

FROM STEIN MANIFOLDS TO OKA MANIFOLDS: THE H-PRINCIPLE IN COMPLEX ANALYSIS

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ABSTRACT. This introduction to the homotopy principle in complex analysis and geometry, better known as the Oka theory, is aimed at wide mathematical audience. After a brief historical survey of the h-principle in smooth analysis and geometry, I present the key notions of Oka manifolds and Oka maps, which developed from the Oka–Grauert principle and Gromov’s theory of elliptic complex manifolds and elliptic holomorphic submersions. I discuss recent and ongoing developments, open problems, and mention some applications. The paper also includes a brief survey of the recently developed h-principles in the classical theory of minimal surfaces.

1. INTRODUCTION.

The homotopy principle, or the h-principle for short, is said to hold in an analytic problem if a solution exists in the absence of topological or homotopy theoretic obstructions. Most instances concern solutions of underdetermined partial differential equations, PDEs, and more generally of partial differential relations, PDRs. In the holomorphic case, we can talk of holomorphic partial differential relations, HPDRs. A PDE or (H)PDR is said to satisfy the h-principle if any formal (non-holonomic) solution can be deformed through formal solutions to a genuine solution. Thus in the presence of the h-principle, a differential topological or a complex analytic problem reduces to a homotopy theoretic problem. We speak of the basic h-principle, pertaining to individual solutions, and of the parametric h-principle. The latter involves a pair of parameter spaces $Q \subset P$, which are typically compact Hausdorff spaces, and a continuous family of formal solutions parametrised by points $p \in P$, which are genuine solutions for $p \in Q$. The parametric h-principle is said to hold if any such family can be deformed to a family of genuine solutions while preserving the solutions for values $p \in Q$. This implies that the inclusion of the space of genuine solutions in the space of formal solutions is a weak homotopy equivalence, that is, it induces a bijection of path components and an isomorphism of k -th homotopy groups of the two spaces for every $k \in \mathbb{N} = \{1, 2, \dots\}$. Sometimes there is even a genuine homotopy equivalence between the two spaces. On the other hand, the failure of the h-principle means that the objects being studied have nontrivial geometry which cannot be reduced to purely topological considerations.

The first known h-principle is the Whitney–Graustein theorem [128] which says that two immersions of a circle in the plane are regularly homotopic (connected by a path of immersions) if and only if they have the same winding number. Two decades later, Smale [122] classified regular homotopy classes of immersions of the k -sphere S^k in the Euclidean space $\mathbb{R}^n$ for $1 \leq k < n$ by elements of the k -th homotopy group $\pi_k(V_k(\mathbb{R}^n))$ of the Stiefel manifold of k -frames in $\mathbb{R}^n$. It turns out that for $n \in \{2, 6\}$ any two immersions $S^n \rightarrow \mathbb{R}^{n+1}$ are regularly homotopic, so one can turn these spheres inside-out through immersions in $\mathbb{R}^{n+1}$. Hirsch [84] proved the h-principle for immersions of

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smooth k -manifolds to $\mathbb{R}^n$ for $k < n$, and also for $k = n$ in the case of open manifolds. Phillips developed the h-principle for smooth submersions and foliations [116, 117, 118].

In another direction, Nash [112, 113] and Kuiper [97] proved that a smooth compact Riemannian manifold (M, g) of dimension k which admits an embedding in $\mathbb{R}^n$ for some $n > k$ also admits a $\mathcal{C}^1$ isometric embedding in $\mathbb{R}^n$ with its Euclidean metric. In fact, every short embedding can be uniformly approximated by isometric embeddings. This is false for $\mathcal{C}^2$ isometric embeddings due to curvature obstructions in low codimension.

A major conceptual development of the theory was made by Mikhail Gromov [79, 77], and the term h-principle entered into general use following his work. Gromov formulated the h-principle in very general terms and developed important new techniques for solving it, some of them together with Yakov Eliashberg. We refer to the surveys [33, 77, 123] for these developments.

The h-principle in complex analysis. The first instance of the h-principle in complex analysis is the result of Kiyoshi Oka [114] from 1939, saying that on a domain of holomorphy in $\mathbb{C}^n$ every topological complex line bundle admits a compatible structure of a holomorphic line bundle, and any two holomorphic line bundles which are topologically isomorphic are also holomorphically isomorphic. A far reaching generalisation was obtained by Hans Grauert [70, 71, 72] in 1958. Recall that a *Stein manifold* is a complex manifold which admits many global holomorphic functions; see Sect. 2. A Stein space is a complex space (with singularities) which enjoys the analogous function-theoretic properties as Stein manifolds. A domain in $\mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy. A Riemann surface is Stein if and only if it is not compact. Grauert proved that complex vector bundles and, more generally, principal fibre bundles with complex Lie group fibres on Stein spaces have the same classification in the topological and the holomorphic categories. See also the expositions by Cartan [28] and Henkin and Leiterer [83, 109], and the subsequent works of Forster and Ramspott [36, 37, 38, 39]. These results led to the heuristic *Oka principle*, a term coined by J-P. Serre stating that *analytic problems on Stein spaces which can be cohomologically formulated have only topological obstructions*.

A homotopy-theoretic viewpoint on Oka theory was initiated by Gromov [77, 78] in the late 1980s. Modern Oka theory focuses on properties of a complex manifold Y which imply that any continuous map $X \rightarrow Y$ from a Stein space X is homotopic to a holomorphic map; the same problem is considered for sections of holomorphic maps onto Stein spaces. (All Stein spaces will be assumed reduced.) Adding the approximation and interpolation conditions modelled on properties of holomorphic functions on Stein spaces, one obtains several Oka properties (see Def. 3.1) which a given complex manifold may or may not have. One of Gromov's main results in [78] is that a geometric condition called *ellipticity* — the existence of a dominating holomorphic spray on the manifold, see Def. 4.1 — implies all these Oka properties. The analogous result holds for sections of elliptic submersions onto Stein spaces; see [78, 63] and [51, Theorem 6.2.2]. The Oka–Grauert theory fits in this framework by considering maps to classifying spaces, which are complex homogeneous and hence elliptic. However, Gromov's theorem, with detailed presentations in [62, 63, 50] and [51, Chaps. 5–6], applies to a considerably wider class of problems.

Subsequent research was motivated by the question whether ellipticity is also a necessary condition for the Oka principle, in finding new examples of elliptic manifolds and submersions, and in attempts to unify the theory by finding simpler conditions characterising the Oka property. The first question remained open until recently and we shall say more about it Sect. 4, where we also mention examples of elliptic manifolds. The last question was resolved by the introduction of the following Runge-type approximation property in [48].

Definition 1.1. A complex manifold Y enjoys the *Convex Approximation Property*, *CAP*, if every holomorphic map from an open neighborhood of a compact convex set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, to Y can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \rightarrow Y$.

The following is a cumulative result of [46, 48, 49, 50]; see also [51, Theorem 5.4.4 and Prop. 5.15.1].

Theorem 1.2. A complex manifold Y which enjoys *CAP* has all Oka properties in Def. 3.1 for maps $X \rightarrow Y$ from any Stein space X . Furthermore, all these Oka properties are pairwise equivalent.

A complex manifold Y satisfying these equivalent conditions is called an *Oka manifold* [49, 105]. This gives an affirmative answer to Gromov's problem [78, 3.4 (D)], asking whether the Oka property can be characterised by a simple Runge-type approximation property. Every elliptic manifold in the sense of Gromov (see Def. 4.1) enjoys *CAP* and hence is an Oka manifold; see Theorem 4.3. This extends Grauert's result that every complex homogeneous manifold is Oka. Essentially the same proof shows the following; see [51, Theorem 5.4.4].

Theorem 1.3. If $h : Z \rightarrow X$ is a stratified holomorphic fibre bundle over a Stein space X whose fibres are Oka manifolds, then sections $X \rightarrow Z$ of h satisfy all forms of the Oka principle.

It is also important to consider the Oka principle for lifting holomorphic maps in the following diagram.

$$\begin{array}{ccc} & & Y \\ & \nearrow F & \downarrow h \\ X & \xrightarrow{f} & Z \end{array}$$

A holomorphic map $h : Y \rightarrow Z$ of complex manifolds is said to be an *Oka map* if it is a topological (Serre) fibration and it satisfies the following Oka property.

The Oka property for liftings: Given a holomorphic map $f : X \rightarrow Z$ from a Stein space X , any continuous lifting $X \rightarrow Y$ of f is homotopic to a holomorphic lifting $F : X \rightarrow Y$, with approximation and interpolation conditions in Def. 3.1, and the corresponding result holds in the parametric setting.

We refer to [51, Definition 7.4.1] for the precise definition. A holomorphic fibre bundle with Oka fibre is an Oka map (see [51, Corollary 7.4.8]). In particular, a holomorphic covering map is an Oka map. The importance of Oka maps lies in the following result; see [40, Theorem 3.15].

Theorem 1.4. If $Y \rightarrow Z$ is an Oka map of connected complex manifolds, then Y is an Oka manifold if and only if Z is an Oka manifold.

Lárusson [105, 106] constructed a model category in which Stein manifolds are cofibrant, Stein inclusions are cofibrations, Oka manifolds are fibrant, and Oka maps are fibrations. It is thus fair to say that Stein manifolds are the natural sources of holomorphic maps while Oka manifolds are their natural targets.

Recently, Kusakabe showed in [99, Theorem 1.3] that the following property also characterises Oka manifolds.

