and Gong and Shi [21]. It was shown by Hill [28] that the result fails in general for domains with Levi degenerate boundaries.

A simpler proof of Hamilton's theorem on strongly pseudoconvex domains was given by Greene and Krantz [24, Theorem 1.13] by using the $\overline{\partial}$–Neumann method of Kohn [32, 33, 10] for solving the $\overline{\partial}$-equation. Their approach, together with stability results for Kohn's solutions of the $\overline{\partial}$-equation with respect to a family of complex structures (see [24, Sect. 3] and Theorem 3.2), will be used to give the following parametric version of Hamilton's theorem.

Theorem 3.1. *Let (X, J) be a Stein manifold of complex dimension n. Let $k \geq 1$ and $r \geq 2k+2n+9$ be integers, and let $D \Subset X$ be a relatively compact, strongly pseudoconvex domain with boundary bD of class $\mathscr{C}^r$. Assume that T is a topological space and $\mathscr{J} = \{J_t\}_{t\in T}$ is a family of integrable complex structures on $\bar{D}$ of class $\mathscr{C}^{0,r}(\bar{D})$ such that for some $t_0 \in T$, J_{t_0} is the restriction of J to $\bar{D}$. Then there exist a neighbourhood $T_0 \subset T$ of t_0 and a family of diffeomorphisms $F_t : D \to D_t = F_t(D) \subset X$ in $\mathscr{C}^k(D, X)$, depending continuously on $t \in T_0$, such that F_t is a biholomorphic map from (D, J_t) onto (D_t, J_{t_0}) for every $t \in T_0$ and F_{t_0} is the identity on D. If $bD \in \mathscr{C}^\infty$ and $\mathscr{J}$ is of class $\mathscr{C}^{0,\infty}(T \times \bar{D})$ then the family $\mathscr{F} = \{F_t\}_{t\in T_0}$ can be chosen to be of class $\mathscr{C}^{0,\infty}$ on $T_0 \times \bar{D}$.*

Theorem 3.1 is likely not optimal in terms of regularity. For relatively compact domains in open Riemann surfaces, a more precise result [14, Theorem 4.3] was obtained via the Beltrami equation.

We begin with preliminaries. Choose a smooth Riemannian metric g on X and let dV be the associated volume form. Fix a relatively compact domain $D \Subset X$ with $\mathscr{C}^1$ boundary. Let $L^2(D)$ denote the space of measurable functions f on D with $\|f\|^2_{L^2(D)} = \int_D |f|^2 dV < +\infty$. For $s \in \mathbb{Z}_+$ we denote by $H_s(D) = W^{s,2}(D)$ the Sobolev (Hilbert) space of functions on D whose derivatives of order up to s belong to $L^2(D)$. In particular, $H_0(D) = L^2(D)$. (For a discussion of Sobolev spaces for any real $s \in \mathbb{R}$, see Adams [1] or Folland and Kohn [10, Appendix].) When $X = \mathbb{R}^N$ with the Euclidean metric g, the norm on $H_s(D)$ is given by $\|f\|^2_{H_s(D)} = \sum_{|\alpha|\leq s} \|D^\alpha f\|^2_{L^2(D)}$, where D^α for a multiindex $\alpha \in \mathbb{Z}^N_+$ denotes a partial derivative with respect to the coordinates on $\mathbb{R}^N$. On a smooth manifold X we introduce these and other norms mentioned in the sequel by using a finite covering of $\bar{D}$ by smooth charts; see [10, Appendix, p. 122]. By $\mathscr{C}^s(D)$ we denote the Banach space of functions having continuous bounded partial derivatives of order $\leq s$, with $\|f\|_{\mathscr{C}^s(D)} = \sum_{|\alpha|\leq s} \sup_{x\in D} |D^\alpha f(x)|$ when $D \subset \mathbb{R}^N$. Given an integrable almost complex structure J on X, we have the induced metrics on the bundles $\Lambda^{p,q}(X)$ of differential (p,q)-forms (see Section 2). We denote by $H^{p,q}_s(D, J)$ the Sobolev space $W^{s,2}(D)$ of (p,q)-forms on D with respect to J, endowed with the inner product (2.3). The norms on these space are introduced by a system of local charts covering $\bar{D}$.

The following result is [24, Theorem 3.10] by Greene and Krantz. The regularity statements for a single complex structure are due to Kohn [32]; see also [10, Proposition 3.1.15, p. 52].

Theorem 3.2. *Assume that X is a smooth Riemannian manifold of real dimension $2n \geq 2$, $s \geq 1$ is an integer, $D \Subset X$ is a relatively compact domain with boundary of class $\mathscr{C}^{2s+5}$, T is a topological space, and $\mathscr{J} = \{J_t\}_{t\in T}$ is a continuous family of integrable Stein structures of class $\mathscr{C}^{2s+5}$ on $\bar{D}$ (i.e., $\mathscr{J}$ is of class $\mathscr{C}^{0,2s+5}$ on $T \times \bar{D}$) such that D is strongly J_t-pseudoconvex with Stein interior for every $t \in T$. Then the following assertions hold.*

(a) *For every $\alpha \in H_0^{p,q}(D, J_t)$ ($p \geq 0,\ q \geq 1,\ t \in T$) with $\overline{\partial}_{J_t}\alpha = 0$ there is a unique (Kohn) solution $K_t\alpha \in H_0^{p,q-1}(D, J_t)$ of the equation $\overline{\partial}_{J_t}(K_t\alpha) = \alpha$ satisfying $K_t\alpha \perp \ker(\overline{\partial}_{J_t})$ with respect to the inner product $(\cdot\,,\cdot)_{J_t}$ given by (2.3).*
(b) *If $\alpha \in H_s^{p,q}(D, J_t)$ then $K_t\alpha \in H_s^{p,q-1}(D, J_t)$, and $\|K_t\alpha\|_s \leq C\|\alpha\|_s$ for some $C > 0$ which can be chosen independent of t in any compact subset of T.*
(c) *If the forms $\alpha_t \in H_s^{p,q}(D, J_t)$ depend continuously on $t \in T$, then $K_t\alpha_t \in H_{s-1}^{p,q-1}(D, J_t)$ also depend continuously on $t \in T$.*
(d) *If $s > k+n+1$ and the forms $\alpha_t \in H_s^{0,1}(D, J_t)$ depend continuously on $t \in T$, then the functions $K_t\alpha_t \in \mathscr{C}^k(D)$ depend continuously on $t \in T$.*
(e) *If bD is $\mathscr{C}^\infty$ smooth, $\mathscr{J}$ is of class $\mathscr{C}^{0,\infty}$, and $\alpha_t \in \mathscr{D}_\infty^{p,q}(\bar{D}, J_t)$ are smooth and continuous in t, then $K_t\alpha_t \in \mathscr{D}_\infty^{p,q-1}(\bar{D}, J_t)$ are also smooth and continuous in $t \in T$.*

The Kohn solution $\phi = K_t\alpha$ of the equation $\overline{\partial}_{J_t}\phi = \alpha$, subject to $\overline{\partial}_{J_t}\alpha = 0$, is given by $\phi = \vartheta_t N_t \alpha$, where N_t is the $\overline{\partial}$–Neumann operator associated to J_t and ϑ_t is the Hilbert space adjoint of $\overline{\partial}_{J_t}$ on D; see [10, Theorem 3.1.14]. The same result holds if $\bar{D}$ is a compact smooth manifold with boundary that is not necessarily embedded in an ambient manifold. Part (d) follows from (c) and the following Sobolev embedding theorem; see Adams [1, p. 97ff] or Folland and Kohn [10, Proposition A.1.2, p. 115] for $X = \mathbb{R}^N$; the general case follows by using charts (see [10, p. 122]). Part (e) holds because the forms $K_t\alpha_t$ in (a) are independent of the smoothness class (see [24, p. 55]).

Proposition 3.3 (Sobolev embedding theorem)**.** *Let D be a relatively compact domain with $\mathscr{C}^1$ boundary in a smooth manifold X of dimension N. Then, $H_s(D) \subset \mathscr{C}^k(D)$ and $\|\cdot\|_{\mathscr{C}^k(D)} \leq C\|\cdot\|_{H_s(D)}$ for some $C > 0$ if and only if $s > k + N/2$. If this holds then the weak derivatives of $u \in H_s(D)$ up to order k are, after correction on a set of measure zero, classical derivatives.*

Proof of Theorem 3.1. We follow [24, proof of Theorem 1.13]. Choose a proper J-holomorphic embedding $f : X \hookrightarrow \mathbb{C}^{2n+1}$ (see [4] and [12, Theorem 2.4.1]). By Docquier and Grauert [8] (see also [26, Theorem 8, p. 257] or [12, Theorem 3.3.3, p. 74]) there are an open neighbourhood $U \subset \mathbb{C}^{2n+1}$ of $f(X)$ and a holomorphic retraction $\tau : U \to f(X)$. Set $s = k+n+2$, so $2s+5 = 2k+2n+9 = r$. (If $k = \infty$, we take $s = r = \infty$.) Recall that $\mathscr{J} = \{J_t\}_{t\in T}$ is of class $\mathscr{C}^{0,r}$ and J_{t_0} is the restriction of J to $\bar{D}$. Note that $\alpha_t := \overline{\partial}_{J_t}(f|_{\bar{D}})$ for $t \in T$ is a $\mathbb{C}^{2n+1}$-valued $(0,1)$-form with respect to J_t, of class $\mathscr{C}^r(\bar{D})$ and hence in $H_s^{0,1}(\bar{D}, J_t)$, depending continuously on $t \in T$. By Theorem 3.2 (d), for every $t \in T$ close to t_0 there is a unique solution ϕ_t of $\overline{\partial}_{J_t}\phi_t = \alpha_t$ and $\phi_t \perp \ker(\overline{\partial}_{J_t})$, with $\phi_t \in \mathscr{C}^k(D)$ depending continuously on $t \in T$. The map $f_t = f - \phi_t : D \to \mathbb{C}^{2n+1}$ is then J_t-holomorphic and continuous t as an element of the space $\mathscr{C}^k(D)^{2n+1}$. For $t = t_0$ we have $J_{t_0} = J$ and hence $\alpha_{t_0} = \overline{\partial}_J f = 0$, $\phi_{t_0} = 0$, and $f_{t_0} = f|_D$. It follows that for t close enough to t_0 the map f_t is so close to $f|_D$ in $\mathscr{C}^k(D)^{2n+1}$ that its image belongs to U. For such t, the map $F_t = f^{-1}(\tau(f_t)) : D \to X$ is well-defined, (J_t, J)-holomorphic, it depends continuously on t as an element of $\mathscr{C}^k(D, X)$, and $F_{t_0} = \mathrm{Id}_D$. It follows that F_t is (J_t, J)-biholomorphic on D for t close to t_0. □

4. A WILD FAMILY OF COMPLEX STRUCTURES ON $\mathbb{R}^4$

In this section, we construct a smooth family $\{J_t\}_{t\in\mathbb{R}}$ of integrable complex structures on $\mathbb{R}^{2n}$ for any $n > 1$ with wild behaviour of holomorphic hulls near $t = 0$. It is built by using a Fatou–Bieberbach map $\mathbb{C}^2 \to \mathbb{C}^2$ with non-Runge image, constructed by Wold [52]. This example motivates the definition of a tame family of complex structures; see Definition 5.1.

A compact set K in a complex manifold (X, J) is said to be *holomorphically convex* or *J-convex* if K equals its holomorphically convex hull (also called J-convex hull), defined by

$$\widehat{K}_J = \big\{p \in X : |f(p)| \le \max_{x\in K} |f(x)| \ \text{ for all } f \in \mathscr{O}_J(X)\big\}.$$

Here, $\mathscr{O}_J(X)$ denotes the algebra of J-holomorphic functions on X. When J is the standard complex structure on $X = \mathbb{C}^n$ then $\widehat{K}_J$ is the polynomial hull of K. See Hörmander [29, 30] and Stout [48] for further information on holomorphic convexity.

If X is an open Riemann surface then a compact subset $K \subset X$ is holomorphically convex if and only if $X \setminus K$ has no relatively compact connected components. This is a topological condition independent of the choice of the complex structure. This fact plays an important role in the proof of the Oka principle in [14, Theorem 1.6] for maps from families of complex structures on a smooth open surface to an Oka manifold. When attempting to obtain analogous results for families of integrable Stein structures $\{J_t\}_{t\in T}$ on a smooth open manifold X of dimension $2n \ge 4$, one of the problems concerns the behaviour of J_t-convex hulls $\widehat{K}_{J_t}$ of a compact set $K \subset X$ with respect to the parameter t. The following result shows that when $X = \mathbb{R}^{2n}$, $n > 1$, the hulls can explode when $t \in T$ approaches a limit value $t_0 \in T$.

Theorem 4.1. *Given a compact set $K \subset \mathbb{R}^{2n}$ $(n > 1)$ with nonempty interior, there is a family of integrable smooth complex structures $\{J_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}^{2n}$, depending smoothly on $t \in \mathbb{R}$, such that J_0 is the standard structure on $\mathbb{C}^n$, $(\mathbb{R}^{2n}, J_t)$ is biholomorphic to $(\mathbb{R}^{2n}, J_0) \cong \mathbb{C}^n$ for every $t \in \mathbb{R}$, and for any neighbourhood $U \subset \mathbb{R}$ of $0 \in \mathbb{R}$ the set $\bigcup_{t\in U} \widehat{K}_{J_t} \subset \mathbb{R}^{2n}$ is unbounded.*

Proof. It suffices to consider the case when K is the closed unit ball in $\mathbb{R}^4 \cong \mathbb{C}^2$.

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. By Wold [52], there is an injective holomorphic map $\Phi : \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ such that $\Phi(\mathbb{C}^2) \subset \mathbb{C}^* \times \mathbb{C}$ but the polynomial hull $\widehat{\Phi(K)}$ of $\Phi(K)$ contains the origin $0 \in \mathbb{C}^2$. In particular, $\widehat{\Phi(K)} \not\subset \Phi(\mathbb{C}^2)$ and hence $\Phi(\mathbb{C}^2)$ is not Runge in $\mathbb{C}^2$. We shall construct a family of smooth diffeomorphisms $\Psi_t : \mathbb{C}^2 \to \mathbb{C}^2$, depending smoothly on $t \in (-1, 1)$, such that $\Psi_t = \Phi$ holds on a neighbourhood of $t^{-1}K$ for every $t \ne 0$. Since the balls $t^{-1}K$ increase to $\mathbb{C}^2$ as t decreases to 0, we obtain a smooth family $\{\Psi_t\}_{t\in(-1,1)}$ by setting $\Psi_0 = \Phi$. One can extend the parameter space to $\mathbb{R}$ by applying a diffeomorphism from $\mathbb{R}$ onto $(-1, 1)$.

