THE CALABI–YAU AND SULLIVAN CONJECTURES FOR MINIMAL SURFACES WITH SYMMETRIES

ANTONIO ALARCÓN AND FRANC FORSTNERIČ

Abstract. Let G be a finite group acting on a connected open Riemann surface X by holomorphic automorphisms and acting on a Euclidean space $\mathbb{R}^n$, $n \geq 3$, by orthogonal transformations. In a previous paper, the second author gave a necessary and sufficient condition for the existence of a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$, and he asked whether such F can be chosen to be a proper map. Assuming that this condition holds, we give an affirmative answer when X is a bordered Riemann surface and G acts freely on the boundary of X. This contributes to the Sullivan conjecture concerning the existence of proper hyperbolic minimal surfaces in $\mathbb{R}^3$ of finite topology. Under the same conditions, we also construct a G-equivariant proper conformal minimal immersion of X to the unit ball of $\mathbb{R}^n$ which is complete in the induced metric, thereby opening a new direction in the Calabi–Yau theory regarding the study of complete bounded minimal surfaces. Analogous results hold for null curves in $\mathbb{C}^n$.

1. Introduction

The classical Calabi-Yau and Sullivan conjectures concern the existence of complete bounded minimal surfaces in $\mathbb{R}^3$, and of proper hyperbolic minimal surfaces in $\mathbb{R}^3$ of finite topology, respectively. Since the pioneering positive results of Jorge-Xavier [19] and Nadirashvili [24] concerning the former, and of Morales [23] regarding the latter, a primary topic of interest has been to understand the topological, geometric, and conformal properties of such surfaces; we refer to [7, Sec. 3.10 and Ch. 7] for surveys on these questions. Concerning possible conformal restrictions, Alarcón and López proved that every open Riemann surface is the underlying complex structure of a proper minimal surface in $\mathbb{R}^3$ [8], while the authors showed that every bordered Riemann surface is the complex structure of a complete bounded minimal surface in $\mathbb{R}^n$ [5].

In this paper, we establish analogues of these existence results for minimal surfaces with a prescribed finite group of symmetries that are normalised by bordered Riemann surfaces. Our main new results are summarised in Theorems 1.3, 1.7, and 1.8. To the best of our knowledge, these are the first general existence results regarding the Calabi-Yau and Sullivan conjectures for minimal surfaces with symmetries in Euclidean spaces.

Assume that X is an open Riemann surface and G is a finite group acting effectively on X by holomorphic automorphisms. For any point $x \in X$, the stabiliser

$$G_x = \{g \in G : gx = x\} \tag{1.1}$$

is a cyclic subgroup of G (see Miranda [22, Proposition 3.1, p. 76]), and

$$X_0 = \{x \in X : G_x \neq \{1\}\} \tag{1.2}$$

Date: 22 October 2026.

2020 *Mathematics Subject Classification.* Primary 53A10; Secondary 32E30, 32H02, 32Q56.

Key words and phrases. Riemann surface, minimal surface, equivariant conformal minimal immersion, Calabi–Yau problem, Sullivan conjecture.

is a closed, discrete, G-invariant subset of X (see [22, Proposition 3.3, p. 76]). The stabiliser G_x acts on the tangent space $T_xX \cong \mathbb{C}$ as a cyclic group of rotations by

$$gv = dg(v), \quad g \in G_x,\ v \in T_xX.$$

Furthermore, there is a local holomorphic coordinate z on X, with the point x corresponding to $z = 0$, in which a generator g of G_x is the rotation

$$gz = \mathrm{e}^{\imath\phi} z, \quad \phi = 2\pi/|G_x|,\ \imath = \sqrt{-1} \tag{1.3}$$

(see [22, Corollary 3.5, p. 93]). Assume that the group G also acts on the Euclidean space $\mathbb{R}^n$. A map $F : X \to \mathbb{R}^n$ is said to be *G-equivariant* if

$$F(gx) = gF(x) \ \text{ for all } x \in X \text{ and } g \in G. \tag{1.4}$$

The problem of existence, behaviour, and classification of G-equivariant maps has been studied in many geometries; see the discussion and references in [10, 20], among other sources. Recently, the second author gave the following necessary and sufficient condition for the existence of a G-equivariant conformal minimal immersion $X \to \mathbb{R}^n$, $n \geq 3$; see [13, Theorem 1.1] and Remark 1.2 below.

Theorem 1.1 (Corrected form of Theorem 1.1 in [13])**.** *Let G be a finite group acting effectively on a connected open Riemann surface X by holomorphic automorphisms and acting on the Euclidean space $\mathbb{R}^n$, $n \geq 3$, by orthogonal transformations. The following conditions are equivalent:*

(a) *For every point $x \in X$ with nontrivial stabiliser G_x there exists a G_x-equivariant conformal linear embedding $T_xX \to \mathbb{R}^n$.*
(b) *There exists a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$.*

If (a) holds then F in (b) can be chosen such that $F(X)$ is not contained in any affine hyperplane of $\mathbb{R}^n$.

Condition (a) is equivalent to the following. Let $x \in X$ be a point with a nontrivial stabiliser group G_x, and let g be a generator of G_x such that $dg_x : T_xX \to T_xX$ is the rotation R_ϕ through the angle $\phi = 2\pi/|G_x|$ in the positive direction. Then there exists a two-dimensional real linear subspace $\Lambda \subset \mathbb{R}^n$ on which g also acts as the rotation R_ϕ in one of the two orientations of Λ.

A more precise version of this result is given by [13, Theorem 5.1]; see Theorem 2.4. Recall that a conformal minimal immersion $F : X \to \mathbb{R}^n$ is called *nondegenerate* if $F(X)$ is not contained in an affine hyperplane of $\mathbb{R}^n$, and *nonflat* if its image is not contained in an affine two-plane. Thus, when $n > 3$, nondegeneracy is strictly stronger than nonflatness.

Remark 1.2. Condition (a) in Theorem 1.1 is stronger than the one in [13, Theorem 1.1], which reads

(a') For every nontrivial stabiliser G_x, $x \in X$, there is a two-dimensional linear subspace $\Lambda_x \subset \mathbb{R}^n$ on which G_x acts effectively as a cyclic group of rotations.

This condition does not suffice for the existence of a G_x-equivariant conformal linear isomorphism $T_xX \xrightarrow{\cong} \Lambda_x \subset \mathbb{R}^n$, which is used in [13]. The point is that a generator of G_x must act as a rotation by the same angle on both T_xX and Λ_x. Hence, condition (a') must be replaced by condition (a) in Theorem 1.1 in all existence theorems in [13]. On the other hand, condition (a') ensures the existence of G-equivariant conformal minimal surfaces $F : X \to \mathbb{R}^n$ which are branched at those points of X_0 (1.2) where condition (a) fails; see Theorem 2.5.

Assuming that condition (a) in Theorem 1.1 holds, several questions about possible additional properties of a conformal minimal immersion F in part (b) of the theorem were posed in [13]. It was asked in particular whether F can be chosen to be a *proper map* to $\mathbb{R}^n$; see [13, Problem 1.13]. We give an affirmative answer in the important special case when X is a bordered Riemann surface and G acts freely on its boundary bX. (Throughout the paper, a bordered Riemann surface means the interior X of

a connected compact domain $\overline{X} = X \cup bX$ with nonempty smooth boundary bX in a Riemann surface R.) Since every such surface is of finite topology and hyperbolic conformal type, the following result pertains to the study of the aforementioned Sullivan conjecture.

Theorem 1.3. *Let X be a bordered Riemann surface and G be a finite group acting effectively on X by holomorphic automorphisms such that no element $g \in G \setminus \{1\}$ has any fixed points on bX, i.e. G acts freely on bX. Assume that G also acts on $\mathbb{R}^n$, $n \geq 3$, by orthogonal transformations. If condition (a) in Theorem 1.1 holds then there exists a proper, nondegenerate, G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$.*

Remark 1.4. We may assume without loss of generality that the bordered Riemann surface X in Theorem 1.3 is a relatively compact G-invariant domain with real analytic boundary in an open Riemann surface R and that G acts on R by holomorphic automorphisms. Indeed, by [26, Theorem 8] the surface X is biholomorphic to the complement $X = X^* \setminus \bigcup_{j=1}^m \Delta_j$ of finitely many closed pairwise disjoint discs Δ_j with real analytic boundaries in a compact Riemann surface X^*. By the Schwarz reflection principle, every holomorphic automorphism of X extends to an injective holomorphic map on a neighbourhood of $\overline{X}$ in X^*. Since the group G is finite, there is a neighbourhood U of $\overline{X}$ on which all these maps are defined, and they satisfy the group law. The connected component R of the G-invariant domain $\bigcap_{g\in G} gU$ containing $\overline{X}$ is then an open Riemann surface on which G acts by holomorphic automorphisms. More generally, Maskit [21] proved that every open Riemann surface X of finite genus embeds in a compact Riemann surface X^* of the same genus such that every holomorphic automorphism of X extends to a holomorphic automorphism of X^*.

Theorem 1.3 is proved by combining the technique from [13], which was used to prove Theorem 1.1 (see the exposition in Section 2), with the method of constructing proper conformal minimal immersions from a bordered Riemann surface X to any convex domain in $\mathbb{R}^n$ (see [2, Theorem 1.4 and Corollary 1.5]) by using the technique for exposing boundary points and the Riemann–Hilbert modification method for conformal minimal surfaces; see Subsections 3.1 and 3.2. In the case at hand, the target convex domain is $\mathbb{R}^n$ itself, which simplifies the construction.

Another method for constructing proper minimal immersions of an arbitrary open Riemann surface to $\mathbb{R}^n$ for any $n \geq 3$, relying on the Runge-Mergelyan approximation theorems for conformal minimal immersions, was first introduced in [8] for $n = 3$ and developed in full generality in [6]; see also [7, Theorem 3.10.3]. However, the induction step in this method requires one to fix some of the components of the map under construction to be harmonic functions on X, and we were unable to adapt it to the equivariant situation. Nevertheless, combining the proofs of Theorems 1.1 and 1.3 implies the following corollary.

Corollary 1.5. *Let X and G be as in Theorem 1.1, satisfying condition (a). Then there exists an almost proper G-equivariant conformal minimal immersion $X \to \mathbb{R}^n$.*

Recall that a continuous map $F : X \to Y$ of topological spaces is called almost proper if for every compact set $K \subset Y$, every connected component of the preimage $F^{-1}(K)$ is compact.

Another question related to the Sullivan conjecture that has been a focus of interest is that by Schoen and Yau, asking whether there are hyperbolic minimal surfaces in $\mathbb{R}^3$ with a proper projection into a plane (see [25, p. 18] and [7, Sec. 3.10]). This problem was settled in [8], where it was shown that projecting properly into a plane does not impose any restriction on the underlying complex structure of a minimal surface in $\mathbb{R}^3$. The following remains as an open question.

Question 1.6. Let X and G be as in Theorem 1.1, satisfying condition (a). Does there exist a G-equivariant conformal minimal immersion $X \to \mathbb{R}^3$ having a proper projection into a plane?

Let ds^2 denote the Euclidean metric on $\mathbb{R}^n$. Let us call a $\mathscr{C}^1$ immersion $F : X \to \mathbb{R}^n$ bounded if its image $F(X)$ is contained in a ball, and complete if the induced Riemannian metric F^*ds^2 on X is a complete metric. With X and G as in Theorem 1.3, it is natural to ask whether there exists a G-equivariant conformal minimal immersion $F : X \to \mathbb{R}^n$ which is bounded and complete (see [13, Problem 1.12]). We provide the following affirmative answer.

Theorem 1.7. *Let X and G be as in Theorem 1.3. If condition (a) in Theorem 1.1 holds then there exists a continuous G-equivariant map $F : \overline{X} \to \mathbb{R}^n$ whose restriction $F|_X : X \to \mathbb{R}^n$ is a complete, nondegenerate conformal minimal immersion with a bounded image. Furthermore, any nonflat G-equivariant conformal minimal immersion $F_0 : \overline{X} \to \mathbb{R}^n$ can be approximated uniformly on $\overline{X}$ by maps $F : \overline{X} \to \mathbb{R}^n$ as above.*

Without the G-equivariance condition, Theorem 1.7 is a special case of [2, Theorem 1.1]. Combining the methods of proof of Theorems 1.3 and 1.7 we obtain the following.

Theorem 1.8. *Assume that X and G are as in Theorem 1.3 and condition (a) in Theorem 1.1 holds. Then there exists a continuous G-equivariant map $F : \overline{X} \to \overline{\mathbb{B}}^n$ to the closed unit ball of $\mathbb{R}^n$ such that the restriction $F|_X : X \to \mathbb{B}^n$ is a complete, proper, nondegenerate G-equivariant conformal minimal immersion.*

Theorems 1.7 and 1.8 contribute to the circle of results on the famous Calabi–Yau problem for minimal surfaces; see [7, Chap. 7] for a discussion of this topic. Their proof provides a general construction of complete bounded minimal surfaces in $\mathbb{R}^n$ with a prescribed conformal structure and a given group of symmetries, thereby opening a new direction in the theory. Without G-equivariance, Theorem 1.8 is a special case of [2, Theorem 1.2] which gives the analogous result with the ball replaced by an arbitrary strongly convex domain with $\mathscr{C}^2$ boundary in $\mathbb{R}^n$, $n \geq 3$. See also [3] and the exposition in [7, Secs. 8.3–8.5] where similar results were proved for minimal surfaces in mean convex domains in $\mathbb{R}^3$ and $(n-2)$-convex domains in $\mathbb{R}^n$ for $n \geq 4$.

