EVERY PROJECTIVE OKA MANIFOLD IS ELLIPTIC

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Abstract. We show that every projective Oka manifold is elliptic in the sense of Gromov. This gives an affirmative answer to a long-standing open question.

1. Introduction

A complex manifold Y is said to be an Oka manifold if it satisfies all forms of the homotopy principle (also called the Oka principle in this context) for holomorphic maps $X \to Y$ from any Stein manifold X. One of the simplest characterisations of this class of manifolds is the convex approximation property introduced in [10]; see also [12, Sect. 5.4]. In Gromov's terminology [17, 3.1, p. 878], Oka manifolds are called Ell_∞ manifolds.

A complex manifold Y is said to be elliptic if it admits a dominating holomorphic spray $s : E \to Y$ defined on the total space of a complex vector bundle $\pi : E \to Y$ [17, 0.5, p. 855]. This means that s restricts to the identity map on the zero section $E_0 \cong Y$ of E, and for every $y \in Y$, the differential ds_{0_y} at the origin $0_y \in E_y = \pi^{-1}(y)$ maps the fibre E_y onto T_yY. An ostensibly weaker condition, called subellipticity, was introduced by the first-named author in [8, Definition 2]. It asks for the existence of finitely many holomorphic sprays (E_j, π_j, s_j) on Y $(j = 1, \dots, m)$ that are dominating, in the sense that

$$(ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) + \cdots + (ds_m)_{0_y}(E_{m,y}) = T_yY \quad \text{for all } y \in Y. \tag{1.1}$$

One of the main results of Oka theory is that every elliptic manifold is an Oka manifold (see Gromov [17, 0.6, p. 855] and [15]), and every subelliptic manifold is an Oka manifold (see [8, Theorem 1.1]). For a comprehensive survey, see [12, Chap. 5]. Examples of elliptic and subelliptic manifolds can be found in [12, Sect. 6.4] and in the surveys [7, 11, 14]. Every complex homogeneous manifold is elliptic but not conversely. Several recent results are mentioned below.

In this paper we prove the following main result.

Theorem 1.1. *Every projective Oka manifold is elliptic.*

It follows that, for projective manifolds, the Oka property, ellipticity, and subellipticity are equivalent. The spray bundle on a projective Oka manifold Y that emerges in the proof of the theorem is easily described: it is the direct sum of some number of copies of the dual of a sufficiently ample line bundle on Y. In a suitable

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embedding of Y into complex projective space, the spray bundle is therefore the direct sum of copies of the universal line bundle.

Theorem 1.1 solves a long-standing open problem, originating in Gromov's 1989 paper [17, 3.2.A" Question], whether every Oka manifold is elliptic; see also [12, Problem 6.4.21], where the analogous question was posed for subellipticity. The first counterexamples to both questions for noncompact manifolds were found only very recently. In 2024, Kusakabe showed that the complement $\mathbb{C}^n \setminus K$ of any compact polynomially convex set $K \subset \mathbb{C}^n$ for $n \geq 2$ is an Oka manifold [24, Theorem 1.6]. A few years earlier it was shown by Andrist, Shcherbina, and Wold [2] that if K is a compact set with nonempty interior in $\mathbb{C}^n$ for $n \geq 3$, then $\mathbb{C}^n \setminus K$ fails to be subelliptic. Taking K to also be polynomially convex, it follows that $\mathbb{C}^n \setminus K$ is Oka but not subelliptic. These examples are not Stein. As observed by Gromov, every Stein Oka manifold is elliptic [17, 3.2.A, p. 879].

In light of Theorem 1.1, the main remaining question on this topic is the following.

Problem 1.2. Is there a compact non-projective Oka manifold that fails to be elliptic or subelliptic?

A much-studied property of algebraic manifolds is the algebraic version of ellipticity. A complex algebraic manifold Y is said to be algebraically elliptic if it admits an algebraic dominating spray $s : E \to Y$ defined on the total space of an algebraic vector bundle $\pi : E \to Y$; see [12, Definition 5.6.13 (e)]. Similarly, Y is algebraically subelliptic if it admits finitely many algebraic sprays (E_j, π_j, s_j) satisfying (1.1). It was recently shown by Kaliman and Zaidenberg [20] that every algebraically subelliptic manifold is algebraically elliptic; the converse is a tautology. Algebraic ellipticity is a Zariski-local condition as shown by Gromov [17, 3.5.B, 3.5.C]; see also [12, Proposition 6.4.2]. No such results are known in the holomorphic category. A recent result of Banecki [4] is that every rational projective manifold is algebraically elliptic. This generalises the result of Arzhantsev, Kaliman, and Zaidenberg [3, Theorem 1.3] that every uniformly rational projective manifold is algebraically elliptic. See also the recent examples of algebraically elliptic projective manifolds in [19, 21] and the survey [32]. Every algebraically elliptic manifold Y satisfies the algebraic homotopy approximation theorem for maps $X \to Y$ from affine algebraic manifolds X, showing in particular that every holomorphic map that is homotopic to an algebraic map is a limit of algebraic maps in the compact-open topology; see [9, Theorem 3.1], [12, Theorem 6.15.1], and the recent generalisations in [1, Sect. 2]. As shown by Lárusson and Truong [26], this is the closest analogue of the Oka principle in the algebraic category. However, there are examples of projective Oka manifolds that fail to be algebraically elliptic, for example, abelian varieties. Hence, the algebraic counterpart to Theorem 1.1 is not true, and the GAGA principle of Serre [30] fails for ellipticity of projective manifolds.

