THE UNIVERSAL FAMILY OF PUNCTURED RIEMANN SURFACES IS STEIN

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ABSTRACT. We show that for any pair of integers $g \geq 0$ and $n \geq 1$ the universal family of n-punctured compact Riemann surfaces of genus g is a Stein manifold. We describe its basic function theoretic properties and pose several challenging questions.

The notion of a Teichmüller space originates in the papers [33, 34, 35] of Oswald Teichmüller, who defined a complex manifold structure on the set of isomorphism classes of marked closed Riemann surfaces of genus g. Ahlfors [2] showed in 1960 that this complex structure can be defined by periods of holomorphic abelian differentials. In [35], Teichmüller also introduced the universal Teichmüller curve – a space V over a Teichmüller space T whose fibre above $t \in T$ is a Riemann surface (M, J_t) representing that point, also called the universal family of Riemann surfaces over T – and showed that it has the structure of a complex manifold. Teichmüller's theory was developed by Ahlfors and Bers [3] and by Grothendieck, who gave a series of lectures in Cartan's seminar 1960–1961; see the discussion and references in [1]. Grothendieck asked whether every finite dimensional Teichmüller space is a Stein manifold [22, p. 14]. An affirmative answer was given by Bers and Ehrenpreis [8, Theorem 2] who showed that any finite dimensional Teichmüller space embeds as a domain of holomorphy in a complex Euclidean space, hence is Stein. (Another proof was given by Wolpert [36]; see also the surveys by Bers [7] and Nag [28].) The Teichmüller space $T(M)$ of a Riemann surface M is finite dimensional if and only if $M = \widehat{M} \setminus \{p_1, \ldots, p_n\}$ is a compact Riemann surface $\widehat{M}$ of some genus $g \geq 0$ with $n \geq 0$ punctures. Such M is said to be of finite conformal type and its Teichmüller space is denoted $T(g, n)$. The universal family $\pi : \widehat{V}(g, n) \to T(g, n)$ is a holomorphic submersion whose fibre over any point $t \in T(g, n)$ is the compact Riemann surface $(\widehat{M}, J_t)$ with the complex structure J_t determined by t, and with n holomorphic sections $s_1, \ldots, s_n : T(g, n) \to \widehat{V}(g, n)$ with pairwise disjoint images representing the punctures. (See Nag [28, pp. 322–323].) The open subset

$$V(g, n) = \widehat{V}(g, n) \setminus \bigcup_{i=1}^{n} s_i(T(g, n)) \tag{1}$$

of $\widehat{V}(g, n)$ is the universal family of n-punctured compact Riemann surfaces of genus g. If $2g + n \geq 3$ then the Teichmüller family $\pi : V(g, n) \to T(g, n)$ is the universal object in the complex analytic category of topologically marked holomorphically varying families of n-punctured genus g Riemann surfaces (see [28, Theorem 5.4.3]). In this paper we prove the following result.

Theorem 1. *If $n \geq 1$ then the universal family $V(g, n)$ is a Stein manifold.*

This is a special case of Theorems 8 and 9 given in the sequel. The theorem is trivial if $g = 0$ and $n \in \{1, 2, 3\}$ since $T(0, n)$ is then a singleton and $V(0, n)$ equals $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\mathbb{C} \setminus \{0, 1\}$, respectively. If $2g + n \geq 3$ then $T(g, n)$ is biholomorphic to a bounded topologically contractible Stein domain in $\mathbb{C}^{3g-3+n}$ [28, p. 161], but it cannot be holomorphically realised as a convex domain in $\mathbb{C}^{3g-3+n}$ if $g \geq 2$ (see Marković [27]). The n holomorphic sections $s_1, \ldots, s_n$ of

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the submersion $\pi : \widehat{V}(g,n) \to T(g,n)$ in (1), corresponding to the punctures of $V(g,n)$, are called *canonical sections*. The problem of describing the global holomorphic sections of this fibration was first considered by Hubbard, who showed in [25] that $V(g,0) \to T(g,0)$ has no holomorphic sections if $g \geq 3$ and six sections if $g = 2$. See also [26]. Earle and Kra proved in [10, p. 50] that for $n \geq 1$, $\pi : \widehat{V}(g,n) \to T(g,n)$ has exactly n sections if $g \geq 3$ and exactly $2n + 6$ sections if $g = 2$.