Remark 1.9. Let $H_1(X, \mathbb{Z})$ denote the first homology group of X. The flux homomorphism $\mathrm{Flux}_F : H_1(X, \mathbb{Z}) \to \mathbb{R}^n$ of a conformal minimal immersion $F : X \to \mathbb{R}^n$ is defined by

$$\mathrm{Flux}_F([C]) = \oint_C d^c F, \quad [C] \in H_1(X, \mathbb{Z}), \tag{1.5}$$

where d^c is the conjugate differential on X. (See [7, Definition 2.3.2].) An action of a group G on X induces an action of G on $H_1(X, \mathbb{Z})$. If a conformal minimal immersion $F : X \to \mathbb{R}^n$ is G-equivariant then clearly so is its flux homomorphism. In Theorems 1.3, 1.7, and 1.8 one can also control the flux homomorphism of a G-equivariant conformal minimal immersion to be any given G-equivariant homomorphism $H_1(X, \mathbb{Z}) \to \mathbb{R}^n$. This will be clear from the proofs; see also [13, Remark 5.2].

The analogues of Theorems 1.3 and 1.7 also hold for *holomorphic null curves*, that is, holomorphic immersions $H = (H_1, \ldots, H_n) : X \to \mathbb{C}^n$, $n \geq 3$, whose differential dH satisfies the nullity condition

$$\sum_{i=1}^n (dH_i)^2 = 0.$$

Let $O(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : AA^t = I\}$ denote the complex orthogonal group on $\mathbb{C}^n$. The null quadric $\mathbf{A} \subset \mathbb{C}^n$ (2.7) is $O(n, \mathbb{C})$-invariant, and the group $O(n, \mathbb{C})$ acts transitively on $\mathbf{A}_* = \mathbf{A} \setminus \{0\}$. For reference, we recall the following analogue of Theorem 1.1; see [13, Theorem 5.3]. The special case with G the trivial group is given by [7, Theorem 3.6.1].

Theorem 1.10 (Theorem 5.3 in [13])**.** *Let G be a finite group acting effectively on a connected open Riemann surface X by holomorphic automorphisms, and acting on $\mathbb{C}^n$ ($n \geq 3$) by complex orthogonal*

transformations in $O(n,\mathbb{C})$. Assume that for every point $x \in X$ with nontrivial stabiliser G_x there is a G_x-invariant complex line $L_x \subset \mathbf{A} \subset \mathbb{C}^n$ and a G_x-equivariant $\mathbb{C}$-linear isomorphism $T_xX \to L_x$. Then there exists a nondegenerate G-equivariant holomorphic null immersion $H : X \to \mathbb{C}^n$.

Part (a) of the following result is an analogue of Theorem 1.3 for G-equivariant holomorphic null curves. Part (b) is an analogue of a special case of Theorem 1.7.

Theorem 1.11. *Let X and G be as in Theorem 1.3. Assume that G also acts on $\mathbb{C}^n$, $n \geq 3$, by elements of $O(n,\mathbb{C})$ and the condition on the stabilisers in Theorem 1.10 is satisfied. Then the following hold.*

(a) *There exists a proper G-equivariant holomorphic null immersion $H : X \to \mathbb{C}^n$ whose image $H(X)$ is not contained in any affine hyperplane of $\mathbb{C}^n$.*
(b) *There exists a continuous G-equivariant map $H : \overline{X} \to \mathbb{C}^n$ whose restriction $H|_X : X \to \mathbb{C}^n$ is a bounded complete holomorphic null immersion such that $H(X)$ is not contained in any affine hyperplane of $\mathbb{C}^n$.*

Part (b) contributes to the *Yang problem* concerning the existence and properties of complete bounded complex submanifolds of complex Euclidean spaces. See [1] for a recent survey.

The paper is organised as follows. In Section 2 we outline the proof of Theorem 1.1 given in [13]. The same general scheme but with suitable technical additions is then used to prove Theorems 1.3, 1.7, and 1.8 in Section 5 via recursive applications of the main properness and completeness lemmas in Section 4. The same methods yield the proof of Theorem 1.11 also given in Section 5. The necessary additions to establish the main lemmas include G-equivariant versions of the technique of exposing boundary points of bordered Riemann surfaces (see Theorem 3.1 in Subsection 3.1), and of the Riemann–Hilbert deformation method for conformal minimal surfaces and holomorphic null immersions (see Theorem 3.3 and Corollary 3.4 in Subsection 3.2). This classical complex analytic technique for solving certain boundary value problems for holomorphic maps was first introduced to the subject of conformal minimal surfaces by the authors in [5], where it was shown that every bordered Riemann surface admits a complete bounded conformal minimal immersion in $\mathbb{R}^3$. In the subsequent paper [2], the Riemann–Hilbert technique was extended to minimal surfaces in $\mathbb{R}^n$ for arbitrary $n \geq 3$, and it was used to give complete bordered minimal surfaces with boundaries consisting of Jordan curves. (The boundary curves are necessarily nonrectifiable at every point.) Soon thereafter, the technique was improved for $n = 3$ and applied in the construction of complete, proper conformal minimal immersions of bordered Riemann surfaces to strongly minimally convex domains in $\mathbb{R}^3$ (in particular, to strongly convex domains); see [3]. A detailed presentation of this method, and of its applications to the Calabi–Yau problem, can be found in [7, Chaps. 6–7].

2. Outline of proof of Theorem 1.1

In this section we outline the proof of Theorem 1.1; details can be found in [13]. Together with the additions in subsequent sections, this will serve to prove the new results stated in the Introduction.

We begin by establishing the notation. Let X be a bordered Riemann surface, and assume that G is a finite group acting on X by holomorphic automorphisms and acting on $\mathbb{R}^n$ by orthogonal transformations. By $\mathrm{CMI}(X,\mathbb{R}^n)$ we denote the space of all conformal minimal immersions $X \to \mathbb{R}^n$. (This notation is used in [7] and several other sources.) It contains the subset

$$\mathrm{CMI}_G(X,\mathbb{R}^n) \subset \mathrm{CMI}(X,\mathbb{R}^n)$$

of all G-equivariant conformal minimal immersions. Given $r \in \mathbb{N} = \{1,2,\ldots\}$, we denote by

$$\mathrm{CMI}^r_G(\overline{X},\mathbb{R}^n) \subset \mathrm{CMI}^r(\overline{X},\mathbb{R}^n) \tag{2.1}$$

the spaces of maps $X \to \mathbb{R}^n$ of this type which extend to immersions $\overline{X} \to \mathbb{R}^n$ of class $\mathscr{C}^r$. Adding a star as a subscript will denote the spaces of nonflat maps of the respective types:

$$\mathrm{CMI}^r_{G,*}(\overline{X}, \mathbb{R}^n) \subset \mathrm{CMI}^r_*(\overline{X}, \mathbb{R}^n). \tag{2.2}$$

We now recall the setup used in the proof of Theorem 1.1 in [13]. As explained in the introduction,

$$X_0 = \{x \in X : G_x \neq \{1\}\}$$

is a closed, discrete, G-invariant subset of X, and G acts freely on its complement

$$X_1 := X \setminus X_0 = \{x \in X : gx \neq x \text{ for all } g \in G \setminus \{1\}\}.$$

The orbit space X/G is an open Riemann surface, the quotient projection

$$\pi : X \to X/G$$

is a holomorphic map with branch locus X_0, and the restriction $\pi|_{X_1} : X_1 \to X_1/G$ is a holomorphic covering projection of degree $|G|$. Choose a holomorphic immersion $\tilde{h} : X/G \to \mathbb{C}$ (such exists by [18]). The holomorphic function

$$h = \tilde{h} \circ \pi : X \to \mathbb{C} \tag{2.3}$$

is G-invariant (that is, $h \circ g = h$ for all $g \in G$) and its branch locus equals X_0. By the chain rule, the holomorphic 1-form $\theta = dh$ on X satisfies the following condition for every $g \in G$:

$$\theta_{gx} \circ dg_x = \theta_x \text{ for all } x \in X, \quad \{\theta = 0\} = X_0. \tag{2.4}$$

The implication (b)⇒(a) in Theorem 1.1 is seen as follows. Let $F : X \to \mathbb{R}^n$ be a G-equivariant immersion. Differentiating the equation $g \circ F = F \circ g$ at $x \in X$ and using $gx = x$ gives

$$g \circ dF_x = dF_{gx} \circ dg_x = dF_x \circ dg_x : T_xX \to \mathbb{R}^n. \tag{2.5}$$

(The first equality holds for every $x \in X$ and $g \in G$.) Hence, the differential $dF_x : T_xX \to \mathbb{R}^n$ is a G_x-equivariant linear embedding onto its image $dF_x(T_xX) \subset \mathbb{R}^n$, and dF_x is conformal if F is such. Taking the $\mathbb{C}$-linear parts gives

$$g \circ \partial F_x = \partial F_{gx} \circ dg_x = \partial F_x \circ dg_x. \tag{2.6}$$

Here, ∂ is the $\mathbb{C}$-linear component of the exterior differential d, so $d = \partial + \overline{\partial}$.

In the remainder of this section, we outline the proof of the implication (a)⇒(b) in Theorem 1.1. We introduce the following notation:

$$\mathbf{A} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\} \text{ the null quadric;} \tag{2.7}$$

$$\mathbf{A}_* = \mathbf{A} \setminus \{0\} \text{ the punctured null quadric;} \tag{2.8}$$

$$\overline{\mathbf{A}} = \text{the closure of } \mathbf{A} \subset \mathbb{C}^n \text{ in the projective compactification } \mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}; \tag{2.9}$$

$$Y = \overline{\mathbf{A}} \setminus \{0\} = \mathbf{A}_* \cup Y_0; \tag{2.10}$$

$$Y_0 = Y \setminus \mathbf{A}_* = \big\{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\big\}. \tag{2.11}$$

The projection

$$p : \mathbf{A}_* \to Y_0, \quad p(z_1, \dots, z_n) = [z_1 : \cdots : z_n]$$

is a holomorphic $\mathbb{C}^*$-bundle. Extending p to Y as the identity on Y_0 gives a holomorphic line bundle projection $p : Y \to Y_0$ with the zero section Y_0. The actions of the real and the complex orthogonal groups $O(n, \mathbb{R}) \subset O(n, \mathbb{C})$ on $\mathbb{C}^n$ extend to $\mathbb{CP}^n$, with $\mathbf{A}_*$ and the hyperplane at infinity $\mathbb{CP}^n \setminus \mathbb{C}^n \cong \mathbb{CP}^{n-1}$ as invariant submanifolds. To every oriented two-dimensional linear subspace $\Lambda \subset \mathbb{R}^n$ we associate

a complex null line $L = L(\Lambda) \subset \mathbf{A} \subset \mathbb{C}^n$ by choosing an oriented basis (u, v) of Λ such that $\|u\| = \|v\| \neq 0$ and $u \cdot v = 0$ (such a pair (u, v) is called a conformal frame) and setting

$$L(\Lambda) = \mathbb{C}(u - \imath v) \subset \mathbf{A} \subset \mathbb{C}^n. \tag{2.12}$$

A rotation R_ϕ on Λ for the angle ϕ in the positive direction corresponds to the multiplication by $\mathrm{e}^{\imath\phi}$ on the complex null line $L(\Lambda)$.

Fix a point $x_0 \in X$. By the Weierstrass representation formula, every conformal minimal immersion $F : X \to \mathbb{R}^n$ is of the form

$$F(x) = v + \Re \int_{x_0}^{x} f\theta, \qquad x \in X, \tag{2.13}$$

where $v = F(x_0) \in \mathbb{R}^n$ and

$$f = 2\partial F/\theta : X \to Y = \mathbf{A}_* \cup Y_0$$

is a holomorphic map such that $f^{-1}(Y_0) = X_0$, the holomorphic 1-form $f\theta = 2\partial F$ has no zeros or poles, and it satisfies the real period vanishing conditions

$$\Re \int_C f\theta = 0 \quad \text{for every } [C] \in H_1(X, \mathbb{Z}). \tag{2.14}$$

If in addition F is G-equivariant then the map $f : X \to Y$ is also G-equivariant:

$$f(gx) = \frac{2\partial F_{gx}}{\theta_{gx}} = \frac{2\partial F_{gx} \circ dg_x}{\theta_{gx} \circ dg_x} \overset{(2.6),(2.4)}{=} \frac{g\, 2\partial F_x}{\theta_x} = g\, f(x), \quad g \in G. \tag{2.15}$$

The G-equivariance condition on the map F (2.13) on the orbit Gx_0 also gives

$$gv = gF(x_0) = F(gx_0) = v + \Re \int_{x_0}^{gx_0} f\theta, \quad g \in G, \tag{2.16}$$

where the integral is taken over any path from x_0 to gx_0. Conversely, if $f : X \to Y$ is a G-equivariant holomorphic map such that $f\theta$ is a holomorphic $\mathbb{C}^n$-valued 1-form on X without zeros (this means that the zeros of $\theta = dh$ (2.3) at the points of X_0 exactly cancel the poles of f) and the period vanishing conditions (2.14) hold, then the map F given by (2.13) is a well-defined conformal minimal immersion. In particular, the real part of the integral in (2.13) is independent of the choice of a path from x_0 to x. Assuming in addition that f satisfies condition (2.16) (ignoring the two middle terms involving F), the map F is also G-equivariant. Indeed, given a path $\gamma : [0, 1] \to X$, we have for any $g \in G$ that

$$\int_{g\gamma} f\theta = \int_0^1 f(g\gamma(t))\, \theta_{g\gamma(t)}(dg_{\gamma(t)}\dot\gamma(t))\, dt \overset{(2.4)}{=} \int_0^1 g f(\gamma(t))\, \theta_{\gamma(t)}(\dot\gamma(t))\, dt = g \int_\gamma f\theta, \tag{2.17}$$

and hence

$$\begin{aligned} F(gx) &\overset{(2.13)}{=} v + \Re \int_{x_0}^{gx} f\theta = \left(v + \Re \int_{x_0}^{gx_0} f\theta\right) + \Re \int_{gx_0}^{gx} f\theta \\ &\overset{(2.16),(2.17)}{=} gv + \Re\left(g \int_{x_0}^{x} f\theta\right) = g\left(v + \Re \int_{x_0}^{x} f\theta\right) \\ &\overset{(2.13)}{=} gF(x). \end{aligned}$$

We now recall the two main steps in the proof of Theorem 1.1 given in [13].