Definition 1.5. A complex manifold Y enjoys *Convex Relative Ellipticity*, *CRE*, if for every compact convex set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, and holomorphic map $f : U' \rightarrow Y$ from an open neighbourhood of K there exist a neighbourhood $U \subset U'$ of K and a holomorphic map $F : U \times \mathbb{C}^N \rightarrow Y$ for some $N \in \mathbb{N}$ such that for every $z \in U$ we have $F(z, 0) = f(z)$ and the differential $dF(z, \cdot)|_0 : T_0\mathbb{C}^N \rightarrow T_{f(z)}Y$ at $0 \in \mathbb{C}^N$ is surjective.

A map F as in the above definition is called a *dominating holomorphic spray with the core* $F(\cdot, 0) = f$. This condition, for maps from any Stein manifold in place of a convex domain $U \subset \mathbb{C}^n$, was introduced by Gromov [78] as condition Ell_1 (see also [40, Definition 3.1]). It has also been called *relative ellipticity* in the literature to emphasise that it concerns sprays of holomorphic maps to the given manifold, as opposed to sprays on the manifold itself. It is easily seen that every holomorphic map $f : U' \rightarrow Y$ in Def. 1.5 admits a local dominating spray $F : U \times \mathbb{B} \rightarrow Y$ with the core $f|_U$, where $K \subset U \subseteq U'$, U is convex, and $\mathbb{B}$ is a ball in some $\mathbb{C}^N$. The main point is to find a global spray $U \times \mathbb{C}^N \rightarrow Y$ which agrees with F to the second order along $U \times \{0\}$. This shows that relative ellipticity, and hence CRE, is implied by the basic Oka property with jet interpolation (BOPJI); see Def. 3.1 (iv'). By definition, every Oka manifold satisfies BOPJI and hence also CRE. To complete the picture, Kusakabe proved in [99] that CRE implies CAP. (An exposition of his proof can also be found in [40, Theorem 3.3].) Summarising, the following properties of a complex manifold are equivalent.

Theorem 1.6. $\text{OKA} \iff \text{CAP} \iff \text{CRE}$.

The characterisation of Oka manifolds by CRE is sometimes easier to apply than CAP. By using it, Kusakabe showed in [99, Theorem 1.4] that the Oka property is Zariski local in the following sense.

Theorem 1.7. *If every point in a complex manifold Y admits a Zariski open* Oka neighbourhood, then Y is an Oka manifold.*

This result has been an important source of new examples of Oka manifolds in recent times. In [100], Kusakabe also characterised the Oka property of sections of holomorphic submersions by a fibred version of CRE.

Oka manifolds have a number of special analytic properties. It is immediate that an Oka manifold Y is dominable by the Euclidean space $\mathbb{C}^n$ with $n = \dim Y$ at every point $y \in Y$, in the sense that there exists a holomorphic map $f : \mathbb{C}^n \rightarrow Y$ with $f(0) = y$ such that $df_0 : T_0\mathbb{C}^n \rightarrow T_yY$ is an isomorphism. Conversely, dominability at most points implies the Oka property for maps from open Riemann surfaces (see Theorem 3.12), but it is not known whether it implies the full Oka property. With some more work, one can see that a connected Oka manifold Y admits a surjective holomorphic map $f : \mathbb{C}^n \rightarrow Y$, $n = \dim Y$, such that for every $y \in Y$ there is a point $z \in f^{-1}(y) \in \mathbb{C}^n$ such that $df_z : T_z\mathbb{C}^n \rightarrow T_yY$ is an isomorphism (see [52, Theorem 1.1]). It follows that a connected Oka manifold does not admit any nonconstant bounded plurisubharmonic function. Furthermore, all pseudometrics on complex manifolds which are decreasing under holomorphic maps and vanish on Euclidean spaces (such as the Kobayashi metric, the Caratheodory metric, the Eisenmann metrics, the Sibony metric, to name some of the best known ones) also vanish on any Oka manifold. Since a compact complex manifold of general type (i.e. of maximal Kodaira dimension equal to its complex dimension) is not dominable by a Euclidean space (see Kobayashi and Ochiai [96]), no such manifold is Oka. There are many examples of compact Oka manifolds among Fano manifolds (Kodaira dimension $-\infty$), although it is not known whether every Fano manifold is Oka. We refer to [51, Sect. 7.3] for a review of known Oka manifolds among compact complex surfaces. For Riemann surfaces there is a precise dichotomy — either the surface is Oka, and then it is one of the surfaces $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or a torus, or it is Kobayashi hyperbolic and hence a quotient of the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Elliptic manifolds (see Def. 4.1) remain an important source of examples of Oka manifolds, especially among the algebraic ones. The theory of algebraically elliptic manifolds is an active area of contemporary research; see Sect. 4. We have recently shown with Lárusson [60] that every projective Oka manifold is elliptic; see Theorem 4.5. This gives an affirmative answer to a conjecture of Gromov [78, 3.2.A']. On the other hand, due to Kusakabe [102] we now have many examples of noncompact

*A domain in a complex manifold Y is Zariski open if its complement is a closed complex subvariety of Y .

Oka manifolds which are not elliptic; see Corollary 4.4. It remains an open question whether there exist compact nonprojective Oka manifolds which fail to be elliptic.

There are several open problems concerning the relationship between the Oka property and topological, analytical, and metrical properties of complex manifolds. A Stein manifold X has the homotopy type of a CW complex of dimension at most $\dim_{\mathbb{C}} X = \frac{1}{2} \dim_{\mathbb{R}} X$. Conversely, given a smooth almost complex manifold (X, J_0) of real dimension $2n \neq 4$ with the homotopy type of a CW complex of dimension $\leq n$, J_0 is homotopic to a Stein structure J on X ; see Eliashberg [31]. The analogous result holds if $\dim_{\mathbb{R}} X = 4$ after a suitable change of the $\mathcal{C}^\infty$ structure on X ; see Gompf [68, 69]. Furthermore, every continuous map $f : X \rightarrow Y$ from a Stein manifold X to an arbitrary complex manifold Y is homotopic to a holomorphic map with respect to a Stein structure J on X which is homotopic to the original one (and up to a change of the $\mathcal{C}^\infty$ structure on X if $\dim_{\mathbb{R}} X = 4$); see [66, 65]. This is called the *soft Oka principle*. See also the monograph by Jöricke [91] which studies this phenomenon in the context of braids. On the other hand, almost nothing is known concerning the homotopy types of Oka manifolds. In particular, we do not know which groups arise as fundamental groups of Oka manifolds. Another intriguing question concerns the geometric shape of Oka domains in complex manifolds; see Problem 3.10.

A challenging problem is to understand the relationship between the Oka property and metrical properties of complete hermitian and Kähler manifolds. Since negativity of the holomorphic sectional curvature is related to Kobayashi hyperbolicity (see [74, 76, 23]), which is the very opposite of the Oka property, one might expect that positivity together with completeness is related to the Oka property. An important result in this direction is given by Mok's solution of the generalised Frankel conjecture [111], which implies that every compact Kähler manifold of nonnegative holomorphic bisectional curvature is an Oka manifold [40, Theorem 11.4]. Our recent result with Kusakabe [55] on the Oka property of disc tubes in semipositive ample line bundles on flag manifolds (see Theorem 3.11) also contributes to the heuristic principle that metric positivity is related to the Oka property.

Another problem is the relationship between Oka manifolds and special manifolds in the sense of Campana [24, 25]. It was shown by Campana and Winkelmann [26] that every projective manifold Y which satisfies the basic Oka property (every continuous map from a Stein manifold to Y is homotopic to a holomorphic map) is special. In particular, every projective Oka manifold is special but the converse is an open problem.

There is a growing list of applications of Oka theory, some of which are discussed in my book [51]. There are the dimensionwise optimal embedding and immersion theorems for Stein manifolds into Euclidean spaces, due to Eliashberg and Gromov [32] and Schürmann [120], the h-principle for holomorphic immersions of Stein manifolds $X^n \rightarrow \mathbb{C}^N$ for $N > n$ (see Eliashberg and Gromov [80] and [77, Sect. 2.1.5]), and the h-principle for holomorphic submersions $X^n \rightarrow \mathbb{C}^q$ with $n > q \geq 1$ (see Forstnerič [44]). There exist proper holomorphic maps, immersions and embeddings $X \rightarrow \mathbb{C}^n$ of Stein manifolds for suitable values of $n > \dim X$ whose images avoid any given unbounded strictly convex set [30] and have small limit sets at infinity [41]. Kutzschebauch, Lárusson, and Schwarz developed G -equivariant Oka theory and the notion of a G -Oka manifold, where G is a complex Lie group; see their survey [104]. There are major recent developments on factorisation problems for holomorphic matrix-valued functions on Stein spaces [88, 89, 86, 87], which rely on the Oka principle for sections of stratified elliptic submersions onto Stein spaces; see [51, Theorem 6.2.2]. Oka theory combined with methods of convex integration theory was used to establish h-principles in the classical theory of minimal surfaces in Euclidean spaces (see [3, 58, 2, 8, 9] and the monograph [7]), and in the theory of holomorphic Legendrian curves [57, 59]. A brief discussion of the h-principles in the theory of minimal surfaces is included in Sect. 5.

