FAMILIES OF PROPER MINIMAL SURFACES

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ABSTRACT. Assume that X is a connected, open, oriented smooth surface, B is a compact Euclidean neighbourhood retract, and $\mathscr{J} = \{J_b\}_{b\in B}$ is a continuous family of complex structures on X of local Hölder class $\mathscr{C}^\alpha$ for some $0 < \alpha < 1$. We construct a continuous family of J_b-conformal minimal immersions $u_b : X \to \mathbb{R}^3$, $b \in B$, properly projecting to $\mathbb{R}^2$ and having an arbitrary given family of flux homomorphisms $\mathrm{Flux}_{u_b} : H_1(X,\mathbb{Z}) \to \mathbb{R}^3$. In particular, there are continuous families of proper J_b-holomorphic null immersions $X \to \mathbb{C}^3$ and of proper J_b-holomorphic immersions $X \to \mathbb{C}^2$, $b \in B$.

1. INTRODUCTION

In this paper, we establish the existence of families of proper minimal surfaces in Euclidean space $\mathbb{R}^3$ with varying but prescribed complex structures, depending continuously on a parameter in a compact Euclidean neighbourhood retract.

The global theory of minimal surfaces in $\mathbb{R}^3$ has been a major focus of interest since the fundamental developments by Osserman in the 1960s [31], with emphasis on the conformal properties of such surfaces under global assumptions such as completeness or properness; see e.g. [28]. Until not too long ago, it was widely believed that properness of a minimal surface in $\mathbb{R}^3$ imposes strong restrictions on its underlying complex structure. In this direction, it was conjectured that a proper minimal surface in $\mathbb{R}^3$ having either finite topology (Sullivan) or a proper projection into a plane (Schoen-Yau) must have parabolic conformal type in the sense of Ahlfors-Nevanlinna [30, 1] (that is, it does not carry any nonconstant negative subharmonic function [3, Section IV.6]); see [32, p. 18] and [7, Section 3.10]. A counterexample to Sullivan's conjecture was given in 2003 by Morales [29], who constructed a proper minimal surface with the conformal type of the disc. Later, in 2012, Alarcón and López proved that every open Riemann surface is the complex structure of a minimal surface in $\mathbb{R}^3$ properly projecting into a plane [11], thereby settling both conjectures in the most general possible way. Nevertheless, almost all minimal surfaces in $\mathbb{R}^3$ (in the natural topological sense of Baire category) are nonproper [10, Corollary 1.5], so it is fair to say that proper ones are hard to find.

In this paper, X is a connected, open, orientable smooth surface with a countable base of topology, B is a compact subset of a Euclidean space $\mathbb{R}^m$ which admits a retraction from an open neighbourhood $U \subset \mathbb{R}^m$ (every finite CW complex is such; see [21, Definition 1.5] and the subsequent discussion), and $\mathscr{J} = \{J_b\}_{b\in B}$ is a continuous family of complex structures on X of local Hölder class $\mathscr{C}^\alpha$ for some $0 < \alpha < 1$. See Section 2 for the details. Here is our main result; see also Remark 3.6. This gives an affirmative answer to [21, Problem 8.7 (c)].

Theorem 1.1. *Assume that X, B, and $\mathscr{J}$ are as above. Then there exists a continuous map $u = (u_1, u_2, u_3) : B \times X \to \mathbb{R}^3$ such that the map $u_b = u(b,\cdot) = (u_{b,1}, u_{b,2}, u_{b,3}) : X \to \mathbb{R}^3$ is a proper J_b-conformal minimal immersion for every $b \in B$. Furthermore, u can be chosen such that every map $(u_{b,1}, u_{b,2}) : X \to \mathbb{R}^2$ is proper and the flux Flux_{u_b} of u_b equals any given continuous family of homomorphisms $\mathcal{F}_b : H_1(X,\mathbb{Z}) \to \mathbb{R}^3$ for $b \in B$.*

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Let us clarify the technical notions in the statement of the theorem. Assume that J is a complex structure on X, so (X,J) is an open Riemann surface. Let $J_{\rm st}$ denote the standard complex structure on $\mathbb{C}$, given by multiplication by $\mathrm{i}=\sqrt{-1}$. The exterior differential d on X then splits in the sum $d=\partial+\overline\partial$ of the $(J,J_{\rm st})$-linear part $\partial=\partial_J$ and the $(J,J_{\rm st})$-antilinear part $\overline\partial=\overline\partial_J$, with J-holomorphic functions being the kernel of $\overline\partial$. The operator $d^c=d^c_J=\mathrm{i}\,(\overline\partial-\partial)=2\,\Im(\partial)$ is the conjugate differential, and $dd^c=2\mathrm{i}\,\partial\overline\partial=-2\mathrm{i}\overline\partial\partial$ is the Laplace operator whose kernel are the J-harmonic functions. A smooth immersion $u=(u_1,u_2,\ldots,u_n):X\to\mathbb{R}^n$, $n\ge 2$, is J-conformal if and only if the Riemannian metric on X obtained by pulling back the Euclidean metric on $\mathbb{R}^n$ via the immersion u, together with the chosen orientation on X, induces the given complex structure J; equivalently, if $\partial u=(\partial u_1,\partial u_2,\ldots,\partial u_n)$ satisfies the nullity condition

$$(\partial u_1)^2+(\partial u_2)^2+\cdots+(\partial u_n)^2=0.$$

Assume now that $n\ge 3$. A J-conformal immersion $u:X\to\mathbb{R}^n$ is minimal, in the sense that it parameterises a minimal surface in $\mathbb{R}^n$, if and only if it is J-harmonic; equivalently, ∂u is a J-holomorphic 1-form. The *flux* of such a map u is the homomorphism $\mathrm{Flux}_u:H_1(X,\mathbb{Z})\to\mathbb{R}^n$ defined on any closed curve $C\subset X$ by $\mathrm{Flux}_u(C)=\oint_C d^c u$, where $d^c u$ is the conjugate differential (see [7, Definition 2.3.2]). Since $dd^c u=0$, $\oint_C d^c u$ only depends on the homology class $[C]\in H_1(X,\mathbb{Z})$. (Recall that $H_1(X,\mathbb{Z})\cong\mathbb{Z}^\ell$ for some $\ell\in\{0,1,\ldots,\infty\}$.) In particular, $\mathrm{Flux}_u=0$ if and only if u has a J-harmonic conjugate $v:X\to\mathbb{R}^n$, and in this case, $F=u+\mathrm{i}v:X\to\mathbb{C}^n$ is a J-holomorphic immersion whose differential $dF=(dF_1,dF_2,\ldots,dF_n)$ satisfies $(dF_1)^2+(dF_2)^2+\cdots+(dF_n)^2=0$. Such F is called a J-holomorphic null curve in $\mathbb{C}^n$.

Let $\mathscr{J}=\{J_b\}_{b\in B}$ be a family of complex structures on X as above. A continuous map $F:B\times X\to\mathbb{C}^n$ is said to be *$\mathscr{J}$-holomorphic* if the map $F_b=F(b,\cdot):X\to\mathbb{C}^n$ is J_b-holomorphic for every $b\in B$. Taking $\mathcal{F}_b=0$ for all $b\in B$ in Theorem 1.1 gives the following corollary, which provides an affirmative answer to [21, Problem 8.7 (b)] for families of holomorphic null immersions.

Corollary 1.2. *If X, B, and $\mathscr{J}$ are as in Theorem 1.1, then there exists a $\mathscr{J}$-holomorphic map $F=(F_1,F_2,F_3):B\times X\to\mathbb{C}^3$ such that for every $b\in B$, the map $F_b=F(b,\cdot)=(F_{b,1},F_{b,2},F_{b,3}):X\to\mathbb{C}^3$ is a J_b-holomorphic null immersion and the map $(\Re F_{b,1},\Re F_{b,2}):X\to\mathbb{R}^2$ is proper. In particular, the map $F_b:X\to\mathbb{C}^3$ is proper for every $b\in B$.*

For B a singleton, Theorem 1.1 and Corollary 1.2 were proved by Alarcón and López [11] in a more precise form with approximation on compact Runge sets in X. The higher dimensional case of minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$ for $n>3$ was treated in [4, 6] and [7, Sec. 3.10]. The h-principle for families of such maps (not necessarily proper) from a single open Riemann surface was obtained in [23], while the parametric h-principle for the inclusion of the subset of complete immersions was established in [9]. Families of not necessarily proper or complete minimal surfaces and null curves from a variable family of open Riemann surfaces were first constructed by the second named author in [21, Sec. 8]. The nontrivial addition in Theorem 1.1 and Corollary 1.2 is to ensure properness of maps in such families.

Taking the first two components of a map in Corollary 1.2 gives the following corollary, which provides an affirmative answer to [21, Problem 8.7 (a)].

Corollary 1.3. *If X, B, and $\mathscr{J}$ are as in Theorem 1.1, then there exists a $\mathscr{J}$-holomorphic map $F=(F_1,F_2):B\times X\to\mathbb{C}^2$ such that for every $b\in B$, the J_b-holomorphic map $F_b:X\to\mathbb{C}^2$ is an immersion and the J_b-harmonic map $\Re F_b=(\Re F_{b,1},\Re F_{b,2}):X\to\mathbb{R}^2$ is proper. Hence, the immersion F_b is proper for every $b\in B$.*

The special case of Corollary 1.3 for a single map is related to the more ambitious conjecture by Schoen and Yau [32, p. 18] that *no hyperbolic open Riemann surface admits a proper harmonic* *map into* $\mathbb{R}^2$. The first counterexample to their conjecture was given by Božin [17], who found an explicit example of a proper harmonic map $\mathbb{D} \to \mathbb{R}^2$ from the disc. Another counterexample was given by Forstnerič and Globevnik [22], who constructed a proper holomorphic map $(f_1, f_2) : \mathbb{D} \to \mathbb{C}^2$ with nowhere vanishing components, so $(\log|f_1|, \log|f_2|) : \mathbb{D} \to \mathbb{R}^2$ is a proper harmonic map. Counterexamples with arbitrary finite topology were obtained by Alarcón and Gálvez [8]. Finally, a construction of a proper harmonic map from any open Riemann surface into $\mathbb{R}^2$, different from the one in [11], was carried out by Andrist and Wold [14]. See also the discussion in [7, Sec. 3.10].

Recently, Drinovec Drnovšek and Kališnik [18] proved that for every X and $\mathscr{J} = \{J_b\}_{b\in B}$ as above there exists a $\mathscr{J}$-holomorphic map $F : B \times X \to \mathbb{C}^2$ such that the real part $\Re F_b : X \to \mathbb{R}^2$ of F_b is a proper J_b-harmonic map for every $b \in B$. (Their result holds for all metric parameter spaces B.) For parameter spaces B used in this paper, Corollary 1.3 improves their result in that the map $F_b : X \to \mathbb{C}^2$ is a proper *immersion* for every $b \in B$. Nevertheless, assuming that B is as in [21, Theorem 1.6] and using the existence of continuous families of J_b-holomorphic immersions $X \to \mathbb{C}$ [21, Corollary 8.3], it is possible to upgrade the proof in [18] to obtain a continuous family of proper J_b-holomorphic maps $F_b = (F_{b,1}, F_{b,2}) : X \to \mathbb{C}^2$ such that every component function $F_{b,i} : X \to \mathbb{C}$ for $b \in B$ and $i = 1, 2$ is an immersion.

The restriction to compact parameter spaces B in this paper is mainly out of convenience since it enables a geometrically simpler construction in the proofs. We believe that, with more work and using some ideas and technical arguments from the construction in [18], the results could be extended to countably compact spaces of the type used in [21, Theorem 1.6], including noncompact ones. Furthermore, we expect that one can prove a Runge approximation theorem for families of proper conformal minimal immersions in $\mathbb{R}^n$ for any $n \geq 3$.

Note that the families of open Riemann surfaces $\{(X, J_b)\}_{b\in B}$ in the above results are considerably more general than the Teichmüller families, in which the individual surfaces are quasiconformally equivalent to one another. Let us look at the topologically simplest examples when the surface X is either simply or doubly connected. Every complex structure on $X = \mathbb{R}^2$ is biholomorphically equivalent to the standard complex structure J_{st} on $\mathbb{C}$ or to its restriction to the unit disc $\mathbb{D} \subset \mathbb{C}$. We can put both examples in a compact connected family parameterised by $B = [0, 1]$. Every complex structure on $X = \mathbb{R}^2 \setminus \{0\}$ is equivalent to J_{st} restricted to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or to an annulus $A_r = \{z \in \mathbb{C} : r < |z| < 1\}$ for $0 \leq r < 1$. For any $0 < r_0 < 1$ one can realise all these structures with $0 \leq r \leq r_0$ in a smooth family with the parameter space $B = [0, 1]$.