Step 1: There is a G-equivariant conformal minimal immersion $F_0 : V \to \mathbb{R}^n$ from a neighbourhood V of the closed discrete subset $X_0 \subset X$ (1.2).

To see this, fix a point $x_0 \in X_0$ and set $k = |G_{x_0}| > 1$. As explained in the introduction, there is a holomorphic coordinate z on a disc $x_0 \in \Delta \subset X$, with $z(x_0) = 0$, such that a generator g_0 of G_{x_0} acts by

$$g_0 z = \mathrm{e}^{\imath\phi} z, \quad \phi = 2\pi/k, \quad k = |G_{x_0}|.$$

The function $w = z^k$ is a local holomorphic coordinate on X/G at $\pi(x_0)$, with $\pi(z) = z^k$ and

$$h(z) - h(0) = \tilde{h}(z^k) - \tilde{h}(0), \qquad \theta = dh = \tilde{h}'(z^k) k z^{k-1}\, dz.$$

By condition (a) in Theorem 1.1 there is a G_{x_0}-equivariant conformal linear isomorphism $F_0 : \mathbb{R}^2 \to \Lambda$ onto a 2-plane $\Lambda \subset \mathbb{R}^n$, so g_0 acts on Λ as the rotation through the angle $\phi = 2\pi/k$. It follows that g_0 acts on the null line $L = L(\Lambda) \subset \mathbb{C}^n$ (2.12) as multiplication by $\mathrm{e}^{\imath\phi}$. Note that $2\partial F_0 = f_0\theta$ where

$$f_0(z) = \frac{y_0}{\tilde{h}'(z^k) z^{k-1}}, \qquad y_0 \in L \setminus \{0\},\ z \in \Delta. \tag{2.18}$$

Although this function has a pole at $\{z = 0\} = \{x_0\}$ in the affine trivialisation of the null line, it defines a holomorphic map $f_0 : \Delta \to Y$ satisfying

$$f_0(x_0) \in Y_0, \qquad f_0(\Delta \setminus \{x_0\}) \subset \mathbf{A}_*.$$

Here, $\mathbf{A}_*$ is the punctured null quadric (2.8). Moreover, the map f_0 is G_{x_0}-equivariant by (2.15). We extend F_0 and f_0 by G-equivariance to the orbit $G\cdot\Delta$ and perform the same construction on all G-orbits of X_0. This defines a G-equivariant map $f_0 : V \to Y$ on a G-invariant neighbourhood $V \subset X$ of X_0, with $f_0^{-1}(Y_0) = X_0$, and a G-equivariant conformal minimal immersion $F_0 : V \to \mathbb{R}^n$ with $2\partial F_0 = f_0\theta$.

Step 2: With F_0 and f_0 as above, we construct a G-equivariant holomorphic map $f : X \to Y$, which agrees with f_0 to sufficiently high order along X_0, such that $f(X_1) \subset \mathbf{A}_*$ (2.8) and the conditions (2.14) and (2.16) hold. The interpolation order at the points of X_0 is chosen so that the holomorphic 1-form $f\theta$ extends across X_0 without zeros. The map $F : X \to \mathbb{R}^n$ defined by (2.13) is then a G-equivariant conformal minimal immersion. In addition, we ensure that $F|_{X_0} = F_0|_{X_0}$.

Let Y be given by (2.10). Consider the action of G on $X \times Y$ given by

$$g(x, y) = (gx, gy), \quad x \in X,\ y \in Y,\ g \in G.$$

The natural projection $X \times Y \to X$ is then G-equivariant, so it induces a projection

$$\rho : Z = (X \times Y)/G \to X/G. \tag{2.19}$$

Clearly, G-equivariant maps $f : X \to Y$ are naturally identified with sections $\tilde{f} : X/G \to Z$ of $\rho : Z \to X/G$. Note that $Z = (X\times Y)/G$ is a reduced complex space, the map ρ (2.19) is holomorphic, it is branched over the closed discrete subset X_0/G of X/G, and the restricted map

$$\rho : Z_1 = \rho^{-1}(X_1/G) \to X_1/G,$$

where $X_1 = X \setminus X_0$, is a holomorphic fibre bundle with the fibre $Y = \mathbf{A}_* \cup Y_0$; see (2.8)–(2.11). (The transition maps of this bundle are given by the action of G on Y.) Since the submanifold $Y_0 \subset Y$ and its complement $Y \setminus Y_0 = \mathbf{A}_*$ are both G-invariant, Z_1 is the disjoint union

$$Z_1 = [(X_1 \times \mathbf{A}_*)/G] \sqcup [(X_1 \times Y_0)/G]\,.$$

The subset

$$\Omega := (X_1 \times \mathbf{A}_*)/G$$

of Z_1 is a G-invariant Zariski open domain in Z without singularities, and its complement $Z \setminus \Omega$ is a closed complex subvariety of Z containing $\mathrm{sing}(Z) \cup \mathrm{br}(\rho)$. The restricted projection

$$\rho : \Omega = (X_1 \times \mathbf{A}_*)/G \to X_1/G \tag{2.20}$$

is a holomorphic G-bundle with fibre $\mathbf{A}_*$. The main point of the above construction is summarised in the following result.

Lemma 2.1 (Lemma 3.1 in [13])**.** *There is a natural bijective correspondence between (continuous or holomorphic) sections $\tilde{f} : X/G \to Z$ of the map $\rho : Z \to X/G$ (2.19) satisfying*

$$\tilde{f}(X_1/G) \subset \Omega = (X_1 \times \mathbf{A}_*)/G \ \text{ and } \ \tilde{f}(X_0/G) \subset (X_0 \times Y_0)/G$$

and (continuous or holomorphic) G-equivariant maps $f : X \to Y$ satisfying

$$f(X_1) \subset \mathbf{A}_* \ \text{ and } \ f(X_0) \subset Y_0.$$

The map f_0 from step 1 of the proof corresponds to a local holomorphic section $\tilde{f}_0$ of $Z \to X/G$ on an open neighbourhood $V/G \subset X/G$ of X_0/G such that

$$\tilde{f}_0\big((V \setminus X_0)/G)\big) \subset \Omega.$$

The fibre $\mathbf{A}_*$ of the holomorphic fibre bundle (2.20) is $O(n, \mathbb{C})$-homogeneous, hence an Oka manifold by a theorem of Grauert [17]. (See also [15, Proposition 5.6.1].) Therefore, sections of the ramified holomorphic map $\rho : Z = (X \times Y)/G \to X/G$ mapping X_1/G to Ω satisfy the Oka principle. (See [15, Theorem 6.14.6, p. 310], which originates from [14], along with [12, Theorem 1.3].) This gives global holomorphic sections $\tilde{f} : X/G \to Z$ satisfying $\tilde{f}(X_1/G) \subset \Omega$ which agree with $\tilde{f}_0$ at every point $p \in X_0/G$ to any given finite order $k(p) \in \mathbb{N}$. Furthermore, $\tilde{f}$ can be chosen such that the corresponding G-equivariant map $f : X \to Y$ integrates to a G-equivariant conformal minimal immersion; the necessary period vanishing conditions are obtained using period dominating sprays. The key technical ingredient in the construction of a section $\tilde{f}$ with the desired properties is provided by [13, Lemma 4.5], and the construction is completed in [13, Sect. 5].

A more precise version of Theorem 1.1, with approximation on admissible Runge sets and interpolation on closed discrete subsets, is [13, Theorem 5.1]. We state it here for convenience of the reader since we will refer to it in the sequel. We begin by recalling the relevant definitions.

Definition 2.2 (Definition 1.12.9 in [7])**.** An *admissible set* in a Riemann surface X is a compact set of the form $S = K \cup E$, where K is a (possibly empty) finite union of pairwise disjoint compact domains with piecewise $\mathscr{C}^1$ boundaries in X and $E = \overline{S \setminus K}$ is a union of finitely many pairwise disjoint, smoothly embedded Jordan arcs and closed Jordan curves meeting K only at their endpoints (if at all) and such that their intersections with the boundary bK of K are transverse.

Definition 2.3 (Definition 3.1.2 in [7])**.** Let $S = K \cup E$ be an admissible set in a Riemann surface X, let θ be a holomorphic 1-form on a neighbourhood of S in X without zeros on bS, and let $Y = \overline{\mathbf{A}} \setminus \{0\}$ (2.10). A *generalised conformal minimal immersion* $S \to \mathbb{R}^n$ ($n \geq 3$) of class $\mathscr{C}^r$ ($r \in \mathbb{N}$) is a pair $(F, f\theta)$, where $F : S \to \mathbb{R}^n$ is a $\mathscr{C}^r$ map whose restriction to $\mathring{S} = S \setminus bS = \mathring{K}$ is a conformal minimal immersion and the map $f \in \mathscr{A}^{r-1}(S, Y)$ satisfies the following conditions:

(a) $f\theta = 2\partial F$ holds on K (in particular, the zeros of θ cancel the poles of f), and
(b) for any smooth path $\alpha : [0, 1] \to X$ parameterizing a connected component of $E = \overline{S \setminus K}$ we have that $\Re(\alpha^*(f\theta)) = \alpha^* dF = d(F \circ \alpha)$.

Given an admissible set $S \subset X$ and integers $r \geq 1$, $n \geq 3$, we denote by

$$\mathrm{GCMI}^r(S, \mathbb{R}^n)$$

the space of all generalised conformal minimal immersions $S \to \mathbb{R}^n$ of class $\mathscr{C}^r$. An element $(F, f\theta) \in \mathrm{GCMI}^r(S, \mathbb{R}^n)$ is called *nonflat* if, on every connected component of K and of E, the image of f is not contained in the completion in Y of a complex line through the origin in $\mathbf{A}$.

Let G be a finite group as in Theorem 1.1, $S \subset X$ be a G-invariant admissible set, and θ be a G-invariant 1-form on X (see (2.4)). The pair $(F, f\theta) \in \mathrm{GCMI}^r(S, \mathbb{R}^n)$ is called G-equivariant if

$$F(gx) = gF(x), \quad f(gx) = gf(x), \quad x \in S, \ g \in G.$$

If $S = K$ (i.e., there are no attached arcs) then G-equivariance of F implies G-equivariance of f by (2.15); however, this is not necessarily the case on the attached arcs. Conversely, G-equivariance of f implies G-equivariance of its integral after the integration constants have been chosen compatibly. On a connected set, this is ensured by the orbit-integral conditions (2.16). We denote by

$$\mathrm{GCMI}^r_{G,*}(S,\mathbb{R}^n) \subset \mathrm{GCMI}^r_G(S,\mathbb{R}^n) \tag{2.21}$$

the spaces of (nonflat) G-equivariant generalised conformal minimal immersions $S \to \mathbb{R}^n$ of class $\mathscr{C}^r$. We recall the following more precise version of Theorem 1.1 which will be used.

Theorem 2.4 (Theorem 5.1 in [13])**.** *Assume that G is a finite group acting effectively on a connected open Riemann surface X by holomorphic automorphisms and acting on $\mathbb{R}^n$, $n \geq 3$, by orthogonal transformations. Let $\pi : X \to X/G$ be the quotient projection and $X_0 \subset X$ be the set* (1.2) *of its branch points. Assume that $S \subset X$ is a G-invariant admissible set (see Definition 2.2) which is Runge in X, $A \subset X$ is a closed G-invariant discrete set such that $X_0 \subset A \subset X \setminus bS$, $V \subset X \setminus S$ is an open G-invariant neighbourhood of $A \setminus S$, and $F_0 : S \cup V \to \mathbb{R}^n$ is such that $F_0|_S \in \mathrm{GCMI}^r_{G,*}(S,\mathbb{R}^n)$ $(r \geq 1)$ and $F_0|_V \in \mathrm{CMI}_G(V,\mathbb{R}^n)$. Then, F_0 can be approximated in the $\mathscr{C}^r$ topology on S by nonflat G-equivariant conformal minimal immersions $F : X \to \mathbb{R}^n$ which agree with F_0 to a given finite order $k(a) \in \mathbb{N}$ at every point $a \in A$.*

The construction described above, which was developed in [13], also yields the following theorem concerning the existence of G-equivariant conformal minimal surfaces with branch points. This result uses the weaker hypothesis (a') in Remark 1.2 on the points $x \in X$ with nontrivial stabiliser G_x.