Besides its intrinsic importance in Oka theory, Theorem 1.1 is interesting for the following reason. For a long time, essentially the only known examples of Oka manifolds were the elliptic and subelliptic manifolds. Thanks to recent developments, we now have at our disposal several other methods to discover Oka manifolds. In particular, Kusakabe's localisation theorem [23, Theorem 1.4] says that a complex manifold covered by Zariski-open (in the holomorphic sense) Oka domains is an Oka manifold. No such localisation result is available for holomorphic ellipticity or subellipticity, so it is interesting that we nevertheless get ellipticity from the Oka property in the class of projective manifolds.

2. Proof of Theorem 1.1

Let Y be a projective manifold, embedded in n-dimensional complex projective space $\mathbb{CP}^n$. Let $z = [z_0 : z_1 : \cdots : z_n]$ be homogeneous coordinates on $\mathbb{CP}^n$. Set $\Lambda_\alpha = \{z_\alpha = 0\}$ for $\alpha = 0, 1, \ldots, n$, and let $U_\alpha = \mathbb{CP}^n \setminus \Lambda_\alpha \cong \mathbb{C}^n$ be the affine chart with coordinates $(z_0/z_\alpha, \ldots, z_n/z_\alpha)$, where the term $z_\alpha/z_\alpha = 1$ is omitted. Denote the affine coordinates on U_0 by $x = (x_1, \ldots, x_n)$, with $x_i = z_i/z_0$. Let $\partial_{x_i} = \partial/\partial x_i$ denote the coordinate vector fields on $U_0 \cong \mathbb{C}^n$. Since $Y_0 = Y \cap U_0$ is an algebraic submanifold of $U_0 \cong \mathbb{C}^n$, Serre's Theorem A gives finitely many polynomial vector fields

$$W_j(x) = \sum_{i=1}^n V_{i,j}(x)\partial_{x_i}, \quad j = 1, \ldots, m, \tag{2.1}$$

on $U_0 \cong \mathbb{C}^n$ whose restrictions to Y_0 are tangent to Y_0 and span the tangent space T_yY at every point $y \in Y_0$. To this collection we associate the polynomial vector field V on the total space $U_0 \times \mathbb{C}^m \cong \mathbb{C}^{n+m}$ of the trivial vector bundle $\pi : U_0 \times \mathbb{C}^m \to U_0$, defined by

$$V(x,t) = \sum_{i=1}^n \sum_{j=1}^m t_j\, V_{i,j}(x)\partial_{x_i}, \tag{2.2}$$

where $x \in U_0$ and $t = (t_1, \ldots, t_m) \in \mathbb{C}^m$ is the fibre variable. Note that V is horizontal in the sense that its t-component equals zero. Furthermore, V vanishes on the zero section $U_0 \times \{0\}^m = \{t = 0\}$, and for every $(x,t) \in Y_0 \times \mathbb{C}^m$ we have that

$$d\pi_{(x,t)}V(x,t) = \sum_{i=1}^n \sum_{j=1}^m t_j\, V_{i,j}(x)\partial_{x_i} = \sum_{j=1}^m t_j\, W_j(x) \in T_xY. \tag{2.3}$$

The formula without the last inclusion holds for all $(x,t) \in U_0 \times \mathbb{C}^m$.

Recall that $\Lambda_0 = \mathbb{CP}^n \setminus U_0 = \{z_0 = 0\}$. Denote by $\mathbb{U} \to \mathbb{CP}^n$ the universal line bundle.

Lemma 2.1. *Let k_0 denote the maximum of the degrees of the polynomials $V_{i,j}(x)$. For every $k \geq k_0$, the vector field V (2.2) extends to an algebraic vector field on the total space of the vector bundle $E = (\mathbb{CP}^n \times \mathbb{C}^m) \otimes \mathbb{U}^k = m\mathbb{U}^k$ on $\mathbb{CP}^n$ (the direct sum of m copies of the k-th tensor power of $\mathbb{U}$), which vanishes on the zero section E_0 of E and on $E|_{\Lambda_0}$.*

Proof. For every $\alpha = 0, 1, \ldots, n$ we have a vector bundle trivialisation $\theta_\alpha : E|U_\alpha \xrightarrow{\cong} U_\alpha \times \mathbb{C}^m$ with transition maps $\theta_{\alpha,\beta} = \theta_\alpha \circ \theta_\beta^{-1}$ on $(U_\alpha \cap U_\beta) \times \mathbb{C}^m$ given by

$$\theta_{\alpha,\beta}([z], t) = \big([z], (z_\alpha/z_\beta)^k t\big), \quad t \in \mathbb{C}^m,\ 0 \leq \alpha, \beta \leq n. \tag{2.4}$$

In particular, $\theta_{\alpha,0}([z], t) = \big([z], (z_\alpha/z_0)^k t\big)$. We analyse the behaviour of the vector field V (2.2) near the hyperplane $\Lambda_0 \setminus \Lambda_\alpha$ for $\alpha = 1, \ldots, n$. It suffices to consider the case