Our proof of Theorem 1.1 strongly depends on the recent work [21] by the second named author. For suitable parameter spaces B (more precisely, for local Euclidean neighbourhood retracts, a class of topological spaces which includes all finite CW complexes and more generally all countable locally compact CW-complexes of finite dimension), the main result of that paper is an Oka principle with approximation for $\mathscr{J}$-holomorphic maps $B \times X \to Y$ to any Oka manifold Y; see [21, Theorem 1.6]. Applications include the construction of continuous families of holomorphic immersions $X \to \mathbb{C}^n$ for any $n \geq 1$, of holomorphic null immersions $X \to \mathbb{C}^n$ for any $n \geq 3$, and of conformal minimal immersions $X \to \mathbb{R}^n$ with any given family of flux homomorphisms for $n \geq 3$; see [21, Sec. 8]. The main novelty of the results in this paper is that we construct families of proper maps of the given types; in the case of holomorphic immersions $X \to \mathbb{C}^n$, this requires $n \geq 2$. Properness is a nontrivial condition which is not easily achieved even for single maps. In fact, proper maps form a meagre (topologically small) subset in any class of maps considered above, so it is not surprising that it is difficult to construct families of proper maps. With the approximation results from [21] in hand, our proof broadly follows the original construction in [11] (see also [7, Section 3.11]) of a minimal surface in $\mathbb{R}^3$ with given complex structure and proper projection into a plane. However, our approach introduces several major differences required to adapt the argument to the parametric setting. In particular, to compensate the lack of a parametric version of the Runge approximation theorem for conformal minimal immersions with fixed component functions (see [11, Corollary 4.8 and Theorem 4.9] or [7, Section 3.7]), which has been a crucial tool in all previous constructions of proper minimal surfaces with arbitrary complex structure, we employ a parametric version of the López-Ros deformation for minimal surfaces [27], developed by Alarcón and Lárusson in [9], together with improved parametric versions of Gromov's convex integration lemma [25] (see Section 2.3).

We mention several open problems related to the results of this paper.

The analogous results for single maps in the above mentioned works hold with approximation on compact Runge subsets of X. We expect that this generalisation is also possible in the present setting, but the proofs become more complicated from the technical viewpoint.

The second question is whether an h-principle holds in these results. It has recently been shown by Vrhovnik [34] that every nonflat conformal minimal immersion $X \to \mathbb{R}^n$, $n \geq 3$, is homotopic through a family of such immersions to a proper one. Does the analogous result hold for families?

The maps in our main results are immersions. Every open Riemann surface X admits many proper conformal minimal *embeddings* $X \hookrightarrow \mathbb{R}^n$ for any $n \geq 5$, and proper holomorphic null embeddings $X \hookrightarrow \mathbb{C}^n$ for any $n \geq 3$ (see [4, 6], and [7, Sec. 3.10]). What can be said in this respect for families?

Assuming that the parameter space B is a smooth manifold, one may expect that refinements of the techniques used in this paper, which are available in [21], yield families of immersions of the given types depending smoothly on the parameter. In such a case, one may also hope that the use of transversality methods would yield families of embeddings of the given types when the target Euclidean space has sufficiently big dimension. We also expect that the analogue of Theorem 1.1 holds for continuous families of conformal structures on a nonorientable smooth open surface X. Results in this direction for a single conformal structure on X can be found in [13, 5].

The final problem concerns the universal family $V(g, n)$ of n-punctured compact Riemann surfaces of genus g with $n \geq 1$. The corresponding parameter space is the Teichmüller space $T(g, n)$, which can be holomorphically realised as a bounded contractible Stein domain in a complex Euclidean space, and the natural Teichmüller projection $\pi : V(g, n) \to T(g, n)$ is a holomorphic submersion whose fibre over any point $b \in T(g, n)$ is the n-puncture compact Riemann surface (X, J_b) of genus g with the complex structure determined by b. It has recently been proved by the second named author that $V(g, n)$ is a Stein manifold [19, Theorem 1.1]. Furthermore, there is a holomorphic map $F : V(g, n) \to \mathbb{C}^N$ for some $N \in \mathbb{N}$ whose restriction $F_b = F(b, \cdot\,) : (X, J_b) \to \mathbb{C}^N$ to any fibre is a proper algebraic embedding [19, Corollary 3.5]. Can one choose F such that F_b is a proper algebraic null embedding for every $b \in T(g, n)$? If so then the real part $\Re F : V(g, n) \to \mathbb{R}^N$ restricted to any fibre is a complete proper minimal surface with finite total curvature.

2. Preliminaries

2.1. **Families of complex structures on surfaces.** Let X be a smooth, connected, orientable surface endowed with a smooth Riemannian metric. A complex structure on X is given by an endomorphism $J : TX \to TX$ of its tangent bundle satisfying $J^2 = -\mathrm{Id}$. Thus, J is a section of the smooth vector bundle $T^*X \otimes TX \to X$ whose fibre over $x \in X$ is the space $\mathrm{Hom}(T_xX, T_xX)$ of linear maps $T_xX \mapsto T_xX$. (Note that every orientable vector bundle on an open surface X is trivial. When X is orientable, this holds in particular for the tangent bundle TX and its derived vector bundles.) The multiplication by $\mathfrak{i} = \sqrt{-1}$ defines the standard complex structure J_{st} on $\mathbb{C}$. A differentiable function $f : X \to \mathbb{C}$ is *J-holomorphic* if it satisfies the Cauchy–Riemann equation

$$df_x \circ J_x = \mathfrak{i}\, df_x = J_{\mathrm{st}} df_x, \quad x \in X.$$

A complex structure J is said to be of local Hölder class $\mathscr{C}^{(k,\alpha)}$ for some $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $0 < \alpha < 1$ if, for every relatively compact domain $\Omega \Subset X$, the restriction $J|_\Omega \in \Gamma^{(k,\alpha)}(\Omega, T^*\Omega \otimes T\Omega)$ is a section of class $\mathscr{C}^{(k,\alpha)}(\Omega)$ of the restricted vector bundle $T^*\Omega \otimes T\Omega \to \Omega$. (The Hölder norms are computed with respect to the given Riemannian metric on X; see Gilbarg and Trudinger [24, Sect. 4.1].) For such J, there is an atlas $\{(U_i, \phi_i)\}_i$ of open sets $U_i \subset X$ covering X and J-holomorphic charts $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}$ of class $\mathscr{C}^{(k+1,\alpha)}(U_i)$. (See [15, Theorem 5.3.4] among other references.) It follows that the smooth structure on X determined by a complex structure J of local class $\mathscr{C}^{(k,\alpha)}$ is $\mathscr{C}^{(k+1,\alpha)}$ compatible with the given smooth structure on X; see [16, Theorem 2.1].

A family $\mathscr{J} = \{J_b\}_{b\in B}$ of complex structures on X is said to be of class $\mathscr{C}^{0,(k,\alpha)}$ if for any relatively compact domain $\Omega \Subset X$ the map $B \ni b \mapsto J_b|_\Omega \in \Gamma^{(k,\alpha)}(\Omega, T^*\Omega \otimes T\Omega)$ is continuous. If $k = 0$, we write $\mathscr{C}^{(0,\alpha)}(\Omega) = \mathscr{C}^{\alpha}(\Omega)$. The proofs of our main results immediately extend to the case when $\mathscr{J}$ is of class $\mathscr{C}^{0,(k,\alpha)}$ for any $k \in \mathbb{Z}_+$, and they yields maps as in Theorem 1.1 and Corollary 1.2 which are of class $\mathscr{C}^{0,(k+1,\alpha)}$ in the reference complex structure on X. (See [21, Theorem 8.2 and Corollary 8.6] for the construction of families of not necessarily proper maps of these classes.)

If the parameter space B is a manifold of class $\mathscr{C}^l$ for some $l > 0$, we can also introduce the notion of a family $\mathscr{J}$ of class $\mathscr{C}^{l,(k,\alpha)}$; see [21]. In view of [21, Theorem 1.6], one may expect that our main results can be extended to families of conformal minimal immersions and null immersions of class $\mathscr{C}^{l,(k+1,\alpha)}$ whenever $0 \le l \le k+1$ and $0 < \alpha < 1$. However, for simplicity we shall only consider the case $l = 0$ in this paper, that is, continuous dependence on the parameter.

Fix a reference smooth complex structure J on X. By a theorem of Gunning and Narasimhan [26], the open Riemann surface (X, J) admits a holomorphic immersion $z : X \to \mathbb{C}$, which therefore provides a local J-holomorphic coordinate on X at every point. A family $\mathscr{J} = \{J_b\}_{b\in B}$ of complex structures in the same orientation class as J can then equivalently be given by a family of maps $\mu_b : X \to \mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ of the same local smoothness class as J_b and the same regularity in the parameter $b \in B$, with $\mu_b = 0$ corresponding to the reference structure $J_b = J$. (See [21, Sec. 2] for the details.) Such a function μ_b is called a Beltrami multiplier. Given an open subset $U \subset X$, any solution $f : U \to \mathbb{C}$ of the Beltrami equation

$$f_{\bar z} = \mu_b f_z \tag{2.1}$$

is a holomorphic map from (U, J_b) to $(\mathbb{C}, J_{\rm st})$. (If U is a domain in $\mathbb{C}$ and $\|\mu_b\|_{U,\infty} < 1$, where $\|\cdot\|_\infty$ is the essential supremum, a map f satisfying (2.1) is also called μ_b-quasiconformal.) The theory of quasiconformal maps on domains in $\mathbb{C}$ was developed by Ahlfors and Bers [2], also for Beltrami multipliers in L^p spaces for $p > 1$. (See the survey and references in [21, Sec. 2].) When $2 < p < \infty$, any local solution of (2.1) is of Hölder class $\mathscr{C}^\alpha$ with $\alpha = 1 - 2/p$, while for Beltrami multipliers of class $\mathscr{C}^{(k,\alpha)}$ with $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, the equation (2.1) has local solutions of class $\mathscr{C}^{(k+1,\alpha)}$. Furthermore, solutions in these function spaces can be chosen to depend locally analytically on μ, so they are as regular in the parameter b as the family μ_b (or, equivalently, J_b).

Ahlfors and Bers also developed the global theory of quasiconformal maps on planar domains. By [2, Theorem 6, p. 396], for any μ on the plane or the disc with $\|\mu\|_\infty < 1$ there exist unique μ-conformal homeomorphisms of the whole plane and the unit disk onto themselves with fixed points at 0, 1, ∞ and 0,1 respectively. We state the following special case for later reference.

Proposition 2.1. *Given a continuous family $\{J_b\}_{b\in B}$ of complex structures of class $\mathscr{C}^\alpha(\bar{\mathbb{D}})$ on the closed disc $\bar{\mathbb{D}}$, there is a unique continuous family of diffeomorphisms $\phi_b : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ of class $\mathscr{C}^{(1,\alpha)}(\bar{\mathbb{D}})$ such that $\phi_b : \mathbb{D} \to \mathbb{D}$ is J_b-holomorphic and satisfies $\phi_b(0) = 0,\ \phi_b(1) = 1$ for every $b \in B$.*

A similar statement holds for a continuous family $\{J_b\}_{b\in B}$ of complex structures of class $\mathscr{C}^\alpha$ on $\mathbb{R}^2$ which are quasiconformally equivalent to the standard structure $J_{\rm st}$ on $\mathbb{C}$.

The Ahlfors–Bers theory extends to smoothly bounded relatively compact domains in smooth open surfaces. If $\mathscr{J} = \{J_b\}_{b\in B}$ is a continuous family of complex structures of local class $\mathscr{C}^{(k,\alpha)}$ on a smooth open surface X and Ω is a smoothly bounded relatively compact domain in X, then every point $b_0 \in B$ has a neighbourhood $B_0 \in B$ and a continuous family of (J_b, J_{b_0})-holomorphic diffeomorphisms $\Phi_b : \Omega \to \Phi_b(\Omega) \subset X$, $b \in B_0$, of class $\mathscr{C}^{(k+1,\alpha)}(\Omega)$ (see [21, Corollary 4.5]). This result plays a major role in the proof of the Oka principle for $\mathscr{J}$-holomorphic maps $F : B \times X \to Y$ to any Oka manifold Y (see [21, Theorem 1.6]). In turn, this Oka principle is the underlying tool to prove the results in this paper. It will be applied for maps to the punctured null quadric

$$\mathbf{A} = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n_* = \mathbb{C}^n \setminus \{0\} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\big\}, \quad n \geq 3, \tag{2.2}$$

and also to some other Oka manifolds. (For the theory of Oka manifolds, see [20, Chap. 5].)