Theorem 2.5. *Let G be a finite group acting effectively on a connected open Riemann surface X by holomorphic automorphisms and acting on the Euclidean space $\mathbb{R}^n$, $n \geq 3$, by orthogonal transformations. Assume that for every nontrivial stabiliser G_x, $x \in X$, there is a two-dimensional linear subspace $\Lambda_x \subset \mathbb{R}^n$ on which G_x acts effectively as a cyclic group of rotations. Then, there exists a G-equivariant conformal harmonic map $F : X \to \mathbb{R}^n$ which is branched at every point $x \in X$ at which the stabiliser group G_x is nontrivial but does not satisfy condition (a) in Theorem 1.1.*

Proof. The only difference concerns the local construction at the points $x_0 \in X_0$ (1.2) with a nontrivial stabiliser group G_{x_0}; see Step 1 above. Choose a generator g_0 of G_{x_0} such that, in a suitable local holomorphic coordinate z on X with $z(x_0) = 0$, we have $g_0 z = \mathrm{e}^{\imath\phi} z$ where $\phi = 2\pi/k$, $k = |G_{x_0}|$ (see (1.3)). If condition (a) in Theorem 1.1 holds at x_0, choose the unbranched local model from Step 1 above. Otherwise, choose a two-dimensional linear subspace $\Lambda \subset \mathbb{R}^n$ on which G_{x_0} acts effectively by rotations. There is an orientation of Λ in which the generator g_0 acts on Λ as the rotation through the angle $m\phi$ for some integer m satisfying

$$1 \leq m \leq k/2, \qquad \gcd(m,k) = 1.$$

Choose an oriented conformal frame $(\mathbf{u},\mathbf{v})$ for Λ. The map

$$F_0(z) = F_0(r\mathrm{e}^{\imath t}) = r^m\,(\cos(mt)\mathbf{u} + \sin(mt)\mathbf{v}) = \mathfrak{R}\big(z^m(\mathbf{u} - \imath\mathbf{v})\big)$$

is then a G_{x_0}-equivariant conformal harmonic map whose branch order at $z = 0$ is $m - 1$. In particular, F_0 is unbranched (a linear isomorphism onto Λ) if and only if $m = 1$. Extending this construction by G-equivariance to the orbit Gx_0 and doing the same on every G-orbit in X_0 gives a conformal G-equivariant harmonic map $F_0 : V \to \mathbb{R}^n$ from a G-invariant neighbourhood $V \subset X$ of X_0 whose branch locus is precisely the subset of X_0 where condition (a) in Theorem 1.1 fails. In Step 2 of the proof, impose sufficiently high jet interpolation on the local maps $f_0 = 2\partial F_0/\theta$ at the points of X_0. This preserves their pole orders and nonzero leading coefficients. Consequently, the resulting differential $f\theta$ has the prescribed zero orders at these points and is nowhere zero on $X \setminus X_0$. The remaining details in Step 2 are the same as above. □

3. Main new tools

In this section, we present the main new tools that, together with the results from [13] in the previous section, will be used in the proof of the main theorems. Namely, we establish G-equivariant versions of the method for exposing boundary points of bordered Riemann surfaces, due to Forstnerič and Wold [16] (see Theorem 3.1 and Corollary 3.2), as well as of the Riemann-Hilbert method for minimal surfaces developed by the authors [5], and in a more complete form by Alarcón, Drinovec Drnovšek, Forstnerič, and López [2] (see Theorem 3.3).

3.1. **G-equivariant exposure of boundary points of bordered Riemann surfaces.** The following result is an equivariant version of [7, Theorem 6.7.1]. The technique used in the proof originates in the result of Forstnerič and Wold [16, Theorem 2.3] on exposing boundary points of bordered Riemann surfaces; see also [15, Theorem 9.9.1]. This method was first used to prove that if $\overline{X}$ is a compact bordered Riemann surface which admits a holomorphic embedding in $\mathbb{C}^2$, then the interior X admits a proper holomorphic embedding in $\mathbb{C}^2$ [16]. Later, it played a fundamental role in showing that every bordered Riemann surface immerses properly holomorphically in the ball of $\mathbb{C}^2$ as a complete complex curve [4], as well as in all constructions of complete bounded minimal surfaces normalised by given bordered Riemann surfaces (see the discussion in Section 1).

Theorem 3.1. *Let G be a finite group acting effectively on a connected open Riemann surface R by holomorphic automorphisms. Assume that $X \Subset R$ is a smoothly bounded G-invariant domain such that G acts freely on bX, $A = \{a_1, \ldots, a_k\} \subset X$ and $P = \{p_1, \ldots, p_m\} \subset bX$ are G-invariant subsets, and $E_1, \ldots, E_m$ are smoothly embedded pairwise disjoint compact arcs in R such that for each $i = 1, \ldots, m$, E_i is attached to $\overline{X}$ transversely at the point $p_i \in bX$, $E_i \cap \overline{X} = \{p_i\}$, and the set $E = \bigcup_{i=1}^m E_i$ is G-invariant. (See* [7, Fig. 6.4, p. 292].*) Let $q_i \in E_i$ denote the other endpoint of E_i, set $Q = \{q_1, \ldots, q_m\}$, and let b_i be a point in the relative interior of E_i such that the set $B = \{b_1, \ldots, b_m\}$ is G-invariant. Suppose that for every $i = 1, \ldots, m$ we are given a neighbourhood $V_i \subset R$ of the arc E_i and a neighbourhood $U_i \Subset V_i$ of the point p_i such that the sets $\overline{V}_i$ $(i = 1, \ldots, m)$ are pairwise disjoint and the sets $U = \bigcup_{i=1}^m U_i$ and $V = \bigcup_{i=1}^m V_i$ are G-invariant. Given $d \in \mathbb{N}$ there exists a smooth G-equivariant diffeomorphism $\phi : \overline{X} \to \phi(\overline{X}) \subset R$ which is holomorphic in X and satisfies the following conditions.*

(a) *ϕ approximates the identity to a desired precision in the smooth topology on $\overline{X} \setminus U$.*
(b) *ϕ is tangent to the identity map to order d at each point $a_1, \ldots, a_k$.*
(c) *For every $i = 1, \ldots, m$ we have that*

$$\phi(\overline{X} \cap U_i) \subset V_i, \quad \phi(p_i) = q_i, \quad b_i \in \phi(X). \tag{3.1}$$

Proof. Let $\pi : R \to R/G$ be the quotient projection, and set

$$X_0 = \{x \in X : G_x \neq \{1\}\}.$$

First observe that G acts freely on the union $E = \bigcup_{i=1}^m E_i$. Indeed, if $gx = x$ for some $x \in E_i$, then the disjointness of the arcs implies $gE_i = E_i$. Since $E_i \cap \overline{X} = \{p_i\}$ and $\overline{X}$ is G-invariant, this gives $gp_i = p_i$. The freeness of the action on bX then implies $g = 1$. Consequently, after replacing R by a sufficiently small G-invariant neighbourhood of $\overline{X} \cup E$, we may assume that the ramification locus of π is precisely the finite set X_0. Enlarging the finite set A if necessary, we may assume that $X_0 \subset A$. The domain $X/G \Subset R/G$ has smooth boundary, and the projection

$$\pi : R \setminus X_0 \longrightarrow (R \setminus X_0)/G$$

is an unbranched covering with deck group G. Choose one arc E_i from each G-orbit of the family $\{E_1, \ldots, E_m\}$. Their images under π are pairwise disjoint embedded arcs attached transversely to

$b(X/G)$. By shrinking the neighbourhoods used in the construction, we may choose, for these representative arcs, pairwise disjoint simply connected neighbourhoods $T_i \subset (R \setminus X_0)/G$ such that $\pi(E_i) \subset T_i$ and the component of $\pi^{-1}(T_i)$ containing E_i is contained in V_i. We also choose sufficiently small boundary neighbourhoods of $\pi(p_i)$ whose lifts containing p_i lie in U_i. These choices can be made disjoint from $\pi(A)$. Applying [7, Theorem 6.7.1] to the domain $X/G \Subset R/G$, the projected arcs, and the interpolation set $\pi(A) = A/G$ gives a smooth embedding

$$\varphi : \overline{X}/G \longrightarrow R/G$$

which is holomorphic in X/G, is as close to the identity as desired away from the chosen boundary neighbourhoods, and satisfies

$$\varphi(\pi(p_i)) = \pi(q_i), \qquad \pi(b_i) \in \varphi(X/G).$$

The images of the chosen boundary neighbourhoods lie in the corresponding sets T_i. We require, in addition, that φ agree with the identity to order N at every point of $\pi(A) = A/G$, where $N \geq d + 1$. We explain how to choose the lift of φ. There is a homotopy

$$H_t : \overline{X}/G \longrightarrow R/G, \qquad 0 \leq t \leq 1,$$

from $H_0 = \mathrm{Id}$ to $H_1 = \varphi$ such that $H_t|_{X_0/G} = \mathrm{Id}$ and $H_t^{-1}(X_0/G) = X_0/G$ for all $t \in [0, 1]$, the homotopy is uniformly close to the identity away from the chosen boundary neighbourhoods, and

$$H_t(\pi(p_i)) \in T_i, \qquad 0 \leq t \leq 1.$$

Furthermore, near every point $c \in \pi(A)$, H_t may be chosen holomorphic and tangent to the identity to order N. Here is a direct construction of this homotopy. On the complement of small boundary half-discs around the points $\pi(p_i)$, the map φ is arbitrarily close to the identity, so the two maps can be joined by a small homotopy. In a coordinate w centred at $c \in \pi(A)$, choose this homotopy to be

$$H_t(w) = w + t\big(\varphi(w) - w\big).$$

On a small coordinate disc this is holomorphic, fixes c, and has no other preimage of c. On the remaining part the homotopy can be chosen to agree with these local homotopies. Over each boundary half-disc, extend the homotopy inside the corresponding contractible set T_i. As the sets T_i avoid X_0/G, this construction has all the stated properties. The covering homotopy property now gives a unique lift

$$\phi_t : \overline{X} \setminus X_0 \longrightarrow R \setminus X_0, \quad t \in [0, 1],$$

such that

$$\phi_0 = \mathrm{Id}, \qquad \pi \circ \phi_t = H_t \circ \pi, \quad t \in [0, 1].$$

For every $g \in G$, the maps $\phi_t \circ g$ and $g \circ \phi_t$ lift the same homotopy and agree at $t = 0$. Uniqueness of homotopy lifting therefore gives

$$\phi_t \circ g = g \circ \phi_t, \qquad g \in G,\ t \in [0, 1].$$

In particular, the terminal lift $\phi = \phi_1$ is G-equivariant. We next check its extension across X_0 and the interpolation condition. Let $a \in X_0$ and $k = |G_a|$. Choose local coordinates z at a and w at $\pi(a)$ in which π is given by $w = z^k$. The prescribed interpolation gives

$$\varphi(w) = w + w^{N+1}\alpha(w)$$

for a holomorphic function α near 0. The local homotopy above is

$$H_t(w) = w\big(1 + tw^N\alpha(w)\big), \quad t \in [0, 1].$$

Its lift starting at the identity is

$$\phi_t(z) = z\big(1 + tz^{kN}\alpha(z^k)\big)^{1/k}, \quad t \in [0, 1],$$

where the root is chosen continuously from the value 1 at $t = 0$. On a sufficiently small disc this root is holomorphic and equals 1 at $z = 0$. Hence ϕ extends holomorphically across a and satisfies

$$\phi(z) = z + O(z^{kN+1}).$$

In particular, it is locally biholomorphic there and agrees with the identity to order at least d. At a point of $A \setminus X_0$, the same conclusion follows from the fact that π is locally biholomorphic and the lift is the one close to the identity. Thus condition (b) holds. Since φ is holomorphic in X/G and smooth up to its boundary, its lift ϕ is holomorphic in X and smooth on $\overline{X}$. We claim that it is injective. Indeed, if $\phi(x) = \phi(y)$ then $\varphi(\pi(x)) = \varphi(\pi(y))$, and hence $\pi(x) = \pi(y)$. Thus $y = gx$ for some $g \in G$. Equivariance of ϕ gives $g\phi(x) = \phi(gx) = \phi(y) = \phi(x)$. Away from X_0 the action is free, so $g = 1$. At points of X_0, injectivity follows because the lift fixes every such point. Therefore ϕ is a smooth diffeomorphism onto its image. The smallness of the homotopy away from the boundary neighbourhoods selects the lift close to the identity there. Smooth dependence of local inverse branches of π, together with the preceding local formula at ramification points, shows that ϕ approximates the identity as closely as desired in the smooth topology on $\overline{X} \setminus U$. This proves (a). Finally, the path $t \mapsto H_t(\pi(p_i))$ lies in T_i. Its lift starting at p_i remains in the component of $\pi^{-1}(T_i)$ containing E_i. Since π is injective on this component and $\varphi(\pi(p_i)) = \pi(q_i)$, we obtain $\phi(p_i) = q_i$. The same choice of sheets and the neighbourhood control in the exposing construction give $\phi(\overline{X} \cap U_i) \subset V_i$, using smaller boundary neighbourhoods inside U_i and sufficiently close approximation on their complement. The image $\phi(X)$ is G-invariant and satisfies $\pi(\phi(X)) = \varphi(X/G)$. Consequently,

$$\phi(X) = \pi^{-1}\big(\varphi(X/G)\big).$$

Since $\pi(b_i) \in \varphi(X/G)$, this implies $b_i \in \phi(X)$. Thus all three conditions in (3.1) hold. □

Recall that $E = \bigcup_{i=1}^m E_i$. In the situation of Theorem 3.1, the set

$$S = \overline{X} \cup E \subset R \tag{3.2}$$

is a G-invariant admissible set (see Definition 2.2). Recall that q_i denotes the endpoint of the arc E_i which is not contained in $\overline{X}$. Combining Theorems 2.4 and 3.1 gives the following corollary, which is used in the proof of all main results in the paper.

Corollary 3.2. *Assume the hypothesis of Theorem 3.1, let $S \subset R$ be the admissible set* (3.2)*, and suppose that G acts by orthogonal transformations on $\mathbb{R}^n$, $n \geq 3$. Given*

- $F_0 \in \mathrm{GCMI}^r_{G,*}(S, \mathbb{R}^n)$*, $r \geq 1$ (see* (2.21)*),*
- *a neighbourhood $W_i \subset \mathbb{R}^n$ of the arc $F_0(E_i)$ and a neighbourhood $U_i' \subset R$ of the point p_i for every $i = 1, \ldots, m$, and*
- *numbers $\epsilon > 0$ and $d \in \mathbb{N}$,*

there exist pairwise disjoint neighbourhoods $U_i \subset U_i'$ of p_i $(i = 1, \ldots, m)$ and a nonflat G-equivariant conformal minimal immersion $F : \overline{X} \to \mathbb{R}^n$ such that $U = \bigcup_{i=1}^m U_i$ is G-invariant and the following conditions hold:

(a) $\|F - F_0\|_{\mathscr{C}^r(\overline{X} \setminus U)} < \epsilon$.
(b) *For every $i = 1, \ldots, m$ we have $F(\overline{X} \cap U_i) \subset W_i$, $F(p_i) = F_0(q_i)$ and $F_0(b_i) \in F(X)$.*
(c) *F agrees with F_0 to order d on the finite set $A \subset X$.*
(d) $\mathrm{Flux}_F = \mathrm{Flux}_{F_0}$.