Assume for a moment that such a family Ψ_t exists. Let J_t denote the complex structure on $\mathbb{R}^4 \cong \mathbb{C}^2$ obtained by pulling back by Ψ_t the standard complex structure $J_{\rm st}$ on $\mathbb{C}^2$. In other words, $\Psi_t : \mathbb{C}^2 \to \mathbb{C}^2$ is a biholomorphism from $(\mathbb{C}^2, J_t)$ onto $(\mathbb{C}^2, J_{\rm st})$. Note that J_t depends smoothly on t since Ψ_t does, and it agrees with $J_{\rm st}$ on a neighbourhood of $t^{-1}K \supset K$ since on this set we have that $\Psi_t = \Phi$, which is $J_{\rm st}$-holomorphic. Thus, the family J_t extends smoothly to the point $t = 0$ by taking $J_0 = J_{\rm st}$. For $t \ne 0$, the J_t-convex hull of K equals

$$\widehat{K}_{J_t} = \Psi_t^{-1}(\widehat{\Psi_t(K)}) = \Psi_t^{-1}(\widehat{\Phi(K)}) \tag{4.1}$$

where the second equality follows from the fact that $\Psi_t = \Phi$ on $t^{-1}K \supset K$. We claim that the set $\widehat{K}_{J_t} \setminus t^{-1}K$ is nonempty for every $t \ne 0$. Indeed, if $\widehat{K}_{J_t} \subset t^{-1}K$ then, since $\Phi = \Psi_t$ on $t^{-1}K$, it follows from (4.1) that $\Phi(\widehat{K}_{J_t}) = \Psi_t(\widehat{K}_{J_t}) = \widehat{\Phi(K)}$, a contradiction to $\widehat{\Phi(K)} \not\subset \Phi(\mathbb{C}^2)$. As $t \to 0$, the sets $t^{-1}K$ increase to $\mathbb{C}^2$, and hence the hulls $\widehat{K}_{J_t}$ are not contained in any bounded subset of $\mathbb{C}^2$ for t in a neighbourhood of 0.

It remains to explain the construction of the family of diffeomorphisms $\Psi_t : \mathbb{C}^2 \to \mathbb{C}^2$ with the stated properties. It suffices to consider the parameter values $t \in (0,1)$. Choose a smooth isotopy of injective holomorphic maps $\Phi_s : \mathbb{C}^2 \to \mathbb{C}^2$ for $s \in [0,1]$ such that Φ_0 is the identity map on $\mathbb{C}^2$ and $\Phi_1 = \Phi$. Explicitly, we can take $\Phi_0(z) = z$ and

$$\Phi_s(z) = s\,\Phi(0) + s^{-1} A_s\big(\Phi(sz) - \Phi(0)\big), \quad s \in (0,1],$$

where $s \mapsto A_s \in GL_2(\mathbb{C})$ is a smooth path with $A_0 = \Phi'(0)^{-1}$ and $A_1 = I$. Note that $\{\Phi_s\}_{s\in[0,1]}$ is the flow of the holomorphic time-dependent vector field V on $\mathbb{C}^2$ defined on the open set

$$\Sigma = \big\{\big(s, \Phi_s(z)\big) : s \in [0,1],\ z \in \mathbb{C}^2\big\} \subset [0,1] \times \mathbb{C}^2$$

(the trace of the isotopy $\{\Phi_s\}_{s\in[0,1]}$) by

$$V(s, \Phi_s(z)) = \frac{\partial}{\partial u}\Big|_{u=s} \Phi_u(z).$$

For a fixed $t \in (0,1)$ consider the compact set

$$\Sigma_t = \big\{\big(s, \Phi_s(z)\big) : s \in [0,1],\ z \in t^{-1}K\big\} \subset [0,1] \times \mathbb{C}^2.$$

Pick a smooth function $\chi : (0,1) \times [0,1] \times \mathbb{C}^2 \to [0,1]$ such that for every $t \in (0,1)$ the function $\chi(t,\cdot,\cdot) : [0,1] \times \mathbb{C}^2 \to [0,1]$ equals 1 on a neighbourhood of Σ_t and has compact support. For $(t,s) \in (0,1) \times [0,1]$ we define a vector field $W_{t,s}$ on $\mathbb{C}^2$ by

$$W_{t,s}(z) = \chi(t,s,z) V(s,z), \quad z \in \mathbb{C}^2.$$

Note that $W_{t,s}$ is smooth in all variables, it agrees with $V(s,\cdot)$ on a neighbourhood of Σ_t, and has compact support in $[0,1] \times \mathbb{C}^2$ for every fixed $t \in (0,1)$. It follows that the flow $\Psi_{t,s}$ of $W_{t,s}$ with respect to the variable $s \in [0,1]$, with $t \in (0,1)$ as a parameter, solving the initial value problem

$$\frac{\partial}{\partial u}\Big|_{u=s} \Psi_{t,u}(z) = W_{t,s}(\Psi_{t,s}(z)), \quad \Psi_{t,0}(z) = z,$$

exists for all $s \in [0,1]$ and $z \in \mathbb{C}^2$, it agrees with the flow of V for $z \in t^{-1}K$ (which is $\Phi_s(z)$), and is fixed near infinity in the z variable since $W_{t,s}$ has compact support. It follows that every map $\Psi_{t,s} : \mathbb{C}^2 \to \mathbb{C}^2$ for $t \in (0,1),\ s \in [0,1]$ is a diffeomorphism onto $\mathbb{C}^2$. Setting $s = 1$ gives a family of diffeomorphisms $\Psi_t = \Psi_{t,1} : \mathbb{C}^2 \to \mathbb{C}^2$, $t \in (0,1)$, with the stated properties. □

The following implies that one cannot do any serious analysis for families of J_t-holomorphic functions for $\mathscr{J} = \{J_t\}_{t\in\mathbb{R}}$ in Theorem 4.1. See Corollary 6.4 for a more general result.

Lemma 4.2. *(Notation as above.) If f is a holomorphic function on $\Omega = \Phi(\mathbb{C}^2) \subset \mathbb{C}^2$ such that $f \circ \Phi \in \mathscr{O}(\mathbb{C}^2)$ extends to a continuous family $f_t \in \mathscr{O}_{J_t}(\mathbb{C}^2)$ for t near 0, then f is bounded on the set $\widehat{\Phi(K)} \cap \Omega$, which is not relatively compact in Ω.*

Proof. From (4.1) we get $\widehat{K}_{J_t} \cap t^{-1}K = \Psi_t^{-1}(\widehat{\Phi(K)}) \cap t^{-1}K$. Since $\Psi_t = \Phi$ on $t^{-1}K$, it follows that

$$\Psi_t(\widehat{K}_{J_t} \cap t^{-1}K) = \widehat{\Phi(K)} \cap \Phi(t^{-1}K).$$

When $t \to 0$, the set on the right hand side increases to $\widehat{\Phi(K)} \cap \Omega$. Choose a point $p \in \widehat{\Phi(K)} \cap \Omega$; hence $p \in \widehat{\Phi(K)} \cap \Phi(t^{-1}K)$ for all small enough $t \neq 0$. Note that $p_t := \Psi_t^{-1}(p) \in \widehat{K}_{J_t} \cap t^{-1}K$ converges to $p_0 = \Phi^{-1}(p)$ as $t \to 0$. Let $f \in \mathscr{O}(\Omega)$. Suppose that there is a continuous family of holomorphic functions $f_t \in \mathscr{O}_{J_t}(\mathbb{C}^2)$ for t near 0 such that $f_0 = f \circ \Phi$. Since $p_t \in \widehat{K}_{J_t}$, we have $|f_t(p_t)| \le \max_K |f_t|$. Letting $t \to 0$ gives $|f_0(p_0)| \le \max_K |f_0|$, which is equivalent to $|f(p)| \le \max_{\Phi(K)} |f|$. This shows that f is bounded on the set $\widehat{\Phi(K)} \cap \Omega$ as claimed. □

Remark 4.3. The construction in the proof of Theorem 4.1 works on any contractible Stein manifold X which admits an injective holomorphic map $\Phi : X \to X$ such that, for some compact subset $K \subset X$ with nonempty interior, we have that $\widehat{\Phi(K)} \not\subset \Phi(X)$. Besides $\mathbb{C}^n$, an example is any bounded convex domain X in $\mathbb{C}^n$ for $n > 1$. Indeed, assume that X is such, and let $K \subset X$ be a compact set with nonempty interior. By translation we may assume that $0 \in \mathring{K}$. Let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be an injective holomorphic map as in the proof of Theorem 4.1, satisfying $\widehat{\Phi(K)} \not\subset \Phi(\mathbb{C}^n)$. Set $L = \Phi(K)$. For any $s > 0$ we then have $\widehat{sL} = s\,\widehat{L} \not\subset s\Phi(\mathbb{C}^n)$. Replacing Φ by $s\Phi$ for a suitable $s > 0$ we ensure that $\Phi(X) \subset X$. It follows that $\widehat{\Phi(K)} \not\subset \Phi(X)$. However, we do not know whether the phenomenon in Theorem 4.1 can occur on every Stein manifold X with $\dim_{\mathbb{C}} X > 1$.

5. Tame families of Stein structures

Assume that T is a topological space, X is a smooth open manifold of even dimension, $\pi : T \times X \to T$ is the projection $\pi(t,x) = t$, and $\mathscr{J} = \{J_t\}_{t\in T}$ a continuous family of integrable complex structures on X. We introduce a tameness condition on $\mathscr{J}$ which excludes the pathology in Theorem 4.1; see Definition 5.1. If T is locally compact and Hausdorff then tameness is characterised in terms of properness over T of the family of J_t-convex hulls of any compact set in X; see Proposition 5.2. Assuming that the complex structures J_t are Stein and sufficiently regular, tameness is equivalent to local boundedness of the family of J_t-convex hulls of any compact set; see Proposition 5.3. If T is locally compact, paracompact and Hausdorff then tameness is characterised by the existence of a continuous family of strongly J_t-plurisubharmonic exhaustion functions on X; see Theorem 5.5. We conclude the section with examples and constructions of tame families of Stein structures.

Definition 5.1. A family $\mathscr{J} = \{J_t\}_{t\in T}$ of complex structures on X is *tame* at a point $t_0 \in T$ if for every compact set $K \subset X$ and open set $U \subset X$ containing $\widehat{K}_{J_{t_0}}$ there is a neighbourhood $T_0 \subset T$ of t_0 such that $\widehat{K}_{J_t} \subset U$ holds for all $t \in T_0$. The family $\mathscr{J}$ is tame if it is tame at every point $t_0 \in T$.

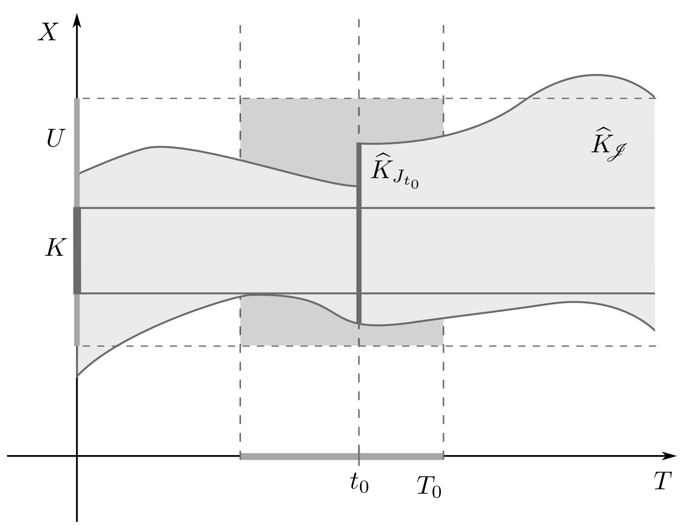

Figure 1. An upper semicontinuous family of hulls $\widehat{K}_{J_t}$.

Any family of complex structures on a smooth surface X is tame since the hull of a compact set does not depend on the choice of a complex structure on X. The same holds if X is compact and connected. Theorem 4.1 gives smooth nontame families of Stein structures on $\mathbb{R}^{2n}$ for any $n > 1$.

Recall that a continuous map $S \to T$ of topological spaces is said to be proper if the preimage of any compact set in T is compact.

Proposition 5.2. *Assume that T is a locally compact Hausdorff space. The following conditions on a continuous family $\mathscr{J} = \{J_t\}_{t\in T}$ of complex structures on X are equivalent.*

(a) *The family $\mathscr{J}$ is tame.*
(b) *For every compact subset $K \subset X$, its $\mathscr{J}$-convex hull*

$$\widehat{K}_{\mathscr{J}} = \bigcup_{t\in T} \{t\} \times \widehat{K}_{J_t} \subset T \times X \tag{5.1}$$

is such that the projection $\pi : \widehat{K}_{\mathscr{J}} \to T$ is proper.

Proof. Assume that $\mathscr{J}$ is tame. It is easily seen that $\widehat{K}_{\mathscr{J}}$ is then closed in $T \times X$. Let $T' \subset T$ be compact. Given $t \in T'$, pick a neighbourhood $U_t \subset X$ of $\widehat{K}_{J_t}$ with compact closure $\overline{U}_t$. Tameness gives a compact neighbourhood $T_t \subset T$ of t such that the $\pi^{-1}(T_t) \cap \widehat{K}_{\mathscr{J}}$ is a closed subset of $T_t \times \overline{U}_t$, hence compact. The compact set T' is covered by finitely many sets T_{t_j} obtained in this way, and it follows that $\pi^{-1}(T') \cap \widehat{K}_{\mathscr{J}}$ is compact. This proves (a) $\Rightarrow$ (b). Conversely, assume that $\pi : \widehat{K}_{\mathscr{J}} \to T$ is proper. Let $t_0 \in T$ and $U \subset X$ be an open set containing $\widehat{K}_{J_{t_0}}$. Choose a compact neighbourhood $T_0 \subset T$ of t_0. Then, $\pi^{-1}(T_0) \cap \widehat{K}_{\mathscr{J}}$ is compact, and hence closed in $T_0 \times X$. If the condition in Definition 5.1 fails at t_0, there is a net $\{(t_j, x_j)\}_{j\in A} \subset \widehat{K}_{\mathscr{J}}$ with $\lim_j t_j = t_0$ such that the net $\{x_j\}_{j\in A}$ has an accumulation point $x_0 \in X \setminus U$. Since $\pi^{-1}(T_0) \cap \widehat{K}_{\mathscr{J}}$ is closed, it contains (t_0, x_0) which contradicts the initial assumption. Hence, $\mathscr{J}$ is tame. □

The nontame families $\mathscr{J}$ of Stein structures in Theorem 4.1 are such that the family of hulls $\widehat{K}_{J_t}$ of some compact K is not locally bounded at some $t_0 \in T$, and $\widehat{K}_{\mathscr{J}}$ fails to be closed. If $\mathscr{J}$ is sufficiently regular and the hulls of any compact set are locally bounded, we show that the family is tame.