2.2. **Holomorphic null immersions and conformal minimal immersions.** We recall the basics on holomorphic null immersions and conformal minimal immersions of open Riemann surfaces to $\mathbb{C}^n$ and $\mathbb{R}^n$, respectively. For more information, see the monographs [31, 7], among many other sources.

A holomorphic immersion $F = (F_1, F_2, \dots, F_n) : X \to \mathbb{C}^n$, $n \geq 3$, from an open Riemann surface X is said to be a null immersion if its differential dF satisfies the nullity condition

$$(dF_1)^2 + (dF_2)^2 + \cdots + (dF_n)^2 = 0.$$

Choosing a nowhere vanishing holomorphic 1-form θ on X (such exists by [26]), we have $dF = f\theta$ where $f = (f_1, \dots, f_n) : X \to \mathbf{A}$ is a holomorphic map to the punctured null quadric (2.2). Conversely, fixing a point $x_0 \in X$, a holomorphic map $f : X \to \mathbf{A}$ such that $f\theta$ is an exact 1-form on X determines a holomorphic null immersion $F : X \to \mathbb{C}^n$ by

$$F(x) = F(x_0) + \int_{x_0}^{x} f\theta, \qquad x \in X. \tag{2.3}$$

An immersion $u = (u_1, u_2, \dots, u_n) : X \to \mathbb{R}^n$, $n \geq 3$, is conformal if and only if its $(1,0)$-differential $\partial u = (\partial u_1, \partial u_2, \dots, \partial u_n)$ satisfies

$$(\partial u_1)^2 + (\partial u_2)^2 + \cdots + (\partial u_n)^2 = 0.$$

A conformal immersion u is minimal if and only if it is harmonic,

$$dd^c u = -2\mathrm{i}\overline{\partial}\partial u = 0.$$

In this case, $2\partial u = f\theta$ where $f : X \to \mathbf{A}$ is a holomorphic map, and we recover u from f by

$$u(x) = u(x_0) + \Re \int_{x_0}^{x} f\theta, \qquad x \in X. \tag{2.4}$$

The formulas (2.3) and (2.4) are known as the *Weierstrass representation* of holomorphic null curves and conformal minimal surfaces, respectively, and $f\theta$ is called the Weierstrass data of F or u.

In the special case $n = 3$ considered in this paper, a more explicit Weierstrass representation formula for holomorphic null immersions $F : X \to \mathbb{C}^3$ with $dF = f\theta = (f_1, f_2, f_3)\theta$ is given by

$$F(x) = F(x_0) + \int_{x_0}^{x} \Big(1, \frac{1}{2}\Big(g - \frac{1}{g}\Big), \frac{\mathrm{i}}{2}\Big(g + \frac{1}{g}\Big)\Big) f_1\theta, \tag{2.5}$$

where $g : X \to \mathbb{CP}^1$ is, up to a change of coordinates, the (holomorphic) Gauss map of F. (See [7, p. 101 and Sec. 2.5] for the details.) The real part of the above integral gives a formula reproducing conformal minimal surfaces $u : X \to \mathbb{R}^3$ from its Weierstrass data.

Given a continuous family $\mathscr{J} = \{J_b\}_{b\in B}$ of complex structures on X, there is a continuous family $\{\theta_b\}_{b\in B}$ of nowhere vanishing J_b-holomorphic 1-forms on X (see [21, Theorem 7.1]). Using such a family, the analogous representation formulas hold for families of $\mathscr{J}$-holomorphic null curves and conformal minimal surfaces with continuous dependence on $b \in B$.

Let us record the following consequence of Proposition 2.1.

Proposition 2.2. *Assume that $u : \mathbb{D} \to \mathbb{R}^n$ is a conformal minimal immersion. Given a continuous family of complex structures $\mathscr{J} = \{J_b\}_{b\in B}$ of class $\mathscr{C}^\alpha(\bar{\mathbb{D}})$ on $\bar{\mathbb{D}}$, there is a continuous family of J_b-conformal harmonic immersions $u_b : \mathbb{D} \to \mathbb{R}^n$ satisfying $u_b(\mathbb{D}) = u(\mathbb{D})$ for all $b \in B$.*

Proof. Letting $\phi_b : \mathbb{D} \to \mathbb{D}$ be a family of J_b-holomorphic diffeomorphisms given by Proposition 2.1, the family $u_b = u \circ \phi_b : \mathbb{D} \to \mathbb{R}^n$, $b \in B$, clearly satisfies Proposition 2.2. The analogous conclusion holds for null curves $\mathbb{D} \to \mathbb{C}^n$. □

2.3. **Convex integration lemmas.** We shall use the parametric version of Gromov's *convex integration lemma*; see Gromov [25, 2.1.7. One-Dimensional Lemma] for the basic case and Spring [33, Theorem 3.4, p. 39] for the parametric case. By $\mathrm{Co}(A)$ we denote the convex hull of a subset $A \subset \mathbb{R}^m$. The first lemma concerns the punctured null quadric $\mathbf{A} \subset \mathbb{C}^n$, $n \ge 3$ (2.2).

Lemma 2.3. *Let B be a compact Hausdorff space, $f_0 : B \times [0,1] \to \mathbf{A}$ a continuous map, and $F_0 : B \times [0,1] \to \mathbb{C}^n$ a continuous map such that $F_0(b,\cdot) : [0,1] \to \mathbb{C}^n$ is of class $\mathscr{C}^1([0,1])$, its t-derivative $\dot F_0(b,t) = \frac{\partial}{\partial t} F_0(b,t)$ depends continuously on $(b,t) \in B \times [0,1]$, and*

$$\dot F_0(b,0) = f_0(b,0), \quad \dot F_0(b,1) = f_0(b,1) \quad \text{for all } b \in B.$$

Given $\epsilon > 0$ there is a continuous map $F : B \times [0,1] \to \mathbb{C}^n$ satisfying the following for all $b \in B$:

(a) *In the endpoints $t = 0, 1$ of $[0,1]$, the values of $F(b,\cdot)$ and its first t-derivatives coincide with those of $F_0(b,\cdot)$.*
(b) $\|F - F_0\|_\infty < \epsilon$.
(c) $f(b,t) := \dot F(b,t) \in \mathbf{A}$ *for all $t \in [0,1]$.*

Proof. When $B = \{b_0\}$ is a singleton and $\mathbf{A}$ is replaced by an open subset $\Omega \subset \mathbb{C}^n$ with $\mathrm{Co}(\Omega) = \mathbb{C}^n$, the lemma coincides with [25, 2.1.7]. The parametric case follows from [33, Theorem 3.4, p. 39]. See also [23, proof of Lemma 3.1], where it is shown how to replace an open subset $\Omega \subset \mathbb{C}^n$ in the stated result with the submanifold $\mathbf{A} \subset \mathbb{C}^n$. The idea of proof is to let Ω be a small open tubular neighbourhood of $\mathbf{A}$, with a smooth retraction $\rho : \Omega \to \mathbf{A}$. Hence, $\mathrm{Co}(\Omega) = \mathbb{C}^n$. Applying [33, Theorem 3.4, p. 39] to maps $B \times [0,1] \to \Omega$ and postcomposing them by the retraction $\rho : \Omega \to \mathbf{A}$ gives a map $f : B \times [0,1] \to \mathbf{A}$, obtained by suitably deforming the map f_0 with fixed ends at $t = 0, 1$, such that the map $F : B \times [0,1] \to \mathbb{C}^n$ defined by

$$F(b,t) = F_0(b,0) + \int_0^t f(b,s)\, ds, \quad t \in [0,1],\ b \in B$$

satisfies the lemma, except that $F(b,1)$ is only close to $F_0(b,1)$ but not necessarily equal to it. As shown in [23, Proof of Theorem 1.1], the identity $F(b,1) = F_0(b,1)$ for all $b \in B$ can be obtained by a suitable correction, using period dominating sprays on a family of nondegenerate curves $t \mapsto f(b,t) \in \mathbf{A}$. It is also explained in [23, Remark 3.2] why we can use any compact Hausdorff space B as the parameter space, as opposed to merely manifolds of class $\mathscr{C}^1$ in the hypothesis of [33, Theorem 3.4, p. 39]. □

Our second lemma pertains to maps $F(b,t)$ whose t-derivatives belong to a variable family of closed algebraic submanifolds of $\mathbb{C}^n$.

Lemma 2.4. *Let B be a compact Hausdorff space, $\widetilde{B} = B \times [0,1]$, and $\{A_{b,t} : b \in B,\ t \in [0,1]\}$ a continuous family of connected, closed, algebraic submanifolds of $\mathbb{C}^n$, $n \geq 2$, none of them contained in any affine hyperplane of $\mathbb{C}^n$. Denote by $\pi : Z = \widetilde{B} \times \mathbb{C}^n \to \widetilde{B}$ the natural projection, and let $\mathcal{A} \subset Z$ be the subset whose fibre over $(b,t) \in \widetilde{B}$ equals $A_{b,t}$. Let $f_0 : \widetilde{B} \to \mathcal{A} \subset Z$ and $F_0 : \widetilde{B} \to Z$ be continuous sections of $\pi : Z \to \widetilde{B}$ such that $F_0(b,\cdot)$ is of class $\mathscr{C}^1([0,1])$, its t-derivative $\dot{F}_0(b,t) = \frac{\partial}{\partial t} F_0(b,t) \in \mathbb{C}^n$ depends continuously on $(b,t) \in \widetilde{B}$, and*

$$\dot{F}_0(b,0) = f_0(b,0), \quad \dot{F}_0(b,1) = f_0(b,1) \quad \textit{for all } b \in B.$$

Given $\epsilon > 0$ there is a section $F : B \times [0,1] \to Z$ of the same class as F_0 and satisfying the following conditions for all $b \in B$.

(a) $F(b,0) = F_0(b,0)$, $\dot{F}(b,0) = f_0(b,0)$, *and* $\dot{F}(b,1) = f_0(b,1)$.
(b) $\|F - F_0\|_\infty < \epsilon$.
(c) $f(b,t) := \dot{F}(b,t) \in A_{b,t}$ *for all* $t \in [0,1]$.

Proof. By [7, Lemma 3.5.1], the conditions on the submanifolds $A_{b,t} \subset \mathbb{C}^n$ imply $\mathrm{Co}(A_{b,t}) = \mathbb{C}^n$ for every $(b,t) \in \widetilde{B}$. Hence, the lemma is a special case of [33, Theorem 3.4, p. 39], except that we must replace the subset $\mathcal{A} \subset Z$ by a small open neighbourhood $\Omega \subset Z$ with a fibre preserving retraction $\Omega \to A$. The idea of proof is the same as for Lemma 2.3. (In this case, we do not need to achieve the exact conditions $F(b,1) = F_0(b,1)$, $b \in B$.) □

Remark 2.5. In the present paper, Lemma 2.4 will be used for families of nonsingular quadric curves

$$A_{b,t} = \{(x,y) \in \mathbb{C}^2 : x^2 + y^2 = c(b,t)\}$$

with a continuous function $c : B \times [0,1] \to \mathbb{C}^*$; see Claim 3.4.