Theorem 1 has interesting consequences for function theory and complex geometry. By classical results of Remmert, Bishop, and Narasiman (see [15, Theorem 2.4.1] and the references therein), the Stein manifold $V(g,n)$ for $n \geq 1$ admits a proper holomorphic embedding in $\mathbb{C}^N$ with $N = 2\dim V(g,n) + 1$, which equals $6g - 6 + 2n + 1$ if $2g + n \geq 3$. When $\dim V(g,n) \geq 2$, it also embeds properly holomorphically in $\mathbb{C}^N$ with $N = \left[\frac{3\dim V(g,n)}{2}\right] + 1$ (see Eliashberg and Gromov [11] and Schürmann [31], or the exposition in [15, Secs. 9.3–9.4]). Since every fibre of $V(g,n)$ is biholomorphic to a closed affine algebraic curve in $\mathbb{C}^3$, the following question is natural.

Problem 2. Does $V(g,n)$ for $n \geq 1$ admit a proper holomorphic embedding in some $\mathbb{C}^N$ which is algebraic on every fibre?

Except in the trivial cases, $T(g,n)$ is not affine algebraic, and it is not reasonable to expect that $V(g,n)$ is affine algebraic.

Since $T(g,n)$ is contractible, the holomorphic submersion $\pi : V(g,n) \to T(g,n)$ is smoothly trivial and the inclusion of any fibre of π in $V(g,n)$ is a homotopy equivalence. It follows that for any manifold Y, the restriction of a continuous map $V(g,n) \to Y$ to any fibre of $V(g,n)$ lies in the same homotopy class of maps $M \to Y$, where M is the underlying n-punctured smooth compact surface of genus g. The following is an immediate corollary to this observation, Theorem 1, and the main result of Oka theory [15, Theorem 5.4.4]. For Oka manifolds, see [15, Chap. 5] and the surveys [13, 12, 17].

Corollary 3. *Let $\pi : V(g,n) \to T(g,n)$ be as above, $n \geq 1$, and let Y be an Oka manifold. There is a holomorphic map $V(g,n) \to Y$ in every homotopy class. Furthermore, a holomorphic map $M_t \to Y$, $t \in T(g,n)$, from any fibre of $V(g,n)$ extends to a holomorphic map $V(g,n) \to Y$. More generally, given a closed complex subvariety T' of $T(g,n)$, every continuous map $F_0 : V(g,n) \to Y$ which is holomorphic on $\pi^{-1}(T')$ is homotopic to a holomorphic map $F : V(g,n) \to Y$ by a homotopy which is fixed on $\pi^{-1}(T')$.*

If $n \geq 1$ and $(g,n) \neq (0,1)$ then $V(g,n)$ is not simply connected, and its homotopy type is that of a finite bouquet of circles. In this case, homotopically nontrivial maps $V(g,n) \to Y$ exist whenever the manifold Y is not simply connected. This gives the following corollary. The last statement follows by taking the Oka manifold $Y = \mathbb{C}^*$. The result obviously holds for $g = 0,\ n = 1$ since $V(0,1) = \mathbb{C}$.

Corollary 4. *If $n \geq 1$ and Y is an Oka manifold which is not simply connected, there is a holomorphic map $V(g,n) \to Y$ which is nonconstant on every fibre. In particular, $V(g,n)$ admits a nowhere vanishing holomorphic function which is nonconstant on every fibre.*

Another way to obtain fibrewise nonconstant holomorphic maps from $V(g,n)$ to an Oka manifold is to inductively use the Oka property with approximation on compact holomorphically convex subsets of the Stein manifold $V(g,n)$; see [15, Theorem 5.4.4]. The Oka principle can be used to obtain many further properties of holomorphic universal families $V(g,n) \to Y$ in any Oka manifold.