$\alpha = 1$ since the same argument will apply to every α. For simplicity we first make the calculation in the case when $m = 1$, so $E = \mathbb{U}^k$. The calculation is made in two steps. In the first step, we express V in the fibre coordinate t' on the line bundle chart $E|U_1 \cong \mathbb{C}^n \times \mathbb{C}$ over the domain $U_0 \cap U_1 = \{x = (x_1, \ldots, x_n) \in \mathbb{C}^n : x_1 \neq 0\}$. The transition map $\theta_{1,0}$ is given by $\theta_{1,0}(x, t) = \big(x, x_1^k t\big)$, so $t' = x_1^k t$. Its differential has the block form

$$D\theta_{1,0}(x,t) = \begin{pmatrix} I_n & 0 \\ B & x_1^k \end{pmatrix} \tag{2.5}$$

where I_n is the $n \times n$ identity matrix and $B = (kx_1^{k-1}t, 0, \ldots, 0)$. It follows that the vector field $V' = D\theta_{1,0} \cdot V$ equals

$$\begin{aligned} V' &= t\sum_{i=1}^{n} V_i(x)\partial_{x_i} + kt^2 x_1^{k-1} V_1(x)\partial_{t'} \\ &= t' x_1^{-k} \sum_{i=1}^{n} V_i(x)\partial_{x_i} + k(t')^2 x_1^{-k-1} V_1(x)\partial_{t'}, \end{aligned}$$

where we used that $t = x_1^{-k}t'$. In the second step, we express the vector field V' in the standard affine coordinates $x' = (x'_1, x'_2, \ldots, x'_n)$ on U_1. Note that

$$\begin{aligned} x'_1 &= \frac{z_0}{z_1} = \frac{1}{x_1}, \\ x'_i &= \frac{z_i}{z_1} = \frac{x_i}{x_1}, \quad i = 2, \ldots, n. \end{aligned}$$

Write $x' = \psi(x)$ and $(x', t') = \tilde{\psi}(x, t') = (\psi(x), t')$. We have that

$$D\psi(x) = \begin{pmatrix} -\frac{1}{x_1^2} & 0 & 0 & \cdots & 0 \\ -\frac{x_2}{x_1^2} & \frac{1}{x_1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{x_n}{x_1^2} & 0 & 0 & \cdots & \frac{1}{x_1} \end{pmatrix}.$$

Hence, the vector field $\widetilde{V} = D\tilde{\psi} \cdot V'$ equals

$$\begin{aligned} \widetilde{V}(x, t') = -\frac{1}{x_1^{k+2}} V_1(x) t' \partial_{x'_1} &+ \sum_{i=2}^{n} \left[-\frac{x_i}{x_1^{k+2}} V_1(x) + \frac{1}{x_1^{k+1}} V_i(x) \right] t' \partial_{x'_i} \\ &+ \frac{k}{x_1^{k+1}} V_1(x)(t')^2 \partial_{t'}. \end{aligned}$$

Note that $x = \psi^{-1}(x') = (1/x'_1, x'_2/x'_1, \cdots, x'_n/x'_1)$. Inserting this in the above expression gives

$$\begin{aligned} \widetilde{V}(x', t') &= -(x'_1)^{k+2} V_1(\psi^{-1}(x'))\, t' \partial_{x'_1} \\ &\quad + \sum_{i=2}^{n} (x'_1)^{k+1} \left[-x'_i V_1(\psi^{-1}(x')) + V_i(\psi^{-1}(x')) \right] t' \partial_{x'_i} \\ &\quad + k(x'_1)^{k+1} V_1(\psi^{-1}(x'))(t')^2 \partial_{t'}. \end{aligned}$$

Note that the affine hyperplane $\{z_0 = 0,\ z_1 \neq 0\}$ corresponds to $\{x'_1 = 0\}$. Since ψ^{-1} is a fractional linear map with a simple pole along $x'_1 = 0$, the functions $V_i(\psi^{-1}(x'))$ are rational in x' with a pole of degree at most k_0 along $x'_1 = 0$ and no other singularities. It follows from the above expression that for $k \geq k_0$ the vector field $\widetilde{V}(x', t')$ is polynomial in (x', t') and it vanishes on $\{x'_1 = 0\} \cup \{t' = 0\}$.

In the general case when V is given by (2.2) and $m \in \mathbb{N}$ is arbitrary, the calculation is similar. Using the fibre variables $t = (t_1, \ldots, t_m)$ on $E|U_0$ and $t' = (t'_1, \ldots, t'_m) = x_1^k t$ on $E|U_1$, the differential $D\theta_{0,1}$ has the block form (2.5), where B is now an $m \times n$ matrix and the lower right entry is replaced by $x_1^k I_m$. The vector field $V' = D\theta_{1,0} \cdot V$ equals

$$\begin{aligned} V' &= \sum_{i=1}^{n} \sum_{j=1}^{m} t_j \, V_{i,j}(x) \partial_{x_i} + k x_1^{k-1} \sum_{j,l=1}^{m} t_j t_l V_{1,j}(x) \partial_{t'_l} \\ &= x_1^{-k} \sum_{i=1}^{n} \sum_{j=1}^{m} t'_j \, V_{i,j}(x) \partial_{x_i} + k x_1^{-k-1} \sum_{j,l=1}^{m} t'_j t'_l V_{1,j}(x) \partial_{t'_l}. \end{aligned}$$

The second step, expressing V' in the affine coordinates x' on U_1, is the same as before, and we leave the details to the reader. The new vector field $\widetilde{V} = D\widetilde{\psi} \cdot V'$ is polynomial in (x', t'), of second order in t', and it vanishes on $\{x'_1 = 0\} \cup \{t' = 0\}$. Since this argument holds on every chart U_α for $\alpha = 1, \ldots, n$, the lemma is proved. $\square$