3. Proof of Theorem 1.1

3.1. **Preparations.** For $l \in \mathbb{N} = \{1,2,\ldots\}$, $l \geq 2$, we denote $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z} = \{0,1,\ldots,l-1\}$. Let $C \subset X$ be a (closed) Jordan curve. By a *division* of C we mean a family of compact connected subarcs $\mathscr{D} = \{\alpha_a : a \in \mathbb{Z}_l\}$ ($l \geq 3$) of C such that $\bigcup_{a \in \mathbb{Z}_l} \alpha_a = C$, the subarcs α_a and α_{a+1} have a common endpoint and are otherwise disjoint for every $a \in \mathbb{Z}_l$, and $\alpha_a \cap \alpha_e = \varnothing$ for every $a, e \in \mathbb{Z}_l$ with $e \notin \{a-1, a, a+1\}$. Assume now that $\mathcal{C} = \bigcup_{i=1}^k C_i \subset X$ is a union of finitely many pairwise disjoint Jordan curves. By a division of $\mathcal{C}$, we mean a family $\mathscr{D} = \bigcup_{i=1}^k \mathscr{D}_i$ where $\mathscr{D}_i$ is a division of C_i for $i = 1,\ldots,k$; here the number of subarcs in $\mathscr{D}_i$ might depend on i. Finally, assume that $K \subset X$ is a smoothly bounded compact domain with the boundary components $C_1,\ldots,C_k$, so $bK = \mathcal{C}$. If $\delta > 0$ is a number and $v : B \times K \to \mathbb{R}^3$ is a continuous map such that $v_b = v(b,\cdot) = (v_{b,1}, v_{b,2}, v_{b,3}) : K \to \mathbb{R}^3$ is a J_b-conformal minimal immersion (on a neighbourhood of K) such that

$$\max\{v_{b,1}(p), v_{b,2}(p)\} > \delta \quad \text{for every } p \in bK \text{ and } b \in B,$$

then we say that a division $\mathscr{D} = \bigcup_{i=1}^k \mathscr{D}_i$ of $bK = \bigcup_{i=1}^k C_i$ is *compatible with* (v,δ) if the following conditions are satisfied for every $i \in \{1,\ldots,k\}$.

($\mathfrak{D}$1) The division $\mathscr{D}_i$ of C_i is of the form $\mathscr{D}_i = \{\alpha_{i,a} : a \in \mathbb{Z}_{l_i}\}$ for an even integer $l_i \geq 4$.
($\mathfrak{D}$2) $v_{b,1}(p) > \delta$ for every $p \in \alpha_{i,a}$, $a \in \mathbb{Z}_{l_i}$ odd, and $b \in B$.
($\mathfrak{D}$3) $v_{b,2}(p) > \delta$ for every $p \in \alpha_{i,a}$, $a \in \mathbb{Z}_{l_i}$ even, and $b \in B$.
($\mathfrak{D}$4) If $a \in \mathbb{Z}_{l_i}$ and $p_{i,a}$ denotes the only point in $\alpha_{i,a} \cap \alpha_{i,a+1}$, then $\partial_{J_b} v_{b,1}(p_{i,a}) \neq 0$ and $\partial_{J_b} v_{b,2}(p_{i,a}) \neq 0$ for every $b \in B$.

These conditions imply $\min\{v_{b,1}(p_{i,a}), v_{b,2}(p_{i,a})\} > \delta$ for every $i \in \{1,\ldots,k\}$, $a \in \mathbb{Z}_{l_i}$, and $b \in B$. In general, given v and δ as above, a division of bK compatible with (v,δ) need not exist, but it always exists when B is a singleton. This is an important extra difficulty with respect to the construction of isolated proper minimal surfaces.

We shall need the following notions.

Definition 3.1 ([7, Definitions 1.12.9 and 3.1.2])**.** An *admissible* set in X is a compact set of the form $S = K \cup E$, where K is a (possibly empty) finite union of pairwise disjoint compact domains with piecewise $\mathscr{C}^1$ boundaries in X and $E = \overline{S \setminus K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K only at their endpoints (if at all) and such that their intersections with the boundary bK of K are transverse.

Let J be a complex structure on X and θ be a nowhere vanishing holomorphic 1-form on X. A *generalized J-conformal minimal immersion* $S \to \mathbb{R}^3$ of class $\mathscr{C}^r$ $(r \in \mathbb{N})$ is a pair $(u, f\theta)$, where $u : S \to \mathbb{R}^3$ is a $\mathscr{C}^r$ map whose restriction to $\mathring{S} = \mathring{K}$ is a J-conformal minimal immersion, and $f : S \to \mathbf{A} \subset \mathbb{C}^3$ (2.2) is a map of class $\mathscr{C}^{r-1}$, J-holomorphic on $\mathring{S} = \mathring{K}$, such that $f\theta = 2\partial_J u$ everywhere on K, and for any smooth path β in X parametrizing a component of E we have that $\Re(\beta^*(f\theta)) = \beta^*(du) = d(u \circ \beta)$. Similarly, a *generalized J-holomorphic null immersion* $S \to \mathbb{C}^3$ of class $\mathscr{C}^r$ is a pair $(F, f\theta)$, where f is as above and $F : S \to \mathbb{C}^n$ is a $\mathscr{C}^r$ map $(r \in \mathbb{N})$ satisfying $dF = f\theta$ which is J-holomorphic on $\mathring{S}$.

3.2. **Outline of proof of Theorem 1.1.** Let $\mathcal{F}_b : H_1(X, \mathbb{Z}) \to \mathbb{R}^3$, $b \in B$, be a continuous family of homomorphisms. We shall construct a map $u = (u_1, u_2, u_3) : B \times X \to \mathbb{R}^3$ satisfying the theorem, with $\mathrm{Flux}_{u_b} = \mathcal{F}_b$ and $(u_{b,1}, u_{b,2}) : X \to \mathbb{R}^2$ being proper for every $b \in B$, in an inductive process. To begin with, choose an exhaustion

$$X_1 \subset X_2 \subset \cdots \subset \bigcup_{j \geq 0} X_j = X \tag{3.1}$$

of X by connected, smoothly bounded Runge compact domains such that X_1 is a disc, $X_j \subset \mathring{X}_{j+1}$ for every $j \in \mathbb{N}$, and the Euler characteristic $\chi(X_{j+1} \setminus \mathring{X}_j)$ of $X_{j+1} \setminus \mathring{X}_j$ is either 0 or -1 for every $j \in \mathbb{N}$ (such exists by standard topological arguments; see e.g. [12, Lemma 4.2]). The latter condition simply means that the smoothly bounded compact domain $X_{j+1} \setminus \mathring{X}_j$ consists of finitely many connected components all of them being annuli, except perhaps one that might be a pair of pants, that is, a sphere with three discs removed. Hence, X_j is a strong deformation retract of X_{j+1} when $\chi(X_{j+1} \setminus \mathring{X}_j) = 0$ (the noncritical case), while the number of boundary components of X_j and X_{j+1} differ by one when $\chi(X_{j+1} \setminus \mathring{X}_j) = -1$ (the critical case).

We claim that there is a sequence $\Xi_j = \{u^j, \mathscr{D}^j, \epsilon_j\}$, $j \in \mathbb{N}$, where

- $u^j : B \times X_j \to \mathbb{R}^3$ is a continuous map such that $u^j_b = u^j(b, \cdot) = (u^j_{b,1}, u^j_{b,2}, u^j_{b,3}) : X_j \to \mathbb{R}^3$ is a nonflat J_b-conformal minimal immersion (on a neighbourhood of X_j) with

$$\max\{u^j_{b,1}(p), u^j_{b,2}(p)\} > j \quad \text{for every } p \in bX_j \text{ and } b \in B, \tag{3.2}$$

- $\mathscr{D}^j$ is a division of bX_j compatible with (u^j, j), and
- $0 < \epsilon_j < 1$ is a number,

such that the following conditions hold for every $j \geq 2$:

(a$_j$) $\max\{u^j_{b,1}(p), u^j_{b,2}(p)\} > j - 1$ for every $p \in X_j \setminus \mathring{X}_{j-1}$ and $b \in B$.
(b$_j$) $|u^j_b(p) - u^{j-1}_b(p)| < \epsilon_{j-1}$ for every $p \in X_{j-1}$ and $b \in B$.
(c$_j$) $\epsilon_j < \epsilon_{j-1}/2$, and if $v : X \to \mathbb{R}^3$ is a J_b-conformal harmonic map such that $|v(p) - u^j_b(p)| < 2\epsilon_j$ for every $p \in X_j$ and some $b \in B$, then v is a nonflat immersion on X_j.
(d$_j$) $\mathrm{Flux}_{u^j_b}(C) = \mathcal{F}_b(C)$ for every closed curve $C \subset X_j$ and $b \in B$.

We emphasize that the existence of each division $\mathscr{D}^j$ is established as a part of the induction.

Assume that such a sequence exists. By (3.1), (b$_j$), and the first part of (c$_j$), there is a limit continuous map

$$u = \lim_{j\to\infty} u^j : B \times X \to \mathbb{R}^3$$

such that $u_b = u(b,\cdot) = (u_{b,1}, u_{b,2}, u_{b,3}) : X \to \mathbb{R}^3$ is a J_b-conformal harmonic map with

$$|u_b(p) - u_b^j(p)| < 2\epsilon_j \quad \text{for every } p \in X_j,\ j \in \mathbb{N}, \text{ and } b \in B. \tag{3.3}$$

The second part of (c$_j$) then guarantees that u_b is a nonflat J_b-conformal minimal immersion, while conditions (d$_j$) ensure that $\mathrm{Flux}_{u_b} = \mathcal{F}_b$ holds for every $b \in B$; take (3.1) into account. Furthermore, given $b \in B$ and $j \geq 2$, we have by (a$_j$), (b$_j$), and (3.3) that

$$\max\{u_{b,1}(p), u_{b,2}(p)\} > j - 1 - 2\epsilon_j > j - 3 \quad \text{for every } p \in X_{j+1} \setminus \mathring{X}_j.$$

This and (3.1) imply that $\max\{u_{b,1}, u_{b,2}\} : X \to \mathbb{R}$ is a proper map, hence so is $(u_{b,1}, u_{b,2}) : X \to \mathbb{R}^2$. Therefore, the map u satisfies the conclusion of the theorem.

To complete the proof, it remains to construct a sequence Ξ_j with the desired properties.

3.3. **The induction.** Set $\epsilon_0 = 1$ and $X_0 = \varnothing$. For the base case when $j = 1$, choose any nonflat J_{st}-conformal minimal immersion $v = (v_1, v_2, v_3) : \bar{\mathbb{D}} \to \mathbb{R}^3$ such that

$$v_i(p) > 1 \text{ and } \partial_{J_{\mathrm{st}}} v_i(p) \neq 0 \text{ for every } p \in \bar{\mathbb{D}} \text{ and } i \in \{1, 2\}. \tag{3.4}$$

Since X_1 is a disc, Proposition 2.2 furnishes a continuous map $u^1 : B \times X_1 \to \mathbb{R}^3$ such that $u_b^1 = u^1(b,\cdot) = (u_{b,1}^1, u_{b,2}^1, u_{b,3}^1) : X_1 \to \mathbb{R}^3$ is a nonflat J_b-conformal minimal immersion with $u_b^1(X_1) = v(\bar{\mathbb{D}})$ for every $b \in B$. It follows from (3.4) that condition (a$_1$) and inequality (3.2) for $j = 1$ are satisfied, while any division $\mathscr{D}^1$ of the Jordan curve bX_1 consisting of (for instance) 4 subarcs is compatible with $(u^1, 1)$; see conditions ($\mathfrak{D}$1) to ($\mathfrak{D}$4) in Subsection 3.1. Furthermore, (d$_1$) holds true since X_1 is simply connected. Finally, by compactness of B, the Cauchy estimates provide a number $\epsilon_1 > 0$ satisfying condition (c$_1$), whereas condition (b$_1$) is void.

For the inductive step, fix an integer $j \geq 2$ and assume that we already have a triple $\Xi_{j-1} = \{u^{j-1}, \mathscr{D}^{j-1}, \epsilon_{j-1}\}$ as above, satisfying condition (d$_{j-1}$) and inequality (3.2) with $j-1$ in place of j. We shall construct a suitable triple $\Xi_j = \{u^j, \mathscr{D}^j, \epsilon_j\}$ satisfying conditions (a$_j$) to (d$_j$) as well as inequality (3.2). For this, we distinguish cases depending on the Euler characteristic of $X_j \setminus \mathring{X}_{j-1}$.

The noncritical case. Assume that $\chi(X_j \setminus \mathring{X}_{j-1}) = 0$, so X_{j-1} is a strong deformation retract of X_j. For simplicity of exposition, we assume that bX_j is connected, hence bX_{j-1} is connected as well and $X_j \setminus \mathring{X}_{j-1}$ is a smoothly bounded compact annulus. For the general case, we apply the same procedure in each component of $X_j \setminus \mathring{X}_{j-1}$.

Write $\mathscr{D}^{j-1} = \{\alpha_a : a \in \mathbb{Z}_l\}$, $l \geq 4$ even, for the given division of the closed Jordan curve bX_{j-1} compatible with $(u^{j-1}, j-1)$, and denote by p_a the only point in $\alpha_a \cap \alpha_{a+1}$, $a \in \mathbb{Z}_l$. Set

$$\mathbb{Z}_l^{\mathrm{odd}} = \{a \in \mathbb{Z}_l : a \text{ odd}\} \quad \text{and} \quad \mathbb{Z}_l^{\mathrm{even}} = \{a \in \mathbb{Z}_l : a \text{ even}\}. \tag{3.5}$$

It is obvious that $\mathbb{Z}_l = \mathbb{Z}_l^{\mathrm{odd}} \cup \mathbb{Z}_l^{\mathrm{even}}$ and $\mathbb{Z}_l^{\mathrm{odd}} \cap \mathbb{Z}_l^{\mathrm{even}} = \varnothing$. Conditions ($\mathfrak{D}$2) to ($\mathfrak{D}$4) then give the following properties for every $b \in B$.