Proposition 5.3. *Let X be a smooth manifold of dimension $2n$, $\mathscr{J} = \{J_t\}_{t\in T}$ be a continuous family of Stein structures of class $\mathscr{C}^r$ on X where $r \geq n + 6$, and $K \subset X$ be a compact set. If for every point $t_0 \in T$ there are a neighbourhood $T_0 \subset T$ of t_0 and a relatively compact domain $\Omega \Subset X$ such that $\widehat{K}_{J_t} \subset \Omega$ holds for all $t \in T_0$, then $\widehat{K}_{\mathscr{J}}$ is closed in $T \times X$. If in addition T is locally compact Hausdorff and the above condition holds for every compact set $K \subset X$, then $\mathscr{J}$ is tame.*

Proof. Let $(t_0, x_0) \notin \widehat{K}_{\mathscr{J}}$. Then there is a J_{t_0}-holomorphic function f on X such that $|f(x_0)| > 1+3\varepsilon$ and $\max_{x\in K} |f(x)| \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Let $U \subset X$ be a compact neighbourhood of x_0 such that $|f(x)| > 1 + 3\varepsilon$ for all $x \in U$. Let $\Omega \Subset X$ and $T_0 \subset T$ be as in the proposition. Enlarging Ω, we may assume that it is a smoothly bounded strongly J_{t_0}-pseudoconvex domain which also contains U. Shrinking T_0 around t_0 if necessary, Theorem 3.2 and Proposition 3.3 give a continuous family of functions $\{u_t\}_{t\in T_0}$ on Ω such that $\overline{\partial}_{J_t} u_t = \overline{\partial}_{J_t} f$ and $u_{t_0} = 0$. Then $f_t = f - u_t$ is J_t-holomorphic on Ω and continuous in $t \in T_0$, with $f_{t_0} = f$. Hence, there is a neighbourhood $T_1 \subset T_0$ of t_0 such that $\min_{x\in U} |f_t(x)| > 1 + 2\varepsilon$ and $\max_{x\in K} |f_t(x)| \leq 1$ for all $t \in T_1$. We claim that $T_1 \times U$ is disjoint from $\widehat{K}_{\mathscr{J}}$. If not, choose $(t', x') \in (T_1 \times U) \cap \widehat{K}_{\mathscr{J}}$, so $x' \in \widehat{K}_{J_{t'}} \subset \Omega$. The Oka-Weil theorem gives a $J_{t'}$-holomorphic function F on X such that $|F - f_{t'}| < \varepsilon$ on $\widehat{K}_{J_{t'}} \supset K \cup \{x'\}$, which implies $|F(x')| > \max_{x\in K} |F(x)|$, a contradiction to $x' \in \widehat{K}_{J_{t'}}$. This proves the claim and shows that $\widehat{K}_{\mathscr{J}}$ is closed. If T is locally compact Hausdorff, it follows that the projection $\pi : \widehat{K}_{\mathscr{J}} \to T$ is proper, so the last statement follows from Proposition 5.2. □

In the remainder of the section, we assume that the parameter T is locally compact Hausdorff. A closed subset $K \subset T \times X$ is called *proper over T*, or simply *proper*, if the restricted projection $\pi|_K : K \to T$ is proper. The proof of Proposition 5.2 shows that K is proper if and only if the fibres $K_t = \{x \in X : (t,x) \in K\}$, $t \in T$, are compact and upper semicontinuous. Given a subset $K \subset T \times X$ with compact fibres K_t, we define its $\mathscr{J}$-convex hull by

$$\widehat{K}_{\mathscr{J}} = \big\{(t,x) \in T \times X : x \in (\widehat{K_t})_{J_t}\big\}. \tag{5.2}$$

(In (5.1) we used the same notation for $K \subset X$ to mean $(\widehat{T\times K})_{\mathscr{J}}$, but this should not cause any confusion.) A proper subset $K \subset T \times X$ is said to be $\mathscr{J}$-convex if $K = \widehat{K}_{\mathscr{J}}$.

Lemma 5.4. *If $K \subset T \times X$ is proper and $\mathscr{J}$ is tame then the hull $\widehat{K}_{\mathscr{J}}$ (5.2) is also proper.*

Proof. Fix $t_0 \in T$ and an open set $U \subset X$ with $(\widehat{K}_{t_0})_{J_{t_0}} \subset U$. By [29, Theorem 5.1.6] there is a strongly J_{t_0}-plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$ such that $\rho < 0$ on $(\widehat{K}_{t_0})_{J_{t_0}}$ and $\rho > 0$ on $X \setminus U$. The compact set $L = \{\rho \le 0\}$ is then J_{t_0}-convex and satisfies $K_{t_0} \subset \mathring{L} \subset L \subset U$. Since K is proper, there is a neighbourhood $T_0 \subset T$ of t_0 such that $K_t \subset L$ for all $t_0 \in T$. Since $\mathscr{J}$ is tame, we have that $(\widehat{K}_t)_{J_t} \subset \widehat{L}_{J_t} \subset U$ for all t near t_0, so $\widehat{K}_{\mathscr{J}}$ is proper. $\square$

Under a stronger regularity assumption on a family $\mathscr{J} = \{J_t\}_{t\in T}$ of Stein structures on X, we have the following characterisation of tameness in terms of families of strongly J_t-plurisubharmonic exhaustion functions on X for $t \in T$.

Theorem 5.5. *Assume that X is a smooth manifold, T is a locally compact Hausdorff space, and $\mathscr{J} = \{J_t\}_{t\in T}$ is a continuous family of integrable Stein structures on X of local Hölder class $\mathscr{C}^{0,(k,\alpha)}(T \times X)$ for some $k \in \mathbb{N}$ and $0 < \alpha < 1$. The following conditions are equivalent.*

(a) *The family $\mathscr{J}$ is tame.*
(b) *For every $t_0 \in T$ there are a neighbourhood $T_0 \subset T$ of t_0 and a function $\rho : T_0 \times X \to \mathbb{R}$ of class $\mathscr{C}^{0,k+1}$ such that $\rho(t,\cdot\,)$ is a strongly J_t-plurisubharmonic exhaustion on X for every $t \in T_0$.*

If in addition T is paracompact then (a) and (b) are also equivalent to the following:

(c) *There is a function $\rho : T \times X \to \mathbb{R}$ of class $\mathscr{C}^{0,k+1}$ such that $\rho_t = \rho(t,\cdot\,)$ is a strongly J_t-plurisubharmonic exhaustion function on X for every $t \in T$.*

If T is a $\mathscr{C}^l$ manifold and $\mathscr{J}$ is of local class $\mathscr{C}^{l,(k,\alpha)}$, then ρ can be chosen to be of class $\mathscr{C}^{l,k+1}$.

Note that the $\pm$i-eigenspaces of J_t depend algebraically on the coefficients of J_t in a given smooth frame on TX, and hence the operators ∂_{J_t}, $\overline{\partial}_{J_t}$, and $d^c_{J_t}$ are as regular in $t \in T$ as the family $\mathscr{J} = \{J_t\}_{t\in T}$. In the operator $dd^c_{J_t}$, the coefficients of $d^c_{J_t}$ get differentiated, and hence $dd^c_{J_t}$ depends continuously on $t \in T$ if the family $\mathscr{J}$ is of class $\mathscr{C}^{0,1}$. This implies the following observation.

Lemma 5.6. *Assume that the family of complex structures $\mathscr{J} = \{J_t\}_{t\in T}$ on X is of local class $\mathscr{C}^{0,(1,\alpha)}$, $0 < \alpha < 1$. Let ϕ be a $\mathscr{C}^2$ strongly J_{t_0}-plurisubharmonic function on a domain $V \subset X$ for some $t_0 \in T$. Given an open relatively compact subset $U \Subset V$, there is a neighbourhood $T_0 \subset T$ of t_0 such that ϕ is strongly J_t-plurisubharmonic on U for every $t \in T_0$.*

Proof of Theorem 5.5. For simplicity of notation we assume that $k = 1$ and $l = 0$; the proof is the same in the general case.

We first prove that (b) $\Rightarrow$ (a). Fix $t_0 \in T$, a compact set $K \subset X$, and an open relatively compact set $U \Subset X$ containing $\widehat{K}_{J_{t_0}}$. Choose a neighbourhood $T_0 \subset T$ of t_0 and a function $\rho : T_0 \times X \to \mathbb{R}$ satisfying condition (b). By adding a constant to ρ_{t_0} we can ensure that $\rho_{t_0} < -1$ on K. Since ρ_{t_0} is an exhaustion function on X, there is a relatively compact domain $V \Subset X$ containing $\overline{U}$ such that $\rho_{t_0} > 1$ on $X \setminus V$. Choose a strongly J_{t_0}-plurisubharmonic function $\psi : X \to \mathbb{R}$ such that $\psi < 0$ on $\widehat{K}_{J_{t_0}}$ and $\psi > 0$ on $X \setminus U$ (see [29, Theorem 5.1.6]). Replacing ψ by $c\psi$ for a suitable $c > 0$ we may assume that $-1 < \psi < 0$ on K, $\psi > 0$ on $X \setminus U$, and $\psi < \rho_{t_0}$ on bV. Since ρ_t is continuous in t, we can shrink T_0 around t_0 to ensure that for every $t \in T_0$ we have $\rho_t < -1$ on K and $\psi < \rho_t$ on bV. By Lemma 5.6 we can further shrink T_0 to ensure that ψ is strongly J_t-plurisubharmonic on V for every $t \in T_0$. For $t \in T_0$ we define the function $\phi_t : X \to \mathbb{R}$ by

$$\phi_t(x) = \begin{cases} \max\{\psi(x), \rho_t(x)\}, & x \in V; \\ \rho_t(x), & x \in X \setminus V. \end{cases}$$

Note that ϕ_t is a piecewise $\mathscr{C}^2$ strongly J_t-plurisubharmonic exhaustion function on X satisfying

$$\phi_t = \psi < 0 \text{ on } K \text{ and } \phi_t > 0 \text{ on } X \setminus U. \tag{5.3}$$

(To obtain $\mathscr{C}^2$ strongly plurisubharmonic exhaustion functions satisfying (5.3) we can use the regularized maximum; see [12, p. 69]. However, this is inessential.) Since the holomorphic hull of K equals its plurisubharmonic hull (see [29, Theorems 4.3.4 and 5.2.10]), it follows from (5.3) that $\widehat{K}_{J_t} \subset U$ for all $t \in T_0$. This shows that $\mathscr{J}$ is tame.

Next, we prove that (a) $\Rightarrow$ (b). Since the statement in (b) is local in t, we may assume that T is compact. Tameness of $\mathscr{J}$ and compactness of T imply that for every compact set $K \subset X$, the hull $\widehat{K}_{\mathscr{J}}$ (5.1) is also compact. Hence, we can find an exhaustion $K^0 \subset K^1 \subset K^2 \subset \cdots \subset \bigcup_{i=0}^{\infty} K^i = X$ by compact sets such that $\widehat{K^i}_{\mathscr{J}} \subset T \times \mathring{K}^{i+1}$ holds for every $i = 0, 1, 2, \ldots$. Choose an increasing sequence $0 < c_1 < c_2 < \cdots$ with $\lim_{i\to\infty} c_i = +\infty$. We proceed inductively.

In the initial step, fix a neighbourhood $U^1 \Subset X$ of K^1 and choose an open subset $\widetilde{U}^1 \Subset X$ such that $\overline{U^1} \subset \widetilde{U}^1$. We shall find a function $\rho^1 : T \times X \to \mathbb{R}_+$ of class $\mathscr{C}^{0,2}$ such that ρ^1_t is strongly plurisubharmonic on U^1 for all $t \in T$, ρ^1 has compact support contained in $T \times \widetilde{U}_1$, and $\rho^1 > c_1$ on $T \times K^1$. To do this, fix $t \in T$ and pick a strongly J_t-plurisubharmonic function $\phi_t : X \to \mathbb{R}_+$ such that $\phi_t > c_1$ on K^1. Lemma 5.6 gives a neighbourhood $T_t \subset T$ of t such that ϕ_t is strongly J_s-plurisubharmonic on $\widetilde{U}^1$ for every $s \in T_t$. By compactness of T we obtain a finite covering $T = \bigcup_{j=1}^m T_j$ and for each $j = 1, \ldots, m$ a $\mathscr{C}^2$ function $\phi_j : \widetilde{U}^1 \to \mathbb{R}_+$ such that $\phi_j > c_1$ on K^1 and ϕ_j is strongly J_t-plurisubharmonic for every $t \in T_j$. Let $\{\chi_j\}_{j=1}^m$ be a continuous partition of unity on T with $\mathrm{supp}\chi_j \subset T_j$. Also, let $\xi : X \to [0,1]$ be a smooth function with compact support contained in $\widetilde{U}^1$ which equals 1 on U^1. The function $\rho^1(t,x) = \xi(x) \sum_{j=1}^m \chi_j(t)\phi_j(x)$ is then fibrewise strongly plurisubharmonic on $T \times U_1$ and has compact support contained in $T \times \widetilde{U}^1$.

In the second step, we pick a neighbourhood $U^2 \Subset X$ of K^2 and find a function $\rho^2 : T \times X \to \mathbb{R}_+$ of class $\mathscr{C}^{0,2}$ with compact support such that $\rho^2 = 0$ on $\widehat{K^0}_{\mathscr{J}}$, $\rho^1 + \rho^2$ is fibrewise strongly plurisubharmonic on $T \times U^2$, and $\rho^1 + \rho^2 > c_2$ on $T \times (K^2 \setminus K^1)$. (We also have $\rho^1 + \rho^2 > c_1$ on $T \times K^1$.) To do this, fix $t \in T$ and apply [29, Theorem 5.1.6] to find a smooth J_t-plurisubharmonic function $\phi_t : X \to \mathbb{R}_+$ which vanishes on a neighbourhood of $\widehat{K^0}_{J_t}$, it is positive strongly J_t-plurisubharmonic on $X \setminus K^1$ (recall that $\widehat{K^0}_{J_t}$ is contained in the interior of K^1), and $\rho^1_t + \phi_t$ is strongly J_t-plurisubharmonic on X and satisfies $\rho^1_t + \phi_t > c_2$ on $K^2 \setminus K^1$. By tameness of $\mathscr{J}$ and Lemma 5.6 there is a neighbourhood $T_t \subset T$ of t such that the function $\rho^1_s + \phi_t : U^2 \to \mathbb{R}_+$ satisfies the same conditions for all $s \in T_t$, and ϕ_t vanishes on a neighbourhood of $\widehat{K^0}_{J_s}$ for all $s \in T_t$. As in the first step, this gives a finite open covering $T = \bigcup_{j=1}^m T_j$, functions $\phi_j : X \to \mathbb{R}_+$ $(j = 1, \ldots, m)$, a partition of unity $\{\chi_j\}_{j=1}^m$ on T with $\mathrm{supp}\chi_j \subset T_j$, and a smooth cut-off function $\xi : X \to [0,1]$ such that the function $\rho^2(t,x) = \xi(x)\sum_{j=1}^m \chi_j(t)\phi_j(x)$ enjoys the stated properties.

This process can be continued inductively to yield a sequence of nonnegative functions $\rho^1, \rho^2, \ldots$ of class $\mathscr{C}^{0,2}(T \times X)$ with compact supports such that their partial sums $\tilde\rho^i = \rho^1 + \cdots + \rho^i$ are of class $\mathscr{C}^{0,2}$ and satisfy the following conditions for every $i = 1, 2, \ldots$:

(i) $\tilde\rho^i$ is fibrewise strongly plurisubharmonic on a neighbourhood of $T \times K^i$ and has compact support.
(ii) $\tilde\rho^i > c_1$ on $T \times K^1$ and $\tilde\rho^i > c_j$ on $T \times (K^j \setminus K^{j-1})$ for $j = 2, \ldots, i$.