Since the extended vector field V on E, given by Lemma 2.1, vanishes on the zero section E_0 of E, there is a neighbourhood $\Omega \subset E$ of E_0 with convex fibres such that the flow $\phi_\tau(e)$ of V, starting at time $\tau = 0$ in any point $e \in \Omega$, exists for all $\tau \in [0, 1]$. The map

$$s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n$$

is then a local holomorphic spray on $\mathbb{CP}^n$. On the zero section $E_0 \cong \mathbb{CP}^n$ we have a natural splitting $TE|E_0 = E \oplus T\mathbb{CP}^n$. Identifying a vector $e \in E_x = \pi^{-1}(x)$ with $e \in T_{0_x} E_x$, we let

$$(Vds)_x(e) = (ds)_{0_x}(e) \in T_x \mathbb{CP}^n$$

denote the vertical derivative of s at $x \in \mathbb{CP}^n$ applied to the vector e. We claim that for every $e = (x, t) \in \Omega$, with $x \in U_0$, we have

$$(Vds)_x(t_1, \ldots, t_m) = \sum_{j=1}^{m} t_j \, W_j(x). \tag{2.6}$$

To see this, note that in the vector bundle chart on $E|U_0$ the vector field V is of the form (2.2), that is, it is horizontal and its coefficients are linear in the fibre variable t. It follows that

$$\pi \circ \phi_\tau(x, \delta t) = \pi \circ \phi_{\delta\tau}(x, t) \tag{2.7}$$

holds for every $(x, t) \in E|U_0$, $0 \leq \delta \leq 1$, and all τ for which the flow exists. Taking $(x, t) \in \Omega$, this holds for all $\tau \in [0, 1]$. At $\tau = 1$ we obtain

$$s(x, \delta t) = \pi \circ \phi_1(x, \delta t) = \pi \circ \phi_\delta(x, t), \quad 0 \leq \delta \leq 1.$$

Differentiating with respect to δ at $\delta = 0$ and noting that $\left.\frac{d}{d\delta}\right|_{\delta=0} \phi_\delta(x, t) = V(x, t)$ and $d\pi_{(x,t)} V(x, t) = \sum_{j=1}^{m} t_j \, W_j(x)$ (see (2.3)) gives (2.6).

Set $E|Y = \pi^{-1}(Y)$. Condition (2.3) implies that the spray $s = \pi \circ \phi_1$ maps the domain $\Omega \cap E|Y$ to Y, so it is a local holomorphic spray on Y. Since the vector fields $W_1, \ldots, W_m$ generate the tangent space T_xY every point $x \in Y_0 = Y \cap U_0$, we see from (2.6) that the restricted spray $s : \Omega \cap E|Y \to Y$ is dominating on Y_0. On the other hand, since V vanishes on $E|\Lambda_0$, ϕ_1 is the identity on this set and the spray $s = \pi$ is trivial over Λ_0.

In order to find a local dominating spray on Y, we proceed as follows. For $\alpha \in \{0, 1, \ldots, n\}$ set $Y_\alpha = Y \cap U_\alpha$; this is an algebraic submanifold of $U_\alpha \cong \mathbb{C}^n$. Choose $m \in \mathbb{N}$ big enough that the tangent bundle TY_α is pointwise generated by m polynomial vector fields W_j^α of the form (2.1) on U_α for every $\alpha \in \{0, 1, \ldots, n\}$. In the affine coordinates $x = (x_1, \ldots, x_n) = (z_0/z_\alpha, \ldots, z_n/z_\alpha)$ on U_α we have $W_j^\alpha(x) = \sum_{i=1}^n V_{i,j}^\alpha(x)\partial_{x_i}$ where $V_{i,j}^\alpha$ are polynomials. Let

$$k_0 := \max_{\alpha,i,j} \deg V_{i,j}^\alpha. \tag{2.8}$$

For every $k \geq k_0$ the above argument gives an algebraic vector field V^α on the vector bundle $E^\alpha = m\mathbb{U}^k$ that vanishes on the zero section E_0^α and is of the form (2.2) in the chart $E^\alpha|U_\alpha \cong U_\alpha \times \mathbb{C}^m$. Explicitly, in the affine coordinates $x = (z_0/z_\alpha, \ldots, z_n/z_\alpha)$ on U_α and fibre coordinates $t^\alpha = (t_1^\alpha, \ldots, t_m^\alpha)$ on $E^\alpha|U_\alpha$ we have

$$V^\alpha(x, t^\alpha) = \sum_{i=1}^n \sum_{j=1}^m t_j^\alpha \, V_{i,j}^\alpha(x)\partial_{x_i}.$$

We can take V^0 to be the vector field V in (2.2).

Let $E = E^0 \oplus E^1 \oplus \cdots \oplus E^n = (n+1)m\mathbb{U}^k$ and denote the vector bundle projection by $\pi : E \to \mathbb{CP}^n$. The algebraic vector field V^α on E^α constructed above can be extended to an algebraic vector field on E by first extending it trivially (horizontally) to each of the summands $E^\beta|U_\alpha$ of $E|U_\alpha$ for $\beta \neq \alpha$ (note that these are trivial bundles), and then observing that the resulting vector field on $E|U_\alpha$ extends to an algebraic vector field on E taking into account the condition (2.8) (see Lemma 2.1). With these extensions in place, we consider the vector field $V = \sum_{\alpha=0}^n V^\alpha$ on E. The construction implies that

$$d\pi_e V(e) \in T_yY \quad \text{for every } y \in Y \text{ and } e \in E_y = \pi^{-1}(y).$$