(I$_1$) $u_{b,1}^{j-1}(p) > j - 1$ for every $p \in \alpha_a$, $a \in \mathbb{Z}_l^{\mathrm{odd}}$.
(I$_2$) $u_{b,2}^{j-1}(p) > j - 1$ for every $p \in \alpha_a$, $a \in \mathbb{Z}_l^{\mathrm{even}}$.
(I$_3$) $\partial_{J_b} u_{b,1}^{j-1}(p_a) \neq 0$ and $\partial_{J_b} u_{b,2}^{j-1}(p_a) \neq 0$ for every $a \in \mathbb{Z}_l$.

In particular,

$$\min\{u_{b,1}^{j-1}(p_a), u_{b,2}^{j-1}(p_a)\} > j-1 \quad \text{for every } a \in \mathbb{Z}_l, b \in B. \tag{3.6}$$

The first deformation. In the first step of the construction, we shall approximate u^{j-1} uniformly on $B \times X_{j-1}$ by a continuous family $\hat{u} : B \times X_j \to \mathbb{R}^3$ of J_b-conformal minimal immersions with some control of their images. For each $a \in \mathbb{Z}_l$ we choose a smooth embedded arc $\gamma_a \subset X_j \setminus \mathring{X}_{j-1}$ with the initial point $p_a \in bX_{j-1}$, the final point $q_a \in bX_j$, and otherwise disjoint from $bX_j \cup bX_{j-1}$. We choose the family of arcs γ_a, $a \in \mathbb{Z}_l$, to be pairwise disjoint and such that the set

$$S = X_{j-1} \cup \bigcup_{a \in \mathbb{Z}_l} \gamma_a$$

is admissible in the sense of Definition 3.1. Also, for each $a \in \mathbb{Z}_l$ we denote by β_a the arc in bX_j connecting q_{a-1} and q_a without meeting any of the other points q_e for $e \in \mathbb{Z}_l \setminus \{a-1, a\}$. We denote by $D_a \subset X_j \setminus \mathring{X}_{j-1}$ the closed disc bounded by γ_{a-1}, α_a, γ_a, and β_a, $a \in \mathbb{Z}_l$. (See Figure 3.1.) Note that $D_a \cap D_{a+1} = \gamma_a$, $D_a \cap D_e = \varnothing$ for $e \in \mathbb{Z}_l \setminus \{a-1, a, a+1\}$,

$$X_j \setminus \mathring{X}_{j-1} = \bigcup_{a \in \mathbb{Z}_l} D_a = \bigcup_{a \in \mathbb{Z}_l^{\text{odd}}} D_a \cup \bigcup_{a \in \mathbb{Z}_l^{\text{even}}} D_a, \tag{3.7}$$

and

$$bX_j = \bigcup_{a \in \mathbb{Z}_l} \beta_a = \bigcup_{a \in \mathbb{Z}_l^{\text{odd}}} \beta_a \cup \bigcup_{a \in \mathbb{Z}_l^{\text{even}}} \beta_a. \tag{3.8}$$

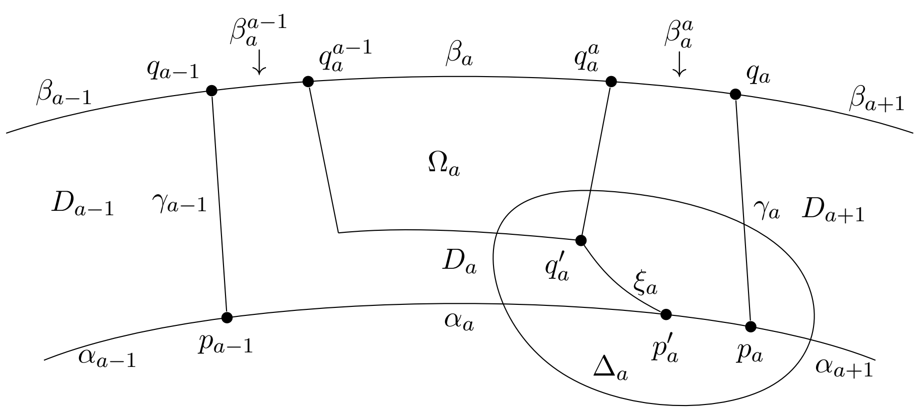


Figure 3.1. Configuration of sets in $X_j \setminus \mathring{X}_{j-1}$.

Consider the $\mathscr{J}$-holomorphic map $f : B \times X_{j-1} \to \mathbf{A}$ to the punctured null quadric (2.2) given by

$$f(b, \cdot) = 2\partial_{J_b} u_b^{j-1}/\theta_b \quad \text{for every } b \in B,$$

and set $f_b = f(b, \cdot) = (f_{b,1}, f_{b,2}, f_{b,3}) : X_{j-1} \to \mathbf{A}$.

The following claim is an immediate consequence of Lemma 2.3; for granting the second condition, take into account (3.6).

Claim 3.2. *The pair of maps $u^{j-1} : B \times X_{j-1} \to \mathbb{R}^3$ and $f : B \times X_{j-1} \to \mathbf{A}$ can be extended to continuous maps $u^{j-1} : B \times S \to \mathbb{R}^3$ and $f : B \times S \to \mathbf{A}$, respectively, such that, setting $u_b^{j-1} = u^{j-1}(b, \cdot) = (u_{b,1}^{j-1}, u_{b,2}^{j-1}, u_{b,3}^{j-1})$ and $f_b = f(b, \cdot) = (f_{b,1}, f_{b,2}, f_{b,3})$ for every $b \in B$, the following conditions are satisfied.*

- *The pair $(u_b^{j-1}, f_b\theta_b)$ is a generalized J_b-conformal minimal immersion $S \to \mathbb{R}^3$ of class $\mathscr{C}^1$ for every $b \in B$ (see Definition 3.1).*
- $\min\{u_{b,1}^{j-1}(p), u_{b,2}^{j-1}(p)\} > j-1$ *for every* $p \in \bigcup_{a\in\mathbb{Z}_l} \gamma_a$ *and* $b \in B$.
- $\min\{u_{b,1}^{j-1}(q_a), u_{b,2}^{j-1}(q_a)\} > j$ *for every* $a \in \mathbb{Z}_l$ *and* $b \in B$.
- $f_{b,1}(q_a) f_{b,2}(q_a) \neq 0$ *for every* $a \in \mathbb{Z}_l$ *and* $b \in B$.

By the Mergelyan theorem for families of conformal minimal immersions (see [21, Theorem 8.2 and Corollary 8.6]), we can approximate u^{j-1} uniformly on $B \times S$ by a continuous map $\hat{u} : B \times X_j \to \mathbb{R}^3$ satisfying the following conditions for every $b \in B$.

(II_1) The map $\hat{u}_b = \hat{u}(b,\cdot) = (\hat{u}_{b,1}, \hat{u}_{b,2}, \hat{u}_{b,3}) : X_j \to \mathbb{R}^3$ is a J_b-conformal minimal immersion.
(II_2) $|\hat{u}_b(p) - u_b^{j-1}(p)| < \epsilon_{j-1}/3$ for every $p \in X_{j-1}$.
(II_3) $\min\{\hat{u}_{b,1}(p), \hat{u}_{b,2}(p)\} > j-1$ for every $p \in \bigcup_{a\in\mathbb{Z}_l} \gamma_a$.
(II_4) $\min\{\hat{u}_{b,1}(q_a), \hat{u}_{b,2}(q_a)\} > j$ for every $a \in \mathbb{Z}_l$.
(II_5) $\hat{f}_{b,1}(q_a)\hat{f}_{b,2}(q_a) \neq 0$ for every $a \in \mathbb{Z}_l$, where

$$\hat{f}_b = (\hat{f}_{b,1}, \hat{f}_{b,2}, \hat{f}_{b,3}) = 2\partial_{J_b}\hat{u}_b/\theta_b : X_j \to \mathbf{A}.$$

(II_6) $\mathrm{Flux}_{\hat{u}_b}(C) = \mathcal{F}_b(C)$ for every closed curve $C \subset X_j$; take into account (d_{j-1}).
(II_7) $\hat{f}_{b,1}(p_a)\hat{f}_{b,2}(p_a) \neq 0$ for every $a \in \mathbb{Z}_l$; see condition (I_3).

Taking a look into the conditions required in the induction, (II_3) and (II_4) show that the immersions $\hat{u}_b$ assume suitable values on $\bigcup_{a\in\mathbb{Z}_l} \gamma_a$ (see inequality (3.2) and condition (a_j)), but they need not do so on the complement of $\bigcup_{a\in\mathbb{Z}_l} \gamma_a$ in $X_j \setminus X_{j-1}$, so we have to keep working.

The second deformation I: a spray of López–Ros transformations. We shall now deform the family $\hat{u}_b$ on $X_j \setminus X_{j-1}$ in order to obtain more control on its image over the sets D_a for $a \in \mathbb{Z}_l^{\mathrm{odd}}$; see (3.7).

Consider the $\mathscr{J}$-holomorphic map $\hat{f} : B \times X_j \to \mathbf{A}$ given by $\hat{f}(b,\cdot) = \hat{f}_b : X_j \to \mathbf{A}$ for every $b \in B$. Our next task is to embed $\hat{f}$ as the core of a period dominating spray of maps $B \times X_j \to \mathbf{A}$, keeping fixed the first component function; this will enable us to kill the periods in a subsequent step. In the nonparametric case, this has typically been done using the Oka principle for sections of ramified holomorphic maps with Oka fibres (see [20, Section 6.14]); a tool that is not available in the parametric framework. Instead, we shall use a parametric version of the López-Ros deformation for minimal surfaces (see [27]) as in [9]. Set

$$\psi_b = \frac{\hat{f}_{b,1}}{\hat{f}_{b,2} - \mathfrak{i}\hat{f}_{b,3}} : X_j \to \mathbb{CP}^1, \quad b \in B. \tag{3.9}$$

A calculation shows that

$$\hat{f}_{b,2} = \frac{1}{2}\Big(\frac{1}{\psi_b} - \psi_b\Big)\hat{f}_{b,1} \quad \text{and} \quad \hat{f}_{b,3} = \frac{\mathfrak{i}}{2}\Big(\frac{1}{\psi_b} + \psi_b\Big)\hat{f}_{b,1}, \quad b \in B. \tag{3.10}$$

(See (2.5).) Moreover,

$$\frac{\hat{f}_{b,1}}{\psi_b} = \hat{f}_{b,2} - \mathfrak{i}\hat{f}_{b,3} \quad \text{and} \quad \psi_b\hat{f}_{b,1} = -\hat{f}_{b,2} - \mathfrak{i}\hat{f}_{b,3} \tag{3.11}$$

are J_b-holomorphic functions on X_j for every $b \in B$. Note that for any $\mathscr{J}$-holomorphic map $(\hat{f}_1, \tilde{f}_2, \tilde{f}_3) : B \times X_j \to \mathbf{A}$ sharing the first component $\hat{f}_1$ with the map $\hat{f}$, there is a $\mathscr{J}$-holomorphic function $\mu : B \times X_j \to \mathbb{C}^*$ such that

$$\tilde{f}_{b,2} = \frac{1}{2}\Big(\frac{1}{\mu_b\psi_b} - \mu_b\psi_b\Big)\hat{f}_{b,1}, \quad \tilde{f}_{b,3} = \frac{\mathfrak{i}}{2}\Big(\frac{1}{\mu_b\psi_b} + \mu_b\psi_b\Big)\hat{f}_{b,1}, \quad b \in B. \tag{3.12}$$

Fix a point $x_0 \in \mathring{X}_{j-1}$ and let $\Gamma_1, \ldots, \Gamma_m$ be a family of smooth oriented Jordan curves in $\mathring{X}_{j-1}$ such that any two of them only intersect at x_0, they form a basis of the homology group $H_1(X_{j-1}, \mathbb{Z})$

(and hence also of $H_1(X_j,\mathbb{Z})$), and the set $\Gamma=\bigcup_{i=1}^m \Gamma_i$ is Runge in X [7, Lemma 1.12.10]. Denote by $\mathcal{P}: B\times \mathscr{C}(\Gamma,\mathbb{C}^*)\to (\mathbb{C}^2)^m$ the period map that sends for each $b\in B$ a map $h\in\mathscr{C}(\Gamma,\mathbb{C}^*)$ to

$$\mathcal{P}(b,h)=\left(\int_{\Gamma_i}\Big(\frac{\hat f_{b,1}}{h\psi_b}\,,\,h\psi_b\hat f_{b,1}\Big)\theta_b\right)_{i=1,\ldots,m}\in(\mathbb{C}^2)^m. \tag{3.13}$$

Since $\hat f_b$ is nonflat for each $b\in B$, the functions $\hat f_{b,1}/\psi_b$ and $\psi_b\hat f_{b,1}$ in (3.11) are complex linearly independent. We can therefore use [9, Claim 2.3] to obtain a continuous function

$$\sigma:\mathbb{B}\times B\times X_j\to\mathbb{C}^*,$$

depending holomorphically on a parameter $\zeta=(\zeta_1,\ldots,\zeta_N)$ in a ball $\mathbb{B}\subset\mathbb{C}^N$ centred at the origin for some large $N\in\mathbb{N}$ and satisfying the following conditions for every $b\in B$.