Another topic intimately related to Oka theory is the Andersén–Lempert theory and Varolin’s density property concerning Stein manifolds with a large group of holomorphic automorphisms; see Subsec. 3.2. Their main feature is the approximation theorem for isotopies of biholomorphic maps between pseudoconvex Runge domains by isotopies of holomorphic automorphisms (see Forstnerič and Rosay [64, Theorem 1.1], [51, Theorem 4.10.5]). Surprisingly, such highly symmetric objects exist in big classes and were found by applying criteria developed by Kaliman, Kutzschebauch and coauthors over the last two decades; see the surveys [42, 103] and [51, Chap. 4]. This theory had a great impact on the solution of famous old problems, among them the holomorphic linearisation problem, the embedding problem for Riemann surfaces, the h-principle for holomorphic submersions of Stein manifolds to Euclidean spaces, and many others. A Stein manifold Y with the density property is Oka at infinity (see Theorem 3.6 due to Kusakabe [102]), and it contains properly embedded copies of any Stein manifold X with $2 \dim X < \dim Y$ in every homotopy class of maps $X \rightarrow Y$ [13]. Arosio and Lárusson studied the dynamics of generic endomorphisms of Oka–Stein manifolds [15] and of Stein manifolds with the density property [16, 17].

Reader, follow me to the continuation of this story.

2. STEIN MANIFOLDS.

In this section we recall the basic properties of Stein manifolds which are relevant for this survey, referring to [81, 85, 75, 51] for more information. Stein manifolds were introduced in the literature by Karl Stein in 1951 [124] under the name *holomorphically complete manifolds* by the following axioms.

- (a) Holomorphic functions on X separate any distinct pair of points.
- (b) For every point $p \in X$ there is a holomorphic map $f = (f_1, \dots, f_n) : X \rightarrow \mathbb{C}^n$ with $n = \dim_{\mathbb{C}} X$ whose differential at p has complex rank n .
- (c) For every compact set K in X , its holomorphically convex (or $\mathcal{O}(X)$ -convex) hull

$$(2.1) \quad \widehat{K} = \{p \in X : |f(p)| \leq \max_{x \in K} |f(x)| \text{ for all } f \in \mathcal{O}(X)\}$$

is also compact. (By $\mathcal{O}(X)$ we denote the space of holomorphic functions $X \rightarrow \mathbb{C}$.)

A compact set $K \subset X$ is said to be holomorphically convex in X , or $\mathcal{O}(X)$ -convex, if and only if $K = \widehat{K}$. When $X = \mathbb{C}^n$, such a set is called polynomially convex. The same axioms define Stein spaces if (b) is replaced by the condition that holomorphic functions on X generate the ring $\mathcal{O}_{X,x}$ of germs of holomorphic functions at every point $x \in X$. In fact, (b) is a consequence of (a) and (c). A complex space satisfying condition (c) is said to be holomorphically convex. A domain in $\mathbb{C}^n$ is Stein if and only if it is holomorphically convex if and only if it is a domain of holomorphy. An open Riemann surface is Stein according to Behnke and Stein [21].

Stein manifolds have a number of characterisations. A connected complex manifold X is Stein if and only if it is biholomorphic to a closed complex submanifold of a Euclidean space $\mathbb{C}^N$; we can take $N = 2 \dim X + 1$. Equivalently, X admits a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^{2n+1}$. See [51, Theorem 2.4.1] for a summary and references. Thus, Stein manifolds are complex analytic analogues of affine algebraic manifolds. A more precise embedding theorem of Eliashberg and Gromov [32] and Schürmann [120] is that a Stein manifold of dimension $n > 1$ embeds properly holomorphically in $\mathbb{C}^N$ with $N = \lfloor \frac{3n}{2} \rfloor + 1$; this dimension is optimal by examples of Forster [35]. Similar embedding results hold for Stein spaces with bounded local embedding dimension; see [51, Theorem 9.3.7]. The proofs rely on Oka theory; see the detailed exposition in [51, Secs. 9.3–9.4]. The problem whether every open Riemann surface embeds as a closed nonsingular complex curve in $\mathbb{C}^2$ is still wide open; see the summary of known results in [51, Secs. 9.10–9.11] and the recent paper [29].

Another important characterisation is that a complex manifold X is Stein if and only if it admits a smooth strongly plurisubharmonic exhaustion function $\rho : X \rightarrow [0, +\infty)$, i.e., one satisfying

$dd^c \rho > 0$; see Grauert [73]. The analogous condition characterises Stein spaces with singularities, see Fornæss and Narasimhan [34].

One of the highlights of complex analysis on Stein spaces are Cartan's Theorems *A* and *B*. Let $\mathcal{F}$ be a coherent analytic sheaf on a Stein space X . Theorem A says that every stalk $\mathcal{F}_x$ for $x \in X$ is generated as an $\mathcal{O}_{X,x}$ -module by germs of global sections of $\mathcal{F}$, with Runge approximation of sections on compact $\mathcal{O}_X$ -convex subsets of X . Theorem B says that $H^q(X, \mathcal{F}) = 0$ for all $q = 1, 2, \dots$. Furthermore, the Dolbeault cohomology groups of a Stein manifold vanish, that is, the Cauchy–Riemann equation $\bar{\partial}u = \alpha$ is solvable on X for any (p, q) -form α ($p \geq 0, q \geq 1$) with $\bar{\partial}\alpha = 0$.

An important role in Oka theory is played by the theorem of Siu [121] saying that a locally closed Stein subvariety X in a complex space Z admits an open Stein neighbourhood. See also [51, Secs. 3.1–3.2] and the generalisation of Siu's theorem in [51, Theorem 3.2.1]. A related result of Poletsky [119] concerns Stein neighbourhoods of graphs of holomorphic mappings that are continuous up to the boundary of a domain.

The following classical results on holomorphic functions on a Stein space X motivate the definition of Oka properties and of Oka manifolds; see Def. 3.1 and Theorem 3.2.

- (i) *The Oka–Weil approximation theorem*: If K is a compact $\mathcal{O}(X)$ -convex subset of X then any holomorphic function $U \rightarrow \mathbb{C}$ on a neighbourhood $U \subset X$ of K is a uniform limit on K of holomorphic functions $X \rightarrow \mathbb{C}$.
- (ii) *The Cartan–Oka extension theorem*: If X' is a closed complex subvariety of X then any holomorphic function $X' \rightarrow \mathbb{C}$ extends to a holomorphic function $X \rightarrow \mathbb{C}$.
- (iii) If $K \subset U \subset X$ and $X' \subset X$ are as above and $f : U \cup X' \rightarrow \mathbb{C}$ is a continuous function which is holomorphic on U and on X' , we can approximate f uniformly on K by functions $F \in \mathcal{O}(X)$ satisfying $F|_{X'} = f|_{X'}$.
- (iv) If a function f in (iii) is holomorphic on a neighbourhood of $K \cup X'$, then $F \in \mathcal{O}(X)$ in (iii) can be chosen to approximate f on K and to agree with f to any given finite order along X' .

Note that (iii) is a combination of (i) and (ii), while (iv) upgrades the interpolation on X' to jet interpolation. The analogous results hold in the parametric case; see [51, Theorem 2.8.4].

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Results (i)–(iv) above, and their parametric analogues in [51, Theorem 2.8.4], describe the essential function-theoretic properties of Stein spaces. We now consider the analogous conditions for maps $X \rightarrow Y$ from a Stein space X to a complex manifold Y .

3.1. Oka properties characterising Oka manifolds.

Definition 3.1. Let Y be a complex manifold.

- (i') Y has the *basic Oka property with approximation*, *BOPA*, if for any Stein space X , compact holomorphically convex subset $K \subset X$, and continuous map $f : X \rightarrow Y$ which is holomorphic on a neighbourhood of K there is a homotopy $f_t : X \rightarrow Y$ ($t \in I = [0, 1]$) such that $f_0 = f$, each map f_t is holomorphic on a neighbourhood of K (independent of $t \in I$) and approximates f uniformly on K and uniformly in $t \in I$ to a given precision, and the map f_1 is holomorphic on X .
- (ii') Y has the *basic Oka property with interpolation*, *BOPI*, if for any Stein space X , closed complex subvariety $X' \subset X$, and continuous map $f : X \rightarrow Y$ such that $f|_{X'}$ is holomorphic there is a homotopy $f_t : X \rightarrow Y$ ($t \in I$) such that $f_0 = f$, $f_t|_{X'} = f|_{X'}$ for every $t \in I$, and f_1 is holomorphic on X .
- (iii') Y has the *basic Oka property with approximation and interpolation*, *BOPAI*, if the combination of BOPA and BOPI holds (cf. property (iii) in Sect. 2).

(iv') Y has the *basic Oka property with approximation and jet interpolation*, *BOPAJI*, if the analogue of property (iv) in Sect. 2 holds for maps $X \rightarrow Y$. Omitting the approximation condition defines the *basic Oka property with jet interpolation*, *BOPJI*.

In order to avoid topological obstructions, the initial continuous map $f : X \rightarrow Y$ (or a continuous family of maps in the parametric case) is assumed to be defined on all of X . Note that CAP (see Def. 1.1) is a restricted version of BOPA, and CRE (see Def. 1.5) is a restricted version of BOPJI. Similarly one defines the parametric versions of these properties called POPA, POPI, POPAI, POPAJI and POPJI, respectively; see [51, Sect. 5.15]. This gives several ostensibly different Oka-type conditions on a complex manifold. It turns out that they are pairwise equivalent [51, Proposition 5.15.1], although several implications are highly nontrivial.