Intuitively speaking, we make a small modification of $F_0 : \overline{X} \to \mathbb{R}^n$ away from the points $p_i \in bX$ and a big modification in a small neighbourhood U_i of every p_i to obtain a G-equivariant conformal minimal immersion $F : \overline{X} \to \mathbb{R}^n$ which maps $U_i \cap \overline{X}$ into a thin tube around the arc $F_0(E_i)$ and sends the point $p_i \in bX$ to the other endpoint $F_0(q_i)$ of $F_0(E_i)$ for every $i = 1, \ldots, m$.

Proof. Choose a sufficiently small G-invariant open neighbourhood $\widetilde{S}$ of S such that S/G is a strong deformation retract of $\widetilde{S}/G$. By [13, Lemma 4.5], we can approximate F_0 in the $\mathscr{C}^r(S)$ topology, with value interpolation on $B \cup P \cup Q$ and interpolation to order d on A, by a G-equivariant conformal minimal immersion $\widetilde{F}_0 : \widetilde{S} \to \mathbb{R}^n$, defined on an open neighbourhood $\widetilde{S} \subset R$ of S, with $\mathrm{Flux}_{\widetilde{F}_0} = \mathrm{Flux}_{F_0}$. Assuming that the approximation is sufficiently close on S, $\widetilde{F}_0$ is nonflat and satisfies

$$\widetilde{F}_0(E_i) \subset W_i, \qquad i = 1, \ldots, m.$$

Choose pairwise disjoint neighbourhoods $V_i \subset \widetilde{S}$ of the arcs E_i such that $V = \bigcup_{i=1}^m V_i$ is G-invariant and

$$\widetilde{F}_0(V_i) \subset W_i, \qquad i = 1, \ldots, m.$$

For every i, choose a neighbourhood $U_i \Subset U_i' \cap V_i$ of p_i such that $U = \bigcup_{i=1}^m U_i$ is G-invariant. Let

$$\phi : \overline{X} \to \phi(\overline{X}) \subset \widetilde{S}$$

be a G-equivariant diffeomorphism furnished by Theorem 3.1, approximating the identity map on $\overline{X} \setminus U$. Assuming that the approximations in both steps are sufficiently close, the map

$$F = \widetilde{F}_0 \circ \phi : \overline{X} \to \mathbb{R}^n$$

is a well-defined G-equivariant conformal minimal immersion satisfying the corollary. The last condition in (b) is seen as follows. For every i, condition (3.1) gives a point $x_i \in X$ such that $\phi(x_i) = b_i$. Since $\widetilde{F}_0$ agrees with F_0 at b_i, we have $F(x_i) = \widetilde{F}_0(b_i) = F_0(b_i)$ and hence $F_0(b_i) \in F(X)$. □

3.2. **G-equivariant Riemann–Hilbert deformations.** The classical Riemann–Hilbert boundary value problem asks for continuous or smooth functions on a compact bordered Riemann surface $\overline{X}$ which are holomorphic in the interior X and whose value at any boundary point $x \in bX$ lies in a given real curve $\lambda_x \subset \mathbb{C}$ depending continuously or smoothly on x. A general result of this type is due to Černe [9] whose paper contains references to many earlier works. However, in geometric applications it often suffices to use approximate solutions of a Riemann–Hilbert problem, which are typically easier to obtain and which may exist even when exact solutions do not. Such approximate solutions have been used in particular for moving the boundary of the image toward the boundary of a target domain, which leads by an inductive process to existence of proper maps (see [11] and the references therein), and to increase intrinsic distances of the image by inserting oscillatory behaviour near the boundary, which inductively leads to completeness of the limit map (see [4]). The Riemann–Hilbert method was first adapted for use in the theory of conformal minimal surfaces in $\mathbb{R}^n$ and holomorphic null curves by the authors in [5]. A comprehensive discussion of this topic can be found in [7, Chap. 6].

We shall adapt the Riemann–Hilbert deformation method to G-equivariant conformal minimal immersions. As before, $\mathbf{A}$ denotes the null quadric (2.7) in $\mathbb{C}^n$ for $n \geq 3$. Set

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} = b\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

The following result is a G-equivariant analogue of [7, Theorem 6.6.2]; see also [7, Theorem 6.6.1] for the corresponding result concerning holomorphic null curves.

Theorem 3.3 (Equivariant Riemann–Hilbert deformations of minimal surfaces)**.** *Let X be a bordered Riemann surface on which a finite group G acts by holomorphic automorphisms such that the action is free on bX. Assume that G also acts on $\mathbb{R}^n$ by orthogonal transformations. Let $I_1, \ldots, I_k$ be pairwise disjoint closed arcs in bX, none of which is a connected component of bX, such that $I = \bigcup_{j=1}^k I_j$ is G-invariant. Choose a G-invariant annular neighbourhood $A \subset \overline{X}$ of bX and a smooth G-equivariant retraction $\rho : A \to bX$. Assume the following:*

- *$F : \overline{X} \to \mathbb{R}^n$ is a nonflat G-equivariant conformal minimal immersion of class $\mathscr{C}^1(\overline{X})$.*

- *For each $i = 1, \dots, k$, $(\mathbf{u}_i, \mathbf{v}_i)$ is a pair of orthogonal vectors in $\mathbb{R}^n$ satisfying $|\mathbf{u}_i| = |\mathbf{v}_i| > 0$, and if for some $g \in G$ we have $g(I_i) = I_j$ then $g\mathbf{u}_i = \mathbf{u}_j$ and $g\mathbf{v}_i = \mathbf{v}_j$.*
- *$r : bX \to [0, 1]$ is a continuous G-invariant function supported on I.*
- *$\sigma : I \times \overline{\mathbb{D}} \to \mathbb{C}$ is a $\mathscr{C}^1$ function which is G-invariant in the first variable (that is, $\sigma(gx, \xi) = \sigma(x, \xi)$ for all $g \in G$, $x \in I$, and $\xi \in \overline{\mathbb{D}}$), for every $x \in I$ the function $\sigma(x, \cdot)$ is holomorphic in $\mathbb{D}$ and satisfies $\sigma(x, 0) = 0$, and $\frac{\partial \sigma}{\partial \xi}$ is nowhere vanishing on $I \times \overline{\mathbb{D}}$.*
- *$\Lambda \subset X$ is a finite G-invariant set containing X_0 (1.2).*

Consider the continuous map $\varkappa : bX \times \overline{\mathbb{D}} \to \mathbb{R}^n$ given by

$$\varkappa(x, \xi) = \begin{cases} F(x), & \text{if } x \in bX \setminus I; \\ F(x) + \Re \sigma(x, r(x)\xi)\mathbf{u}_i + \Im \sigma(x, r(x)\xi)\mathbf{v}_i, & \text{if } x \in I_i,\ i \in \{1, \dots, k\}. \end{cases} \tag{3.3}$$

Given $\epsilon > 0$, $d \in \mathbb{N}$, and a neighbourhood $U \subset \overline{X}$ of I, there exist a neighbourhood $\Omega \subset U \cap A$ of I and a G-equivariant conformal minimal immersion $\widetilde{F} : \overline{X} \to \mathbb{R}^n$ satisfying the following conditions.

(i) $\operatorname{dist}(\widetilde{F}(x), \varkappa(x, \mathbb{T})) < \epsilon$ *for all* $x \in bX$.
(ii) $\operatorname{dist}(\widetilde{F}(x), \varkappa(\rho(x), \overline{\mathbb{D}})) < \epsilon$ *for all* $x \in \Omega$.
(iii) $\widetilde{F}$ *is ϵ-close to F in the $\mathscr{C}^1$ norm on* $\overline{X} \setminus \Omega$.
(iv) $\widetilde{F} - F$ *vanishes to order d at every point in* Λ.
(v) $\operatorname{Flux}_{\widetilde{F}} = \operatorname{Flux}_F$ *(see* (1.5)*).*

Observe that since the action of G on bX is free and every element of G preserves the boundary orientation, the induced action of G on the set of arcs $\{I_1, \dots, I_k\}$ in the above theorem is free. Indeed, if $g \in G \setminus \{1\}$ preserved a proper boundary arc setwise, then it would preserve the ordered pair of its endpoints. Hence both endpoints would be fixed by g, contrary to the freeness of the action on bX.

Proof. Let $(\mathbf{u}_i, \mathbf{v}_i)$ for $i = 1, \dots, k$ be as in the theorem. Then,

$$\mathbf{w}_i = \mathbf{u}_i - \imath \mathbf{v}_i \in \mathbf{A}_* \subset \mathbb{C}^n \setminus \{0\}, \qquad i = 1, \dots, k$$

Furthermore, if $g \in G$ and $g(I_i) = I_j$ then

$$g\mathbf{w}_i = g\mathbf{u}_i - \imath g\mathbf{v}_i = \mathbf{u}_j - \imath \mathbf{v}_j = \mathbf{w}_j.$$

Thus the family of null directions $\{\mathbf{w}_i\}_{i=1}^k$ is compatible with the action of G.

We will adapt the proof of the Riemann–Hilbert theorem for conformal minimal immersions (see [2, Theorem 3.6] or [7, Theorem 6.6.2]), performing the construction simultaneously on the G-orbits of the boundary arcs. We explain the modifications.

By Remark 1.4, we may assume that X is a relatively compact G-invariant domain with real analytic boundary in an open Riemann surface R, that G acts on R by holomorphic automorphisms, and its stabiliser locus is a finite set $X_0 \subset X$. Choose a G-invariant holomorphic 1-form θ on R as in (2.4). Thus, $\{\theta = 0\} = X_0$. Let $Y = \mathbf{A}_* \cup Y_0 \subset \mathbb{CP}^n$ be the manifold (2.10). The map

$$f = 2\partial F/\theta : \overline{X} \to Y \tag{3.4}$$

is of class $\mathscr{A}(\overline{X}, Y)$, holomorphic on X, and G-equivariant. Since the action of G is free in a neighbourhood of bX, after shrinking the annulus A around bX we have $A \cap X_0 = \varnothing$ and hence

$$f(A) \subset \mathbf{A}_*.$$

Let $I_{i_1}, \dots, I_{i_s}$ be representatives of the G-orbits in the set of arcs $\{I_1, \dots, I_k\}$. Choose pairwise disjoint closed boundary discs

$$D_{i_\nu} \subset U \cap A, \qquad \nu = 1, \dots, s$$

with piecewise smooth boundaries such that for every $\nu = 1, \dots, s$ the arc

$$I_{i_\nu} \subset bX \cap bD_{i_\nu}$$

lies in the relative interior of $bX \cap bD_{i_\nu}$ and

$$\rho(D_{i_\nu}) \subset bX \cap D_{i_\nu}.$$

These discs may be chosen so small that the sets

$$D_{g,i_\nu} := g(D_{i_\nu}), \qquad g \in G, \quad \nu = 1, \dots, s,$$

are pairwise disjoint. If $g(I_{i_\nu}) = I_j$, we write $D_j = g(D_{i_\nu})$. In this way we obtain a G-invariant family of pairwise disjoint discs $D_1, \dots, D_k$.

Let σ be as in the theorem. On every representative disc D_{i_ν} we apply the local Riemann–Hilbert construction used in the proof of [2, Theorem 3.6] (see also [7, Theorem 6.6.2]) with the null direction

$$\mathbf{w}_{i_\nu} = \mathbf{u}_{i_\nu} - \iota \mathbf{v}_{i_\nu} \in \mathbf{A}_*$$

and the attached family of conformal minimal discs

$$\overline{\mathbb{D}} \ni \xi \longmapsto \varkappa(x, \xi) = F(x) + \Re\sigma(x, r(x)\xi)\mathbf{u}_i + \Im\sigma(x, r(x)\xi)\mathbf{v}_i, \quad x \in I_{i_\nu},\ \xi \in \overline{\mathbb{D}}$$

(see (3.3)). This yields holomorphic null data

$$f_{i_\nu} : D_{i_\nu} \times \overline{\mathbb{D}} \to Y, \quad \nu = 1, \dots, s$$

on the representative discs D_{i_ν} such that for some point $p_{i_\nu} \in \mathring{D}_{i_\nu}$, the integrals

$$F(p_{i_\nu}) + \Re \int_{p_{i_\nu}}^{x} f_{i_\nu}(x, \xi)\theta, \qquad x \in D_{i_\nu},\ \xi \in \overline{\mathbb{D}}$$

satisfy the estimates (i)–(iii). (The integral from p_{i_ν} to x is taken within the disc D_{i_ν}. See the proofs of Theorems 6.6.1 and 6.6.2 in [7] for the details.) We extend this construction to every disc in the orbit by G-equivariance. More precisely, for any $g \in G$ we define the null data on the disc $D_j = g(D_{i_\nu})$ by

$$f_j(x, \xi) = g f_{i_\nu}(g^{-1}x, \xi), \qquad x \in D_j,\ \xi \in \overline{\mathbb{D}}. \tag{3.5}$$

The definition is unambiguous since the action of G on the set of arcs, and hence on the chosen family of discs, is free. The identities

$$\sigma(gx, \xi) = \sigma(x, \xi), \qquad r(gx) = r(x), \qquad g\mathbf{w}_{i_\nu} = \mathbf{w}_j$$

show that the resulting integrals satisfy the same Riemann–Hilbert estimates on all the discs D_j for $j = 1, \dots, k$, and the family is G-equivariant.