(III_1) The function $\sigma(\zeta,b,\cdot):X_j\to\mathbb{C}^*$ is J_b-holomorphic for every $\zeta\in\mathbb{B}$.
(III_2) $\sigma(0,b,p)=1$ for every $p\in X_j$.
(III_3) σ is period dominating, in the sense that the map

$$\mathbb{B}\ni\zeta\longmapsto\mathcal{P}\big(b,\sigma(\zeta,b,\cdot)\big)\in(\mathbb{C}^2)^m$$

is submersive at $\zeta=0$; see (3.13).

Remark 3.3. Note that [9, Claim 2.3] provides such a function in the special case when J_b does not depend on $b\in B$, so it deals with a single open Riemann surface instead of with a family. Nevertheless, using also the argument in [21, proof of Theorem 8.2, the noncritical case], the proof in [9] extends in a straightforward way to a variable family of complex structures on X.

Set $\sigma_{\zeta,b}=\sigma(\zeta,b,\cdot)$ for $\zeta\in\mathbb{B}$ and $b\in B$, and let $\varphi:\mathbb{B}\times B\times X_j\to\mathbf{A}$ be the continuous map

$$\varphi(\zeta,b,\cdot)=\left(\hat f_{b,1}\,,\,\frac12\Big(\frac{1}{\sigma_{\zeta,b}\psi_b}-\sigma_{\zeta,b}\psi_b\Big)\hat f_{b,1}\,,\,\frac{\mathfrak{i}}{2}\Big(\frac{1}{\sigma_{\zeta,b}\psi_b}+\sigma_{\zeta,b}\psi_b\Big)\hat f_{b,1}\right),\quad(\zeta,b)\in\mathbb{B}\times B. \tag{3.14}$$

It follows that φ is well defined (that is, it takes its values in $\mathbf{A}$), and for every $\zeta\in\mathbb{B}$ and $b\in B$ the map $\varphi(\zeta,b,\cdot):X_j\to\mathbf{A}$ is J_b-holomorphic and satisfies $\varphi(0,b,\cdot)=\hat f_b$.

The second deformation II: replacement of the core. We shall now replace the functions ψ_b in (3.14) by suitable functions that are obtained by another application of the López–Ros transformation with parameters. The aim is to create a strong but controlled deformation of $\hat u_b$ on the sets D_a for $a\in\mathbb{Z}_l^{\mathrm{odd}}$.

By (II_7) and compactness of B, for each $a\in\mathbb{Z}_l^{\mathrm{odd}}$ there is an open, smoothly bounded disc neighborhood Δ_a of p_a in X_j, with $\overline{\Delta}_a\cap(bX_j\cup\bigcup_{e\in\mathbb{Z}_l\setminus\{a\}}\gamma_e)=\varnothing$, such that

$$\hat f_{b,1}(p)\hat f_{b,2}(p)\neq0\quad\text{for every } p\in\overline{\Delta}_a \text{ and } b\in B. \tag{3.15}$$

Further, by (II_3), (II_4), (II_5), and compactness of B, for each $a\in\mathbb{Z}_l^{\mathrm{odd}}$ there is a closed disc $\Omega_a\subset D_a\setminus(\gamma_{a-1}\cup\alpha_a\cup\gamma_a)$ such that the following conditions hold.

(IV_1) $\Omega_a\cap\beta_a$ is a compact connected Jordan arc.
(IV_2) $\hat u_{b,1}(p)>j-1$ for every $p\in\overline{D_a\setminus\Omega_a}$ and $b\in B$.
(IV_3) $\hat u_{b,1}(p)>j$ for every $p\in\overline{\beta_a\setminus\Omega_a}$ and $b\in B$.
(IV_4) $\Delta_a\cap b\Omega_a\neq\varnothing$ and $D_a\cap(\Delta_a\setminus\mathring{\Omega}_a)$ is path connected. (The latter condition might require to replace Δ_a by a smaller neighbourhood of p_a.)
(IV_5) $\hat f_{b,1}(p)\hat f_{b,2}(p)\neq0$ for every $p\in\overline{\beta_a\setminus\Omega_a}$ and $b\in B$.

Denote by β_a^{a-1} the connected component of $\overline{\beta_a\setminus\Omega_a}$ with q_{a-1} as an endpoint, and β_a^a the one containing q_a. Denote by q_a^{a-1} and q_a^a the other endpoint of β_a^{a-1} and β_a^a, respectively. Clearly,

$$\overline{\beta_a\setminus\Omega_a}=\beta_a^{a-1}\cup\beta_a^a. \tag{3.16}$$

In view of (IV_4), for each $a \in \mathbb{Z}_l^{\mathrm{odd}}$ there is a smooth embedded arc $\xi_a \subset (D_a \setminus \mathring{\Omega}_a) \cap \Delta_a$ with initial point $p'_a \in \alpha_a \setminus \{p_a\}$ and endpoint $q'_a \in b\Omega_a$, and otherwise disjoint from $bD_a \cup b\Omega_a$. Choose these arcs so that the Runge compact set

$$S^{\mathrm{odd}} = X_{j-1} \cup \Big(\bigcup_{a \in \mathbb{Z}_l^{\mathrm{even}}} D_a \Big) \cup \Big(\bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \Omega_a \cup \xi_a \Big) \subset X_j$$

is admissible (Definition 3.1); see Figure 3.1. Condition (3.15) ensures that

$$\hat{f}_{b,1}(p) \neq 0 \quad \text{for every } b \in B,\, p \in \xi_a, \text{ and } a \in \mathbb{Z}_l^{\mathrm{odd}}. \tag{3.17}$$

Claim 3.4. *For any number $\tau > 0$ there is a continuous function $\mu : B \times S^{\mathrm{odd}} \to \mathbb{C}^*$ such that, setting $\mu_b = \mu(b, \cdot)$, we have that $\mu_b(p) = 1$ for every $p \in S^{\mathrm{odd}} \setminus \bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \xi_a$ and*

$$\Re \int_{\xi_a} \Big(\frac{1}{\mu_b \psi_b} - \mu_b \psi_b \Big) \hat{f}_{b,1} \theta_b > \tau \quad \textit{for every } a \in \mathbb{Z}_l^{\mathrm{odd}} \textit{ and } b \in B.$$

This follows from Lemma 2.4, applied to the family of maps $(\hat{f}_{b,2}, \hat{f}_{b,3}) : \xi_a \to \mathbb{C}^2$ $(a \in \mathbb{Z}_l^{\mathrm{odd}},\ b \in B)$, together with the observation in (3.12). Indeed, let $t \in [0,1]$ be a regular parameter on the arc ξ_a for some $a \in \mathbb{Z}_l^{\mathrm{odd}}$. Since $\hat{f}_{b,1}(t) \neq 0$ for all $b \in B$ and $t \in [0,1]$ (3.17), the equation $x^2 + y^2 = -\hat{f}_{b,1}(t)^2$ defines a noningular quadric curve $A_{b,t} \subset \mathbb{C}^2$, and it remains to apply Lemma 2.4 to families of maps $[0,1] \ni t \mapsto (\tilde{f}_{b,2}(t), \tilde{f}_{b,3}(t)) \in A_{b,t}$ with $\tilde{f}_{b,i}(0) = \hat{f}_{b,i}(0)$ and $\tilde{f}_{b,i}(1) = \hat{f}_{b,i}(1)$ for $b \in B$ and $i = 2, 3$. The observation (3.12) gives for each arc ξ_a, with $a \in \mathbb{Z}_l^{\mathrm{odd}}$, a continuous family of multipliers $\mu_{b,a} : \xi_a \to \mathbb{C}^*$ $(b \in B)$ assuming the value 1 at the endpoints of ξ_a. Hence, we can extend these functions to $\mu_b : S^{\mathrm{odd}} \to \mathbb{C}^*$ taking the value 1 on the complement of $\bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \xi_a$.

Consider a function μ given by the claim for a fixed large number $\tau > 0$ to be specified later. Since $\mathbb{C}^*$ is Oka, the Mergelyan theorem for families of holomorphic functions into Oka manifolds in [21, Theorem 1.6] furnishes a continuous function $\hat{\mu} : B \times X_j \to \mathbb{C}^*$ satisfying the following.

(V_1) The function $\hat{\mu}_b = \hat{\mu}(b, \cdot) : X_{j-1} \to \mathbb{C}^*$ is J_b-holomorphic for every $b \in B$.
(V_2) $|\hat{\mu}_b(p) - 1| < 1/\tau$ for every $p \in S^{\mathrm{odd}} \setminus \bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \xi_a$ and $b \in B$.
(V_3) $\Re \int_{\xi_a} \Big(\frac{1}{\hat{\mu}_b \psi_b} - \hat{\mu}_b \psi_b \Big) \hat{f}_{b,1} \theta_b > \tau$ for every $a \in \mathbb{Z}_l^{\mathrm{odd}}$ and $b \in B$.

Assuming that $\tau > 0$ is chosen sufficiently large, the implicit function theorem provides, in view of (V_2) and the period domination property of the spray σ in (III_3), a continuous map $B \to \mathbb{B}$ sending $b \in B$ to $\zeta_b \in \mathbb{B}$ and satisfying the following conditions.

(VI_1) $\zeta_b \in \mathbb{B}'$ for every $b \in B$, where $\mathbb{B}' \subset \mathbb{B} \subset \mathbb{C}^N$ is any given ball centered at the origin.
(VI_2) Setting $h_b = \sigma(\zeta_b, b, \cdot)\hat{\mu}_b$, we have that $\mathcal{P}\big(b, h_b\big) = \mathcal{P}(b, 1)$ for every $b \in B$; see (3.13).

Consider the continuous map $\hat{\psi} : B \times X_j \to \mathbb{CP}^1$ given by

$$\hat{\psi}_b = \hat{\psi}(b, \cdot) = h_b \psi_b = \sigma(\zeta_b, b, \cdot)\hat{\mu}_b \psi_b, \quad b \in B,$$

see (3.9), and the continuous map $\hat{\varphi} : B \times X_j \to \mathbf{A}$ given by

$$\hat{\varphi}(b, \cdot) = \left(\hat{f}_{b,1}\,,\ \frac{1}{2}\Big(\frac{1}{\hat{\psi}_b} - \hat{\psi}_b \Big) \hat{f}_{b,1}\,,\ \frac{\mathfrak{i}}{2}\Big(\frac{1}{\hat{\psi}_b} + \hat{\psi}_b \Big) \hat{f}_{b,1} \right), \quad b \in B; \tag{3.18}$$

cf. (3.14) and (2.5). It turns out that $\hat{\varphi}$ is well defined (that is, it takes its values in $\mathbf{A}$), and the map $\hat{\varphi}_b = (\hat{\varphi}_{b,1}, \hat{\varphi}_{b,1}, \hat{\varphi}_{b,1}) = \hat{\varphi}(b, \cdot) : X_j \to \mathbf{A}$ is J_b-holomorphic for every $b \in B$; that is, the map $\hat{\varphi}$ is $\mathscr{J}$-holomorphic. Moreover, condition (VI_2) ensures that the J_b-holomorphic 1-form $(\hat{\varphi}_b - \hat{f}_b)\theta_b$ is exact on X_j for every $b \in B$ (see condition (II_5)). Consider the continuous map $u' : B \times X_j \to \mathbb{R}^3$ given by

$$u'_b(p) = u'(b,p) = \hat u_b(x_0) + \Re \int_{x_0}^{p} \hat\varphi_b \theta_b \qquad \text{for } p \in X_j \text{ and } b \in B,$$

where $x_0 \in \mathring{X}_{j-1}$ was fixed above. The following conditions hold for every $b \in B$.