Theorem 3.2. *Every Oka property mentioned above is equivalent to the Convex Approximation Property (CAP) in Def. 1.1, and to Convex Relative Ellipticity (CRE) in Def. 1.5.*

A complex manifold Y is said to be an *Oka manifold* if and only if it satisfies any and hence all of these conditions. Oka manifolds appear in Gromov's paper under the name Ell_∞ manifolds [78, 3.3.C']. However, the mentioned equivalences were proved much later, and the notion of an Oka manifold was introduced in [49].

The following is a well known consequence of the parametric h-principle.

Corollary 3.3. *If X is a Stein space and Y is an Oka manifold then the inclusion $\mathcal{O}(X, Y) \hookrightarrow \mathcal{C}(X, Y)$ of the space of holomorphic maps $X \rightarrow Y$ in the space of continuous maps (both endowed with the compact-open topology) is a weak homotopy equivalence.*

Lárusson showed in [107] that the above inclusion is a genuine homotopy equivalence if X is a Stein manifold which admits a strongly plurisubharmonic exhaustion function $\rho : X \rightarrow [0, \infty)$ with only finitely many critical points. In this case, the space $\mathcal{O}(X, Y)$ is a deformation retract of $\mathcal{C}(X, Y)$ if Y is Oka. This holds in particular if X is a strongly pseudoconvex domain in another Stein manifold, or if it is an affine algebraic manifold.

More precise statements can be found in [51, Theorem 5.4.4 and Proposition 5.15.1]. The proof of Theorem 3.2 (without considering CRE) takes up [51, Secs. 5.7–5.13]. The implication $\text{CRE} \implies \text{CAP}$ is due to Kusakabe [99, Theorem 1.3]; see also the exposition in [40, Theorem 3.3].

The following result (see [40, Theorem 1.3]) gives a more precise control of the image of a holomorphic map. It is particularly useful in conjunction with Theorem 3.6 and is often applied in the constructions of proper holomorphic maps. It is obtained by following [51, proof of Theorem 5.4.4]. We only state the basic case.

Theorem 3.4. *Assume that X is a Stein space, K is a compact $\mathcal{O}(X)$ -convex set in X , X' is closed complex subvariety of X , Ω is an Oka domain in a complex manifold Y , and $f : X \rightarrow Y$ is a continuous map which is holomorphic on a neighbourhood of K and on X' such that $f(\overline{X \setminus K}) \subset \Omega$. Then there is a homotopy $f_t : X \rightarrow Y$ ($t \in [0, 1]$) connecting $f = f_0$ to a holomorphic map $f_1 : X \rightarrow Y$ satisfying the conclusion of Theorem 3.2 such that $f_t(\overline{X \setminus K}) \subset \Omega$ holds for all $t \in [0, 1]$.*

Note that every Oka property in Def. 3.1 includes an approximation or an interpolation condition. A complex manifold Y is said to satisfy the *basic Oka property*, BOP, if every continuous map $X \rightarrow Y$ from a Stein manifold X is homotopic to a holomorphic map. Every Oka manifold clearly satisfies BOP but the converse fails in general. Indeed, any holomorphically contractible complex manifold, such as the disc, satisfies BOP but need not be Oka. Campana and Winkelmann proved in [26] that a projective manifold Y satisfying BOP is Campana special [24, 25].

Problem 3.5. Is every projective manifold satisfying BOP an Oka manifold?

A survey of examples of Oka manifolds which predates their characterisation by CRE can be found in [51, Chaps. 5–7]. The recent survey [40] includes many new examples based on the equivalence $\text{CRE} \iff \text{OKA}$, Theorem 1.7, and the use of the density property. We present some of them in the following subsection.

3.2. Oka properties of complements of holomorphically convex sets. A complex manifold Y is said to enjoy the *density property* (see Varolin [126, 127]) if every holomorphic vector field on Y can be approximated uniformly on compacts by Lie combinations of $\mathbb{C}$ -complete holomorphic vector fields. The Euclidean spaces $\mathbb{C}^n$, $n > 1$, enjoy this property by Andersén and Lempert [12]. One of the main results in this subject is that if Y is a Stein manifold with the density property then every isotopy of biholomorphic maps $\Phi_t : \Omega \rightarrow \Phi_t(\Omega)$, $t \in [0, 1]$, with $\Phi_0 = \text{Id}_\Omega$ between pseudoconvex Runge domains in Y can be approximated by isotopies of holomorphic automorphisms of Y (see Forstnerič and Rosay [64], [51, Theorem 4.10.5]). Every Stein manifold with the density property is an Oka manifold (see [92, Theorem 4]). Surveys can be found in [51, Chap. 4], [42], and [103].

In his seminal paper [102], Kusakabe applied the characterisation of Oka manifolds by CRE (see Theorem 3.2) to establish the Oka property of complements of certain closed holomorphically convex sets in Stein manifolds with the density property. This gives a plethora of new examples of Oka manifolds, most of which are not Gromov elliptic. The following result combines Theorem 1.2 and Corollary 1.3 in [102]. (See also [67].)

Theorem 3.6. *If K is a compact polynomially convex set in $\mathbb{C}^n$, $n > 1$, then $\mathbb{C}^n \setminus K$ is an Oka manifold. More generally, if Y is a Stein manifold with the density property and $K \subset Y$ is a compact $\mathcal{O}(Y)$ -convex set then $Y \setminus K$ is an Oka manifold.*

The idea of proof is that, given a set $K \subset Y$ as in the theorem and a holomorphic map $f : U' \rightarrow Y \setminus K$ from a neighbourhood of a compact convex set $L \subset \mathbb{C}^N$, one can find a neighbourhood $U \subset U'$ of L and a holomorphic map $F : U \times \mathbb{C}^n \rightarrow Y \setminus K$, with $n = \dim Y$, such that for every $z \in U$ the map $F_z = F(z, \cdot) : \mathbb{C}^n \rightarrow Y$ is biholomorphic onto its image in $Y \setminus K$ and satisfies $F_z(0) = f(z)$ (see [67, Theorems 1.1 and 3.1]). Thus, $\{F_z(\mathbb{C}^n)\}_{z \in U}$ is a holomorphically varying family of Fatou–Bieberbach domains in $Y \setminus K$. Clearly, such F is a dominating spray with the core f , so $Y \setminus K$ satisfies CRE and hence is an Oka manifold. The construction of F uses the density property of Y and the aforementioned approximation of isotopies of biholomorphic maps between Stein Runge domains in Y by holomorphic automorphisms of Y (see [51, Theorem 4.10.5]).

A closed unbounded subset S of $\mathbb{C}^n$ is said to be polynomially convex if it is exhausted by an increasing sequence of compact polynomially convex sets. For $Y = \mathbb{C}^n$, $n > 2$, Theorem 3.6 is a special case of the following result of Kusakabe [102, Theorems 1.6 and 4.2].

Theorem 3.7. *If $n > 2$, $c > 0$ and $S \subset \{(z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : |w| \leq c(1 + |z|)\}$ is a closed polynomially convex set, then $\mathbb{C}^n \setminus S$ is an Oka manifold.*

In [54], these techniques were used to prove that most concave domains in $\mathbb{C}^n$ for $n > 1$ are Oka. In particular, the following holds; see [54, Theorem 1.8].

Theorem 3.8. *If E is a closed convex set in $\mathbb{C}^n$, $n > 1$, which does not contain any affine real line, then $\mathbb{C}^n \setminus E$ is an Oka domain.*

In particular, if $\phi : \mathbb{C}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a strictly convex function then the concave domain

$$(3.1) \quad \{z = (z', z_n) \in \mathbb{C}^n : \Im z_n < \phi(z', \Re z_n)\}$$

below its graph is an Oka domain while the domain above the graph is Kobayashi hyperbolic. Note that these domains can be arbitrarily close to a halfspace, the latter being neither Oka nor hyperbolic.

Consider $\mathbb{C}^n$ as an affine domain in the projective space $\mathbb{CP}^n$. Given a subset $E \subset \mathbb{C}^n$, we denote by $\overline{E}$ its topological closure in $\mathbb{CP}^n$. Theorem 3.8 is a special case of the following result [54, Theorem 1.1].

Theorem 3.9. *If E is a closed subset of $\mathbb{C}^n$ for $n > 1$ and $\Lambda \subset \mathbb{CP}^n$ is a complex hyperplane such that $\overline{E} \cap \Lambda = \emptyset$ and $\overline{E}$ is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$, then $\mathbb{C}^n \setminus E$ is Oka.*

Sections 4 and 5 of the survey [40] contains many further examples of Oka domains in Euclidean and projective spaces. In particular, there are compact sets in $\mathbb{C}^n$, $n > 1$, of the form $L = K \cup C$, where K is polynomially convex and C is a finite union of rectifiable curves, such that L fails to be polynomially convex yet $\mathbb{C}^n \setminus L$ is an Oka domain (see [40, Theorem 4.12]). The set L in these examples is rationally convex, and I do not know examples of non-rationally convex compact sets in $\mathbb{C}^n$ with Oka complements.

It seems that all known examples of proper Oka domains in Stein manifolds are weakly pseudoconcave. Since Stein domains in Stein manifolds are exactly the pseudoconvex ones and Oka manifolds are dual to Stein manifolds in the sense explained in the Introduction, this seems natural. However, few explicit results are known.

Problem 3.10. Let K be a closed subset of $\mathbb{C}^n$, $n > 1$, with smooth boundary bK .

- (a) (The inverse Levi problem.) Assuming that $\mathbb{C}^n \setminus K$ is Oka, is bK necessarily pseudoconvex?
- (b) Assuming that K is compact and bK is pseudoconvex, is $\mathbb{C}^n \setminus K$ an Oka domain?
- (c) Is every strongly pseudoconcave domain in $\mathbb{C}^n$ of the form (3.1) an Oka domain?