The fact that $V(g,n)$ for $n \geq 1$ is homotopy equivalent to a bouquet of circles implies that every complex vector bundle on $V(g,n)$ is topologically trivial. Since $V(g,n)$ is Stein, the Oka–Grauert principle ([19] or [15, Theorem 3.2.1]) implies the following.

Proposition 5. *Every holomorphic vector bundle on $V(g,n)$ for $n \geq 1$ is holomorphically trivial.*

Corollary 6. *Assume that $g \geq 0$ and $n \geq 1$.*

(a) *There exists a nowhere vanishing holomorphic vector field ξ on $V(g,n)$ which is tangent to the fibres of the projection $\pi : V(g,n) \to T(g,n)$, that is, $d\pi(\xi) = 0$.*
(b) *With ξ as in (a), there exists a holomorphic 1-form θ on $V(g,n)$ satisfying $\langle \theta, \xi \rangle = 1$. In particular, θ is nowhere vanishing on the tangent bundle to any fibre of π.*

Proof. Part (a) follows by applying Proposition 5 to the holomorphic line bundle $\ker d\pi \to V(g,n)$, the vertical tangent bundle of the holomorphic submersion $\pi : V(g,n) \to T(g,n)$. To see (b), consider the following short exact sequence of vector bundles over $V(g,n)$:

$$0 \longrightarrow \ker d\pi \hookrightarrow TV(g,n) \xrightarrow{\alpha} H := TV(g,n)/\ker \pi \longrightarrow 0.$$

Here, $TV(g,n)$ denote the tangent bundle of $V(g,n)$. By Cartan's Theorem B the sequence splits, i.e. there is a holomorphic vector bundle injection $\sigma : H \hookrightarrow TV(g,n)$ such that $\alpha \circ \sigma = \mathrm{Id}_H$. Hence, $TV(g,n) = \ker d\pi \oplus \sigma(H) = \mathbb{C}\xi \oplus \sigma(H)$, where ξ is as in part (a). The unique holomorphic 1-form θ on $V(g,n)$ satisfying $\langle \theta, \xi \rangle = 1$ and $\xi = 0$ on $\sigma(H)$ clearly satisfies part (b). □

By the Gunning–Narasimhan theorem [23], every open Riemann surface M admits a holomorphic immersion $f : M \to \mathbb{C}$. In view of Corollary 6 (b), the following is a natural question.

Problem 7. Let $n \geq 1$. Does there exist a holomorphic function $f : V(g,n) \to \mathbb{C}$ whose restriction to any fibre of $\pi : V(g,n) \to T(g,n)$ is an immersion?

By [14, Theorem 1] there exists a holomorphic function $f : V(g,n) \to \mathbb{C}$ without critical points. The problem is to find a function f such that $\ker df_z$ is transverse to $\ker d\pi_z$ at every point $z \in V(g,n)$. Note that [16, Corollary 8.3] gives a smooth function $f : V(g,n) \to \mathbb{C}$ whose restriction to any fibre of $\pi : V(g,n) \to T(g,n)$ is a holomorphic immersion. Problem 7 is related to the question whether a holomorphic 1-form θ in Corollary 6 (b) can be made exact on every fibre of π by multiplying it with a suitably chosen nowhere vanishing holomorphic function on $V(g,n)$. However, this is not the only problem. Since the submersion $\pi : V(g,n) \to T(g,n)$ does not admit a holomorphic section when $g \geq 3$, there is no natural way of choosing the initial point for computing the fibrewise integrals of θ, which would give a holomorphic family of immersions on the fibres.

In [16] the Oka principle was established for families of maps from very general families of open Riemann surfaces $\{(M, J_t)\}_{t \in T}$ to any Oka manifold Y, where M is a smooth open surface and J_t are complex structures of some local Hölder class on M depending continuously or smoothly on the parameter $t \in T$ in a suitable topological space. The Riemann surfaces in such families need not belong to the same Teichmüller space. For example, a punctured Riemann surface can be a member of a family in which the punctures develop into boundary curves and vice versa. The J_t-holomorphic maps $F_t : (M, J_t) \to Y$ $(t \in T)$ furnished by [16, Theorem 1.6] are merely continuous or smooth in the parameter. In [18] these results were extended to maps from tame families of Stein manifolds of arbitrary dimension to Oka manifolds. Here, a continuous family $\{J_t\}_{t \in T}$ of Stein structures on a smooth manifold X said to be tame if the holomorphically convex hulls of any compact subset $K \subset X$ are upper semicontinuous with respect to the parameter $t \in T$. This is always the case in a family of open Riemann surfaces but it fails general in higher dimensions, even on affine spaces.