(iii) $\tilde\rho^{i+1} = \tilde\rho^i$ on $\widehat{K^i}_{\mathscr{J}}$.

Condition (iii) implies that the sequence is stationary on any compact subset of $T \times X$. It follows that $\rho = \sum_{i=1}^\infty \rho^i : T \times X \to \mathbb{R}_+$ is a fibrewise strongly plurisubharmonic function of class $\mathscr{C}^{0,2}$ satisfying $\rho > c_i$ on $T \times (K^i \setminus K^{i-1})$ for every $i = 1, 2, \ldots$. In particular, $\rho_t = \rho(t, \cdot\,)$ is an exhaustion function on X for every $t \in T$. It is easy to ensure that the Levi form of ρ_t with respect to J_t grows as fast as desired uniformly in $t \in T$. This proves the implication (a) $\Rightarrow$ (b).

Assume now that T is also paracompact. If (b) holds, we obtain a locally finite open cover $\mathcal{V} = \{V_i\}_i$ of T with compact closures $T_i = \overline{V}_i$ and for every i a fibrewise strongly plurisubharmonic exhaustion function $\rho_i : T_i \times X \to \mathbb{R}$. Pick a partition of unity $\{\chi_i\}_i$ on T subordinate to $\mathcal{V}$. Then, the function $\rho = \sum_i \chi_i \rho_i : T \times X \to \mathbb{R}$ satisfies condition (c). The implication (c) $\Rightarrow$ (b) is a tautology.

If T is a $\mathscr{C}^l$ manifold, the same proof gives a function ρ of class $\mathscr{C}^{l,k+1}(T \times X)$. $\square$

The proof of Theorem 5.5 gives the following analogue of the classical result [29, Theorem 5.1.6] for a tame family of Stein structures. We leave the details to the reader.

Theorem 5.7. *Assume that X, T and $\mathscr{J}$ are as in Theorem 5.5. Given a proper $\mathscr{J}$-convex subset $K = \widehat{K}_{\mathscr{J}} \subset T \times X$ and an open set $U \subset T \times X$ containing K, there is a function $\rho : T \times X \to \mathbb{R}$ as in Theorem 5.5 (b) such that $\rho < 0$ on K and $\rho > 0$ on $(T \times X) \setminus U$. Conversely, if ρ is a function as in Theorem 5.5 (b) then for every $c \in \mathbb{R}$ the sublevel set $\{\rho \le c\} \subset T \times X$ is proper and $\mathscr{J}$-convex.*

We conclude the section with examples and constructions of tame families of Stein structures on a smooth manifold X. The first observation is that every sufficiently regular family $\mathscr{J} = \{J_t\}_{t \in T}$, which is locally constant in t outside of a proper subset of $T \times X$, is tame. Hence, the phenomenon of nontameness can only appear due to the behaviour of the complex structures near infinity in X.

Proposition 5.8. *Let X and $\mathscr{J} = \{J_t\}_{t \in T}$ be as in Theorem 5.5. If for every $t_0 \in T$ there are a compact set $K \subset X$ and a neighbourhood $T_0 \subset T$ of t_0 such that $J_t = J_{t_0}$ holds on $X \setminus K$ for all $t \in T_0$, then the family $\mathscr{J}$ is tame.*

Proof. Pick a strongly J_{t_0}-plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}_+$. Lemma 5.6 gives a neighbourhood $T_0 \subset T$ of t_0 such that ρ is strongly J_t-plurisubharmonic on K for every $t \in T_0$. Up to shrinking T_0, the same is true on $X \setminus K$ since $J_t = J_{t_0}$ there. Hence, Theorem 5.5 shows that the family $\{J_t\}_{t \in T_0}$ is tame. Since tameness is a local condition in the parameter t, $\mathscr{J}$ is tame. $\square$

Proposition 5.9. *If (X, J_0) is a Stein manifold and $\Phi_t : X \to \Phi_t(X) \subset X$ is a continuous family of diffeomorphisms onto Stein Runge domains in X, then the family of Stein structures $J_t = \Phi_t^* J_0$ on X is tame.*

Proof. Let $K \subset X$ be a compact set. Set $\Omega_t = \Phi_t(X)$ and $K_t = \Phi_t(K)$. Denote by J_0^t the restriction of J_0 to $T\Omega_t$. Since $\Phi_t : (X, J_t) \to (\Omega_t, J_0^t)$ is a biholomorphism, we have $\widehat{K}_{J_t} = \Phi_t^{-1}((\widehat{K_t})_{J_0^t})$. Since Ω_t is Runge in X, $(\widehat{K_t})_{J_0^t}$ equals $(\widehat{K_t})_{J_0}$, the hull of K_t in (X, J_0). Since the family K_t is continuous in t, the family of hulls $(\widehat{K_t})_{J_0}$ is upper semicontinuous in t, so the same is true for $\widehat{K}_{J_t}$. $\square$

Theorem 4.1 shows that Proposition 5.9 fails in general if the domains $\Phi_t(X)$ are not Runge in X. In that example, $X = \mathbb{C}^n$ with $n > 1$, $t \in \mathbb{R}$, $\Phi_t(\mathbb{C}^n) = \mathbb{C}^n$ for $t \neq 0$, while $\Phi_0(\mathbb{C}^n)$ is not Runge in $\mathbb{C}^n$. It is easy to find an example of a tame family of Stein structures $\{J_t\}_{t \in \mathbb{R}}$ on $\mathbb{R}^{2n}$ such that $(\mathbb{R}^{2n}, J_0)$ equals $\mathbb{C}^n$ while $(\mathbb{R}^{2n}, J_t)$ for $t \neq 0$ is biholomorphic to the unit ball in $\mathbb{C}^n$.

Example 5.10. Assume that (Y, J_Y) is a Stein manifold, X is a smooth manifold, T is a topological space, and $F : T \times X \to Y$ is a map of class $\mathscr{C}^{0,\infty}$ such that for every $t \in T$, the map $F_t =$

$F(t,\cdot\,): X \to Y$ is a proper immersion whose image $F_t(X)$ is an immersed complex submanifold of Y. Let J_t denote the unique complex structure on X such that the map F_t is (J_t, J_Y)-holomorphic. Since F_t is proper, J_t is Stein. Then, the family $\mathscr{J} = \{J_t\}_{t\in T}$ is tame. Indeed, choosing a smooth strongly plurisubharmonic exhaustion function $\rho : Y \to \mathbb{R}_+$, the function $\rho \circ F : T \times X \to \mathbb{R}_+$ satisfies condition (c) in Theorem 5.5. This situation arises naturally if $\Omega \Subset Y$ is a smoothly bounded Stein domain and $\{F_t(Z)\}_{t\in T}$ is a continuous family of complex submanifolds of Y with a connected parameter space T such that for every $t \in T$, $Z_t = \{z \in Z : F_t(z) \in \Omega\}$ is relatively compact in Z and F_t intersects $b\Omega$ transversely along bZ_t. In this case, the domains Z_t have smooth boundaries and are diffeomorphic to each other, so we can smoothly parametrise them by a fixed smooth manifold X.

Corollary 6.4 shows that tameness of a family $\mathscr{J}$ of sufficiently smooth Stein structures on X is implied by, and hence equivalent to the one-fibre extension property for $\mathscr{J}$-holomorphic functions.

6. The Oka principle for tame families of Stein structures

In this section, we state and prove the main result of the paper, Theorem 6.1. It gives a parametric Oka principle with approximation for maps from tame families of Stein structures to any Oka manifold. Except for the regularity assumptions and statements, this result extends the special case concerning families of open Riemann surfaces in [14, Theorem 1.6].

Let T be a topological space, X be a smooth open manifold, and let $\pi : T \times X \to T$ denote the projection. Assume that $\mathscr{J} = \{J_t\}_{t\in T}$ is a tame family of integrable Stein structures on X (see Definition 5.1). Recall that a closed subset $K \subset T \times X$ is called *proper over* T (or simply *proper*) if the restricted projection $\pi|_K : K \to T$ is proper, and is $\mathscr{J}$-convex if $K = \widehat{K}_{\mathscr{J}}$ (see (5.2)). A continuous map f on an open $U \subset T \times X$ is said to be *$\mathscr{J}$-holomorphic* if the map $f_t = f(t,\cdot)$ is J_t-holomorphic on $U_t = \{x \in X : (t,x) \in U\}$ for every $t \in T$. A topological space is said to be *σ-compact* if it is the union of countably many compact subspaces. Every locally compact and σ-compact Hausdorff space is paracompact [39]. A topological space T is a *Euclidean neighbourhood retract* (ENR) if it admits a topological embedding $\iota : T \hookrightarrow \mathbb{R}^N$ for some N whose image $\iota(T) \subset \mathbb{R}^N$ is a neighbourhood retract, and is a *local ENR* if every point of T has an ENR neighbourhood. (See [14, Definition 1.5] and the references therein.) In particular, every finite CW complex is an ENR, and every countable locally compact CW-complex of finite dimension is an ENR.

Theorem 6.1 (The Oka principle for tame families of Stein structures)**.** *Assume the following:*

(a) *T is a σ-compact Hausdorff local ENR. In particular, T may be a finite CW complex or a countable locally compact CW-complex of finite dimension.*
(b) *X is a smooth open manifold of real dimension $2n$.*
(c) *$r \geq 2n + 11$ is an integer, or $r = +\infty$.*
(d) *$\mathscr{J} = \{J_t\}_{t\in T}$ is a tame family of Stein structures of class $\mathscr{C}^{0,r}$ on X (see Definition 5.1).*
(e) *$K \subset T \times X$ is a proper (over T) $\mathscr{J}$-convex subset.*
(f) *Y is an Oka manifold with a distance function dist_Y inducing the manifold topology.*
(g) *$f : T \times X \to Y$ is a continuous map, and there are an open subset $U \subset T \times X$ containing K and a closed subset $Q \subset T$ such that f is $\mathscr{J}$-holomorphic on $U \cup (Q \times X)$.*

Given a continuous function $\epsilon : T \to (0, +\infty)$, there exist a neighbourhood $U' \subset U$ of K and a homotopy $f_s : T \times X \to Y$ $(s \in I = [0,1])$ satisfying the following conditions.

(i) $f_0 = f$.
(ii) *The map f_s is $\mathscr{J}$-holomorphic on U' for every $s \in I$.*
(iii) $\sup_{x\in K_t} \mathrm{dist}_Y(f_s(t,x), f(t,x)) < \epsilon(t)$ *for every $t \in T$ and $s \in I$.*
(iv) *The map $F = f_1$ is $\mathscr{J}$-holomorphic on $T \times X$.*

(v) *The homotopy* $f_s(t,\cdot)$ $(s \in I)$ *is fixed for every* $t \in Q$*, so* $F = f$ *on* $Q \times X$*.*

Remark 6.2. The choice of the integer r in condition (c) is dictated by Theorem 3.1. If $k \geq 1$ and $r \geq 2k + 2n + 9$ are integers or $k = r = +\infty$, it follows from Theorem 3.1 that every continuous $\mathscr{J}$-holomorphic map $f : U \to Y$ on an open subset $U \subset T \times X$ is of class $\mathscr{C}^{0,k}$. Approximation in the fine $\mathscr{C}^{0,0}$ topology (see (iii) in the theorem) can then be upgraded to approximation in the fine $\mathscr{C}^{0,k}$ topology; see the last paragraph in [14, Theorem 1.6]. We shall not formally state or prove this generalisation since it follows easily from the proof of [14, Theorem 1.6].

We first explain the special case of Theorem 6.1 with $Y = \mathbb{C}$. In the following version of the Oka–Weil theorem for tame families of Stein structures, the parameter space T is more general than in Theorem 6.1. The special case when X is an open surface is given by [14, Theorem 1.1].

Theorem 6.3 (The Oka–Weil theorem for tame families of Stein structures)**.** *Assume that* X *is a smooth manifold of dimension* $2n$*,* T *is a locally compact and paracompact Hausdorff space,* $k \geq 1$ *and* $r \geq 2k + 2n + 9$ *are integers or* $k = r = +\infty$*,* $\mathscr{J} = \{J_t\}_{t\in T}$ *is a tame family of Stein structures of class* $\mathscr{C}^{0,r}$ *on* X*,* $K \subset T \times X$ *is a proper over* T *and* $\mathscr{J}$*-convex subset,* $U \subset T \times X$ *is an open set containing* K*, and* $f : U \to \mathbb{C}$ *is a* $\mathscr{J}$*-holomorphic function. Then,* $f \in \mathscr{C}^{0,k}(U)$ *and it can be approximated in the fine* $\mathscr{C}^{0,k}$ *topology on* K *by* $\mathscr{J}$*-holomorphic functions* $F : T \times X \to \mathbb{C}$*. If in addition* Q *is a closed subset of* T *and* f *is also* $\mathscr{J}$*-holomorphic on* $Q \times X$*, then* F *can be chosen such that* $F = f$ *on* $Q \times X$*.*

Theorem 6.3 has the following corollary which shows that tameness of $\mathscr{J}$ is implied by, and hence equivalent to the one-fibre extension property for $\mathscr{J}$-holomorphic functions.

Corollary 6.4. *Assume that* T *is a locally compact and paracompact Hausdorff space,* X *is a smooth manifold, and* $\mathscr{J} = \{J_t\}_{t\in T}$ *is a continuous family of smooth Stein structures on* X*.*

(a) *If* $\mathscr{J}$ *is tame then every* $\mathscr{J}$*-holomorphic function on* $Q \times X$*, where* Q *is a closed subset of* T*, extends to a* $\mathscr{J}$*-holomorphic function on* $T \times X$*.*
(b) *Conversely, if for every* $f \in \mathscr{O}(X, J_{t_0})$ $(t_0 \in T)$ *there are a neighbourhood* $T_0 \subset T$ *of* t_0 *and a* $\mathscr{J}$*-holomorphic function* $F : T_0 \times X \to \mathbb{C}$ *such that* $F(t_0,\cdot) = f$*, then the family* $\mathscr{J}$ *is tame.*

Proof of Corollary 6.4. Note that (a) is a part of Theorem 6.3. We prove (b) by contradiction. Assume for simplicity that T is first countable; in the general case the same argument works with sequences replaced by nets. Assume that $\mathscr{J}$ is not tame. Then there are a point $t_0 \in T$, a compact J_{t_0}-convex set $K \subset X$, a neighbourhood $U \Subset X$ of K, and a sequence $t_j \in T$ with $\lim_{j\to\infty} t_j = t_0$ such that the hull $\widehat{K}_{J_{t_j}}$ is not contained in U for any j. It follows that $\widehat{K}_{J_{t_j}} \cap bU \neq \varnothing$. Pick a point $x_j \in \widehat{K}_{J_{t_j}} \cap bU$ for every j. Since bU is compact, passing to a subsequence we may assume that x_j converges to a point $x_0 \in bU$ as $j \to \infty$. Assume that $F : T_0 \times X \to \mathbb{C}$ is a $\mathscr{J}$-holomorphic function, where $T_0 \subset T$ is a neighbourhood of t_0. Let $f = F(t_0,\cdot)$. For every sufficiently big j we have $t_j \in T_0$ and hence $|F(t_j, x_j)| \leq \max_{x\in K} |F(t_j, x)|$. Taking the limit as $j \to \infty$ gives $|f(x_0)| \leq \max_{x\in K} |f(x)|$. Since $x_0 \notin \widehat{K}_{J_{t_0}} = K$, there exists a function $f \in \mathscr{O}_{J_{t_0}}(X)$ violating the above inequality, and hence such f does not admit a $\mathscr{J}$-holomorphic extension to $T_0 \times X$ for any neighbourhood T_0 of t_0. □

For a tame family $\mathscr{J}$, Corollary 6.4 also implies the following characterisation of $\mathscr{J}$-convex sets by $\mathscr{J}$-holomorphic functions.