An affirmative answer to (a) is known for $n = 2$, see [40, p. 389]. Model examples of non-pseudoconvex domains are Hartogs figures. For the closed Hartogs figure

$$H = \{(z_1, z') \in \mathbb{C}^n : |z_1| \leq \delta, |z'| \leq 1\} \cup \{(z_1, z') : |z_1| \leq 1, 1 - \delta \leq |z'| \leq 1\}$$

with $0 < \delta < 1$, the complement $\mathbb{C}^n \setminus H$ fails to be Oka [40, Example 4.23]. It is not known whether the complement of the Hartogs triangle $\{(z_1, z_2) \in \mathbb{C}^2 : 0 \leq |z_1| \leq |z_2| \leq 1\}$ is an Oka domain [40, Problem 4.24].

In a related direction, Forstnerič and Kusakabe [55] studied Oka properties of disc tubes in certain hermitian holomorphic line bundles on compact complex manifolds. The following is their main result.

Theorem 3.11. *Let E be a holomorphic line bundle on a compact complex manifold X . Assume that for each point $x \in X$ there exists a divisor D in the complete linear system $|E|$ whose complement $X \setminus D$ is a Stein neighbourhood of x with the density property. Given a semipositive hermitian metric h on E , the disc bundle $\{e \in E : |e|_h < 1\}$ is a pseudoconcave Oka domain while $\{e \in E : |e|_h > 1\}$ is Kobayashi hyperbolic. In particular, the zero section of E admits a basis of Oka neighbourhoods $\{|e|_h < c\}$ with $c > 0$. This holds in particular if E is an ample line bundle on a rational homogeneous manifold X of dimension > 1 .*

When X is a complex Grassmannian, an ample line bundle E on X satisfies the conditions in the theorem with divisors $D \subset X$ such that $X \setminus D \cong \mathbb{C}^n$ with $n = \dim X > 1$. If h is a semipositive hermitian metric on E then the intersection of the unit disc bundle $\{|e|_h < 1\}$ with any such chart is a pseudoconcave Hartogs domain of the form $\Omega = \{(z, t) \in \mathbb{C}^n \times \mathbb{C} : |t| < \phi(z)\}$, where ϕ is a positive continuous function on $\mathbb{C}^n$ such that $\log \phi$ is plurisubharmonic, and there is a constant $c > 0$ such that $\phi(z) \geq c|z|$ holds for all $z \in \mathbb{C}^n$. By using Theorem 3.7 one can show that Ω is an Oka domain, and the proof of Theorem 3.11 is then completed by applying the localisation Theorem 1.7. Similar arguments apply when X is a rational homogeneous manifold.

3.3. Oka-1 manifolds. A complex manifold Y is an *Oka-1 manifold* if it enjoys the Oka property with approximation and jet interpolation for maps $X \rightarrow Y$ from any open Riemann surface X . In other words, Y contains plenty of holomorphic curves parametrised by any open Riemann surface. This class of manifolds was introduced and studied by Alarcón and the author in [5]. The following is a special case of [5, Theorem 2.2].

Theorem 3.12. *If Y is complex manifold of dimension n and $E \subset Y$ is a closed subset with zero Hausdorff $(2n - 1)$ -measure such that Y is dominable by $\mathbb{C}^n$ at every point $y \in Y \setminus E$, then Y is an Oka-1 manifold.*

A complex manifold Y satisfying the condition in the above theorem is said to be *densely dominable* by $\mathbb{C}^n$; it is *strongly dominable* if the condition holds with $E = \emptyset$. It is not known whether dense or strong dominability implies the Oka property for maps from higher dimensional Stein manifolds. There are examples of noncompact Oka-1 manifolds which fail to be Oka, but we do not know any such example among compact manifolds. In [5] we found many examples of Oka-1 manifolds among compact complex surfaces. In particular, every Kummer surface and every elliptic K3 surface is Oka-1 (see [5, Sect. 8]). It is not known whether any or all of these surfaces are Oka manifolds. (A survey of known Oka manifolds among compact complex surfaces can be found in [51, Sect. 7.3].) It is conjectured that every projective rationally connected manifold is Oka-1 [5, Conjecture 9.1]. By Campana and Winkelmann [27], every such manifold admits holomorphic lines $\mathbb{C} \rightarrow X$ with dense images.

New examples and functorial properties of Oka-1 manifolds were presented by Lárusson and the author in [61]. We also formulated and studied the algebraic version of the Oka-1 condition, called aOka-1, which concerns approximation and interpolation of holomorphic maps from open subsets of affine algebraic curves to algebraic manifolds by regular algebraic maps. Every aOka-1 manifold is also an Oka-1 manifold. The aOka-1 property is a birational invariant for compact algebraic manifolds, and it holds for all rational manifolds and all algebraically elliptic projective manifolds (see [61, Theorem 1.6]).

It is easily seen that every projective aOka-1 manifold is rationally connected [61, Proposition 1.7]. Conversely, Benoist and Wittenberg [22, Theorem 1.2] proved that every projective *rationally simply connected manifold* is an aOka-1 manifold, and this class contains every smooth hypersurface of degree d in $\mathbb{CP}^n$ with $n \geq d^2 - 1$ (see [22, Corollary 1.3]). Note that their *tight approximation property* is equivalent to the aOka-1 property.

3.4. The Oka principle on families of open Riemann surfaces. In [53], the Oka principle was established for maps from very general families of open Riemann surfaces to Oka manifolds. Assume that X is a smooth open orientable surface, B is a finite CW complex or a countable locally compact CW-complex of finite dimension, and $\{J_b\}_{b \in B}$ is a continuous family of complex structures on X of local Hölder class $\mathcal{C}^\alpha$, $0 < \alpha < 1$. A continuous map $f : B \times X \rightarrow Y$ to a complex manifold Y is said to be fibrewise holomorphic if the map $f(b, \cdot) : X \rightarrow Y$ is holomorphic from the open Riemann surface (X, J_b) for every $b \in B$. The following Oka principle is a special case of [53, Theorem 1.6].

Theorem 3.13. *(Assumptions as above.) If Y is an Oka manifold then every continuous map $f : B \times X \rightarrow Y$ is homotopic to a continuous fibrewise holomorphic map $F : B \times X \rightarrow Y$. If in addition B is a manifold of class $\mathcal{C}^l$ and the family of complex structures $\{J_b\}_{b \in B}$ is of local class $\mathcal{C}^{l, (k, \alpha)}(X)$, where $0 \leq l \leq k + 1$ and $0 < \alpha < 1$, then F can be chosen of local class $\mathcal{C}^{l, (k+1, \alpha)}$.*

The statement concerning the map F in the second part of the theorem means that F has l derivatives in the parameter $b \in B$ followed by $k + 1$ derivatives in the space variable $x \in X$, and

these derivatives are of local Hölder class $\mathcal{C}^\alpha$ in $x \in X$. The analogous definition applies to the family $\{J_b\}_{b \in B}$.

The more precise result [53, Theorem 1.6] includes approximation of fibrewise holomorphic maps on suitable families of compact Runge subsets of X . Theorem 3.13 relates the Oka theory to Teichmüller spaces. It was also used in the construction of families of immersed conformal minimal surfaces $X \rightarrow \mathbb{R}^n$ and holomorphic null curves $X \rightarrow \mathbb{C}^n$, $n \geq 3$, with a prescribed family of conformal structures on the surface X ; see [53, Sect. 8]. Further applications are expected.

The proof of Theorem 3.13 proceeds by inductively extending the domain in $B \times X$ on which a given map is fibrewise holomorphic, using approximation on closed subsets with compact Runge fibres to ensure convergence to a limit map. It also uses solutions of the Beltrami equation on an open Riemann surface (X, J) to show that a small variation of the complex structure J on a smoothly bounded domain $\Omega \Subset X$ can be realised by a small variation of Ω in X with its original structure J . This is special case of Hamilton's theorem [82] but with more precise regularity of the maps. Finally, the problem is reduced locally in the parameter $b \in B$ and semiglobally in the space variable $x \in X$ to the Oka principle given by Theorem 3.2.

An important fact used in the proof of Theorem 3.13 is that the condition on a compact set K in a surface X to be Runge is purely topological (its complement $X \setminus K$ has no relatively compact connected component), hence independent of the choice of the complex structure on X . This is no longer the case if X is a smooth open manifold of dimension $2n > 2$ endowed with a family of Stein structures $\mathcal{J} = \{J_b\}_{b \in B}$. The following example shows that one must impose a suitable condition on the family $\mathcal{J}$ to get an analogue of Theorem 3.13. The proof will appear in a subsequent publication (in preparation).

Theorem 3.14. *Given a compact set $K \subset \mathbb{R}^{2n}$ ($n > 1$) with nonempty interior, there is a family of smooth integrable complex structures $\{J_t\}_{t \in \mathbb{R}}$ on $\mathbb{R}^{2n}$, depending smoothly on $t \in \mathbb{R}$, such that J_0 is the standard complex structure on $\mathbb{C}^n$, the manifold $(\mathbb{R}^{2n}, J_t)$ is biholomorphic to $\mathbb{C}^n$ for every $t \in \mathbb{R}$, and for any neighbourhood $U \subset \mathbb{R}$ of $0 \in \mathbb{R}$ the set $\bigcup_{t \in U} \widehat{K}_{J_t} \subset \mathbb{R}^{2n}$ is unbounded.*

This phenomenon excludes the possibility of any reasonable analysis on such a family of Stein structures. The condition that one must impose to obtain positive results is that for any compact set K in X , the J_b -convex hull $\widehat{K}_{J_b} \subset X$ (2.1) is an upper semicontinuous set-valued function of the parameter $b \in B$. Equivalently, there is a continuous function $\rho : B \times X \rightarrow [0, \infty)$ such that for every $b \in B$, the function $\rho(b, \cdot) : X \rightarrow [0, \infty)$ is a strongly J_b -plurisubharmonic exhaustion. A paper on this topic is in preparation.