Since each V^α vanishes on the zero section of E_0 of E, so does V. Hence, there is a neighbourhood $\Omega \subset E$ of E_0 with convex fibres such that the flow $\phi_\tau(e)$ of V exists for any initial point $e \in \Omega$ and every $\tau \in [0, 1]$. Consider the holomorphic spray

$$s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n. \tag{2.9}$$

We claim that $s : \Omega \cap \pi^{-1}(Y) \to Y$ is dominating. To see this, consider the vector field $V = \sum_{\alpha=0}^n V^\alpha$ on a chart $E|U_\beta$. For simplicity of notation we assume that $\beta = 0$; the argument will be the same in every case. In the affine coordinates $x = (z_1/z_0, \ldots, z_n/z_0)$ on U_0 and fibre coordinates $t = (t^0, t^1, \ldots, t^n)$ on $E|U_0$, where $t^\alpha = (t_1^\alpha, \ldots, t_m^\alpha)$ are fibre coordinates on the direct summand $E^\alpha|U_0$ of $E|U_0$, we see as above that

$$V(x, t) = \sum_{\alpha=0}^n \sum_{i=1}^n \sum_{j=1}^m t_j^\alpha \, \widetilde{V}_{i,j}^\alpha(x)\partial_{x_i} + \Upsilon(x, t) = \Theta(x, t) + \Upsilon(x, t) \tag{2.10}$$

where each $\widetilde{V}^{\alpha}_{i,j}(x)$ is a polynomial, $\widetilde{V}^{0}_{i,j}(x) = V^{0}_{i,j}(x)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and the vertical component Υ of V vanishes to the second order along the zero section $E_0 = \{t = 0\}$, that is, $|\Upsilon(x,t)| = O(|t|^2)$. Since the vector field $\Theta(x,t)$ in (2.10) is linear in the fibre variable t, its flow ψ_τ satisfies $\pi \circ \psi_\tau(x, \delta t) = \pi \circ \psi_{\delta\tau}(x,t)$ (cf. (2.7)). As before, it follows that the vertical derivative of the spray $\tilde{s} = \pi \circ \psi_1 : \Omega \to \mathbb{CP}^n$ over U_0 equals

$$(Vd\tilde{s})_x(x,t) = \sum_{\alpha=0}^{n} \sum_{j=1}^{m} t^{\alpha}_{j}\, \widetilde{W}^{\alpha}_{j}(x),$$

where each $\widetilde{W}^{\alpha}_{j}(x)$ is a polynomial vector field tangent to Y_0 and $\widetilde{W}^{0}_{j} = W^{0}_{j}$ for $j = 1, \ldots, m$. (Compare with (2.6).) Since the vectors $W^0_j(x)$ for $j = 1, \ldots, m$ span T_xY for every $x \in Y_0$, the spray $\tilde{s}$ is dominating over Y_0. Since the second term Υ in (2.10) is of size $O(|t|^2)$, a standard argument using Grönwall's inequality shows that the flow ϕ_τ of V satisfies

$$\phi_\tau(x,t) = \psi_\tau(x,t) + O(|t|^2) \ \text{ as } |t| \to 0 \text{ and } \tau \in [0,1].$$

It follows that the spray $s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n$ (2.9) satisfies $(Vds)_x(x,t) = (Vd\tilde{s})_x(x,t)$ for $x \in U_0$. The same argument holds on every chart $E|U_\beta$.

This proves that the spray $s : \Omega \cap \pi^{-1}(Y) \to Y$ is dominating as claimed.

We now replace the bundle $E \to \mathbb{CP}^n$ by its restriction $X := E|Y \to Y$ and the spray s (2.9) by its restriction $X \cap \Omega \to Y$. Since X is a direct sum of copies of the negative line bundle $\mathbb{U}^k|Y$, it is a 1-convex manifold with the zero section $X_0 \cong Y$ as the exceptional subvariety; that is, X_0 is the maximal compact complex subvariety of X without point components; see Grauert [16, Satz 1, p. 341]. Such X admits a plurisubharmonic exhaustion function $\rho : X \to [0, \infty)$ which vanishes on X_0 and is positive strongly plurisubharmonic on $X \setminus X_0$, and the Remmert reduction of X is a Stein space.

So far, we have not used the hypothesis that Y is an Oka manifold. Under this additional assumption, we shall prove that there exists a holomorphic spray $\tilde{s} : X \to Y$ which agrees with s to the second order along the zero section $X_0 \cong Y$ of X. Clearly, $\tilde{s}$ is then a dominating spray on Y, thereby proving Theorem 1.1.

Since X is a 1-convex manifold and Y is an Oka manifold, the existence of such $\tilde{s}$ follows from the main result of Prezelj [28] if taken at face value; see also [29] which clarifies a part of the construction in [28]. However, it was brought to our attention that one of the technical tools, [28, Theorem 2.4], claims more than had been proved in the literature, and possibly there exists a counterexample to this statement. (This is quoted from Grauert's paper [16] but certain conditions in his result were not taken into account.) For this reason, we proceed in a different way. The main point is to prove the following lemma.

Lemma 2.2. *(Assumptions as above.) Let $r \geq 2$ be an integer. Assume that $U \subset X$ is an open neighbourhood of the zero section $X_0 \cong Y$ of $\pi : X \to Y$ and $s : U \to Y$ is a local dominating holomorphic spray. There are an open neighbourhood $U' \subset U$ of X_0, a ball $0 \in B \subset \mathbb{C}^N$ for some $N \in \mathbb{N}$, and a holomorphic spray of maps $S : U' \times B \to Y$*

with the core $S(\cdot\,,0) = s|U'$ such that S agrees with s to order r along X_0 for every $\zeta \in B$, and S is dominating over $U' \setminus X_0$ with respect to the variable $\zeta \in B$.