Corollary 6.5. *Assume that* T*,* X*, and* $\mathscr{J} = \{J_t\}_{t\in T}$ *are as in Corollary 6.4. If* $\mathscr{J}$ *is tame then a proper over* T *subset* $K \subset Z = T \times X$ *is* $\mathscr{J}$*-convex if and only if for every point* $z_0 = (t_0, x_0) \in Z \setminus K$ *there exists a* $\mathscr{J}$*-holomorphic function* $f : Z \to \mathbb{C}$ *such that* $|f(z_0)| > \sup_{z\in K} |f(z)|$*.*

Proof. If K is not $\mathscr{J}$-convex then at least one of its fibres K_t ($t \in T$) is not J_t-convex, so a function with the stated property does not exist. Assume now that K is $\mathscr{J}$-convex and let $z_0 = (t_0, x_0) \in Z \setminus K$. Then, $x_0 \in X \setminus (\widehat{K_{t_0}})_{J_{t_0}}$, so there exists $f_{t_0} \in \mathscr{O}_{J_{t_0}}(X)$ with $|f_{t_0}(x_0)| > \max_{x \in K_{t_0}} |f_{t_0}(x)|$. By Corollary 6.4 (a) there exists a $\mathscr{J}$-holomorphic function $F : Z \to \mathbb{C}$ with $F(t_0, \cdot) = f_{t_0}$. Since $\mathscr{J}$ is tame, there is a neighbourhood $T_0 \subset T$ of t_0 such that $|F(t_0, x_0)| > \max_{x \in K_t} |F(t, x)|$ holds for all $t \in T_0$. If $\chi : T \to [0, 1]$ is a continuous function with $\chi(t_0) = 1$ and $\mathrm{supp}\chi \subset T_0$ then the function $f : Z \to \mathbb{C}$ given by $f(t, x) = \chi(t)F(t, x)$ satisfies the conclusion of the corollary. □

Proof of Theorem 6.3. We first consider the case when T is compact and $Q = \varnothing$. Since the set $K \subset T \times X$ is proper over the compact set T, K is compact as well. Choose a compact set $L \subset X$ such that $K \subset T \times L$ and set $L' = \widehat{T \times L}_{\mathscr{J}}$ (5.1). It suffices to show that the function f in the theorem can be approximated as closely as desired in $\mathscr{C}^{0,k}(K)$ by $\mathscr{J}$-holomorphic functions $F : U' \to \mathbb{C}$ on an open neighbourhood $U' \subset T \times X$ of L'. If this holds then the conclusion follows by an induction with respect to an exhaustion of $T \times X$ by an increasing sequence of compact $\mathscr{J}$-convex sets.

Consider the problem for $t \in T$ near a fixed $t_0 \in T$. We denote by $K_t \subset X$ the fibre of K over $t \in T$, and likewise for the other sets. For a subset $T_0 \subset T$ we also write $K_{T_0} = K \cap (T_0 \cap X)$ and $U_{T_0} = U \cap (T_0 \times X)$. Choose a relatively compact strongly J_{t_0}-pseudoconvex domain $\Omega \Subset X$ with smooth boundary such that $L'_{t_0} = \widehat{L}_{J_{t_0}} \subset \Omega$. Theorem 3.1 furnishes a neighbourhood $T_0 \subset T$ of t_0 and a map $\Phi : T_0 \times \Omega \to T_0 \times X$ of class $\mathscr{C}^{0,k}$ such that $\Phi(t, x) = (t, \Phi_t(x))$ and

$$\Phi_t : \Omega \to \Phi_t(\Omega) \subset X \text{ is a } (J_t, J_{t_0})\text{-biholomorphism for every } t \in T_0, \tag{6.1}$$

with $\Phi_{t_0} = \mathrm{Id}_\Omega$. Choose J_{t_0}-Stein domains V, V' in X such that $L'_{t_0} \subset V \Subset V' \Subset \Omega$. Shrinking U around K and T_0 around t_0 if necessary, the following inclusions hold for every $t \in T_0$ (see Fig. 2):

$$U_t \subset \Omega, \quad \Phi_t(K_t) \subset V \subset V' \subset \Phi_t(\Omega), \quad L'_t \subset V \subset \Phi_t^{-1}(V'). \tag{6.2}$$

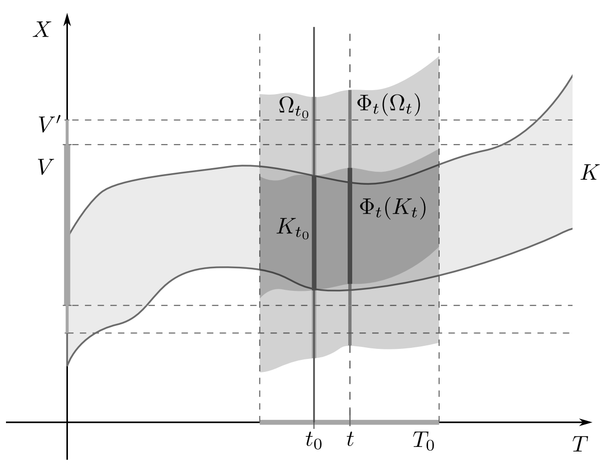

Figure 2. The shaded areas depict the sets $\Phi_t(K_t) \subset \Phi_t(\Omega)$ for $t \in T_0$.

Let $f : U \to \mathbb{C}$ be as in the theorem, so $f_t = f(t, \cdot)$ is J_t-holomorphic on $U_t \supset K_t$ for every $t \in T$. The function $f \circ \Phi^{-1} : \Phi(U_{T_0}) \to \mathbb{C}$ is then continuous and fibrewise J_{t_0}-holomorphic, hence of class $\mathscr{C}^{0,\infty}$. Since K_t is J_t-convex in Ω and the map Φ_t (6.1) is (J_t, J_{t_0})-biholomorphic, $\Phi_t(K_t)$ is J_{t_0}-convex in $\Phi_t(\Omega)$ (and hence in $V' \subset \Phi_t(\Omega)$, see (6.2)) for every $t \in T_0$. Hence, the set $K' := \Phi(K_{T_0}) \subset T_0 \times X$ is proper over T_0 and its fibres are J_{t_0}-convex in the Stein manifold (V', J_{t_0}). By [14, Lemma 5.3] there is a function $F' : T_0 \times V' \to \mathbb{C}$ of class $\mathscr{C}^{0,\infty}$ which is fibrewise J_{t_0}-holomorphic and approximates $f \circ \Phi^{-1}$ as closely as desired in $\mathscr{C}^{0,k}(K')$. Since $\Phi_t(V) \subset V'$ by the third inclusion in (6.2), the function $F := F' \circ \Phi : T_0 \times V \to \mathbb{C}$ is well-defined, of class $\mathscr{C}^{0,k}$, and it approximates f in $\mathscr{C}^{0,k}(K_{T_0})$. By the fifth inclusion in (6.2) we have that $L' \cap (T_0 \times X) \subset T_0 \times V$.

This gives a finite open cover $\{T_j\}_j$ of T and open sets $V_j \subset X$ such that $\{T_j \times V_j\}_j$ is a cover of L', and $\mathscr{J}$-holomorphic functions $F_j : T_j \times V_j \to \mathbb{C}$ approximating f in $\mathscr{C}^{0,k}(K_{T_j})$ for every j. Choose a partition of unity $1 = \sum_j \chi_j$ on T with $\operatorname{supp} \chi_j \subset T_j$ for every j. The function $F(t,x) = \sum_j \chi_j(t) F_j(t,x)$ is then well-defined and $\mathscr{J}$-holomorphic on a neighbourhood of L' in $T \times X$ and it approximates f in $\mathscr{C}^{0,k}(K)$. To conclude the proof, it remains to apply an induction with respect to a normal exhaustion of $T \times X$ by an increasing family of compact $\mathscr{J}$-convex sets.

Suppose now that T is compact and $Q \subset T$ is nonempty. Let $K \subset L' \subset T \times X$ be as above. Choose a strongly pseudoconvex domain $\Omega \Subset X$ such that $L' \cap (Q \times X) \subset Q \times \Omega$. We claim that there is a neighbourhood $T' \subset T$ of Q and a $\mathscr{J}$-holomorphic function $f' : T' \times \Omega \to \mathbb{C}$ which agrees with f on $Q \times \Omega$. If the complex structure J_t is independent of $t \in T'$, this follows from the parametric Oka–Weil theorem [12, Theorem 2.8.4]. In the case at hand, we choose a pair of smoothly bounded strongly pseudoconvex domain $\Omega_1 \Subset \Omega_2 \Subset X$ such that $\overline{\Omega} \subset \Omega_1$, and we cover Q by finitely many open sets $T_1, \ldots, T_m \subset T$ with points $t_j \in T_j$ such that Theorem 3.1 applies on $\overline{T}_j \times \Omega_2$ for every $j = 1, \ldots, m$. This gives maps $\Phi_j : \overline{T}_j \times \Omega_2 \to \overline{T}_j \times X$ of class $\mathscr{C}^{0,k}$ and of the form (6.1) such that $\Phi_{j,t} : \Omega_2 \to \Phi_{j,t}(\Omega_2)$ is a (J_t, J_{t_j})-biholomorphism for every $t \in \overline{T}_j$. Choosing the sets T_j small enough we may assume that the following inclusions hold for $j = 1, \ldots, m$:

$$\overline{T}_j \times \overline{\Omega} \subset \Phi_j^{-1}(\overline{T}_j \times \Omega_1), \quad \overline{T}_j \times \overline{\Omega}_1 \subset \Phi_j(\overline{T}_j \times \Omega_2). \tag{6.3}$$

We apply [12, Theorem 2.8.4] to each function $f \circ \Phi_j^{-1} : (\overline{T}_j \cap Q) \times \Omega_1 \to \mathbb{C}$ (see the second inclusion in (6.3)), which is fibrewise J_{t_j}-holomorphic, to find a fibrewise J_{t_j}-holomorphic function $\tilde{f}_j : \overline{T}_j \times \Omega_1 \to \mathbb{C}$ which agrees with $f \circ \Phi_j^{-1}$ on $(\overline{T}_j \cap Q) \times \Omega_1$. The function $\tilde{f}_j \circ \Phi_j$ is then well-defined and $\mathscr{J}$-holomorphic on $\overline{T}_j \times \Omega$ (see the first inclusion in (6.3)), and it agrees with f on $(\overline{T}_j \cap Q) \times \Omega$. Choose a partition of unity $\{\chi_j\}_{j=1}^m$ on a neighbourhood of Q with $\operatorname{supp}\chi_j \subset T_j$. The function $f' = \sum_{j=1}^m \chi_j (\tilde{f}_j \circ \Phi_j)$ has the desired properties. We now replace f by $(1-\xi) f + \xi f'$ where $\xi : T \to [0,1]$ is a continuous function with compact support contained in a small neighbourhood $Q' \supset Q$ such that $\xi = 1$ on a neighbourhood of Q. This new function is $\mathscr{J}$-holomorphic on $U \supset K$ and on $T' \times \Omega$, and it is close to the original function f on K if the neighbourhood $Q' \supset Q$ was chosen small enough. We apply the previously explained construction to this new function, working on the complement of Q in T to find a $\mathscr{J}$-holomorphic function F on a neighbourhood U' of L' which approximates f on K and agrees with f on $(Q \times X) \cap U'$. This completes the induction step.

When T is a locally compact and paracompact Hausdorff space, the above proof gives an open locally finite cover $\mathcal{T} = \{T_j\}_j$ of T (not necessarily countable) and functions $F_j : \overline{T}_j \times X \to \mathbb{C}$ which approximate f as closely as desired in the $\mathscr{C}^{0,k}$ topology on $K \cap (\overline{T}_j \times X)$ and agree with f on $(\overline{T}_j \cap Q) \times X$. Choosing a partition of unity $\{\xi_j\}_j$ on T subordinate to $\mathcal{T}$ and setting $F = \sum_j \xi_j F_j : T \times X \to \mathbb{C}$ gives functions satisfying the theorem. □

We now turn to the Oka principle in Theorem 6.1 where the target Y is an arbitrary Oka manifold. We shall use the following special case of [14, Lemma 6.3] which we restate in the notation of this paper. In this lemma, the Stein structure on X is independent of the parameter t.

Lemma 6.6. *Assume that $P'' \subset \mathbb{R}^N$ is a neighbourhood retract and $P_0 \subset P_1 \subset P \subset P'$ are compact subsets of P'', each contained in the interior of the next one. Let X be a Stein manifold, $\pi : \mathbb{C}^N \times X \to \mathbb{C}^N$ be the projection, and $K \subset \mathbb{C}^N \times X$ be a compact subset such that $\pi(K) \subset P$ and the fibre $K_t = \{x \in X : (t,x) \in K\}$ is $\mathscr{O}(X)$-convex for every $t \in P$. (The fibre K_t may be empty for some t.) Assume that U is an open neighbourhood of K in $P' \times X$, Y is an Oka manifold endowed with a distance function* dist_Y*, and $f : P' \times X \to Y$ is a continuous map such that for every*

$t \in P$ *the map* $f_t = f(t,\cdot\,) : X \to Y$ *is holomorphic on* $U_t = \{x \in X : (t,x) \in U\}$. *Fix* $\epsilon > 0$. *After shrinking the open set* $U \supset K$ *if necessary, there is a homotopy* $f_s : P \times X \to Y$ $(s \in I = [0,1])$ *satisfying the following conditions.*

(a) $f_0 = f|_{P\times X}$.
(b) $f_s(t,\cdot\,) : X \to Y$ *is holomorphic on* U_t *for every* $s \in I$ *and* $t \in P$.
(c) $\max_{(t,x)\in K} \mathrm{dist}_Y(f_s(t,x), f(t,x)) < \epsilon$ *for every* $s \in I$.
(d) $f_s(t,\cdot\,) = f(t,\cdot\,)$ *for all* $t \in P \setminus P_1$ *and* $s \in I$.
(e) *The map* $f_1(t,\cdot\,) : X \to Y$ *is holomorphic for every* t *in a neighbourhood of* P_0.

Proof of Theorem 6.1. Let the integers $k \geq 1$ and $r \geq 2k+2n+9$ be as in Remark 6.2. We shall follow [14, proof of Theorem 1.6], which treats the case when X is a smooth open surface. The adjustment we have to make is that the J_t-convex hull of a compact set in X may now change with the parameter t. Unlike in the proof of Theorem 6.3, we can not glue partial approximants by partitions of unity on T since the target Y is a manifold, so the problem is nonlinear. Instead, we make all deformations by homotopies and use cut-off functions in the parameters of the homotopy at every inductive step.