4. ELLIPTIC AND SUBELLIPTIC MANIFOLDS.

Gromov introduced the following conditions in [78, 0.5, p. 855].

Definition 4.1. A complex manifold Y is elliptic if there is a holomorphic vector bundle $\pi : E \rightarrow Y$ and a holomorphic map $s : E \rightarrow Y$ such that s restricts to the identity map on the zero section $E(0)$ of E , and for any point $y \in Y$ the differential $ds_{0_y} : T_{0_y}E \rightarrow T_yY$ maps the fibre $E_y = \pi^{-1}(y)$ onto T_yY . An algebraic manifold Y is algebraically elliptic if the vector bundle $\pi : E \rightarrow Y$ and the map $s : E \rightarrow Y$ can be chosen algebraic.

Here, 0_y denotes the origin of the vector space $E_y = \pi^{-1}(y)$. A map s with these properties is called a *dominating spray* on Y . An ostensibly weaker condition, *subellipticity* (see [43, Def. 2]), asks for the existence of a finite family of holomorphic sprays (E_j, π_j, s_j) on Y ($j = 1, \dots, m$) which is dominating in the sense that

$$(4.1) \quad (ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) + \dots + (ds_m)_{0_y}(E_{m,y}) = T_yY \quad \text{for all } y \in Y.$$

An algebraic manifold Y is said to be *algebraically subelliptic* if the same condition holds with algebraic sprays. See also [51, Def. 5.6.13]. Examples of elliptic and subelliptic manifolds can be found in [78], [51, Sect. 6.4], and in the surveys [40, 56]. We recall a few.

Example 4.2. (A) *Every complex homogeneous manifold is elliptic.* Assume that a complex Lie group G acts on a complex manifold Y transitively by holomorphic automorphisms. Let $\mathfrak{g} \cong \mathbb{C}^p$ denote the Lie algebra of G and $\exp : \mathfrak{g} \rightarrow G$ the exponential map. The holomorphic map $s : Y \times \mathfrak{g} \cong Y \times \mathbb{C}^p \rightarrow Y$ given by $s(y, v) = \exp v \cdot y \in Y$ ($y \in Y$, $v \in \mathfrak{g}$) is a dominating spray on Y . Such sprays were used by Grauert [70, 71, 72].

(B) *If Y is a complex manifold whose tangent bundle is spanned by finitely many $\mathbb{C}$ -complete holomorphic vector fields $V_1, \dots, V_m$, then Y is elliptic.* Indeed, let ϕ_j^t , $t \in \mathbb{C}$, be the flow of V_j . The map $s : Y \times \mathbb{C}^m \rightarrow Y$ given by

$$(4.2) \quad s(y, t) = s(y, t_1, \dots, t_m) = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \dots \circ \phi_m^{t_m}(y)$$

clearly satisfies $s(y, 0) = y$ and $\frac{\partial}{\partial t_j} s(y, 0) = V_j(y)$ for any $y \in Y$ and $j = 1, \dots, m$. Thus, s is dominating precisely when the vectors $V_1(y), \dots, V_m(y)$ span the tangent space $T_y Y$ for every $y \in Y$.

(C) Assume that $\pi : E \rightarrow Y$ is a holomorphic vector bundle and $\phi : E \rightarrow E$ is a holomorphic map which restricts to the identity on the zero section $E(0)$ of E . Then, the map $s = \pi \circ \phi : E \rightarrow Y$ is a spray on Y . Let us identify Y with $E(0)$ and the restricted tangent bundle $TE|_Y$ with $TY \oplus E$. The spray s is dominating at $y \in Y$ if for every $v \in T_y Y$ there is a point $e \in E_y$ such that $ds_y(e) = (d\pi)_y \circ (d\phi)_y(e) = v$.

The importance of ellipticity and subellipticity lies in the following result.

Theorem 4.3. (a) (Gromov [78, 0.6, p. 855]) *Every elliptic manifold is an Oka manifold.*
(b) ([43, Theorem 1.1]) *Every subelliptic manifold is an Oka manifold.*

The modern proof of Theorem 4.3 (see [51, Chap. 6]) consists of two parts. The first part is the implication

$$(\text{sub})\text{elliptic} \implies \text{h-Runge approximation} \implies \text{CAP}.$$

Here, a complex manifold Y is said to satisfy the h-Runge approximation condition if the following holds. Given a pair $K \subset L$ of compact $\mathcal{O}(X)$ -convex sets in a Stein space X and a homotopy of holomorphic maps $f_t : U \rightarrow Y$, $t \in I = [0, 1]$, of holomorphic maps from a neighbourhood of K such that f_0 extends to a holomorphic map from a neighbourhood of L in Y , we can approximate the homotopy f_t uniformly on K by a homotopy of holomorphic maps $\tilde{f}_t : L \rightarrow Y$, $t \in I$, such that $\tilde{f}_0 = f_0$. This is a fairly elementary consequence of subellipticity and the parametric Oka–Weil theorem for sections of vector bundles; see [51, Theorem 6.6.1]. Since a compact convex set $K \subset \mathbb{C}^n$ is holomorphically contractible, every holomorphic map f from a convex neighbourhood of K is homotopic to a constant map through holomorphic maps. Hence, the h-Runge approximation property of Y implies that f is a limit of entire maps $\mathbb{C}^n \rightarrow Y$, which means that Y enjoys CAP.

The implication $\text{CAP} \implies \text{OKA}$ is highly nontrivial; see [51, Chap. 5] for the proof.

The notion of (sub)ellipticity is also defined for holomorphic submersions $h : Z \rightarrow X$, and it implies the parametric Oka property for sections $X \rightarrow Z$ provided that X is Stein (see [78, Sect. 4], [63], and [51, Chap. 6]). An even more general case pertains to sections of stratified subelliptic submersions over Stein spaces; see [50] and [51, Theorem 6.2.2]. There is also a relative Oka principle for sections of ramified holomorphic maps onto Stein spaces; see [45] and [51, Sect. 6.14].

It is easily seen that every Stein Oka manifold is elliptic [78, 3.2.A]. Gromov asked whether every Oka manifold is elliptic. The first counterexamples for noncompact manifolds were given by Kusakabe in [98]. He showed in particular that if S is a closed tame countable set in $\mathbb{C}^n$, $n > 1$, whose set of limit

points is discrete then $\mathbb{C}^n \setminus S$ is an Oka domain (see [98, Theorem 1.2]) but there are nonelliptic domains of this type. For example, taking $Z = \{0\} \cup \{1/j : j \in \mathbb{N}\} \subset \mathbb{C}$, the domain $\mathbb{C}^n \setminus (Z^2 \times \{0\}^{n-2})$ for $n \geq 3$ is Oka but not elliptic or subelliptic [98, Corollary 1.4]. Theorem 3.6, together with results of Andrist, Shcherbina and Wold [14], gives a much more abundant class of examples in the following corollary.

Corollary 4.4. *Let $n \geq 3$. For every compact polynomially convex set $K \subset \mathbb{C}^n$ with infinitely many limit points the complement $\mathbb{C}^n \setminus K$ is Oka but not subelliptic. The analogous result holds for compact holomorphically convex sets in any Stein manifold with the density property of dimension ≥ 3 .*

The following recent result of Forstnerič and Lárusson [60] confirms Gromov’s conjecture [78, 3.2.A’].

Theorem 4.5. *Every projective Oka manifold is elliptic.*

From Theorems 4.3 (b) and 4.5 we conclude the following.

Corollary 4.6. *For a projective manifold, being Oka, elliptic, or subelliptic are equivalent conditions.*

There are projective Oka manifolds which fail to be algebraically elliptic; for example, abelian varieties. Hence, the algebraic counterpart to Theorem 4.5 is not true and Serre’s GAGA principle fails in this problem.

The spray bundle $E \rightarrow Y$ in the proof of Theorem 4.5 is the direct sum of copies of the dual of a sufficiently ample line bundle on the projective manifold Y (hence, E is Griffiths negative), and the spray map $s : E \rightarrow Y$ is of the type described in Example 4.2 (C). The construction in [60] first produces a local dominating spray $s_0 : U \rightarrow Y$ defined on an open neighbourhood $U \subset E$ of the zero section $E(0)$; this does not use any additional hypothesis on Y . Since E is negative, it is a 1-convex manifold with the exceptional subvariety $E(0) \cong Y$. Assuming that Y is an Oka manifold, the existence of a global holomorphic map $s : E \rightarrow Y$ which agrees with s_0 to the second order along $E(0)$ then follows from the Oka principle for maps from 1-convex manifolds to Oka manifolds, due to Stopar [125]. Note that such s is a global dominating spray on Y , thus proving that Y is elliptic.

Algebraic analogues of Stein manifolds are affine algebraic manifolds. The properties of holomorphic functions on Stein spaces (see (i)–(iv) in Sect. 2) also hold for (regular) algebraic functions on affine algebraic varieties, and they can be used to define algebraic Oka properties of algebraic manifolds. We discuss them briefly, referring to [40, Sect. 6] for a more comprehensive presentation.