Another interesting question is whether the Riemann surfaces M_t in the universal Teichmüller family $V(g,n)$, $n \geq 1$, admit a representation as a family of conformal minimal surfaces in $\mathbb{R}^k$, $k \geq 3$, whose $(1,0)$-derivatives depends holomorphically on $t \in T(g,n)$. For background, see [29] and [4]. By [16, Corollary 8.6] the answer is affirmative with continuous or smooth dependence on parameter. It remains an open problem whether minimal surfaces in such families can be chosen to be complete and with finite total curvature. Each single surface in the family can be made such by [6].

In view of the description of the Teichmüller submersion $\pi : V(g,n) \to T(g,n)$, Theorem 1 is an immediate consequence of the following result with arbitrary Stein manifold as the base. See also Theorem 9 for a more general result.

Theorem 8. *Assume that X is a Stein manifold, Z is a connected complex manifold with $\dim Z = \dim X + 1$, $\pi : Z \to X$ is a surjective proper holomorphic submersion, and $s_1, \ldots, s_n : X \to Z$ are holomorphic sections with pairwise disjoint images for some $n \geq 1$. Then, the domain $\Omega = Z \setminus \bigcup_{i=1}^n s_i(X)$ is Stein.*

The conclusion fails if the fibres of π have complex dimension > 1 or the sections s_i are not holomorphic. In such case, the domain Ω in the theorem fails to be locally pseudoconvex at some boundary point $s_i(x)$, $x \in X$. Note that Stein complements of smooth complex hypersurfaces in compact Kähler manifolds have recently been studied by Höring and Peternell [24] where the reader can find references to earlier works. In our case, Z is not compact unless X is a point.

Proof of Theorem 8. Let $\pi : Z \to X$ be as in the theorem. Note that every fibre $Z_x = \pi^{-1}(x)$, $x \in X$, is a compact Riemann surface, and the fibres are diffeomorphic but not necessarily biholomorphic to each other. Hence, $\{Z_x\}_{x \in X}$ is a holomorphic family of compact Riemann surfaces and $\Omega_x = Z_x \setminus \bigcup_{i=1}^n s_i(x)$ $(x \in X)$ is a holomorphic family of n-punctured Riemann surfaces. Each $H_i = s_i(X)$ is a closed complex hypersurface in Z whose ideal sheaf is a principal, that is, it is locally near each point of H_i generated by a single holomorphic function.

Recall the following result (see Grauert and Remmert [21, Theorem 5, p. 129]): If Z is a Stein space and H is a closed complex analytic hypersurface in Z (of pure codimension one) whose ideal sheaf is a principal ideal sheaf, then $Z \setminus H$ is also Stein. If Z is nonsingular (a complex manifold) then the ideal sheaf of any closed complex hypersurface in Z is a principal ideal sheaf (see [21, Chap. A.3.5]). Hence, is suffices to prove the theorem in the case $n = 1$, that is, to show that the complement $Z \setminus s(X)$ of a single holomorphic section $s : X \to Z$ is a Stein manifold.

By a theorem of Siu [32], the Stein hypersurface $H = s(X)$ has a basis of open Stein neighbourhoods U in Z. Since $U \setminus H$ is a Stein manifold by the aforementioned theorem [21, Theorem 5, p. 129], it admits a strongly plurisubharmonic exhaustion function $\phi : U \setminus H \to \mathbb{R}_+$. To prove the theorem, we shall construct a strongly plurisubharmonic exhaustion function $Z \setminus H \to \mathbb{R}_+$; a theorem of Grauert [20] will then imply that $Z \setminus H$ is Stein.