It remains to glue the local null data f_j (3.5) to f (3.4) so that the integral of the new G-equivariant holomorphic map $\tilde{f} : \overline{X} \to Y$ is well-defined and satisfies the required interpolation, approximation and period conditions. For this purpose, we use a G-equivariant period-dominating spray with core f. The construction of such sprays, including the periods needed for equivariant integration, is given in [13, Lemma 4.5 and Sect. 5]. We choose the spray so that its members agree with f to order d at every point of the G-invariant finite set Λ. In addition, the orders are chosen to preserve the pole orders and leading coefficients of f at every point of X_0, so that $\tilde{f}\theta$ extends holomorphically and is nowhere zero at these points. Likewise, we construct a G-equivariant family of sprays on the discs D_j, with core f_j (3.5), such that these sprays approximate the one over f as closely as desired outside a small neighbourhood of the arc I_j. To complete the proof, we glue the spray with the core f and the family of sprays over the discs D_j with the core f_j into a new G-equivariant holomorphic spray on $\overline{X}$ with values in Y (2.10) such that a certain member

$$\tilde{f} : \overline{X} \longrightarrow Y$$

approximates f in the $\mathscr{C}^0$ topology off a small neighbourhood $\Omega \subset U \cap A$ of I, and the integral $\widetilde{F} : \overline{X} \to \mathbb{R}^n$ of $\tilde{f}$ is well defined, G-equivariant, and satisfies conditions (i)–(v) in the theorem. Furthermore, the jet interpolation conditions on $\tilde{f}$ at the points of X_0 ensure that

$$\tilde{f}(\overline{X} \setminus X_0) \subset \mathbf{A}_*, \qquad \tilde{f}(X_0) \subset Y_0,$$

so that $\tilde{f}\theta$ extends holomorphically and is nowhere zero. This ensures that $\widetilde{F}$ is an immersion.

For gluing of sprays, and to find a suitable member $\tilde{f}$ of the resulting glued spray, we use the correspondence in Lemma 2.1 as in [13, proof of Theorem 5.1]. Note that G-equivariant maps (and sprays of such maps) from G-invariant subdomains of $\overline{X}$ to Y pass down to the quotient $\overline{X}/G \subset R/G$ as sections (or sprays of sections) of the holomorphic fibration $\rho : Z \to R/G$ (2.19); conversely, sections of ρ yield G-equivariant maps to Y. Hence, we can perform the gluing of sprays over the base $\overline{X}/G$ by using [2, proof of Theorem 3.5], coupled with the technique from [14, Theorem 2.1] (see also [15, Theorem 6.14.6]), which yields the Oka principle for sections of certain ramified holomorphic maps onto a Stein base. Fix a point $x_0 \in X$ and for every $a \in \Lambda$ choose a path γ_a in X from x_0 to a. As shown in [13, proof of Lemma 4.5], the parameter in the dominating G-equivariant spray of maps $\overline{X} \to Y$, obtained by gluing, can be chosen so that the corresponding member $\tilde{f}$ of the spray satisfies

$$\int_C (\tilde{f} - f)\theta = 0 \quad \text{for every } [C] \in H_1(X, \mathbb{Z}), \tag{3.6}$$

$$\Re \int_{x_0}^{g x_0} (\tilde{f} - f)\theta = 0, \qquad g \in G, \tag{3.7}$$

$$\Re \int_{\gamma_a} (\tilde{f} - f)\theta = 0, \qquad a \in \Lambda. \tag{3.8}$$

The real parts of equations (3.6) ensure that the map $\widetilde{F} : \overline{X} \to \mathbb{R}^n$, given by

$$\widetilde{F}(x) = F(x_0) + \Re \int_{x_0}^{x} \tilde{f}\theta, \qquad x \in \overline{X},$$

is a well-defined conformal minimal immersion, whereas the imaginary parts of (3.6) ensure that $\mathrm{Flux}_{\widetilde{F}} = \mathrm{Flux}_F$ (condition (v)). Condition (3.7) implies that (2.16) remains unchanged:

$$g\widetilde{F}(x_0) - \widetilde{F}(x_0) = \Re \int_{x_0}^{g x_0} \tilde{f}\,\theta, \qquad g \in G. \tag{3.9}$$

The G-equivariance of $\tilde{f}$ and (3.9) imply exactly as in (2.17) that $\widetilde{F}$ is G-equivariant. Condition (3.8) implies that $\widetilde{F} - F$ vanishes at every point of Λ. This and the jet-interpolation conditions of sufficiently high order, imposed on the spray at the points of Λ, imply that $\widetilde{F} - F$ vanishes to order d at every point of Λ, so (iv) holds. The estimates in the local Riemann–Hilbert construction are preserved up to a multiplicative constant in the gluing process, and the gluing can be made arbitrarily close to the original spray on $\overline{X} \setminus \Omega$, so we can also ensure that conditions (i)–(iii) hold. □

Corollary 3.4. *The analogue of Theorem 3.3 holds for G-equivariant holomorphic null immersions into $\mathbb{C}^n$ for any $n \geq 3$, where G acts on $\mathbb{C}^n$ by elements of the complex orthogonal group $O(n, \mathbb{C})$. The approximation and jet-interpolation conclusions remain valid, and the deformation may be chosen so that the null differential has the same complex periods as the initial one. In particular, exact null differentials remain exact.*

Proof. The proof is the same as that of Theorem 3.3, except that one does not take real parts when integrating the null differential. The full complex period map is used in the period-correction step. □

4. Main lemmas

In this section, we state and prove the main lemmas concerning properness, completeness, and general position that will lead to the proof of the main theorems via standard recursive procedures. The proofs of these lemmas rely on the new tools established in the previous section.

4.1. **G-equivariant properness lemma.** Our proof of Theorem 1.3 is an adaptation to the G-equivariant case of the boundary-lifting method, used in the construction of proper conformal minimal immersions into convex domains in $\mathbb{R}^n$, $n \geq 3$, and in particular to $\mathbb{R}^n$ itself; see [2, Theorem 1.4]. (Stronger results concerning proper conformal minimal immersions into minimally convex domains in $\mathbb{R}^3$ and $(n-2)$-convex domains in $\mathbb{R}^n$, $n \geq 3$, can be found in [3] and [7, Sect. 8.4].) The proofs of these properness results inductively use the boundary lifting method furnished by exposing boundary points and the Riemann-Hilbert deformation techniques described in the previous section. The induction is based on the following boundary lifting lemma, whose proof relies on Corollary 3.2 and Theorem 3.3. The lemma will be also used in the proof of Theorem 1.8.

Lemma 4.1. *Let X and G be as in Theorem 1.3, let $F \in \mathrm{CMI}^1_{G,*}(\overline{X}, \mathbb{R}^n)$, and assume that for a compact G-invariant set $K \subset \mathring{X}$ and numbers $0 < r < s$ we have*

$$F(\overline{X \setminus K}) \subset s\mathbb{B} \setminus r\overline{\mathbb{B}}, \tag{4.1}$$

where $\mathbb{B} \subset \mathbb{R}^n$ is the unit ball. Given a number δ satisfying $0 < \delta < s - r$, F can be approximated as closely as desired in $\mathscr{C}^1(K, \mathbb{R}^n)$ by $\widetilde{F} \in \mathrm{CMI}^1_{G,*}(\overline{X}, \mathbb{R}^n)$ satisfying the following properties.

(a) $\|\widetilde{F} - F\|_{0,\overline{X}} < \sqrt{s^2 - r^2}$.
(b) $\widetilde{F}(\overline{X \setminus K}) \subset s\mathbb{B} \setminus r\overline{\mathbb{B}}$.
(c) $\widetilde{F}(bX) \subset s\mathbb{B} \setminus (s-\delta)\overline{\mathbb{B}}$.
(d) $\mathrm{Flux}(\widetilde{F}) = \mathrm{Flux}(F)$.

The main point in the lemma is condition (c), which asserts that $\widetilde{F}(bX)$ lies near the boundary of the bigger ball $s\mathbb{B}$ when $\delta > 0$ is small.

Proof. By Remark 1.4, we may assume that X is a relatively compact G-invariant domain with real analytic boundary in an open Riemann surface R, the group G acts on R by holomorphic automorphisms, and the action is free on $R \setminus X$. By (4.1) and compactness of bX, we may also assume by decreasing $\delta > 0$ if necessary that $r + 2\delta < s$ and

$$F(\overline{X \setminus K}) \subset s\mathbb{B} \setminus (r+\delta)\overline{\mathbb{B}}. \tag{4.2}$$

Denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^n$. Fix a number μ with $0 < \mu < \delta$ so small that the following holds:

$$\text{If} \quad p, q \in s\overline{\mathbb{B}} \setminus (r+\delta)\mathbb{B} \quad \text{and} \quad \langle p - q, q \rangle > -\mu, \quad \text{then} \quad |p - q| < \sqrt{s^2 - r^2}. \tag{4.3}$$

Such μ exists by Pythagoras' theorem. To prove the lemma, we shall provide $\widetilde{F} \in \mathrm{CMI}^1_{G,*}(\overline{X}, \mathbb{R}^n)$, as close to F as desired in $\mathscr{C}^1(K, \mathbb{R}^n)$, satisfying conditions (b), (c), (d), and

$$\langle \widetilde{F}(x) - F(x), F(x) \rangle > -\mu \quad \text{for all } x \in bX. \tag{4.4}$$

This, (4.2), and (4.3) imply that $|\widetilde{F}(x) - F(x)| < \sqrt{s^2 - r^2}$ for all $x \in bX$, which ensures (a) by the maximum principle. We will complete the task in two deformation steps.

First deformation. Fix a number $\epsilon_0 > 0$ to be specified later. Let $\mathcal{J}$ be a finite collection of closed arcs in bX satisfying the following conditions.

(i) $bX = \bigcup_{I \in \mathcal{J}} I$ and each connected component of bX contains at least three arcs in $\mathcal{J}$.

(ii) $I \cap I'$ is either empty or a singleton for every pair of arcs $I \neq I' \in \mathcal{J}$.
(iii) $g(I) \cap I = \varnothing$ and $g(I) \in \mathcal{J}$ for every $I \in \mathcal{J}$ and $g \in G \setminus \{1\}$. In particular, $\mathcal{J}$ is G-invariant.
(iv) If $I \in \mathcal{J}$ and $x, y \in I$, then $|F(x) - F(y)| < \epsilon_0$.

Such a collection exists by compactness of bX and continuity of F. It follows that if $x \in I \cap I'$ with $I \neq I' \in \mathcal{J}$, then x is a common endpoint of I and I'. Moreover, each arc in $\mathcal{J}$ intersects exactly two other arcs in $\mathcal{J}$. Denote by $P \subset bX$ the finite set formed by the endpoints of the arcs in $\mathcal{J}$, that is,

$$P = \{x \in bX : \exists I \neq I' \in \mathcal{J},\ I \cap I' = \{x\}\}.$$

Write $P = \{x_1, \dots, x_m\}$ and note that P is a G-invariant set. Let $E_1, \dots, E_m$ be smoothly embedded pairwise disjoint compact arcs in R such that for each $i = 1, \dots, m$, E_i is attached to $\overline{X}$ transversely at x_i, $E_i \cap \overline{X} = \{x_i\}$, and the set $E = \bigcup_{i=1}^m E_i$ is G-invariant. Let y_i denote the other endpoint of E_i, set $Q = \{y_1, \dots, y_m\}$, and consider the G-invariant admissible set

$$S = \overline{X} \cup E \subset R.$$

For each $i \in \{1, \dots, m\}$, choose a number $\lambda_i > 1$ such that

$$s - \delta < \lambda_i |F(x_i)| < s \quad \text{and} \quad \lambda_i |F(x_i)| = \lambda_j |F(x_j)| \quad \text{for all } i, j. \tag{4.5}$$

Next, given $\epsilon_1 > 0$ to be specified later, we extend F, with the same name, to a nonflat G-equivariant generalised conformal minimal immersion $F \in \mathrm{GCMI}^1_{G,*}(S, \mathbb{R}^n)$ such that for every $i \in \{1, \dots, m\}$,

$$d^H\big(F(E_i), [F(x_i), \lambda_i F(x_i)]\big) < \epsilon_1 \quad \text{and} \quad F(y_i) = \lambda_i F(x_i), \tag{4.6}$$

where d^H denotes the Hausdorff distance in $\mathbb{R}^n$ and $[p, q]$ the straight segment connecting a pair of points $p, q \in \mathbb{R}^n$. As usual, we first extend F to a representative arc $E_i \subset E$ in each family (this is possible by [7, Lemma 3.5.4]), and then we extend F to every translate $g(E_i)$ of E_i by equivariance: $F(gx) = gF(x)$ for all $x \in E_i$. Note that

$$g\big([F(x_i), \lambda_i F(x_i)]\big) = [gF(x_i), \lambda_i gF(x_i)] \quad \text{for every } g \in G$$

and, by (4.2),

$$[F(x_i), \lambda_i F(x_i)] \subset s\mathbb{B} \setminus (r + \delta)\overline{\mathbb{B}}. \tag{4.7}$$

In view of (4.2), (4.5), and (4.6), and assuming that $\epsilon_1 > 0$ is sufficiently small, we have that

$$F(E) \subset s\mathbb{B} \setminus (r + \delta)\overline{\mathbb{B}} \quad \text{and} \quad \langle F(x) - F(x_i), F(x_i)\rangle > -\frac{\mu}{2} \quad \text{for all } x \in E_i,\ i = 1, \dots, m. \tag{4.8}$$

Given a number $\epsilon_2 > 0$, Corollary 3.2 then provides a nonflat G-equivariant conformal minimal immersion $F' : \overline{X} \to \mathbb{R}^n$ and a G-invariant collection of (big) closed subarcs $\{J_I \subset \mathring{I} : I \in \mathcal{J}\}$ satisfying the following.