Recall that a complex algebraic manifold Y is said to be algebraically elliptic if it admits an algebraic dominating spray $s : E \rightarrow Y$ on an algebraic vector bundle $\pi : E \rightarrow Y$. Similarly, Y is said to be algebraically subelliptic if it admits finitely many algebraic sprays (E_j, π_j, s_j) satisfying (4.1). Gromov proved that algebraic subellipticity is a Zariski local property [78, 3.5 B], and it is stable with respect to removing algebraic subvarieties of codimension > 1 [78, 3.5 C]. (Removal of hypersurfaces is not allowed in general as it may lead to hyperbolic manifolds.) Examples and properties of such manifolds, with further references, can be found in [51, Sect. 6.4] and [40, Sect. 6]. The following recent result is due to Kaliman and Zaidenberg [94].

Theorem 4.7. *Every algebraically subelliptic manifold is algebraically elliptic.*

Hence these two properties are equivalent, so algebraic ellipticity is a Zariski-local condition. No such result is known in the holomorphic category. See also [40, Theorem 6.2] for some other equivalent properties.

The main Oka-type property of algebraically elliptic manifolds is the algebraic homotopy approximation theorem for maps from affine algebraic varieties; see [47, Theorem 3.1], [51, Theorem 6.15.1], and the recent generalisations in [2, Sect. 2]. In particular, we have the following result.

Theorem 4.8. *Let X be an affine algebraic variety and Y be an algebraically elliptic manifold. Then, every holomorphic map $X \rightarrow Y$ homotopic to an algebraic map is a limit of algebraic maps in the compact-open topology.*

It follows that every algebraically elliptic manifold Y satisfies the algebraic convex approximation property, aCAP: every holomorphic map from a convex set $U \subset \mathbb{C}^n$ to Y is a limit of algebraic maps $\mathbb{C}^n \rightarrow Y$ uniformly on compacts. Theorem 4.8 can be used to prove the following result on algebraic domination. The first part is a special case of [52, Theorem 1.6], while the second part is due to Kusakabe [101, Theorem 1.2].

Theorem 4.9. *Let Y be an algebraically elliptic manifold of dimension n . If Y is compact then it admits an algebraic map $f : \mathbb{C}^n \rightarrow Y$ satisfying $f(\mathbb{C}^n \setminus \text{br}(f)) = Y$, where $\text{br}(f)$ denotes the branch locus of f . If Y is noncompact then it admits an algebraic map $\mathbb{C}^{n+1} \rightarrow Y$ satisfying $f(\mathbb{C}^{n+1} \setminus \text{br}(f)) = Y$.*

There are examples of holomorphic maps from an affine algebraic manifold X to an algebraically elliptic manifold Y which are not homotopic to any algebraic map (see [51, Examples 6.15.7, 6.15.8]). Thus, the algebraic basic Oka property, aBOP, fails in these examples. The following result of Lárusson and Truong [108, Theorem 2] shows that this is a rather common phenomenon, and Theorem 4.8 might be the closest analogue of the Oka principle in the algebraic category.

Theorem 4.10. *If Y is an algebraic manifold which contains a rational curve or is compact, then Y does not have the algebraic basic Oka property, aBOP.*

I am not aware of a single example of an algebraic manifold Y with nontrivial topology satisfying aBOP. The simplest example to consider is $\mathbb{C}^2 \setminus \{0\}$, which is algebraically elliptic.

By [108, Theorem 3] of Lárusson and Truong, together with Theorem 4.7, every smooth nondegenerate toric variety is algebraically elliptic. A result of Banecki [20] says that every rational projective manifold is algebraically elliptic. This generalises the result of Arzhantsev et al. [19, Theorem 1.3] that every uniformly rational projective manifold is algebraically elliptic. See also the recent examples [93, 95] and the survey [129]. A major source of examples of algebraically elliptic manifolds are *flexible manifolds*; see [18]. These are algebraic manifolds whose tangent space at every point is spanned by algebraic vector fields with complete algebraic flows, so they admit dominating algebraic sprays of type (4.2). A list of examples of such manifolds can be found in [40, p. 394].

5. THE H-PRINCIPLES FOR MINIMAL SURFACES IN $\mathbb{R}^n$ AND HOLOMORPHIC NULL CURVES IN $\mathbb{C}^n$.

We begin by recalling the basic facts; see [7, 115]. Let M be a connected open Riemann surface. A smooth immersion $u = (u_1, \dots, u_n) : M \rightarrow \mathbb{R}^n$, $n > 1$, is conformal (angle preserving) if and only if its $(1, 0)$ -differential $\partial u = (\partial u_1, \dots, \partial u_n)$ (the $\mathbb{C}$ -linear part of the differential du) satisfies the nullity condition

$$(5.1) \quad (\partial u_1)^2 + (\partial u_2)^2 + \dots + (\partial u_n)^2 = 0.$$

Let $n \geq 3$. A conformal immersion $u : M \rightarrow \mathbb{R}^n$ is *minimal*, in the sense that it parametrizes a minimal surface in $\mathbb{R}^n$, if and only if it is harmonic, $\bar{\partial} \partial u = 0$; this holds if and only if ∂u is a holomorphic $(1, 0)$ -form. Such an immersion u is said to be *nonflat* if the image $u(M) \subset \mathbb{R}^n$ is not contained in an affine 2-plane. We denote by $\mathfrak{M}(M, \mathbb{R}^n)$ the space of conformal minimal immersions $M \rightarrow \mathbb{R}^n$ with

the compact-open topology, and by $\mathfrak{M}_{\text{nf}}(M, \mathbb{R}^n)$ the subspace of $\mathfrak{M}(M, \mathbb{R}^n)$ consisting of all nonflat conformal minimal immersions.

A holomorphic immersion $F : M \rightarrow \mathbb{C}^n$, $n \geq 3$, is said to be a *null curve* if the differential $dF = \partial F = (dF_1, \dots, dF_1)$ satisfies the nullity condition (5.1). Such F is said to be *nonflat* if $F(M)$ is not contained in an affine complex line. Denote by $\mathfrak{N}(M, \mathbb{C}^n)$ the spaces of null holomorphic immersions $M \rightarrow \mathbb{C}^n$ with the compact-open topology, and by $\mathfrak{N}_{\text{nf}}(M, \mathbb{C}^n)$ the subspace consisting of all nonflat null holomorphic immersions $M \rightarrow \mathbb{C}^n$. Since $dF = 2\partial(\Re F)$, the real part $u = \Re F$ of a (nonflat) null curve is a (nonflat) conformal minimal immersion $M \rightarrow \mathbb{R}^n$; the converse holds if M is simply connected. Hence, we have natural inclusions

$$\Re \mathfrak{N}(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}(M, \mathbb{R}^n), \quad \Re \mathfrak{N}_{\text{nf}}(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_{\text{nf}}(M, \mathbb{R}^n),$$

where $\Re \mathcal{F}$ denotes the space of real parts of maps in a space $\mathcal{F}$.

Choose a nowhere vanishing holomorphic 1-form θ on M (such exists by the Oka–Grauert principle [51, Theorem 5.3.1]). Given a conformal minimal immersion $u : M \rightarrow \mathbb{R}^n$, the map $f = 2\partial u/\theta = (f_1, \dots, f_n) : M \rightarrow \mathbb{C}^n$ is holomorphic since u is harmonic, and by (5.1) it has range in $\mathbb{A}_* = \mathbb{A} \setminus \{0\}$ where $\mathbb{A}$ is the null quadric

$$(5.2) \quad \mathbb{A} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \dots + z_n^2 = 0\}.$$

Conversely, a holomorphic map $f : M \rightarrow \mathbb{A}_*$ such that the 1-form $f\theta$ has vanishing real periods (i.e., $\int_\gamma \Re(f\theta) = 0$ holds for every closed curve γ in M) determines a conformal minimal immersion $u : M \rightarrow \mathbb{R}^n$ by $u(x) = \int^x \Re(f\theta)$, $x \in M$. If $f\theta$ has vanishing complex periods then it integrates to a holomorphic null curve $F(x) = \int^x f\theta$. This is the classical *Weierstrass representation* of minimal surfaces and null curves; see [7, Sec. 2.3] or [115].

Consider the following commuting diagram where the first vertical arrow is the real part projection, the map ϑ is given by $F \mapsto \partial F/\theta$, and ψ is given by $u \mapsto 2\partial u/\theta$.

$$(5.3) \quad \begin{array}{ccccc} \mathfrak{N}_{\text{nf}}(M, \mathbb{C}^n) & \xrightarrow{\vartheta} & \mathcal{O}(M, \mathbb{A}_*) & \hookrightarrow & \mathcal{C}(M, \mathbb{A}_*) \\ \downarrow & & \uparrow \psi & & \\ \Re \mathfrak{N}_{\text{nf}}(M, \mathbb{C}^n) & \hookrightarrow & \mathfrak{M}_{\text{nf}}(M, \mathbb{R}^n) & & \end{array}$$

The punctured null quadric $\mathbb{A}_* = \mathbb{A} \setminus \{0\}$ is a homogeneous space of the complex Lie group $\mathbb{C}^* \times O(n, \mathbb{C})$, where $O(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : AA^t = I\}$ is the orthogonal group over $\mathbb{C}$. Hence, the inclusion $\mathcal{O}(M, \mathbb{A}_*) \hookrightarrow \mathcal{C}(M, \mathbb{A}_*)$ of the space of holomorphic maps $M \rightarrow \mathbb{A}_*$ in the space of continuous maps satisfies the Oka principle and so is a weak homotopy equivalence. By using elements of the convex integration theory, one shows that every holomorphic map $f : M \rightarrow \mathbb{A}_*$ can be deformed to a map for which the 1-form $f\theta$ is exact holomorphic, so it integrates to a null holomorphic immersion $\int f\theta : M \rightarrow \mathbb{C}^n$. Ensuring only that $\Re(f\theta)$ is exact yields a conformally immersed minimal surface $\int \Re(f\theta) : M \rightarrow \mathbb{R}^n$. Furthermore, such deformations can be made in families. This leads to the following result of Forstnerič and Lárusson [58, Theorems 1.1 and 4.1].