Fix a point $x_0 \in X$ and set $z_0 = s(x_0) \in H \subset Z$. Since $\pi : Z \to X$ is a holomorphic submersion with compact one-dimensional fibres, it is a smooth fibre bundle whose fibre M is a compact smooth surface. In particular, there is a neighbourhood $X_0 \subset X$ of x_0 such that the restricted bundle $Z|X_0 = \pi^{-1}(X_0) \to X_0$ can be smoothly identified with the trivial bundle $X_0 \times M \to X_0$. In this identification, $z_0 = (x_0, p_0)$ with $p_0 \in M$. Since ϕ tends to $+\infty$ along H, there are small smoothly bounded open discs $D \Subset D' \Subset M$ with $p_0 \in D$ such that

$$\inf_{p \in bD} \phi(x_0, p) > \max_{p \in bD'} \phi(x_0, p). \tag{2}$$

The set $O = M \setminus D$ is a compact bordered Riemann surface with smooth boundary $bO = bD$, endowed with the complex structure inherited by the identification $M \cong Z_{x_0} = \pi^{-1}(x_0)$. Note that bD' is contained in the interior of O. It follows from (2) and standard results that there is a smooth strongly subharmonic function $u_0 : O \to \mathbb{R}_+$ such that

$$u_0 < \phi(x_0, \cdot\,) \text{ on } bO = bD \text{ and } u_0 > \phi(x_0, \cdot\,) \text{ on } bD'.$$

Shrinking the neighbourhood $X_0 \subset X$ of x_0 if necessary, the following conditions hold for every $x \in X_0$, where we use the smooth fibre bundle isomorphism $Z|X_0 \cong X_0 \times M$:

(a) $s(x) \in D$,
(b) the function $u(x,\cdot\,) = u_0$ is strongly subharmonic on O in the complex structure on $Z_x \cong M$,
(c) $u(x,\cdot\,) < \phi(x,\cdot\,)$ on $bO = bD$, and
(d) $u(x,\cdot\,) > \phi(x,\cdot\,)$ on bD'.

Condition (b) holds since being strongly subharmonic on a compact subset is a stable property under small smooth deformations of the complex structure. We define a function $\rho_0 : (X_0 \times M)\setminus H \to \mathbb{R}_+$ by taking for every $x \in X_0$:

$$\rho_0(x,p) = \begin{cases} \phi(x,p), & p \in D \setminus \{s(x)\}; \\ \max\{\phi(x,p), u(x,p)\}, & p \in D' \setminus D; \\ u(x,p), & p \in M \setminus D'. \end{cases}$$

Note that ρ_0 is well defined, piecewise smooth, strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_0)$, and it agrees with ϕ on $(X_0 \times D)\setminus H$. By using the regularised maximum in the definition of ρ_0 (see [15, Eq. (3.1), p. 69]), we may assume that ρ_0 is smooth and enjoys the other stated properties.

This construction gives an open locally finite cover $\{X_j\}_{j=1}^\infty$ of X with smooth fibre bundle trivialisations $Z|X_j \cong X_j \times M$, discs $D_j \subset M$ such that $s(x) \in D_j$ for all $x \in X_j$, and smooth functions $\rho_j : (Z|X_j)\setminus H = \pi^{-1}(X_j)\setminus H \to \mathbb{R}_+$ such that ρ_j is strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_j)$ and it agrees with ϕ on $(X_j \times D_j)\setminus H$. Let $\{\chi_j\}_j$ be a smooth partition of unity on X with compact supports $\mathrm{supp}(\chi_j) \subset X_j$ for each j. Set

$$\rho = \sum_{j=1}^\infty \chi_j \rho_j : Z \setminus H \to \mathbb{R}_+.$$

By the construction, the restriction of ρ to each fibre $Z_x \setminus \{s(x)\}$ $(x \in X)$ is strongly subharmonic, and there is an open neighbourhood $U_0 \subset U \subset Z$ of H such that $\rho = \phi$ holds on $U_0 \setminus H$. In particular, ρ is strongly plurisubharmonic on $U_0 \setminus H$.