More precisely, the last claim is that for every point $x \in U' \setminus X_0$ the differential of the map $B \ni \zeta \mapsto S(x,\zeta) \in Y$ at $\zeta = 0$ maps $T_0 B = \mathbb{C}^N$ onto $T_{s(x)}Y$. (Note that $S(x,0) = s(x)$.) If Lemma 2.2 holds then the existence of a holomorphic map $\tilde s : X \to Y$ which agrees with s on X_0 to order r follows from [31, Theorem 3.3] by Stopar. Indeed, Lemma 2.2 shows that every map $s : U \to Y$ as in the lemma satisfies Condition $\mathcal{E}$ in [31, p. 4]. Assuming as we may that the domain $U \Subset X$ is relatively compact with smooth strongly pseudoconvex boundary, [31, Theorem 2.5] gives a holomorphic spray $\widetilde S : U \times B \to Y$ with the core $\widetilde S(\cdot\,,0) = s$ which is dominating over $U \setminus X_0$ with respect to the ζ variable and such that $\widetilde S(\cdot\,,\zeta)$ agrees with s to order r along X_0 for every $\zeta \in B$. (Assuming that $N \ge \dim X + \dim Y$, $\widetilde S$ may be chosen to approximate the spray S in the hypotheses of the lemma over a neigbourhood of X_0.) The existence of a local dominating spray $\widetilde S$ with these properties was the main technical issue addressed in [28, 29]. With this result in hand, one can apply the usual inductive procedure in Oka theory, adapted to 1-convex manifolds, to construct a global holomorphic map $\tilde s : X \to Y$ which agrees with s to order r along X_0. See [31, Theorem 3.3] and its proof for the details. (The cited result applies to sections of any holomorphic fibre bundle with an Oka fibre over X, but we apply it to maps $X \to Y$ identified with sections of the trivial bundle $X \times Y \to X$.)

Proof of Lemma 2.2. The idea is to precompose the map $s : U \to Y$ by a suitably chosen spray of holomorphic fibre preserving self-maps of the vector bundle $\pi : X \to Y$. Recall that this bundle is trivial over each domain $Y \cap U_j$ for $j = 0, 1, \ldots, n$, where $U_j \subset \mathbb{CP}^n$ are the standard affine charts. We explain the construction over $Y_0 = Y \cap U_0$; it will apply by symmetry to all other charts. Let $x = (z_1/z_0, \ldots, z_n/z_0)$ be the affine coordinates on U_0 and $t = (t_1, \ldots, t_l)$ the fibre coordinates on the vector bundle chart $X|Y_0 \cong Y_0 \times \mathbb{C}^l$ with $l = m(n+1)$. Let $r \ge 2$ be as in the lemma. Choose $v \in \mathbb{C}^l$ and consider the holomorphic map $\phi : Y_0 \times \mathbb{C}^l \times \mathbb{C} \to Y_0 \times \mathbb{C}^l$ given by

$$\phi(x,t,\zeta) = (x, t + \zeta t_1^r v). \tag{2.11}$$

Note that $\phi(x,t,0) = (x,t)$. We claim that for every $\zeta \in \mathbb{C}$, $\phi(\cdot\,,\cdot\,,\zeta)$ extends to a holomorphic fibre preserving self-map of X which agrees with the identity to order r along X_0. Indeed, consider its expression in the vector bundle chart $X|Y_\alpha$ for some $\alpha \in \{1, \ldots, n\}$. The fibre coordinates t' on this chart are related to t by $t' = (z_\alpha/z_0)^k t$ (see (2.4)), and a calculation gives

$$\phi(x,t',\zeta) = \big(x, t' + \zeta (t_1')^r (z_0/z_\alpha)^{k(r-1)} v\big) = \big(x, t' + \zeta (t_1')^r (1/x_\alpha)^{k(r-1)} v\big)\,.$$

This shows that for every $\zeta \in \mathbb{C}$ the map $\phi(\cdot\,,\cdot\,,\zeta)$ agrees with the identity on X over the affine hyperplane $\{z_0 = 0,\ z_\alpha \ne 0\}$ intersected with Y, and it agrees with the identity to order $r \ge 2$ along $\{t = 0\} = X_0$. Since this holds for every $\alpha = 1, \ldots, n$, the claim follows. Identifying the tangent bundle $T(X|Y_\alpha)$ with $TY_\alpha \times T\mathbb{C}^l$, we have that

$$\frac{\partial}{\partial\zeta}\Big|_{\zeta=0} \phi(x,t,\zeta) = (0, t_1^r v)$$

and hence

$$\frac{\partial}{\partial\zeta}\Big|_{\zeta=0} s\circ\phi(x,t,\zeta) = ds_{(x,t)}(0,t_1^r v) = t_1^r ds_{(x,t)}(0,v). \tag{2.12}$$

Since the spray s is dominating by the assumption, we can choose $v\in\mathbb{C}^l$ such that $ds_{(x,0)}(0,v)$ equals any given tangent vector $w\in T_xY$, and hence the vector (2.12) equals $t_1^r w$. Let $\phi_1,\ldots,\phi_d$ for $d=\dim Y$ be sprays of the form (2.11) for vectors $v_1,\ldots,v_d\in\mathbb{C}^l$ chosen such that the vectors $ds_{(x,0)}(0,v_i)$ for $i=1,\ldots,d$ form a basis of T_xY. Then, there are a neighbourhood $B'\subset\mathbb{C}^d$ of the origin and a neighbourhood $U'\subset U$ of X_0 such that the map $s\circ\phi_1\circ\cdots\circ\phi_d : U'\times B'\to Y$ given by

$$(x,t,\zeta_1,\ldots,\zeta_d)\longmapsto s\circ\phi_1(x,t,\zeta_1)\circ\cdots\circ\phi_d(x,t,\zeta_d)$$