The conditions on T imply that it is locally compact, σ-compact and Hausdorff. Choose a normal exhaustion $T_1 \subset T_2 \subset \cdots \subset \bigcup_{j=1}^\infty T_j = T$ by compact sets (that is, each T_j is contained in the interior of T_{j+1}). Tameness of $\mathscr{J}$ provides an increasing sequence $L_1 \subset L_2 \subset \cdots \subset \bigcup_{j=1}^\infty L_j = X$ of compact sets forming a normal exhaustion of X such that for all $j = 1, 2, \ldots$ we have that

$$K \cap (T_j \times X) \subset T_j \times L_j \quad \text{and} \quad \widehat{(T_j \times L_j)}_{\mathscr{J}} \subset T_j \times L_{j+1}.$$

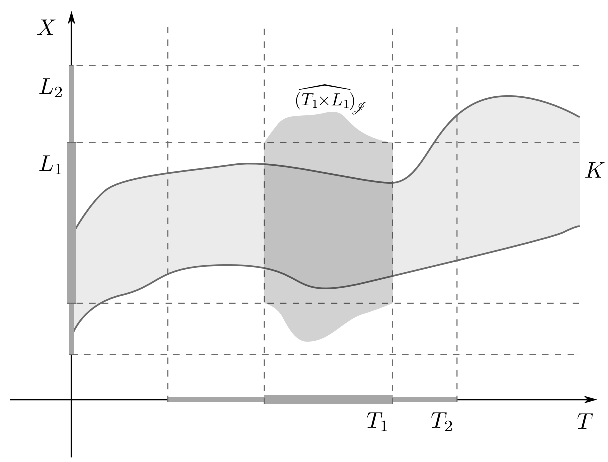

Figure 3. An illustration of the choice of the set L_2.

Define an increasing sequence of subsets $K = K^0 \subset K^1 \subset \cdots \subset \bigcup_{j=0}^\infty K^j = T \times X$ by

$$K^j = \widehat{(T_j \times L_j)}_{\mathscr{J}} \cup K, \quad j = 1, 2, \ldots. \tag{6.4}$$

Note that each K^j is proper over T and $\mathscr{J}$-convex. Let $f^0 = f : T \times X \to Y$ be the given map in the theorem which is $\mathscr{J}$-holomorphic on a neighbourhood of $K = K^0$ and on $Q \times X$. We may assume that the distance function dist_Y is complete. Let $\epsilon : T \to (0, +\infty)$ be the continuous function in the theorem. We shall find a sequence of continuous maps $f^j : T \times X \to Y$ and homotopies $f^j_s : T \times X \to Y$ $(s \in I = [0,1])$ satisfying the following conditions for every $j = 1, 2, \ldots$.

(A) f^j is $\mathscr{J}$-holomorphic on a neighbourhood of the set K^j (6.4).
(B) $f^j_0 = f^{j-1}$ and $f^j_1 = f^j$.

(C) f^j_s is $\mathscr{J}$-holomorphic on a neighbourhood of K^{j-1} for every $s \in I$, where the neighbourhood does not depend on $s \in I$.
(D) $\max_{x\in K^{j-1}_t} \mathrm{dist}_Y(f^{j-1}(t,x), f^j_s(t,x)) < 2^{-j}\epsilon(t)$ for all $s \in I$ and $t \in T$.
(E) The homotopy $f^j_s(t,\cdot)$, $s \in I$, is fixed for all $t \in Q$.

These conditions clearly imply that the homotopies f^j_s $(j \in \mathbb{N},\ s \in I)$ can be assembled into a single homotopy $f_s : T\times X \to Y$ $(s \in I)$ from the initial map $f_0 = f = f^0$ to the limit $\mathscr{J}$-holomorphic map $f_1 = F = \lim_{j\to\infty} f^j : T \times X \to Y$ (condition (iv)) such that for every $s \in I$, f_s is $\mathscr{J}$-holomorphic on a neighbourhood of K (condition (ii)), it approximates f to precision ϵ on K (condition (iii)), and the homotopy is fixed over Q (condition (v)).

Every step in the induction is of the same kind, so it suffices to show the initial step with $j = 1$. This is accomplished by a finite induction which we now explain.

If the subset $Q \subset T$ in condition (v) is nonempty, we choose a small neighbourhood $Q_1 \subset T$ of Q and deform $f^0 = f$ by a homotopy which is fixed for $t \in Q \cup (T \setminus Q_1)$ to another map $\tilde f^0 : Z \to Y$ which approximates f^0 in the fine topology on K^0 such that $\tilde f^0(t,\cdot)$ is holomorphic on a neighbourhood of K^1_t for every t in a closed neighbourhood $\widetilde Q \subset Q_1$ of Q and the other properties of f^0 remain in place. This modification can be done in a similar way as in the last part of the proof of Theorem 6.3 but using the gluing technique in [12, Proposition 5.13.1] instead of partitions of unity. To simplify the notation, we replace f^0 by $\tilde f^0$ and drop the tilde. Define the set

$$\widetilde K^0 := \big[(\widetilde Q \times X) \cap K^1\big] \cup K^0 \subset T \times X. \tag{6.5}$$

Note that $K^0 \subset \widetilde K^0 \subset K^1$, $\widetilde K^0$ is proper over T and $\mathscr{J}$-convex, and f^0 is $\mathscr{J}$-holomorphic on a neighbourhood of $\widetilde K^0$. If $Q = \varnothing$, we take $\widetilde Q = \varnothing$ and $\widetilde K^0 = K^0$.

Fix a point $t_0 \in T_1$. Since T_2 is compact, there are a smoothly bounded strongly J_{t_0}-pseudoconvex domain $\Omega \Subset X$ and J_{t_0}-Stein domains V, V' in X such that

$$\bigcup_{t\in T_2} K^1_t \Subset V \Subset V' \Subset \Omega. \tag{6.6}$$

The conditions on T imply that there is a neighbourhood $P'' \subset T$ of t_0 which is an ENR, so we may assume that $P'' \subset \mathbb{R}^N \subset \mathbb{C}^N$ is a neighbourhood retract. Theorem 3.1 gives a compact neighbourhood $P' \subset T_2$ of t_0, contained in the interior of P'', and a map $\Phi : P' \times \Omega \to X$ of class $\mathscr{C}^{0,k}$ and of the form $\Phi(t,x) = (t, \Phi_t(x))$ such that $\Phi_t : \Omega \to \Phi_t(\Omega) \subset X$ is a (J_t, J_{t_0})-biholomorphism for every $t \in P'$ and $\Phi_{t_0} = \mathrm{Id}_\Omega$. Shrinking P' around t_0 we may assume that for every $t \in P'$ we have

$$U_t \subset \Omega, \quad \Phi_t(\widetilde K^0_t) \subset V \subset V' \subset \Phi_t(\Omega), \quad K^1_t \subset V \subset \Phi_t^{-1}(V'). \tag{6.7}$$

(These are analogues of conditions (6.2).) Pick a compact neighbourhood $P \subset T$ of t_0, contained in the interior of P', and consider the continuous family of maps

$$f'_t := f_t \circ \Phi_t^{-1} : \Phi_t(\Omega) \to Y, \quad t \in P.$$

Since the map $\Phi_t^{-1} : (\Phi_t(\Omega), J_{t_0}) \to (\Omega, J_t)$ is biholomorphic and f_t is J_t-holomorphic on a neighbourhood of $\widetilde K^0_t$, the map f'_t is J_{t_0}-holomorphic on a neighbourhood of $\Phi_t(\widetilde K^0_t)$ for every $t \in P$ (see the second set of inclusions in (6.7)). Pick a pair of smaller neighbourhoods $P_0 \subset P_1 \subset P$ of t_0, each contained in the interior of the next one. Lemma 6.6, applied to the family of J_{t_0}-convex sets $\Phi_t(\widetilde K^0_t)$ in the J_{t_0}-Stein domain $V' \subset X$, gives a homotopy

$$f'_{s,t} : V' \to Y \ \text{ for } t \in P \text{ and } s \in I$$

such that $f'_{s,t} = f'_{0,t} = f'_t$ holds for $t \in P \setminus P_1$ and $s \in I$, the map $f'_{s,t}$ is J_{t_0}-holomorphic on a neighbourhood of $\Phi_t(\widetilde K^0_t)$ and approximates f'_t uniformly on $\Phi_t(\widetilde K^0_t)$ to arbitrary precision for all

$t \in P$ and $s \in I$, and $f'_{1,t}$ is J_{t_0}-holomorphic on V' for t in a neighbourhood of P_0. By the third set of inclusions in (6.7) we have that $\Phi_t(V) \subset V'$ for $t \in P$. It follows that the maps

$$f_{s,t} := f'_{s,t} \circ \Phi_t : V \to Y \quad \text{for } s \in I \text{ and } t \in P \tag{6.8}$$

are J_t-holomorphic on a neighbourhood of $\widetilde{K}^0_t$, $f_{s,t}$ approximates f_t uniformly on $\widetilde{K}^0_t$ (and uniformly in $s \in I$) to arbitrary precision, we have $f_{s,t} = f_{0,t} = f_t$ for $s \in I$ and $t \in P \setminus P_1$, and the map $f^1_t := f_{1,t} : V \to Y$ is J_t-holomorphic for all t in a neighbourhood of P_0. We extend the family of homotopies to all $t \in T$ by setting $f_{s,t} = f_{0,t} = f_t$ for $t \in T \setminus P_1$ and $s \in I$.

Note that for $t \in P_1$ the map $f_{s,t}$ in (6.8) is still defined only on $V \subset X$. In order to extend the homotopy to all of X also for $t \in P_1$, choose a smooth cut-off function $\chi_1 : X \to [0,1]$ such that $\chi_1 = 1$ in a neighbourhood of the compact set $\bigcup_{t \in P} K^1_t$ and $\mathrm{supp}\chi_1 \subset V$. If the sets $Q \subset \widetilde{Q}$ are nonempty, we choose a second cut-off function $\chi_2 : T \to [0,1]$ such that $\chi_2 = 1$ on $T \setminus \widetilde{Q}$ and $\chi_2 = 0$ on a neighbourhood of Q. If Q is empty we simply take $\chi_2 = 1$ on T. We can now extend the maps $f_{s,t} = f_s(t, \cdot\,)$ in (6.8) to all of X without changing their values on a neighbourhood of K^1 by setting

$$\tilde{f}_{s,t}(x) := f_{s\chi_1(x)\chi_2(t),t}(x) \quad \text{for } t \in T,\ x \in X, \text{ and } s \in I.$$

For t in a neighbourhood of Q we have $\chi_2 = 0$ and hence $\tilde{f}_{s,t} = f_{0,t} = f_t$.

Since T_1 is compact, we can find a finite family of triples $P^j_0 \subset P^j_1 \subset P^j$ $(j = 1, 2, \ldots, m)$ of compact sets in T such that $T_1 \subset \bigcup_{j=1}^m P^j_0$ and the construction described above can be made on each of these triples. The induction proceeds as follows. In the first step, we perform the procedure explained above on the first triple (P^1_0, P^1_1, P^1) with the set K^0 and the map $g^0 := f^0 = f$. We obtain a homotopy from g^0 to $g^1 : T \times X \to Y$ such that every map in the homotopy is $\mathscr{J}$-holomorphic on a neighbourhood of K^0, it approximates $g^0 = f^0$ on K^0 to precision $\epsilon/2m$, and the homotopy is fixed for t in a neighbourhood of $\overline{T \setminus P^1_1} \cup Q$. The resulting map g^1 is $\mathscr{J}$-holomorphic on a neighbourhood of the compact $\mathscr{J}$-convex set

$$S^1 := \big[(P^1_0 \times X) \cap K^1\big] \cup K^0 \subset T \times X.$$

Similary we define compact $\mathscr{J}$-convex sets S^ℓ for $\ell = 2, \ldots, m$ by

$$S^\ell = \big[((P^1_0 \cup \cdots \cup P^\ell_0) \times X) \cap K^1\big] \cup K^0 \subset T \times X.$$

(See Fig. 4.) In step $\ell \in \{2, \ldots, m\}$ the same argument is applied to the map $g^{\ell-1}$ on the triple $(P^\ell_0, P^\ell_1, P^\ell)$ with respect to the set $S^{\ell-1}$. The resulting map $g^\ell : T \times X \to Y$ is $\mathscr{J}$-holomorphic on a neighbourhood of S^ℓ. We also obtain a homotopy from $g^{\ell-1}$ to g^ℓ consisting of maps which are $\mathscr{J}$-holomorphic on a neighbourhood of $S^{\ell-1}$, they approximate $g^{\ell-1}$ on $S^{\ell-1}$ to precision $\epsilon/2m$, and the homotopy is fixed for t in a neighbourhood of $\overline{T \setminus P^\ell_1} \cup Q$.

After m steps we obtain a map $g^m : T \times X \to Y$ which is $\mathscr{J}$-holomorphic on a neighbourhood of S^m, which contains K^1 (see (6.4)). We define $f^1 := g^m$. Furthermore, the homotopies between the subsequent maps g^ℓ and $g^{\ell+1}$ for $\ell = 0, 1, \ldots, m-1$ can be assembled into a homotopy f^1_s $(s \in I)$ from the initial map $f^1_0 = f^0 = g^0$ to $f^1_1 = f^1 = g^m$ such that f^1_s satisfies (i)–(v) for $j = 1$. This explains the inductive step and thereby concludes the proof. □

7. The Oka–Weil theorem for sections of fibrewise holomorphic vector bundles

In this section, we assume that T is a locally compact and paracompact Hausdorff space, X is a smooth manifold, and $\mathscr{J} = \{J_t\}_{t \in T}$ is a tame family (see Def. 5.1) of Stein structures on X of class $\mathscr{C}^{0,\infty}$. The main result of this section, Theorem 7.2, is an Oka–Weil approximation theorem for $\mathscr{J}$-holomorphic sections of $\mathscr{J}$-holomorphic vector bundles on $Z = T \times X$. It generalises Theorem 6.3,

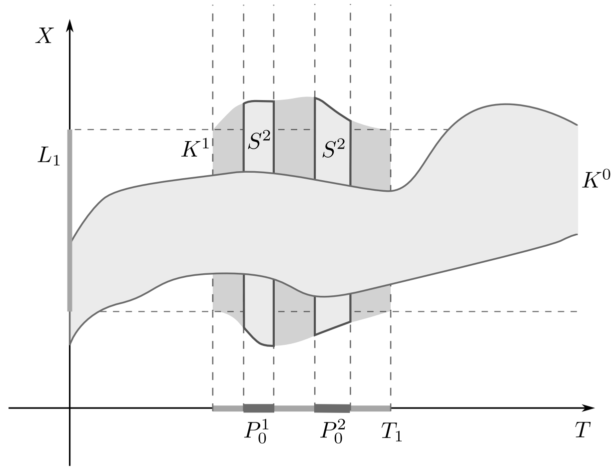

Figure 4. The set S^2.

which pertains to sections of a trivial vector bundle. Theorem 7.2 will be used in the following section to obtain global solvability of the $\overline{\partial}$-equation on tame families of Stein manifolds; see Theorem 8.1.