Note that for every compact set $K \subset X$, the set $\pi^{-1}(K)\setminus U_0 \subset Z \setminus H$ is compact. Hence, by choosing a strongly plurisubharmonic exhaustion function $\tau : X \to \mathbb{R}_+$ whose Levi form $dd^c\tau$ grows fast enough, we can ensure that $\rho + \tau\circ\pi : Z \setminus H \to \mathbb{R}_+$ is a strongly plurisubharmonic exhaustion function. Indeed, denoting by J the almost complex structure operator on Z and by d^c the conjugate differential defined by $(d^c\rho)(z,\xi) = -d\rho(z, J\xi)$ for $z \in Z$ and $\xi \in T_zZ$, ρ is strongly plurisubharmonic at $z \in Z$ if and only if $(dd^c\rho)(z, \xi \wedge J\xi) > 0$ for every tangent vector $0 \neq \xi \in T_zZ$. (Up to a positive factor this equals the Laplace of ρ on the 2-plane $\mathrm{span}(\xi, J\xi) \subset T_zZ$.) Since the function ρ constructed above is strongly subharmonic on every fibre $Z_x \setminus \{s(x)\}$, $x \in X$, we have that

$$(dd^c\rho)(z, \xi \wedge J\xi) > 0 \ \text{ if } z \in Z \setminus H \text{ and } 0 \neq \xi \in \ker d\pi_z.$$

Hence, the eigenvectors of the Levi form $(dd^c\rho)(z,\cdot\,)$ associated to non-positive eigenvalues lie in a closed cone $C_z \subset T_zZ$ which intersects $\ker d\pi_z$ only in the origin. It follows that if $\tau : X \to \mathbb{R}$ is such that $dd^c\tau > 0$ is sufficiently big on T_xX, $x = \pi(z)$, then $dd^c\rho + dd^c(\tau\circ\pi) > 0$ on T_zZ. Furthermore, the estimates are uniform on the compact set $\pi^{-1}(K)\setminus U_0$, where $U_0 \subset Z$ is a neighbourhood of H as above such that $dd^c\rho > 0$ on $U_0 \setminus H$. To see that τ can be chosen such that $dd^c\tau$ grows as fast as desired, note that if $h : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^2$ function then for each point $x \in X$ and tangent vector $\xi \in T_xX$ we have that

$$dd^c(h\circ\tau)(x, \xi\wedge J\xi) = h'(\tau(x))\,(dd^c\tau)(x,\xi\wedge J\xi) + h''(\tau(x))\left(|d\tau(x,\xi)|^2 + |d\tau(x, J\xi)|^2\right).$$

Hence, if τ is a strongly plurisubharmonic exhaustion function on X and the function $h : \mathbb{R} \to \mathbb{R}$ is chosen such that $h'' \geq 0$ and h' grows sufficiently fast, then $dd^c(h\circ u)$ also grows as fast as desired.

This completes the proof of Theorem 8, and it also proves Theorem 1. □

A minor modification of the proof of Theorem 8 gives the following more general result.

Theorem 9. *Assume that X is a Stein manifold, Z is a connected complex manifold with $\dim Z = \dim X + 1$, $\pi : Z \to X$ is a surjective proper holomorphic submersion, and H is a closed complex subvariety of Z of pure codimension one which does not contain any fibre of π. Then, the domain $\Omega = Z \setminus H$ is Stein.*

Proof. The restricted projection $\pi : H \to X$ is a proper holomorphic map whose fibres are compact Riemann surfaces. Since H is of pure codimension one and does not contain any fibre of π, $\pi : H \to X$ is a finite holomorphic map (a branched holomorphic cover). The proper mapping theorem of Remmert [30] (see also Chirka [9, p. 29]) implies that $\pi(H)$ is a closed complex subvariety of X of pure dimension n, hence $\pi(H) = X$ since X is connected, and H is Stein by [21, Theorem 1 (d), p. 125]. By Siu's theorem [32], H admits a basis of open Stein neighbourhoods $U \subset Z$. By the argument in the proof of Theorem 8, the ideal sheaf of H is a principal ideal sheaf, so $U \setminus H$ is Stein for any Stein neighbourhood U of H by [21, Theorem 5, p. 129]. The proof can now be completed by a similar argument as in the proof of Theorem 8, and we leave further details to the reader. □

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