(v) F' is as close to F as desired in $\mathscr{C}^1(K, \mathbb{R}^n)$.
(vi) $F'(x_i) = \lambda_i F(x_i) \in s\mathbb{B} \setminus (s - \delta)\overline{\mathbb{B}}$ for every $i \in \{1, \dots, m\}$; see (4.5) and (4.6).
(vii) $|F'(x) - F(x)| < \epsilon_2$ for all $x \in \bigcup_{I \in \mathcal{J}} J_I$.
(viii) $\langle F'(x) - F(x), F(x)\rangle > -\mu/2$ for all $x \in bX \setminus \bigcup_{I \in \mathcal{J}} J_I$; see (4.8)
(ix) $d^H\big(F'(\Gamma_i), [F(x_i), \lambda_i F(x_i)]\big) < \epsilon_1$ for every $i \in \{1, \dots, m\}$, where $\Gamma_i \subset bX$ is the closure of the connected component of $bX \setminus \bigcup_{I \in \mathcal{J}} J_I$ that contains the point x_i.
(x) $F'(\overline{X} \setminus K) \subset s\mathbb{B} \setminus (r + \delta)\overline{\mathbb{B}}$; see (4.2) and (4.8).
(xi) $\mathrm{Flux}_{F'} = \mathrm{Flux}_F$.

Assuming that $\epsilon_2 > 0$ is sufficiently small, we infer from (vii) and (viii) that

$$\langle F'(x) - F(x), F(x)\rangle > -\frac{\mu}{2} \quad \text{for all } x \in bX. \tag{4.9}$$

On the other hand, condition (vi) guarantees the existence of a G-equivariant collection of (bigger) closed subarcs $\{J_I' \subset \mathring{I} \colon I \in \mathcal{J}\}$ such that $J_I \subset \mathring{J}_I'$ for every $I \in \mathcal{J}$ and

$$F'(x) \in s\mathbb{B} \setminus (s-\delta)\overline{\mathbb{B}} \quad \text{for all } x \in bX \setminus \bigcup_{I\in\mathcal{J}} \mathring{J}_I'. \tag{4.10}$$

Therefore, $F' \in \mathrm{CMI}^1_{G,*}(\overline{X}, \mathbb{R}^n)$ formally satisfies conditions (b), (d), and (4.4), but it only meets (c) at the boundary points outside $\bigcup_{I\in\mathcal{J}} \mathring{J}_I'$.

Second deformation. Set $\Upsilon = \bigcup_{I\in\mathcal{J}} J_I$ and $\Upsilon' = \bigcup_{I\in\mathcal{J}} J_I'$, and denote by $\Gamma_i' \subset bX$ the closure of the connected component of $bX \setminus \bigcup_{I\in\mathcal{J}} \mathring{J}_I'$ that contains the point x_i (hence, $\Gamma_i' \subset \mathring{\Gamma}_i$). For each $I \in \mathcal{J}$ fix a point $z_I \in \mathring{I}$ making sure that the set $\{z_I \colon I \in \mathcal{J}\}$ is G-invariant. Assuming that the fixed positive numbers ϵ_0, ϵ_1, and ϵ_2 are sufficiently small, for every $I \in \mathcal{J}$ there exist

- a pair of orthogonal unit vectors $(\mathbf{u}_I, \mathbf{v}_I)$ in $\mathbb{R}^n$, that are orthogonal to $F'(z_I) \neq 0$, such that if for some $g \in G$ we have $g(I) = I'$ then $g\mathbf{u}_I = \mathbf{u}_{I'}$ and $g\mathbf{v}_I = \mathbf{v}_{I'}$,
- a continuous G-equivariant function $\tau : bX \to [0,1]$ supported on Υ', and
- a $\mathscr{C}^1$ function $\sigma : \Upsilon' \times \overline{\mathbb{D}} \to \mathbb{C}$ which is G-invariant in the first variable, for every $x \in \Upsilon'$ the function $\sigma(x,\cdot)$ is holomorphic in $\mathbb{D}$, $\sigma(x,0) = 0$, and $\frac{\partial\sigma}{\partial\xi}$ is nowhere vanishing on $\Upsilon' \times \overline{\mathbb{D}}$, where ξ is the second variable,

such that the continuous map $\varkappa : bX \times \overline{\mathbb{D}} \to \mathbb{R}^n$ given by

$$\varkappa(x,\xi) = \begin{cases} F'(x), & \text{if } x \in bX \setminus \Upsilon'; \\ F'(x) + \mathfrak{R}\sigma(x,\tau(x)\xi)\mathbf{u}_I + \mathfrak{I}\sigma(x,\tau(x)\xi)\mathbf{v}_I, & \text{if } x \in J_I',\ I \in \mathcal{J} \end{cases} \tag{4.11}$$

satisfies the following conditions.

(I) $\varkappa(x,\xi) \in s\mathbb{B} \setminus (s-\delta)\overline{\mathbb{B}}$ for all $x \in bX$, $\xi \in \mathbb{T}$.
(II) $\varkappa(bX \times \overline{\mathbb{D}}) \subset s\mathbb{B} \setminus (r+\delta)\overline{\mathbb{B}}$.

Indeed, take into account (4.7), (4.10), and that choosing ϵ_0, ϵ_1, and ϵ_2 small enough, if x_i is an endpoint of $I \in \mathcal{J}$ then $F'(I)$ is as close as desired to $[F(x_i), \lambda_i F(x_i)]$ in Hausdorff distance; see conditions (iv), (vii), and (ix). (Note that if these numbers are small and x_j is the other endpoint of I, then $[F(x_i), \lambda_i F(x_i)]$ and $[F(x_j), \lambda_j F(x_j)]$ are close in Hausdorff distance by the aforementioned conditions and the second property in (4.5)). As above, we first define the objects on a representative arc $I \in \mathcal{J}$ in each family, and then on every translate $g(I)$ of I by equivariance.

In this situation, Theorem 3.3 provides a nonflat G-equivariant conformal minimal immersion $\widetilde{F} : \overline{X} \to \mathbb{R}^n$ satisfying the following conditions.

(III) $\widetilde{F}$ is as close as desired to F' in $\mathscr{C}^1(K, \mathbb{R}^n)$.
(IV) $\widetilde{F}(bX) \subset s\mathbb{B} \setminus (s-\delta)\overline{\mathbb{B}}$; see (I).
(V) $\langle \widetilde{F}(x) - F(x), F(x)\rangle > -\mu$ for all $x \in bX$; see (4.9) and (iv), recall that $\mathbf{u}_I$ and $\mathbf{v}_I$ are orthogonal to $F'(z_I)$ for every $I \in \mathcal{J}$, and assume that $\epsilon_0 > 0$ is sufficiently small.
(VI) $\widetilde{F}(\overline{X} \setminus K) \subset s\mathbb{B} \setminus r\overline{\mathbb{B}}$; see (x) and (II).
(VII) $\mathrm{Flux}_{\widetilde{F}} = \mathrm{Flux}_{F'}$.

It follows that $\widetilde{F}$ is as close as desired to F in $\mathscr{C}^1(K, \mathbb{R}^n)$ (see (v) and (III)) and it satisfies conditions (b), (c), (d) (see (xi) and (VII)), and (4.4), thereby proving the lemma. □

4.2. **G-equivariant completeness lemma.** Given a conformal minimal immersion $F : \overline{X} \to \mathbb{R}^n$, we denote by dist_F the distance function on $\overline{X}$ induced by the Riemannian metric F^*ds^2. If $A, B \subset \overline{X}$, we write

$$\mathrm{dist}_F(A,B) = \inf\{\mathrm{dist}_F(x,y) : x \in A,\ y \in B\}.$$

The main ingredient in the proof of Theorems 1.7 and 1.8 (together with Lemma 4.1 for the latter) is the following equivariant version of [2, Lemma 4.1]. It asserts that every G-equivariant conformal minimal immersion $\overline{X} \to \mathbb{R}^n$ of a compact bordered Riemann surface can be approximated as closely as desired in $\mathscr{C}^0(\overline{X})$ by G-equivariant conformal minimal immersions with arbitrarily large intrinsic diameter.

Lemma 4.2. *Let X, G, and n satisfy the hypotheses of Theorem 1.7, and let $F : \overline{X} \to \mathbb{R}^n$ be a nonflat G-equivariant conformal minimal immersion of class $\mathscr{C}^1(\overline{X})$. Fix a point $x_0 \in X$, a compact G-invariant set $K \subset X$, and numbers $\lambda > 0$ (large) and $\epsilon > 0$ (small). Then there exists a nonflat G-equivariant conformal minimal immersion $\widetilde{F} : \overline{X} \to \mathbb{R}^n$ of class $\mathscr{C}^1(\overline{X})$ satisfying the following conditions.*

(a) $\|\widetilde{F} - F\|_{\mathscr{C}^0(\overline{X})} < \epsilon$.
(b) $\|\widetilde{F} - F\|_{\mathscr{C}^1(K)} < \epsilon$.
(c) $\mathrm{dist}_{\widetilde{F}}(x_0, bX) > \lambda$.
(d) $\mathrm{Flux}_{\widetilde{F}} = \mathrm{Flux}_F$.

The main condition is (c), which shows that the intrinsic distance from an interior point to the boundary of X in the metric induced by $\widetilde{F}$ can be made as big as desired. The proof of Lemma 4.2 follows by a finite recursive application of the technical lemma below, in exactly the same way that [2, Lemma 4.1] follows from [2, Lemma 4.2]; hence, we omit the details.

Lemma 4.3. *Let X, G, n, F, x_0, and K be as in Lemma 4.2. Let $\Phi : bX \to \mathbb{R}^n$ be a continuous G-equivariant map and $\delta > 0$ and $d > 0$ be numbers such that*

$$\|F - \Phi\|_{0,bX} < \delta \quad \text{and} \quad 0 < d < \mathrm{dist}_F(x_0, bX). \tag{4.12}$$

Given $\eta > 0$, F can be approximated as closely as desired in $\mathscr{C}^1(K, \mathbb{R}^n)$ by $\widetilde{F} \in \mathrm{CMI}^1_{G,*}(\overline{X}, \mathbb{R}^n)$ satisfying the following properties.

(a) $\|\widetilde{F} - \Phi\|_{0,bX} < \sqrt{\delta^2 + \eta^2}$.
(b) $\mathrm{dist}_{\widetilde{F}}(x_0, bX) > d + \eta$.
(c) $\mathrm{Flux}(\widetilde{F}) = \mathrm{Flux}(F)$.

Proof. The proof consists of adapting that of [2, Lemma 4.2] to the equivariant case by using similar ideas as those in the proof of Lemma 4.1. The proof follows a similar scheme to that of the proof of Lemma 4.1, consisting of two deformation steps. We proceed to explain it.

First of all, by Remark 1.4, we may assume that X is a relatively compact G-invariant domain with real analytic boundary in an open Riemann surface R, that G acts on R by holomorphic automorphisms, and that G acts freely on $R \setminus X$. We may also assume by a slight perturbation of Φ that

$$F - \Phi \text{ vanishes nowhere on } bX. \tag{4.13}$$

Finally, we can also assume without loss of generality that K is a smoothly bounded compact domain that is a strong deformation retract of X with $x_0 \in \mathring{K}$ and, in view of (4.12), there is a number $\mu_0 > 0$ (small) such that

$$\|F - \Phi\|_{0,bX} < \delta - \mu_0 \quad \text{and} \quad \mathrm{dist}_F(x_0, bK) > d + \mu_0. \tag{4.14}$$

First deformation. Fix $\epsilon_0 > 0$ to be specified later and choose a collection $\mathcal{J}$ of closed arcs in bX satisfying conditions (i)–(iv) in the proof of Lemma 4.1. In addition, we take the arcs so small that

$$|\Phi(x) - \Phi(y)| < \epsilon_0 \quad \text{for all } x, y \in I,\ I \in \mathcal{J}. \tag{4.15}$$

We define

$$P = \{x_1, \ldots, x_m\}, \quad E = \bigcup_{i=1}^{m} E_i, \quad Q = \{y_1, \ldots, y_m\}, \quad S = \overline{X} \cup E \subset R$$

as in that proof. Arguing as in the first deformation step in the proof of Lemma 4.1, we extend F to a nonflat G-equivariant generalised conformal minimal immersion $F \in \mathrm{GCMI}^1_{G,*}(S, \mathbb{R}^n)$ such that for every $i \in \{1, \ldots, m\}$, the curve $F(E_i)$ has small extrinsic diameter (that is, it is contained in a small ball in $\mathbb{R}^n$ centred at $F(x_i)$) but its projection onto the vector $F(x_i) - \Phi(x_i) \neq 0$ (see (4.13)) has total variation bigger than η. For this, we choose the extension of F to E_i to be highly oscillating while keeping $F(E_i)$ very close in Hausdorff distance to the straight line segment

$$\big[F(x_i), F(x_i) + \lambda(F(x_i) - \Phi(x_i))/|F(x_i) - \Phi(x_i)|\big]$$

for a small $\lambda > 0$. Next, Corollary 3.2 furnishes a nonflat G-equivariant conformal minimal immersion $F' : \overline{X} \to \mathbb{R}^n$ and a G-equivariant collection of (big) closed subarcs $\{J_I \subset \mathring{I} \colon I \in \mathcal{J}\}$ satisfying the following conditions (cf. properties (v)–(xi) in the proof of Lemma 4.1 as well as [2, Lemma 4.3]):

(v) F' is as close to F as desired in $\mathscr{C}^1(K, \mathbb{R}^n)$.