is a local spray with the core s at $\zeta=0$ which is dominating with respect to $\zeta=(\zeta_1,\ldots,\zeta_d)$ at $\zeta=0$ in a neighbourhood of $(x,0)\in X_0$ in X, except on the hyperplane $t_1=0$, and for every fixed $\zeta\in B'$ it agrees with s to order r along the zero section $X_0\cong Y$. Repeating this construction with t_1^r replaced by t_j^r for $j=1,\ldots,l$ and at other points $x\in Y$ (also in other charts on Y) gives finitely many holomorphic sprays $\phi_1,\ldots,\phi_N$ on X of the form (2.11) and neighbourhoods $U'\subset U$ of X_0 and $0\in B\subset\mathbb{C}^N$ such that the map

$$S=s\circ\phi_1\circ\cdots\circ\phi_N : U'\times B\to Y$$

satisfies the conclusion of the lemma. ☐

Remark 2.3. In the proof of Theorem 1.1, we begin with suitably chosen algebraic (polynomial) vector fields on affine vector bundle charts, which extend to algebraic vector fields on a sufficiently negative vector bundle on the given manifold Y. This is only possible on projective manifolds since a compact complex manifold with a negative (or a positive) line bundle is necessarily projective according to Kodaira [22]. The subsequent techniques using flows of vector fields, and especially the last step of the proof to construct a global dominating spray, are transcendental. Hence, this method does not give algebraic ellipticity. This is not surprising, for we have already mentioned that there are examples of projective Oka manifolds that fail to be algebraically elliptic, for example, abelian varieties.

3. FURTHER RESULTS AND REMARKS ON ELLIPTICITY

In this section we collect some further observations concerning the relationship between the Oka property and ellipticity of a complex manifold.

Remark 3.1. If $L\to Y$ is a negative holomorphic line bundle on a compact (hence projective) manifold Y, then for sufficiently large $k>0$ the vector bundle $\mathrm{Hom}(L^k,TY)\cong L^{-k}\otimes TY$ on Y is generated by finitely many global holomorphic sections $h_1,\ldots,h_N$ (a theorem of Hartshorne; see Lazarsfeld [27, Theorem 6.1.10]). Let $E=NL^k$ denote the direct sum of N copies of L^k. Considering h_i as a homomorphism $h_i : L^k\to TY$, it follows that the holomorphic vector bundle map $h=\oplus_{i=1}^N h_i : E\to TY$ is an epimorphism. Gromov proposed [17, 3.2.A', Step 2, p. 879] that such h is the vertical derivative of a local dominating holomorphic spray $s : U\to Y$ from an open neighbourhood $U\subset E$ of its zero section $E_0\cong Y$. This would give a shorter proof of Theorem 1.1. Although we do not know how to justify Gromov's claim, our proof of Theorem 1.1 follows this idea in spirit if not to the letter. This raises the following question.

Problem 3.2. Which holomorphic vector bundles $E \to Y$ of $\operatorname{rank} E \geq \dim Y$ admit a local dominating spray $s : U \to Y$ from a neighbourhood $U \subset E$ of the zero section of E?

The following observation generalises [13, Proposition 6.2]. Recall that every complex homogeneous manifold is elliptic [12, Proposition 5.6.1], and hence an Oka manifold.

Proposition 3.3. *Assume that a compact complex manifold Y admits a local dominating holomorphic spray (E, π, s). If the vector bundle $\pi : E \to Y$ is generated by global holomorphic sections, then Y is a complex homogeneous manifold.*

The condition on E to be globally generated by holomorphic sections holds for a trivial bundle and for any sufficiently Griffiths positive bundle, but fails for negative bundles.

Proof. Let $s : U \to Y$ be a local dominating spray defined on a neighbourhood $U \subset E$ of the zero section E_0. The vertical derivative $Vds|E_0 : VT(E)|E_0 = E \to TY$ is a vector bundle epimorphism. Given a holomorphic section $\xi : Y \to E$, the map

$$Y \ni y \longmapsto V_\xi(y) := Vds(y)(\xi(y)) \in T_yY$$

is a holomorphic vector field on Y. (We are using the natural identification of the vertical tangent bundle $VT(E)|E_0$ on the zero section E_0 with the bundle E itself.) Applying this argument to sections $\xi_1, \ldots, \xi_m : Y \to E$ generating E gives holomorphic vector fields $V_1, \ldots, V_m$ on Y spanning the tangent bundle TY since Vds is surjective. Thus, Y is holomorphically flexible. Since Y is compact, these vector fields are complete, so their flows are complex 1-parameter subgroups of the holomorphic automorphism group $\operatorname{Aut} Y$, which is a finite-dimensional complex Lie group [6]. The spanning property implies that $\operatorname{Aut} Y$ acts transitively on Y, so Y is homogeneous. $\square$

There are projective Oka manifolds that are not homogeneous, for instance, blowups of certain projective manifolds such as projective spaces, Grassmannians, etc.; see [12, Propositions 6.4.5 and 6.4.6], the papers [18, 25], and the survey [7, Subsect. 6.3]. Many of these manifolds are algebraically elliptic. Another class of non-homogeneous projective surfaces that are algebraically elliptic are the Hirzebruch surfaces H_l for $l = 1, 2, \ldots$; see [5, p. 191] and [12, Proposition 6.4.5]. In view of Proposition 3.3, such manifolds do not admit a local dominating spray from any globally generated holomorphic vector bundle.