(i_1) The map $u'_b = (u'_{b,1}, u'_{b,2}, u'_{b,3}) : X_j \to \mathbb{R}^3$ is a J_b-conformal minimal immersion.
(i_2) $u'_{b,1} = \hat u_{b,1}$ everywhere on X_j.
(i_3) $\mathrm{Flux}_{u'_b} = \mathrm{Flux}_{\hat u_b} = \mathcal{F}_b|_{H_1(X_j,\mathbb{Z})}$; see ($\mathrm{II}_6$).

Furthermore, assuming that the ball $\mathbb{B}'$ in (VI_1) is sufficiently small and $\tau > 0$ is sufficiently large, conditions (II_3) to (II_5), (II_7), (IV_5), (V_2), (V_3), and (VI_1) guarantee the following for every $b \in B$.

(i_4) $|u'_b(p) - \hat u_b(p)| < \epsilon_{j-1}/3$ for every $p \in X_{j-1} \cup \bigcup_{a \in \mathbb{Z}_l^{\mathrm{even}}} D_a$.
(i_5) $u'_{b,2}(p) > j$ for every $p \in \bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \Omega_a$.
(i_6) $\min\{u'_{b,1}(p), u'_{b,2}(p)\} > j-1$ for every $p \in \bigcup_{a \in \mathbb{Z}_l} \gamma_a$.
(i_7) $\min\{u'_{b,1}(q_a), u'_{b,2}(q_a)\} > j$ for every $a \in \mathbb{Z}_l$.
(i_8) $\hat\varphi_{b,1}(q_a)\hat\varphi_{b,2}(q_a) \neq 0$ for every $a \in \mathbb{Z}_l$.
(i_9) $\hat\varphi_{b,1}(p_a)\hat\varphi_{b,2}(p_a) \neq 0$ for every $a \in \mathbb{Z}_l$.
(i_{10}) $\hat\varphi_{b,1}(p) \neq 0$ for every $p \in \overline{\beta_a \setminus \Omega_a}$ and $a \in \mathbb{Z}_l^{\mathrm{odd}}$.
(i_{11}) $\hat\varphi_{b,2}(q_a^{a-1})\hat\varphi_{b,2}(q_a^a) \neq 0$ for every $a \in \mathbb{Z}_l^{\mathrm{odd}}$; see (3.16).

In particular, (i_2) and (i_5) together with (IV_2) and (IV_3) guarantee the following.

(i_{12}) $\max\{u'_{b,1}(p), u'_{b,2}(p)\} > j-1$ for every $p \in \bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} D_a$ and $b \in B$.
(i_{13}) $\max\{u'_{b,1}(p), u'_{b,2}(p)\} > j$ for every $p \in \bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \beta_a$ and $b \in B$.

This shows that the immersions u'_b take suitable values on $\bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} D_a$ (see (3.2) and condition (a_j) in the induction), but need not do so on $\bigcup_{a \in \mathbb{Z}_l^{\mathrm{even}}} D_a$ (see (3.7)), so we have to keep working.

The third deformation. Repeating the arguments in the second deformation but for values $a \in \mathbb{Z}_l^{\mathrm{even}}$ and starting with u' in the place of $\hat u$, and taking into account conditions (i_1) to (i_{13}), we can obtain a continuous map $u^j : B \times X_j \to \mathbb{R}^3$ with the following properties for every $b \in B$, where the sets Ω_a and β_a and the points q_a^{a-1} and q_a^a for $a \in \mathbb{Z}_l^{\mathrm{even}}$ are defined analogously to those for $a \in \mathbb{Z}_l^{\mathrm{odd}}$.

(ii_1) The map $u^j_b = (u^j_{b,1}, u^j_{b,2}, u^j_{b,3}) = u^j(b,\cdot) : X_j \to \mathbb{R}^3$ is a J_b-conformal minimal immersion.
(ii_2) $u^j_{b,2} = u'_{b,2}$ everywhere on X_j.
(ii_3) $\mathrm{Flux}_{u^j_b} = \mathrm{Flux}_{u'_b} = \mathrm{Flux}_{\hat u_b} = \mathcal{F}_b|_{H_1(X_j,\mathbb{Z})}$.
(ii_4) $|u^j_b(p) - u'_b(p)| < \epsilon_{j-1}/3$ for every $p \in X_{j-1} \cup \bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} D_a$.
(ii_5) $u^j_{b,1}(p) > j$ for every $p \in (\bigcup_{a \in \mathbb{Z}_l^{\mathrm{even}}} \Omega_a) \cup (\bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \overline{\beta_a \setminus \Omega_a})$.
(ii_6) $u^j_{b,2}(p) > j$ for every $p \in (\bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \Omega_a) \cup (\bigcup_{a \in \mathbb{Z}_l^{\mathrm{even}}} \overline{\beta_a \setminus \Omega_a})$.
(ii_7) $u^j_{b,1}(p) > j-1$ for every $p \in \bigcup_{a \in \mathbb{Z}_l^{\mathrm{odd}}} \overline{D_a \setminus \Omega_a}$.
(ii_8) $u^j_{b,2}(p) > j-1$ for every $p \in \bigcup_{a \in \mathbb{Z}_l^{\mathrm{even}}} \overline{D_a \setminus \Omega_a}$.
(ii_9) $\partial_{J_b} u^j_{b,1}(p) \neq 0$ and $\partial_{J_b} u^j_{b,2}(p) \neq 0$ for every $p \in \bigcup_{a \in \mathbb{Z}_l} \{q_a, q_a^{a-1}, q_a^a\}$.

It is then clear that u^j satisfies inequality (3.2) (see (ii_5), (ii_6), and (3.8)) as well as conditions (a_j) (see (ii_5) to (ii_8) and (3.8)), (b_j) (see (II_2), (i_4), and (ii_4)), and (d_j) (see (ii_3)) in the induction. (See Figures 3.1 and 3.2.)

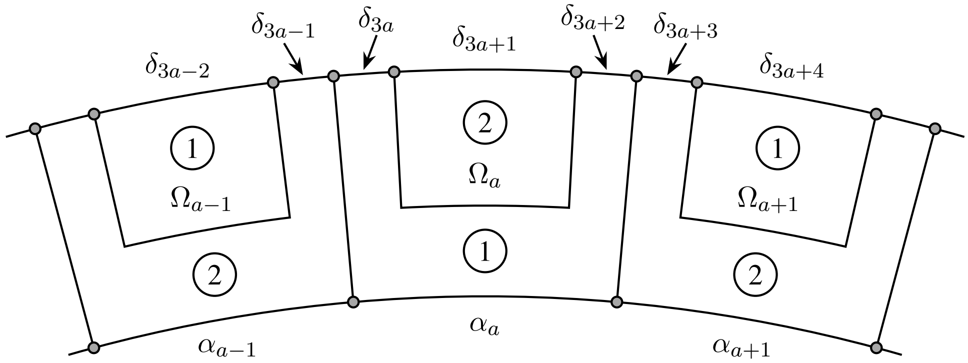


Figure 3.2. The division $\mathscr{D}^j$ of bX_j. Assuming that $a \in \mathbb{Z}_l^{\rm odd}$, the numbers 1 and 2 inside the circles point out the component function $u_{b,1}^j$ or $u_{b,2}^j$, respectively, that is suitably big in each set according to conditions (ii$_5$) to (ii$_8$).

By compactness of B, the Cauchy estimates allow us to choose a number $\epsilon_j > 0$ so small that (c$_j$) holds. Finally, for each $a \in \mathbb{Z}_l = \{0, 1, \ldots, l-1\}$ consider the arcs

$$\delta_{3a} = \beta_a^{a-1}, \quad \delta_{3a+1} = \beta_a \cap \Omega_a, \quad \text{and} \quad \delta_{3a+2} = \beta_a^a,$$

with the endpoints q_{a-1} and q_a^{a-1}, q_a^{a-1} and q_a^a, and q_a^a and q_a, respectively; see (3.16) and Figures 3.1 and 3.2. It follows that $\mathscr{D}^j = \{\delta_a : a \in \mathbb{Z}_{3l}\}$ is a division of bX_j that, in view of properties (ii$_5$), (ii$_6$), and (ii$_9$), is compatible with (u^j, j); see conditions ($\mathfrak{D}$1) to ($\mathfrak{D}$4) in Section 3.1 and recall that $l \geq 4$ is even. This closes the induction in the noncritical case.

The critical case. Assume that $\chi(X_j \setminus \mathring{X}_{j-1}) = -1$, so the number of boundary components of bX_{j-1} and bX_j differ by one. For simplicity of exposition, we assume that $X_j \setminus \mathring{X}_{j-1}$ is connected, hence it is a pair of pants with boundary $bX_{j-1} \cup bX_j$. Otherwise, the other components of $X_j \setminus \mathring{X}_{j-1}$ are all annuli, and we argue on them as in the noncritical case. We distinguish cases.

Case 1: bX_{j-1} is connected. So, bX_j has two connected components. As in the noncritical case, write $\mathscr{D}^{j-1} = \{\alpha_a : a \in \mathbb{Z}_l\}$, $l \geq 4$ even, for the given division of the closed Jordan curve bX_{j-1} compatible with $(u^{j-1}, j-1)$, denote by p_a the only point in $\alpha_a \cap \alpha_{a+1}$, $a \in \mathbb{Z}_l$, and set $\mathbb{Z}_l^{\rm odd}$ and $\mathbb{Z}_l^{\rm even}$ as in (3.5). Thus, conditions (I$_1$) to (I$_3$) in the noncritical case are satisfied.

We may assume without loss of generality that $l \geq 8$ is a multiple of 4. Otherwise, we choose a subarc α_0' of α_0, having p_{l-1} as an endpoint and so small that for every $b \in B$ and $p \in \alpha_0'$ we have

$$\min\{u_{b,1}^{j-1}(p), u_{b,2}^{j-1}(p)\} > j-1, \quad \partial_{J_b} u_{b,1}^{j-1}(p) \neq 0, \quad \partial_{J_b} u_{b,2}^{j-1}(p) \neq 0.$$

Such a subarc exists by compactness of B and conditions (I$_1$) to (I$_3$). Then, splitting α_0' into any family of l adjacent subarcs, say $\beta_1, \ldots, \beta_l$, with $p_{l-1} \in \beta_1$, we obtain that $\{\beta_1, \ldots, \beta_l, \overline{\alpha_0 \setminus \alpha_0'}, \alpha_1, \ldots, \alpha_{l-1}\}$ is a division of bX_{j-1} that is compatible with $(u^{j-1}, j-1)$ and consists of $2l \geq 8$ arcs, which is a multiple of 4 since l is even.

Consider the $\mathscr{J}$-holomorphic map $f : B \times X_{j-1} \to \mathbf{A}$ given by $f(b, \cdot) = 2\partial_{J_b} u_b^{j-1}/\theta_b$ for every $b \in B$, and set $f_b = f(b, \cdot) = (f_{b,1}, f_{b,2}, f_{b,3}) : X_{j-1} \to \mathbf{A}$. Choose a pair of points

$$q \in \alpha_0 \setminus \{p_{l-1}, p_0\} \quad \text{and} \quad q' \in \alpha_{l/2} \setminus \{p_{l/2-1}, p_{l/2}\}.$$

Also, choose a smooth embedded arc $\gamma \subset \mathring{X}_j \setminus \mathring{X}_{j-1}$ with the initial point q, the final point q', and otherwise disjoint from bX_{j-1}. We take γ such that the compact set

$$S = X_{j-1} \cup \gamma \subset \mathring{X}_j$$

is admissible (Definition 3.1) and a strong deformation retract of X_j. (See Figure 3.3.)