We endow each fibre $Z_t = \{t\} \times X$, $t \in T$, with the complex structure J_t. Denote by $\mathscr{C}^{0,\infty} = \mathscr{C}^{0,\infty}_Z$ the sheaf of germs of continuous fibrewise smooth functions on Z, and by $\mathscr{O} = \mathscr{O}_Z$ the sheaf of germs of $\mathscr{J}$-holomorphic functions on Z. These are sheaves of unital abelian rings, with $\mathscr{O}_Z$ a subsheaf of $\mathscr{C}^{0,\infty}_Z$. For a complex Lie group G, $\mathscr{O}^G_Z \subset \mathscr{C}^{0,\infty,G}_Z$ denote the sheaves of germs of maps $Z \to G$ of respective classes; these are sheaves of groups.

Definition 7.1. Let T, X and $\mathscr{J}$ be as above, and let $GL_r(\mathbb{C})$ denote the complex Lie group of invertible $r \times r$ matrices. A complex vector bundle $\pi : E \to Z = T \times X$ of rank r is said to be $\mathscr{J}$-holomorphic if it is defined on an open covering $\{U_i\}_i$ of Z by a 1-cocycle $g = (g_{i,j})$ consisting of $\mathscr{J}$-holomorphic maps $g_{i,j} : U_{i,j} \to GL_r(\mathbb{C})$.

Explicitly, E is obtained by gluing the trivial bundles $U_i \times \mathbb{C}^r$ on the overlaps $U_{i,j} = U_i \cap U_j$ by identifying a point $(z, v) \in U_j \times \mathbb{C}^r$ (where $z = (t, x) \in U_{i,j}$) with $(z, g_{i,j}(z)v) \in U_i \times \mathbb{C}^r$.

Theorem 7.2. *Let T, X and $\mathscr{J}$ be as above and $E \to Z = T \times X$ be a $\mathscr{J}$-holomorphic vector bundle. Given a proper $\mathscr{J}$-convex subset $K \subset Z$ (see* (5.2)*), an open subset $U \subset Z$ containing K, a closed subset $Q \subset T$, a $\mathscr{J}$-holomorphic section $f : U \cup (Q \times X) \to E|U \cup (Q \times X)$, and an integer $k \in \mathbb{Z}_+$, we can approximate f in the fine $\mathscr{C}^{0,k}$ topology on K by $\mathscr{J}$-holomorphic sections $F : Z \to E$ such that $F = f$ on $Q \times X$.*

Proof. We shall assume that T is compact and $Q = \varnothing$; the general case can be dealt with as in the proof of Theorem 6.3, which pertains to sections of trivial bundles.

Since K is proper over the compact set T, it is compact. It suffices to show that, given a $\mathscr{J}$-convex subset $L \subset Z$ with $K \subset L$, we can approximate f in the $\mathscr{C}^{0,k}$ topology on K by $\mathscr{J}$-holomorphic sections of E over a neighbourhood of L. The conclusion of the theorem then follows by induction with respect to an increasing sequence of $\mathscr{J}$-convex subsets exhausting Z. Furthermore, since the problem is linear, we may use partitions of unity on T. This reduces the proof to the approximation problem for parameter values t in a neighbourhood of a given point $t_0 \in T$.

Fix $t_0 \in T$ and smoothly bounded strongly J_{t_0}-pseudoconvex domains $\Omega_i \Subset X$ for $i = 1, 2, 3$ such that $L_{t_0} \subset \Omega_1 \Subset \Omega_2 \Subset \Omega_3$. Theorem 3.1 gives a compact neighbourhood $T_0 \subset T$ of t_0 and a map $\Phi : T_0 \times \Omega_3 \to T_0 \times X$ of class $\mathscr{C}^{0,\infty}$ such that $\Phi(t, x) = (t, \Phi_t(x))$ and $\Phi_t : \Omega_3 \to \Phi_t(\Omega_3) \subset X$ is a (J_t, J_{t_0})-biholomorphism for every $t \in T_0$, with $\Phi_{t_0} = \mathrm{Id}_\Omega$. Shrinking T_0 around t_0, we may assume that the following inclusions hold for every $t \in T_0$:

$$L_t \subset \Omega_1 \subset \Phi_t^{-1}(\Omega_2), \qquad \Omega_2 \Subset \Phi_t(\Omega_3). \tag{7.1}$$

Since the map Φ_t is (J_t, J_{t_0})-biholomorphic, the push-forward vector bundle $\Phi_*(E|(T_0 \times \Omega))$ is continuous in $t \in T_0$ and fibrewise J_{t_0}-holomorphic. Denote by E' the restriction of this bundle to the domain $T_0 \times \overline{\Omega}_2 \subset \Phi(T_0 \times \Omega_3)$ (see (7.1)). Its restriction E'_t to the fibre $\{t\} \times \overline{\Omega}_2$ is a J_{t_0}-holomorphic vector bundle, smooth up to the boundary of Ω_2 and depending continuously on $t \in T_0$. Assuming as we may that T_0 is chosen small enough, the stability result of Leiterer [36, Theorem 2.7] gives a family of J_{t_0}-holomorphic vector bundle isomorphisms over Ω_2,

$$\Psi_t : E'_t \xrightarrow{\cong} E'_{t_0}, \quad t \in T_0, \tag{7.2}$$

smooth up to the boundary and depending continuously on $t \in T_0$. (The cited result is stated in terms of J_{t_0}-holomorphic transition cocycles $g^t = \{g^t_{i,j}\}$ for E'_t for $t \in T_0$, defined on a fixed open cover of $\overline{\Omega}_2$ and continuous up to the boundary of the respective domains, and there are cohomological condition (i), (ii) on the endomorphism bundle $\mathrm{Ad}(E_{t_0})$ of E_{t_0}. As explained in [36, Remark 2.11] and [36, proof of Theorem 2.12], the two cohomology groups appearing in the hypothesis of [36, Theorem 2.7] vanish when the base is a compact strongly pseudoconvex domain with $\mathscr{C}^2$ boundary.) The upshot is that the bundle $E' \to T_0 \times \overline{\Omega}_2$ is fibrewise isomorphic to the trivial (independent of t) extension of the vector bundle $E'_{t_0} := E'|(\{t_0\} \times \overline{\Omega}_2)$.

Denote by E_t the restriction of the initial vector bundle $E \to Z$ to the fibre over $t \in T$. We are given a continuous family of J_t-holomorphic sections $f_t : U_t \to E_t|U_t$, $t \in T$. For every $t \in T_0$, the map $\tilde f_t := f_t \circ \Phi_t^{-1}$ is a J_{t_0}-holomorphic section of the push-forward bundle $E'_t = (\Phi_t)_* E_t$ over the domain $\Phi_t(U_t)$, depending continuously on $t \in T_0$. By using the isomorphisms Ψ_t in (7.2), we may consider $\{\tilde f_t\}_{t \in T_0}$ as a family of J_{t_0}-holomorphic sections of E'_{t_0} over the family of domains $\Phi_t(U_t) \supset \Phi_t(K_t)$. (Here, K_t is the fibre of the set K in the theorem.) Note that for every $t \in T_0$ the set $\Phi_t(K_t)$ is J_{t_0}-convex in $\Phi_t(\Omega_3)$, hence in Ω_2; furthermore $\Phi(K \cap (T_0 \times X))$ is proper over T_0.

By the parametric Oka–Weil theorem for sections of holomorphic vector bundles over Stein manifolds (see [12, Theorem 2.8.4]), we can approximate $\tilde f_t$ in the $\mathscr{C}^k$ topology on $\Phi_t(K_t)$, uniformly in $t \in T_0$, by J_{t_0}-holomorphic sections F'_t of the bundle E'_{t_0} over the Stein domain Ω_2. (Approximation on variable fibres $\Phi_t(K_t)$ is reduced to the case of constant fibres by the same technique as in the proof of Theorem 6.3, using a continuous partition of unity on T_0.) Applying again the vector bundle isomorphisms (7.2), we may consider F'_t as a J_{t_0}-holomorphic section of the bundle E'_t over Ω_2. Finally, $F_t := F'_t \circ \Phi_t$ is a J_t-holomorphic section of the original bundle E_t over the domain $\Phi_t^{-1}(\Omega_2)$, and these sections depend continuously on $t \in T_0$. By (7.1) we have $L_t \subset \Omega_1 \subset \Phi_t^{-1}(\Omega_2)$ for all $t \in T_0$, so $F(t,x) = (t, F_t(x))$ is a $\mathscr{J}$-holomorphic section of $E|(T_0 \times \Omega_1)$ which approximates f on $K \cap (T_0 \times X)$. Note that $L \cap (T_0 \times X) \subset T_0 \times \Omega_1$.

Since T is compact, this gives a finite open cover $\{W_j = T_j \times \Omega_j\}_j$ of L such that $\mathcal{T} = \{T_j\}_j$ is an open cover of T, and $\mathscr{J}$-holomorphic sections F_j of $E|W_j$ approximating f in the $\mathscr{C}^{0,k}$ topology on $K \cap W_j$ as closely as desired for every j. If $\{\chi_j\}_j$ is a continuous partition of unity on T subordinate to $\mathcal{T}$ then $F = \sum_j \chi_j F_j$ is a $\mathscr{J}$-holomorphic section of E over a neighbourhood of L in Z approximating f in the fine $\mathscr{C}^{0,k}$ topology on K. As explained at the beginning, this concludes the proof. □

8. Global solution of the $\overline{\partial}$-equation on tame families of Stein manifolds

In this section, we assume that T is a locally compact and paracompact Hausdorff space, X is a smooth manifold, and $\mathscr{J} = \{J_t\}_{t \in T}$ is a tame family of Stein structures on X (see Def. 5.1) of class $\mathscr{C}^{0,\infty}$. Write $Z = T \times X$. Every fibre $Z_t = \{t\} \times X \cong X$ is endowed with the Stein structure J_t. For each pair of integers $p \geq 0,\ q \geq 1$ we denote by $\mathscr{D}^{p,q}(Z)$ the space of (p,q)-forms on the fibres

(X,J_t) of Z of class $\mathscr{C}^{0,\infty}$. The following is a corollary to Theorems 3.2 and 7.2. A related result for $p=0,\ q=1$ on families of open Riemann surfaces is [15, Corollary 1.2].

Theorem 8.1. *Let $p\ge 0$ and $q\ge 1$. Given a continuous family $\alpha=\{\alpha_t\}_{t\in T}\in\mathscr{D}^{p,q}(Z)$ of smooth (p,q)-forms $\alpha_t\in\mathscr{D}^{p,q}(X,J_t)$ with $\overline\partial_{J_t}\alpha_t=0$ for all $t\in T$, there exists $\beta=\{\beta_t\}_{t\in T}\in\mathscr{D}^{p,q-1}(Z)$ satisfying*

$$\overline\partial_{J_t}\beta_t=\alpha_t \ \text{ on } X \text{ for every } t\in T. \tag{8.1}$$

In the sequel, we shall often write the equation (8.1) in the form $\overline\partial_{\mathscr{J}}\beta=\alpha$ on $Z=T\times X$.

Proof. We begin by showing that the equation (8.1) is solvable on a neighbourhood of any proper $\mathscr{J}$-convex subset $K\subset Z$ (see (5.2)).

Denote by K_t the fibre of K over $t\in T$. Fix a point $t_0\in T$ and a smoothly bounded strongly J_{t_0}-pseudoconvex neighbourhood $D\Subset X$ of K_{t_0}. By tameness of $\mathscr{J}$ and Lemma 5.6, there is a neighbourhood $T_0\subset T$ of t_0 such that D is a strongly J_t-pseudoconvex neighbourhood of K_t for all $t\in T_0$. By Theorem 8.1 (e), there exists $\beta\in\mathscr{D}^{p,q-1}(T_0\times D)$ solving $\overline\partial_{\mathscr{J}}\beta=\alpha$ on $T_0\times D$. In this way, we obtain an open locally finite cover $\mathcal{T}=\{T_i\}_{i\in I}$ of T, smoothly bounded domains $D_i\Subset X$ such that $K\subset\bigcup_{i\in I}T_i\times D_i$, and solutions $\beta_i\in\mathscr{D}^{p,q-1}(T_i\times D_i)$ to $\overline\partial_{\mathscr{J}}\beta_i=\alpha$ on $T_i\times D_i$. Let $\{\chi_i\}_{i\in I}$ be a partition of unity on T subordinate to $\mathcal{T}$. Then, $\beta=\sum_{i\in I}\chi_i\beta_i$ is a solution to (8.1) in a neighbourhood of K.

Choose an exhaustion $K^1\subset K^2\subset\cdots$ of Z by proper $\mathscr{J}$-convex sets (see (5.2)). We shall inductively find solutions β^j to (8.1) in neighbourhoods of K^j such that β^j approximates the solution β^{j-1} from the previous step on a neighbourhood of K^{j-1} (if $q=1$), or agrees with it (if $q>1$).

Denote by Ω^p the sheaf of germs of $\mathscr{J}$-holomorphic $(p,0)$-forms on the fibres of $Z=T\times X$. In particular, $\Omega^0=\mathscr{O}$ is the sheaf of germs of $\mathscr{J}$-holomorphic functions on Z. Since the complex structures $J_t\in\mathscr{J}$ are smoothly compatible, Ω^p is a subsheaf of the sheaf $\mathscr{E}^{p,0}$ of fibrewise smooth $(p,0)$-forms on the fibres of Z. The elements $\beta\in\mathscr{D}^{p,0}(Z)$ satisfying $\overline\partial_{\mathscr{J}}\beta=0$ are precisely the global sections of Ω^p over Z. Equivalently, they are holomorphic sections of the $\mathscr{J}$-holomorphic vector bundle on Z (see Definition 7.1) whose restriction to $Z_t=(X,J_t)$ is $\Lambda^p\,T^{*(1,0)}(X,J_t)$, the p-th exterior power of the $(1,0)$-cotangent bundle of (X,J_t) (see Section 2).

Let β^j be as above, solving $\overline\partial_{\mathscr{J}}\beta^j=\alpha$ on a neighbourhood of K^j for $j=1,2,\ldots$. Then,

$$\overline\partial_{\mathscr{J}}(\beta^j-\beta^{j-1})=0 \ \text{ holds on a neighbourhood } U \text{ of } K^{j-1}. \tag{8.2}$$

If $q=1$, this means that $\beta^j-\beta^{j-1}$ is a section of the sheaf Ω^p on U. By Theorem 7.2, we can approximate it in the fine $\mathscr{C}^{0,j}$ topology on K^{j-1} by a global section γ of Ω^p. Replacing β^j by $\beta^j-\gamma$ ensures that β^j solves $\overline\partial_{\mathscr{J}}\beta^j=\alpha$ on a neighbourhood of K^j and it approximates β^{j-1} on K^{j-1}. Performing this construction inductively gives a sequence β^j converging in the fine $\mathscr{C}^{0,\infty}$-topology to a solution $\beta\in\mathscr{D}^{p,0}(Z)$ of the equation (8.1).