Remark 3.4. Let $\mathscr{S}$ be the largest class of complex manifolds for which the Oka property implies ellipticity, that is, the class of manifolds that are either elliptic or not Oka. As remarked above, it is long known that every Stein manifold belongs to $\mathscr{S}$. By Theorem 1.1, so does every projective manifold. We know of two ways to produce new members of $\mathscr{S}$ from old. If $Y \to X$ is a covering map and X is elliptic, so is Y. Also, X is Oka if and only if Y is. Hence, a covering space of a manifold in $\mathscr{S}$ is in $\mathscr{S}$. Also, it is easily seen that a product of manifolds in $\mathscr{S}$ is in $\mathscr{S}$.

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REFERENCES

[1] A. Alarcón, F. Forstnerič, and F. Lárusson, *Isotopies of complete minimal surfaces of finite total curvature*, `arXiv:2406.04767`, (2024).

[2] R. B. Andrist, N. Shcherbina, and E. F. Wold, *The Hartogs extension theorem for holomorphic vector bundles and sprays*, Ark. Mat. **54** (2016), no. 2, 299–319.

[3] I. Arzhantsev, S. Kaliman, and M. Zaidenberg, *Varieties covered by affine spaces, uniformly rational varieties and their cones*, Adv. Math. **437** (2024), 109449, 18 pp.

[4] J. Banecki, *Retract rational varieties are uniformly retract rational*, `arXiv:2411.17892`, (2024).

[5] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag, Berlin, second edition, 2004.

[6] S. Bochner and D. Montgomery, *Groups on analytic manifolds*, Ann. of Math. (2) **48** (1947), 659–669.

[7] F. Forstnerič, *Recent developments on Oka manifolds*, Indag. Math. (N.S.) **34** (2023), no. 2, 367–417.

[8] F. Forstnerič, *The Oka principle for sections of subelliptic submersions*, Math. Z. **241** (2002), no. 3, 527–551.

[9] F. Forstnerič, *Holomorphic flexibility properties of complex manifolds*, Amer. J. Math. **128** (2006), no. 1, 239–270.

[10] F. Forstnerič, *Runge approximation on convex sets implies the Oka property*, Ann. of Math. (2) **163** (2006), no. 2, 689–707.

[11] F. Forstnerič, *Oka manifolds: from Oka to Stein and back*, Ann. Fac. Sci. Toulouse Math. (6) **22** (2013), no. 4, 747–809. With an appendix by Finnur Lárusson.

[12] F. Forstnerič, *Stein manifolds and holomorphic mappings (The homotopy principle in complex analysis)*, volume 56 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer, Cham, 2017.

[13] F. Forstnerič, *Mergelyan's and Arakelian's theorems for manifold-valued maps*, Mosc. Math. J. **19** (2019), no. 3, 465–484.

[14] F. Forstnerič and F. Lárusson, *Survey of Oka theory*, New York J. Math. **17A** (2011), 11–38.

[15] F. Forstnerič and J. Prezelj, *Oka's principle for holomorphic fiber bundles with sprays*, Math. Ann. **317** (2000), no. 1, 117–154.

[16] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[17] M. Gromov, *Oka's principle for holomorphic sections of elliptic bundles*, J. Amer. Math. Soc. **2** (1989), no. 4, 851–897.

[18] S. Kaliman, F. Kutzschebauch, and T. T. Truong, *On subelliptic manifolds*, Israel J. Math. **228** (2018), no. 1, 229–247.

[19] S. Kaliman and M. Zaidenberg, *Algebraic Gromov ellipticity of cones over projective manifolds*, Math. Res. Lett. **31** (2024), no. 6, 1785–1817.

[20] S. Kaliman and M. Zaidenberg, *Gromov ellipticity and subellipticity*, Forum Math. **36** (2024), no. 2, 373–376.

[21] S. Kaliman and M. Zaidenberg, *Algebraic Gromov's ellipticity of cubic hypersurfaces*, Proc. Steklov Inst. Math. **329** (2025), 79–87.

[22] K. Kodaira, *On Kähler varieties of restricted type (An intrinsic characterization of algebraic varieties)*, Ann. of Math. (2) **60** (1954), 28–48.

[23] Y. Kusakabe, *Elliptic characterization and localization of Oka manifolds*, Indiana Univ. Math. J. **70** (2021), no. 3, 1039–1054.

[24] Y. Kusakabe, *Oka properties of complements of holomorphically convex sets*, Ann. of Math. (2) **199** (2024), no. 2, 899–917.

[25] F. Lárusson and T. T. Truong, *Algebraic subellipticity and dominability of blow-ups of affine spaces*, Doc. Math. **22** (2017), 151–163.

[26] F. Lárusson and T. T. Truong, *Approximation and interpolation of regular maps from affine varieties to algebraic manifolds*, Math. Scand. **125** (2019), no. 2, 199–209.

[27] R. Lazarsfeld, *Positivity in algebraic geometry. II*, volume 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag, Berlin, 2004.

[28] J. Prezelj, *A relative Oka-Grauert principle for holomorphic submersions over 1-convex spaces*, Trans. Amer. Math. Soc. **362** (2010), no. 8, 4213–4228.

[29] J. Prezelj, *Positivity of metrics on conic neighborhoods of 1-convex submanifolds*, Internat. J. Math. **27** (2016), no. 5, 1650047, 24 pp.

[30] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1955–1956), 1–42.

[31] K. Stopar, *Approximation of holomorphic mappings on 1-convex domains*, Internat. J. Math. **24** (2013), no. 14, 1350108, 33 pp.

[32] M. Zaidenberg, *Algebraic Gromov ellipticity: a brief survey*, Taiwanese J. Math. **29** (2025), no. 6, 1681–1705.

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