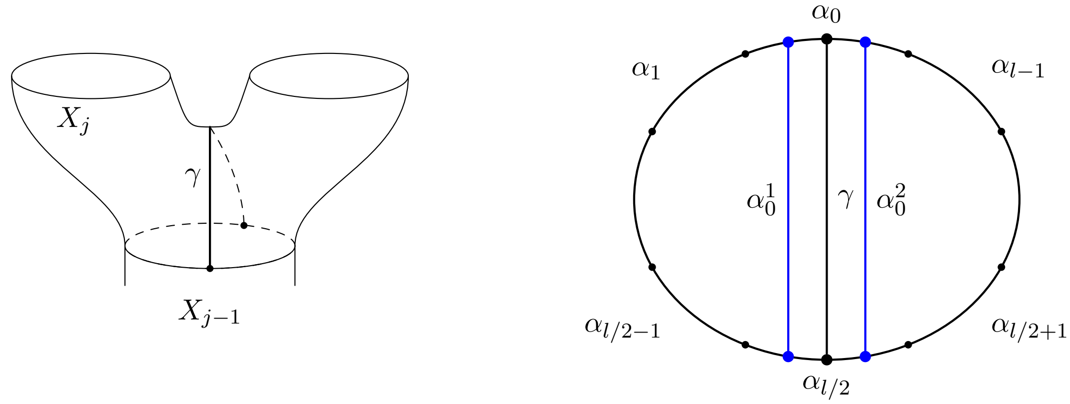


Figure 3.3. Critical case 1

The following claim is an immediate consequence of Lemma 2.3; for the second condition, take into account (I_2) and that $l/2$ is even.

Claim 3.5. *The pair of maps $u^{j-1} : B \times X_{j-1} \to \mathbb{R}^3$ and $f : B \times X_{j-1} \to \mathbf{A}$ can be extended to continuous maps $u^{j-1} : B \times S \to \mathbb{R}^3$ and $f : B \times S \to \mathbf{A}$ such that, setting $u_b^{j-1} = u^{j-1}(b, \cdot) = (u_{b,1}^{j-1}, u_{b,2}^{j-1}, u_{b,3}^{j-1})$ and $f_b = f(b, \cdot) = (f_{b,1}, f_{b,2}, f_{b,3})$ for $b \in B$, the following conditions are satisfied for every $b \in B$.*

- *The pair $(u_b^{j-1}, f_b\theta_b)$ is a generalized J_b-conformal minimal immersion $S \to \mathbb{R}^3$ of class $\mathscr{C}^1$ (see Definition 3.1).*
- *$u_{b,2}^{j-1}(p) > j - 1$ for every $p \in \gamma$.*

By the Mergelyan theorem for families of conformal minimal immersions in [21, Theorem 8.2 and Corollary 8.6], we can approximate u^{j-1} uniformly on $B \times S$ by a continuous map $\hat{u} : B \times X_j \to \mathbb{R}^3$ satisfying the following conditions for every $b \in B$.

(A_1) The map $\hat{u}_b = \hat{u}(b, \cdot) = (\hat{u}_{b,1}, \hat{u}_{b,2}, \hat{u}_{b,3}) : X_j \to \mathbb{R}^3$ is a J_b-conformal minimal immersion.
(A_2) $\hat{u}_{b,1}(p) > j - 1$ for every $p \in \bigcup_{a \in \mathbb{Z}_l^{\rm odd}} \alpha_a$.
(A_3) $\hat{u}_{b,2}(p) > j - 1$ for every $p \in \gamma \cup \bigcup_{a \in \mathbb{Z}_l^{\rm even}} \alpha_a$.
(A_4) $\mathrm{Flux}_{\hat{u}_b}(C) = \mathcal{F}_b(C)$ for every closed curve $C \subset X_j$; take into account (d_{j-1}).
(A_5) $\hat{f}_{b,1}(p_a)\hat{f}_{b,2}(p_a) \neq 0$ for every $a \in \mathbb{Z}_l$, where

$$\hat{f}_b = (\hat{f}_{b,1}, \hat{f}_{b,2}, \hat{f}_{b,3}) = 2\partial_{J_b}\hat{u}_b/\theta_b : X_j \to \mathbf{A}.$$

By (A_3) and compactness of B, we may choose a closed disc neighbourhood D of γ in $\mathring{X}_j$ such that $D \setminus \mathring{X}_{j-1}$ is a closed disc that intersects bX_{j-1} in a pair of small closed arcs around the points $q \in \alpha_0$ and $q' \in \alpha_{l/2}$, $D \cap \alpha_a = \varnothing$ for every $a \in \mathbb{Z}_l \setminus \{0, l/2\}$, and

$$\hat{u}_{b,2}(p) > j - 1 \quad \text{for every } p \in D \text{ and } b \in B. \tag{3.19}$$

Moreover, we choose D such that $X_{j-1} \cup D$ is a smoothly bounded compact domain that is a strong deformation retract of X_j. Set $l' = l/2$, and note by the assumption on l that $l' \geq 4$ and is even. It turns out that $X_{j-1} \cup D$ has two boundary components; one of them, say C_1, containing the arcs α_a for $a = 1, \ldots, l' - 1$, and the other one, say C_2, containing the arcs α_a for $a = l' + 1, \ldots, l - 1$. Moreover,

$$\alpha_0^1 = \overline{C_1 \setminus (\alpha_1 \cup \cdots \cup \alpha_{l'-1})} \quad \text{and} \quad \alpha_0^2 = \overline{C_2 \setminus (\alpha_{l'+1} \cup \cdots \cup \alpha_{l-1})}$$

are connected arcs with endpoints $p_{l'-1}$ and p_0, and p_{l-1} and $p_{l'}$, respectively. (See Figure 3.3.) Setting $\alpha_a^1 = \alpha_a$ and $\alpha_a^2 = \alpha_{l'+a}$ for $a = 1, \ldots, l' - 1$, we have that $\mathcal{D}_i = \{\alpha_a^i : a \in \mathbb{Z}_{l'}\}$ is a division of C_i, $i = 1, 2$, hence $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a division of $b(X_{j-1} \cup D) = C_1 \cup C_2$. Finally, ($A_2$), ($A_3$), ($A_5$), and

(3.19) guarantee that $\mathcal{D}$ is compatible with $(\hat{u}, j-1)$; see conditions ($\mathfrak{D}1$) to ($\mathfrak{D}4$) in Section 3.1. This and (3.19) reduce the proof to the noncritical case.

Case 2: bX_{j-1} has two connected components. So, bX_j is connected. Write C_1 and C_2 for the components of bX_{j-1} and $\mathscr{D}^{j-1} = \mathcal{D}_1 \cup \mathcal{D}_2$ for the given division of bX_{j-1} compatible with $(u^{j-1}, j-1)$, where $\mathcal{D}_i = \{\alpha_a^i : a \in \mathbb{Z}_{l_i}\}$, $l_i \geq 4$ even, is a division of C_i for $i = 1, 2$. Arguing as in the beginning of Case 1, we can assume that $l_1 = l_2 = l$. Denote by p_a^i the only point in $\alpha_a^i \cap \alpha_{a+1}^i$, $i = 1, 2$, $a \in \mathbb{Z}_l$, and set $\mathbb{Z}_l^{\rm odd}$ and $\mathbb{Z}_l^{\rm even}$ as in (3.5). Conditions ($\mathfrak{D}2$) to ($\mathfrak{D}4$) then give the following properties for every $b \in B$.

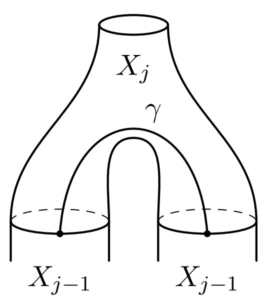
(B$_1$) $u_{b,1}^{j-1}(p) > j-1$ for every $p \in \alpha_a^i$, $i = 1, 2$, $a \in \mathbb{Z}_l^{\rm odd}$.
(B$_2$) $u_{b,2}^{j-1}(p) > j-1$ for every $p \in \alpha_a^i$, $i = 1, 2$, $a \in \mathbb{Z}_l^{\rm even}$.
(B$_3$) $\partial_{J_b} u_{b,1}^{j-1}(p_a^i) \neq 0$ and $\partial_{J_b} u_{b,2}^{j-1}(p_a^i) \neq 0$ for every $i = 1, 2$ and $a \in \mathbb{Z}_l$.

Arguing as in Case 1, we can find a continuous map $\hat{u} : B \times X_j \to \mathbb{R}^3$, close to u^{j-1} uniformly on $B \times X_{j-1}$, and a closed disc $D \subset \mathring{X}_j$ satisfying the following conditions (see Figure 3.4).

(C$_1$) The map $\hat{u}_b = \hat{u}(b, \cdot) = (\hat{u}_{b,1}, \hat{u}_{b,2}, \hat{u}_{b,3}) : X_j \to \mathbb{R}^3$ is a J_b-conformal minimal immersion for every $b \in B$.
(C$_2$) $\hat{u}_{b,1}(p) > j-1$ for every $p \in \bigcup_{a \in \mathbb{Z}_l^{\rm odd}} \alpha_a^i$, $i = 1, 2$, and $b \in B$.
(C$_3$) $\hat{u}_{b,2}(p) > j-1$ for every $p \in D \cup \bigcup_{a \in \mathbb{Z}_l^{\rm even}} \alpha_a^i$, $i = 1, 2$, and $b \in B$.
(C$_4$) $\mathrm{Flux}_{\hat{u}_b}(C) = \mathcal{F}_b(C)$ for every closed curve $C \subset X_j$ and $b \in B$; take into account (d$_{j-1}$).
(C$_5$) $\hat{f}_{b,1}(p_a^i)\hat{f}_{b,2}(p_a^i) \neq 0$ for every $a \in \mathbb{Z}_l$, $i = 1, 2$, and $b \in B$, where

$$\hat{f}_b = (\hat{f}_{b,1}, \hat{f}_{b,2}, \hat{f}_{b,3}) = 2\partial_{J_b}\hat{u}_b/\theta_b : X_j \to \mathbf{A}.$$

(C$_6$) $D \setminus \mathring{X}_{j-1}$ is a closed disc that intersects bX_{j-1} in a pair of small closed arcs, one in the relative interior of α_0^1 and the other in the one of α_0^2.
(C$_7$) $X_{j-1} \cup D$ is a smoothly bounded compact domain that is a strong deformation retract of X_j.



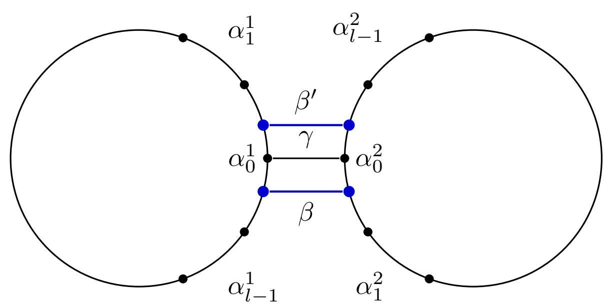


Figure 3.4. Critical case 2

It follows that $\Gamma = b(X_{j-1} \cup D)$ is connected, it contains the arc α_a^i for every $a \in \mathbb{Z}_l \setminus \{0\}$ and $i = 1, 2$, and the set

$$\overline{\Gamma \setminus \cup_{i=1,2}(\alpha_1^i \cup \cdots \cup \alpha_{l-1}^i)}$$

is the disjoint union of two arcs, say, β and β', with endpoints p_{l-1}^1 and p_0^2, and p_{l-1}^2 and p_0^1, respectively. Setting $\alpha_0 = \beta$, $\alpha_a = \alpha_a^2$ for $a \in \mathbb{Z}_l \setminus \{0\}$, $\alpha_l = \beta'$, and $\alpha_{l+a} = \alpha_a^1$ for $a \in \mathbb{Z}_l \setminus \{0\}$, it turns out in view of conditions (C$_2$), (C$_3$), and (C$_5$) that $\mathcal{D} = \{\alpha_a : a \in \mathbb{Z}_{2l}\}$ is a division of $b(X_{j-1} \cup D)$ that is compatible with $(\hat{u}, j-1)$. This and (C$_3$) reduce the proof to the noncritical case.

This closes the induction and completes the proof of the theorem.

Remark 3.6. (a) Our construction shows that the family of proper J_b-conformal minimal immersions $u_b : X \to \mathbb{R}^3$ in Theorem 1.1 can be chosen such that the family of their Weierstrass data $f_b = 2\partial_{J_b} u_b/\theta_b : X \to \mathbf{A}$, $b \in B$, is homotopic to any given continuous map $B \times X \to \mathbf{A}$. The analogous remark holds for families of proper null curves $X \to \mathbb{C}^n$ in Corollary 1.2.

(b) Our construction also gives a map u satisfying Theorem 1.1 such that $\max\{u_{b,1}, u_{b,2}\} : X \to \mathbb{R}$ is a proper map for every $b \in B$. The same holds true in the context of Corollaries 1.2 and 1.3.

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