Theorem 5.1. *All maps in the diagram (5.3) satisfy the parametric h-principle with approximation, and hence they are weak homotopy equivalences.*

The basic h-principles with approximation were proved in this context by Alarcón and Forstnerič in [3] for the maps ϑ and ψ , and in [4, Theorem 1.1] for the inclusion $\Re \mathfrak{N}_{\text{nf}}(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_{\text{nf}}(M, \mathbb{R}^n)$. It is not known whether the inclusion $\Re \mathfrak{N}(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}(M, \mathbb{R}^n)$ satisfies the h-principle.

Let ds^2 denote the Euclidean metric on $\mathbb{R}^n$. A minimal surface $u : M \rightarrow \mathbb{R}^n$ is said to be *complete* if the Riemannian metric $g = u^*ds^2$ induces a complete metric on M . Completeness is an

important and much studied property in the global theory of minimal surfaces. Alarcón and Lárusson [10] proved that the inclusion $\mathfrak{M}_{\text{nf}}^c(M, \mathbb{R}^n) \hookrightarrow \mathfrak{M}_{\text{nf}}(M, \mathbb{R}^n)$ of the space of *complete* nonflat minimal immersions $M \rightarrow \mathbb{R}^n$ in $\mathfrak{M}_{\text{nf}}(M, \mathbb{R}^n)$ also satisfies the parametric homotopy principle, whence is a weak homotopy equivalence. When M is of finite topological type, the inclusion is a genuine homotopy equivalence. The following corollary (see [58, Corollary 1.6]) is seen by analysing the path components of the space $\mathcal{C}(M, \mathbb{A}_*)$, a purely topological issue.

Corollary 5.2. *Let M be a connected open Riemann surface with $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^l$, $l \in \mathbb{Z}_+ \cup \{\infty\}$. Then the path connected components of each of the spaces $\mathfrak{M}_{\text{nf}}(M, \mathbb{R}^3)$, $\mathfrak{M}_{\text{nf}}^c(M, \mathbb{R}^3)$, and $\mathfrak{N}_{\text{nf}}(M, \mathbb{C}^3)$ are in one-to-one correspondence with the elements of the abelian group $(\mathbb{Z}_2)^l$. If $n \geq 4$ then these spaces are path connected.*

The *total curvature* of a conformal minimal surface $u : M \rightarrow \mathbb{R}^n$ is the integral $\text{TC}(u) = \int_M K d\sigma$ of the Gaussian curvature function $K : M \rightarrow (-\infty, 0]$ of u with respect to the area measure $d\sigma$ on M induced by the immersion u . A minimal surface $u : M \rightarrow \mathbb{R}^n$ is said to have *finite total curvature* if $\text{TC}(u) > -\infty$. Complete minimal surfaces of finite total Gaussian curvature are among the most intensively studied minimal surfaces which play an important role in the classical global theory; see [115] and [7, Chap. 4], among other sources. In fact, most classical examples of minimal surfaces are of this kind. Although this family of minimal surfaces has been a focus of interest since the seminal work of Osserman in the 1960s, the theories of approximation and interpolation for complete minimal surfaces of finite total curvature in $\mathbb{R}^n$ have been developed only recently; see [1, 11, 110]. The paper [2] by Alarcón, Forstnerič, and Lárusson provides the first contributions to the homotopy theory of this important class of minimal surfaces, which we now present.

The null quadric $\mathbb{A}$ (5.2) defines a holomorphic subbundle $\mathcal{A}$ of the vector bundle $(T^*M)^{\oplus n}$ with fibre $\mathbb{A}_*$ whose sections are n -tuples $\phi = (\phi_1, \dots, \phi_n)$ of $(1, 0)$ -forms on M without common zeros such that the map $[\phi_1 : \dots : \phi_n] : M \rightarrow \mathbb{CP}^{n-1}$ takes values in the projective quadric

$$(5.4) \quad Q = \{[z_1 : z_2 : \dots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + z_2^2 + \dots + z_n^2 = 0\}.$$

If $u : M \rightarrow \mathbb{R}^n$ is a conformally immersed minimal surface with finite total curvature, then M is the complement in a compact Riemann surface $\overline{M}$ of a nonempty finite set $E = \{x_1, \dots, x_m\}$ whose points are called the ends of M (such M is called an *affine Riemann surface*), the bundle $\mathcal{A} \rightarrow M$ is algebraic (it is holomorphically trivial but not necessarily algebraically trivial), and $\phi = \partial u$ is a meromorphic 1-form on $\overline{M}$ without zeros or poles on M , that is, an algebraic section of $\mathcal{A}$ over M . Furthermore, the immersion u is complete if and only if ∂u has an effective pole at every point of $E = \overline{M} \setminus M$. The surface $u(M)$ is then properly immersed in $\mathbb{R}^n$ and has a fairly simple asymptotic behaviour at every end of M , described by the Jorge–Meeks theorem [90]. Denote by $\mathcal{A}^1(M, \mathbb{A}_*)$ the space of meromorphic 1-forms $\phi = (\phi_1, \dots, \phi_n)$ on $\overline{M}$ having no zeros or poles in M and satisfying (5.1), that is, algebraic sections of the bundle $\mathcal{A} \rightarrow M$. Consider the diagram

$$(5.5) \quad \begin{array}{ccc} \mathfrak{RN}_*(M, \mathbb{C}^n) & \hookrightarrow & \mathfrak{M}_*(M, \mathbb{R}^n) \\ & \searrow \partial & \downarrow \partial \\ & & \mathcal{A}^1(M, \mathbb{A}_*) \end{array}$$

where ∂ is the $(1, 0)$ -differential, $\mathfrak{N}_*(M, \mathbb{C}^n)$ is the space of nonflat complete (proper) algebraic null immersions $M \rightarrow \mathbb{C}^n$, and $\mathfrak{M}_*(M, \mathbb{R}^n)$ is the space of nonflat conformal minimal immersions of finite total curvature. These spaces are endowed with the compact-open topology. The following is [2, Theorem 1.1].

Theorem 5.3. *If M is an affine Riemann surface then the maps in (5.5) are weak homotopy equivalences.*

The proof in [2] builds upon that of Theorem 5.1, using a multiparameter extension of the algebraic approximation theorem in Theorem 4.8, together with a method invented by Alarcón and Lárusson [9] to ensure the existence of effective poles at the ends of M (to get completeness). For the inclusion $\mathcal{RN}_*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$, Theorem 5.3 says in particular that every complete nonflat conformal minimal immersion $M \rightarrow \mathbb{R}^n$ of finite total curvature can be deformed through maps of the same type to the real part of a proper algebraic null curve $M \rightarrow \mathbb{C}^n$. In addition, one can control the flux along an isotopy in $\mathcal{M}_*(M, \mathbb{R}^n)$; see [2, Theorem 1.2].

The Gauss map of a conformal minimal immersion $u = (u_1, u_2, \dots, u_n) : M \rightarrow \mathbb{R}^n$ is the holomorphic map

$$G(u) = [\partial u_1 : \partial u_2 : \dots : \partial u_n] : M \rightarrow Q \subset \mathbb{CP}^{n-1}$$

in the projective hyperquadric (5.4). When $n = 3$, Q is a quadratic rational curve in $\mathbb{CP}^2$, hence biholomorphic to the Riemann sphere $\mathbb{CP}^1$. In this case, the Gauss map is identified with a meromorphic function on M , and it corresponds to the standard Gauss map when identifying $\mathbb{CP}^1$ with the 2-sphere. See [7, Sect. 2.5] for the details. The following result of Alarcón, Forstnerič, and López [6] (see also [7, Theorem 5.4.1]) can be seen as an h-principle for the Gauss map of minimal surfaces.

Theorem 5.4. *Let M be an open Riemann surface and $n \geq 3$ be an integer. For every holomorphic map $\mathcal{G} : M \rightarrow Q \subset \mathbb{CP}^{n-1}$ into the quadric (5.4) there exists a conformal minimal immersion $u : M \rightarrow \mathbb{R}^n$ with the Gauss map $G(u) = \mathcal{G}$ and vanishing flux (i.e., u is the real part of a holomorphic null curve $M \rightarrow \mathbb{C}^n$). If $\mathcal{G}(M)$ is not contained in any proper projective subspace of $\mathbb{CP}^{n-1}$ then u can be chosen to have arbitrary flux and to be an embedding if $n \geq 5$ and an immersion with simple double points if $n = 4$.*

The theory of the Gauss map of minimal surfaces has a long and rich history, and it came as a surprise that every natural candidate map is the Gauss map of some minimal surface. Further results on the space of Gauss maps of complete minimal surfaces were obtained by Alarcón and Lárusson [8].

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