Assume now that $q>1$. As explained earlier, (8.2) implies that $\beta^j-\beta^{j-1}=\overline\partial_{\mathscr{J}}\gamma$ on a neighbourhood U of K^{j-1} for some $\gamma\in\mathscr{D}^{p,q-2}(U)$. Let $\chi:Z\to[0,1]$ be a function of class $\mathscr{C}^{0,\infty}$ with $\mathrm{supp}(\chi)\subset U$ which equals 1 in a neighbourhood of K^{j-1}. Replacing β^j by $\beta^j-\overline\partial_{\mathscr{J}}(\chi\gamma)$ gives a solution to $\overline\partial_{\mathscr{J}}\beta^j=\alpha$ in a neighbourhood of K^j such that $\beta^j=\beta^{j-1}$ holds in a neighbourhood of K^{j-1}. Hence, the sequence β^j is stationary and hence converges to a global solution of (8.1). ☐

Theorem 8.2. *(Assumptions as above.) $H^q(Z,\Omega^p)=0$ for all $q=1,2,\ldots$.*

The groups $H^q(Z,\Omega^p)$ are classically called Dolbeault cohomology groups, although Dolbeault's opinion was that they should in fact be called Grothendieck groups.

Proof. Let $\mathscr{E}^{p,q}$ denote the sheaf of fibrewise smooth (p,q)-forms on $Z = T \times X$ which are continuous in $t \in T$ (i.e. of class $\mathscr{C}^{0,\infty}$). Consider the sequence of homomorphisms of sheaves of abelian groups

$$0 \longrightarrow \Omega^p \hookrightarrow \mathscr{E}^{p,0} \xrightarrow{d_0} \mathscr{E}^{p,1} \xrightarrow{d_1} \mathscr{E}^{p,2} \xrightarrow{d_2} \cdots \tag{8.3}$$

where each d_j is the $\overline{\partial}_{\mathscr{J}}$ operator which equals $\overline{\partial}_{J_t}$ on $Z_t = (X, J_t)$ for every $t \in T$. By Theorem 8.1 the sequence (8.3) is exact. All sheaves in (8.3) except Ω^p are fine sheaves, so their cohomology groups of order ≥ 1 vanish. (See e.g. [26, Chapter VI] or [51] for sheaf cohomology.) Hence, (8.3) is an acyclic resolution of the sheaf Ω^p. It follows that

$$H^q(Z, \Omega^p) = \frac{\mathrm{Ker}\{d_q : \Gamma(Z, \mathscr{E}^{p,q}) \to \Gamma(Z, \mathscr{E}^{p,q+1})\}}{\mathrm{Im}\{d_{q-1} : \Gamma(Z, \mathscr{E}^{p,q-1}) \to \Gamma(Z, \mathscr{E}^{p,q})\}} = \frac{\{\alpha \in \mathscr{D}^{p,q}(Z) : \overline{\partial}_{\mathscr{J}} \alpha = 0\}}{\{\overline{\partial}_{\mathscr{J}} \beta : \beta \in \mathscr{D}^{p,q-1}(Z)\}}.$$

Here, Γ denotes the space of sections. The group on the right hand side vanishes by Theorem 8.1. □

9. The Oka principle for vector bundles on tame families of Stein manifolds

Assume that T is a topological space, X is a smooth manifold, and $\mathscr{J} = \{J_t\}_{t \in T}$ is a tame family of Stein structures on X. The notion of a $\mathscr{J}$-holomorphic vector bundle on $Z = T \times X$ was introduced in Definition 7.1. In this section, we prove the Oka principle for $\mathscr{J}$-holomorphic vector bundles on tame families of Stein manifolds. We begin with line bundles. Denote by $\mathrm{Pic}(Z)$ the set of isomorphism classes of $\mathscr{J}$-holomorphic line bundles on $Z = T \times X$. We have the following Oka principle which was proved for line bundles on Stein manifolds (with T a singleton) by Oka [46].

Theorem 9.1. *Assume that T is a locally compact and paracompact Hausdorff space and $\mathscr{J} = \{J_t\}_{t \in T}$ is a tame family of class $\mathscr{C}^{0,\infty}$ of Stein structures on a smooth manifold X. Then, every topological complex line bundle on $Z = T \times X$ is isomorphic to a $\mathscr{J}$-holomorphic line bundle, and any two $\mathscr{J}$-holomorphic line bundles on Z which are topologically isomorphic are also isomorphic as $\mathscr{J}$-holomorphic line bundles. Furthermore,* $\mathrm{Pic}(Z) \cong H^2(Z, \mathbb{Z})$.

The proof of this result follows the standard cohomological argument for the exponential sheaf sequence on Z, using that $H^1(Z, \mathscr{O}) = 0$ and $H^2(Z, \mathscr{O}) = 0$ (see Theorem 8.2) and $H^1(Z, \mathscr{O}^*) = \mathrm{Pic}(Z)$. We refer to [12, Sect. 5.2] for the classical case of line bundles on Stein manifolds, and to [15, Theorem 2.3] for line bundles on families of open Riemann surfaces.

For vector bundles of arbitrary rank, we have the following Oka principle.

Theorem 9.2. *Assume that T, $\mathscr{J} = \{J_t\}_{t \in T}$ and X are as in Theorem 6.1, with $\mathscr{J}$ of class $\mathscr{C}^{0,\infty}$. Then, every topological vector bundle on $Z = T \times X$ is isomorphic to a $\mathscr{J}$-holomorphic vector bundle, and every pair of $\mathscr{J}$-holomorphic vector bundles which are topologically isomorphic are also isomorphic as $\mathscr{J}$-holomorphic vector bundles.*

Proof. The proof of the first statement follows that of [15, Theorem 2.4], which gives an analogous result on families on open Riemann surfaces. Let $Gr_r(\mathbb{C}^N)$ denote the Grassmann manifold of complex r-dimensional subspaces of $\mathbb{C}^N$, and let $\mathbb{U} \to Gr_r(\mathbb{C}^N)$ denote the universal bundle whose fibre over $\Lambda \in Gr_r(\mathbb{C}^N)$ consists of all vectors $v \in \Lambda \subset \mathbb{C}^N$. Every topological vector bundle of rank r on Z is obtained as the pullback by a continuous map $f : Z \to Gr_r(\mathbb{C}^N)$ of the universal bundle $\mathbb{U}$ for a sufficiently big N; furthermore, homotopic maps induce isomorphic vector bundles, and $\mathscr{J}$-holomorphic maps induce $\mathscr{J}$-holomorphic vector bundles. Since $Gr_r(\mathbb{C}^N)$ is a complex homogeneous manifold, and hence an Oka manifold, every continuous map $Z \to Gr_r(\mathbb{C}^N)$ is homotopic to a $\mathscr{J}$-holomorphic map by Theorem 6.1. This proves the first part.

To prove the second statement, let $E \to Z$ and $E' \to Z$ be $\mathscr{J}$-holomorphic vector bundles of rank r. There is an open cover $\{U_j\}_j$ of Z and $\mathscr{J}$-holomorphic vector bundle isomorphisms

$\theta_j : E|U_j \to U_j \times \mathbb{C}^r$ and $\theta'_j : E'|U_j \to U_j \times \mathbb{C}^r$. Set $U_{i,j} = U_i \cap U_j$ and let

$$g_{i,j} : U_{i,j} \to GL_r(\mathbb{C}), \quad g'_{i,j} : U_{i,j} \to GL_r(\mathbb{C})$$

denote the $\mathscr{J}$-holomorphic transition maps of the two bundles, so that

$$\theta_i \circ \theta_j^{-1}(z,v) = \big(z, g_{i,j}(z)v\big), \quad z \in U_{i,j},\ v \in \mathbb{C}^r,$$

and likewise for E'. A complex vector bundle isomorphism $\Phi : E \to E'$ is given by a collection of complex vector bundle isomorphisms $\Phi_j : U_j \times \mathbb{C}^r \to U_j \times \mathbb{C}^r$ of the form

$$\Phi_j(z,v) = \big(z, \phi_j(z)v\big), \quad z \in U_j,\ v \in \mathbb{C}^r,$$

with $\phi_j(z) \in GL_r(\mathbb{C})$ for $z \in U_j$, satisfying the compatibility conditions

$$\phi_i = g'_{i,j}\phi_j g_{i,j}^{-1} = g'_{i,j}\phi_j g_{j,i} \quad \text{on } U_{i,j}. \tag{9.1}$$

Let $P = \delta(E,E') \to Z$ denote the $\mathscr{J}$-holomorphic fibre bundle with fibre $GL_r(\mathbb{C})$ and transition maps (9.1), so a collection of maps $\phi_j : U_j \to GL_r(\mathbb{C})$ satisfying (9.1) is a section of P over Z. Thus, complex vector bundle isomorphisms $E \to E'$ correspond to sections of $P \to Z$, with $\mathscr{J}$-holomorphic isomorphisms corresponding to $\mathscr{J}$-holomorphic sections. This reduces the problem to proving that every continuous section $f : Z \to P$ is homotopic to a $\mathscr{J}$-holomorphic section.

We proceed as in the proof of Theorem 7.2. By Theorem 3.1, for every $t_0 \in T$ and smoothly bounded strongly pseudoconvex domain $\Omega \Subset X$ there are a neighbourhood $T_0 \subset T$ of t_0 and a map $\Phi : T_0 \times \Omega \to T_0 \times X$ of class $\mathscr{C}^{0,\infty}$ such that $\Phi(t,x) = (t, \Phi_t(x))$ and $\Phi_t : \Omega \to \Phi_t(\Omega) \subset X$ is a (J_t, J_{t_0})-biholomorphism for every $t \in T_0$, with $\Phi_{t_0} = \mathrm{Id}_\Omega$. The push-forward bundles

$$\widetilde{E} = \Phi_*(E|(T_0 \times \Omega)), \quad \widetilde{E}' = \Phi_*(E'|(T_0 \times \Omega))$$

are fibrewise J_{t_0}-holomorphic and depend continuously on $t \in T_0$. Choose a pair of strongly J_{t_0}-pseudoconvex domain $\Omega_1 \Subset \Omega_2 \Subset X$ such that

$$T_0 \times \overline{\Omega}_2 \subset \Phi(T_0 \times \Omega) \ \text{ and } \ T_0 \times \overline{\Omega}_1 \subset \Phi^{-1}(T_0 \times \Omega_2).$$

After shrinking T_0 around t_0, the stability theorem of Leiterer [36, Theorem 2.7] gives a family of J_{t_0}-holomorphic vector bundle isomorphisms $\Psi_t : \widetilde{E}_t \xrightarrow{\cong} E_{t_0}$ over Ω_2 (see (7.2)) depending continuously on $t \in T_0$. We get similar isomorphisms $\Psi'_t : \widetilde{E}'_t \xrightarrow{\cong} E'_{t_0}$ for the bundle $\widetilde{E}'$ over $T_0 \times \Omega_2$. The upshot is that the vector bundles $\widetilde{E}|(T_0 \times \Omega_2)$ and $\widetilde{E}'|(T_0 \times \Omega_2)$ are fibrewise J_{t_0}-isomorphic to the trivial (independent of t) extensions of the vector bundles $E_{t_0}|\Omega_2$ and $E'_{t_0}|\Omega_2$, respectively. In this local picture, a topological isomorphism $E \to E'$ is given by a family of topological isomorphisms $E_{t_0}|\Omega_2 \xrightarrow{\cong} E'_{t_0}|\Omega_2$ depending continuously on $t \in T_0$, and a $\mathscr{J}$-holomorphic isomorphism $E \to E'$ is given by a family of J_{t_0}-holomorphic isomorphisms. Such isomorphisms correspond to sections of a J_{t_0}-holomorphic fibre bundle $H \to \Omega_2$ with fibre $GL_r(\mathbb{C}^N)$ defined as above, see (9.1).

By the parametric Oka principle for sections of holomorphic fibre bundles with Oka fibres over Stein manifolds, a family of topological sections of $H \to \Omega_2$ is isomorphic to a family of holomorphic sections, with approximation on compact holomorphically convex subsets of Ω_2. Going back to the original vector bundles E, E' and $P = \delta(E,E')$, we see that any continuous section of P is homotopic over $T_0 \times \Omega_1$ to a $\mathscr{J}$-holomorphic section, with approximation on a $\mathscr{J}$-convex subset where the section is already holomorphic. The globalisation scheme in the proof of Theorem 6.1 then applies and shows that every continuous section of $P \to Z$ is homotopic to a $\mathscr{J}$-holomorphic section. □

10. Open problems

In this final section we collect some open problems for future investigation. The first problem of technical nature is related to the stability of canonical solutions of the $\overline{\partial}$-equation.

Problem 10.1. Does Theorem 3.2 have an analogue with smooth dependence of solutions on the parameter $t \in T$ when T is a smooth manifold and the family of complex structures $\mathscr{J} = \{J_t\}_{t \in T}$ is smooth in $(t, x) \in T \times X$?

We are not aware of results in the literature concerning Problem 10.1, except when X is a surface (see [14, 15]). An affirmative answer would give a similar generalisation of the parametric Hamilton's theorem (see Theorem 3.1), and hence of all our main results. The corresponding analogue of Theorem 6.3 would show that if $\mathscr{J} = \{J_t\}_{t \in T}$ is a smooth tame family of Stein structures then the manifold $Z = T \times X$, with the complex structure J_t on $\{t\} \times X$ for every $t \in T$, is a *Cartan manifold* in the sense of Jurchescu [31, Sect. 6]; see also the discussion and references in [14, Remark 1.2]. Cartan manifolds are analogues of Stein manifolds in the category of smooth CR manifolds with integrable complex tangent subbundle. For real analytic Cartan manifolds with CR codimension one, the function theory and the Oka principle for vector bundles were treated by Mongodi and Tomassini [40, 41].

Our main result, Theorem 6.1, shows that tame families $\mathscr{J}$ of smooth Stein structures on a given manifold X admit many $\mathscr{J}$-holomorphic maps to any Oka manifold. Theorem 6.3 gives a similar result for functions with more general parameter spaces. Which additional properties can these maps have? The following problem is of particular interest; see [12, Theorem 2.4.1] for the summary of the classical results for Stein manifolds and references to the original papers.

Problem 10.2. Assume that T, X, and $\mathscr{J}$ are as in Theorem 6.3. Is there a $\mathscr{J}$-holomorphic map $F : Z = T \times X \to \mathbb{C}^N$ for a suitable $N \in \mathbb{N}$ such that for every $t \in T$ the J_t-holomorphic map $F(t, \cdot\,) : X \to \mathbb{C}^N$ is proper, an immersion, an embedding? In particular, taking $N = 2\dim_{\mathbb{C}} X + 1$, is there an F such that $F(t, \cdot\,) : X \to \mathbb{C}^N$ is a proper J_t-holomorphic embedding for every $t \in T$?

The Oka principle in Theorem 6.1 only pertains to maps to Oka manifolds. In light of the classical results for a single Stein manifold (see [12, Theorem 5.4.4]), the following is a natural question.

Problem 10.3. Let T, X and $\mathscr{J}$ be as in Theorem 6.1, and let $E \to Z = T \times X$ be a topological $\mathscr{J}$-holomorphic fibre bundle with an Oka fibre. Does the Oka principle hold for sections $Z \to E$?

We expect that this holds true, but the proof would require a suitable reworking of all basic tools used in the proof of the Oka principle for a single Stein manifold; see [12, Chap. 5].

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