(vi) F' is close to F uniformly in $\overline{X}$ (depending on how small the extrinsic diameter of the paths $F(E_i)$ is for $i = 1, \ldots, m$). In particular, we may guarantee in view of (iv) that if $I \in \mathcal{J}$ and $x, y \in I$ then

$$\max\big\{|F(x) - F(y)|, |F'(x) - F'(y)|, |F'(x) - F(y)|\big\} < \epsilon_0.$$

(vii) Let $\Upsilon = \bigcup_{I \in \mathcal{J}} J_I \subset bX$ and for every $i \in \{1, \ldots, m\}$ denote by $\Gamma_i \subset bX$ the closure of the connected component of $bX \setminus \bigcup_{I \in \mathcal{J}} J_I$ containing x_i. For every $i = 1, \ldots, m$ there is a closed smoothly bounded neighbourhood W_i of Γ_i in $\overline{X}$, disjoint from K, such that the sets $W_1, \ldots, W_m$ are pairwise disjoint, and for every path $\gamma \subset \overline{X}$ connecting $bW_i \cap X$ to Γ_i the projection of the path $F'(\gamma) \subset \mathbb{R}^n$ onto the vector $F(x_i) - \Phi(x_i) \neq 0$ has total variation bigger than η.

(viii) $\mathrm{Flux}_{F'} = \mathrm{Flux}_F$.

In constructing F', we impose in part (vii) the partition-based projected-length condition of [2, Lemma 4.3, property (P4)]. The construction is performed on representative attached arcs and extended to their translates by G-equivariance.

Second deformation. Take a G-equivariant collection of closed subarcs $\{J'_I \subset \mathring{I} \colon I \in \mathcal{J}\}$ such that $J_I \subset \mathring{J}'_I$ for every $I \in \mathcal{J}$ and set $\Upsilon' = \bigcup_{I \in \mathcal{J}} J'_I$. Choose points $z_I \in \mathring{I}$, $I \in \mathcal{J}$, as in the proof of Lemma 4.1. For every $I \in \mathcal{J}$ take an orthonormal frame $(\mathbf{u}_I, \mathbf{v}_I)$ in $\mathbb{R}^n$ orthogonal to $F(z_I) - \Phi(z_I) \neq 0$ (see (4.13)), a continuous G-equivariant map $\tau : bX \to [0, 1]$ supported on Υ', and a $\mathscr{C}^1$ function $\sigma : \Upsilon' \times \overline{\mathbb{D}} \to \mathbb{C}$ as in the second deformation step in the proof of Lemma 4.1, such that

$$|\sigma(x, \tau(x)\xi)| \leq \eta \ \text{ for all } (x, \xi) \in bX \times \mathbb{T} \quad \text{and} \quad |\sigma(x, \tau(x)\xi)| = \eta \ \text{ for all } (x, \xi) \in \Upsilon \times \mathbb{T}. \tag{4.16}$$

Given $\epsilon_1 > 0$ to be specified later, Theorem 3.3 applied with these data and the map $\varkappa : bX \times \overline{\mathbb{D}} \to \mathbb{R}^n$ given by (4.11) provides a nonflat G-equivariant conformal minimal immersion $\widetilde{F} : \overline{X} \to \mathbb{R}^n$ satisfying the following conditions.

(I) $\widetilde{F}$ is as close as desired to F' in $\mathscr{C}^1(\overline{X} \setminus \Omega, \mathbb{R}^n)$, where $\Omega \subset \overline{X}$ is an arbitrarily small neighbourhood of Υ' with $K \cap \Omega = \varnothing$. In particular, we can ensure in view of (vi) that if $x \in I \in \mathcal{J}$ and $y \in X \cap b\Omega$ lies in the component of $b\Omega$ intersecting I, then $|\widetilde{F}(y) - F'(x)| < \epsilon_0$.

(II) $\eta - \epsilon_1 < |\widetilde{F}(x) - F'(x)| < \eta + \epsilon_1$ for all $x \in \Upsilon$, $|\widetilde{F}(x) - F'(x)| < \eta + \epsilon_1$ for all $x \in \Upsilon'$, and $|\widetilde{F}(x) - F'(x)| < \epsilon_1$ for all $x \in bX \setminus \Upsilon'$. See (4.16) and (4.11), and recall that the directing vectors $\mathbf{u}_I$ and $\mathbf{v}_I$ are unit vectors for every $I \in \mathcal{J}$.

(III) $|\langle \widetilde{F}(x) - F'(x), F(z_I) - \Phi(z_I)\rangle| < \epsilon_1$ for all $x \in J'_I$, $I \in \mathcal{J}$. Recall that $\mathbf{u}_I$ and $\mathbf{v}_I$ are orthogonal to $F(z_I) - \Phi(z_I)$.

(IV) $\mathrm{Flux}_{\widetilde{F}} = \mathrm{Flux}_{F'}$.

Assuming that the numbers $\epsilon_0 > 0$ and $\epsilon_1 > 0$ are sufficiently small, $\widetilde{F}$ satisfies the conclusion of the lemma. Indeed, concerning condition (a), for every $x \in bX$ we have by (II), (vi), and (4.14) that

$$|\widetilde{F}(x) - F'(x)| < \eta + \epsilon_1, \quad |F'(x) - F(x)| < \epsilon_0, \quad \text{and} \quad |F(x) - \Phi(x)| < \delta - \mu_0. \tag{4.17}$$

On the other hand, (4.15) and (vi) give that

$$|(F(x) - \Phi(x)) - (F(z_I) - \Phi(z_I))| < 2\epsilon_0 \quad \text{for every } x \in I,\ I \in \mathcal{J}. \tag{4.18}$$

By the Schwarz inequality, this, (II), (III), and (4.17) ensure for every $x \in J_I'$, $I \in \mathcal{J}$, that

$$\begin{aligned} |\langle \widetilde{F}(x) - F(x), F(x) - \Phi(x)\rangle| \ &\leq\ |\langle F'(x) - F(x), F(x) - \Phi(x)\rangle| \\ &\quad + |\langle \widetilde{F}(x) - F'(x), F(z_I) - \Phi(z_I)\rangle| \\ &\quad + |\langle \widetilde{F}(x) - F'(x), (F(x) - \Phi(x)) - (F(z_I) - \Phi(z_I))\rangle| \\ &<\ \epsilon_0\delta + \epsilon_1 + 2\epsilon_0(\eta + \epsilon_1), \end{aligned}$$

thus $\widetilde{F}(x) - F(x)$ is approximately orthogonal to $F(x) - \Phi(x)$ when ϵ_0 and ϵ_1 are small. Hence, Pythagoras' theorem implies that

$$|\widetilde{F}(x) - \Phi(x)| < \sqrt{|\widetilde{F}(x) - F(x)|^2 + |F(x) - \Phi(x)|^2} + O(\epsilon_0, \epsilon_1) \quad \text{for every } x \in \Upsilon',$$

where $O(\epsilon_0, \epsilon_1) > 0$ is a constant depending on ϵ_0 and ϵ_1 that goes to 0 as $\epsilon_0, \epsilon_1 \to 0$. In view of (4.17), this guarantees that

$$|\widetilde{F}(x) - \Phi(x)| < \sqrt{(\eta + \epsilon_1 + \epsilon_0)^2 + (\delta - \mu_0)^2} + O(\epsilon_0, \epsilon_1) < \sqrt{\delta^2 + \eta^2} \quad \text{for every } x \in \Upsilon',$$

where the latter inequality holds whenever $\epsilon_0 > 0$ and $\epsilon_1 > 0$ are sufficiently small. Likewise, for every $x \in bX \setminus \Upsilon'$ we have by (II) and (4.17) that

$$|\widetilde{F}(x) - \Phi(x)| \leq |\widetilde{F}(x) - F'(x)| + |F'(x) - F(x)| + |F(x) - \Phi(x)| < \epsilon_1 + \epsilon_0 + \delta - \mu_0 < \sqrt{\delta^2 + \eta^2},$$

where the last inequality holds for small enough ϵ_0 and ϵ_1. This proves (a).

Regarding condition (b), consider a path γ in $\overline{X} \setminus \mathring{K}$ connecting a point $x \in bK$ with a point $y \in bX$, and otherwise disjoint from $bK \cup bX$. If $y \in bX \setminus \mathring{\Upsilon} = \bigcup_{i=1}^m \Gamma_i$, there are $i \in \{1, \ldots, m\}$ and a subpath $\gamma' \subset \gamma \cap W_i$ connecting $bW_i \cap X$ with Γ_i. Taking into account (4.18) and that the vectors $\mathbf{u}_I$ and $\mathbf{v}_I$ are orthogonal to $F(z_I) - \Phi(z_I)$, and assuming that the approximations in Theorem 3.3 to obtain $\widetilde{F}$ from F' are close enough, we have that the projection of $\widetilde{F}|_{\gamma'}$ onto the vector $F(x_i) - \Phi(x_i)$ is very close to that of $F'|_{\gamma'}$ when $\epsilon_0 > 0$ is small. Note that the length of a curve cannot decrease very much under uniform approximation. Applying the argument of [2, Claim 4.4] to the stronger projected-length condition imposed in part (vii) of the first deformation, we obtain

$$\mathrm{length}(\widetilde{F} \circ \gamma) > \eta$$

for every such path γ, provided that the approximation is sufficiently close. If on the contrary $y \in \Upsilon \subset \Omega$, then in view of (I) and the first inequality in (II) there is a point $y' \in \gamma \cap X \cap b\Omega$ near y such that

$$\mathrm{length}(\widetilde{F}(\gamma)) > |\widetilde{F}(y) - \widetilde{F}(y')| \geq |\widetilde{F}(y) - F'(y)| - |F'(y) - \widetilde{F}(y')| > \eta - \epsilon_1 - \epsilon_0.$$

Therefore, $\mathrm{dist}_{\widetilde{F}}(bK, bX) > \eta - \epsilon_1 - \epsilon_0$. Choose the approximation on K sufficiently close that

$$\mathrm{dist}_{\widetilde{F}}(x_0, bK) > d + \frac{\mu_0}{2}$$

(see (4.14)). Taking $\epsilon_0 + \epsilon_1 < \mu_0/2$, the preceding boundary-collar estimate gives $\mathrm{dist}_{\widetilde{F}}(x_0, bX) > d + \eta$, which is condition (b). Finally, (c) follows from (viii) and (IV). □

4.3. **A general position lemma.** As we have already mentioned, the proofs of Theorems 1.3 and 1.8 require a recursive application of Lemma 4.1. For this lemma to be applied, we need to begin with a G-equivariant conformal minimal immersion $\overline{X} \to \mathbb{R}^n$ vanishing nowhere on bX; see (4.1). Such an immersion is provided by the following result.

Lemma 4.4. *If X and G are as in Theorem 1.3, then every $F \in \mathrm{CMI}^1_{G,*}(\overline{X}, \mathbb{R}^n)$ can be approximated as closely as desired uniformly on $\overline{X}$ by $\widetilde{F} \in \mathrm{CMI}^1_{G,*}(\overline{X}, \mathbb{R}^n)$ with $0 \notin \widetilde{F}(bX)$.*

Proof. By Remark 1.4 and Theorem 2.4 we may assume that X is a relatively compact G-invariant domain with real analytic boundary in an open Riemann surface R, that G acts on R by holomorphic automorphisms, that G acts freely on $R \setminus X$, and that F is a nondegenerate G-equivariant conformal minimal immersion $F : R \to \mathbb{R}^n$. Further, after shrinking R around $\overline{X}$, we may assume that the stabiliser locus of the action of G on R equals the finite set

$$X_0 = \{x \in X : G_x \neq \{1\}\} \subset X.$$

Since the action of G is free on a neighbourhood of bX, the quotient projection $\pi : R \to R/G$ maps $\overline{X}$ properly onto the smoothly bounded domain $\overline{X}/G \subset R/G$. Since F is an immersion on $\overline{X}$, its zero set is locally discrete. Choose a neighbourhood N of bX such that

$$B := F^{-1}(0) \cap N = F^{-1}(0) \cap bX;$$

note that this is a finite set. Choose an open neighbourhood $U \subset R/G$ of $\overline{X}/G$ and a biholomorphism

$$\phi : U \to \phi(U) \subset R/G,$$

close to the identity on $\overline{X}/G$, such that ϕ is tangent to the identity at the points in X_0/G and satisfies

$$\phi(bX/G) \subset (R/G) \setminus (B/G).$$

The last condition is achieved by a general position argument. For this, one uses the flow of a holomorphic vector field on the quotient, transverse to the boundary at B/G and vanishing to sufficiently high order at the branch values X_0/G. Set $\widetilde{U} = \pi^{-1}(U) \subset R$. We see as in the proof of Theorem 3.1 that ϕ lifts to a G-equivariant biholomorphism $\varphi : \widetilde{U} \to \varphi(\widetilde{U}) \subset R$ which approximates the identity on $\overline{X}$, it fixes the points of X_0 to a given finite order, and it satisfies

$$\varphi(bX) \subset N \qquad \text{and} \qquad \varphi(bX) \cap B = \varnothing.$$

The map $\widetilde{F} = F \circ \varphi : \widetilde{U} \to \mathbb{R}^n$ then satisfies the lemma. □

5. Proof of the Main Theorems

Proof of Theorem 1.8. In view of Remark 1.4, we regard X as a relatively compact G-invariant domain with real analytic boundary in an open Riemann surface R and assume that G acts on R by holomorphic automorphisms. After shrinking R around $\overline{X}$, we may assume that the stabiliser locus of the action of G on R equals the finite set

$$X_0 = \{x \in X : G_x \neq \{1\}\} \subset X. \tag{5.1}$$

By Theorem 1.1 and Lemma 4.4, there is a nondegenerate G-equivariant conformal minimal immersion $F_0 : R \to \mathbb{R}^n$ with $0 \notin F_0(bX)$. We use the same notation for its restriction to $\overline{X}$. Moreover, up to applying a homothety, we may assume that $F_0(\overline{X}) \subset \mathbb{B}$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$. In particular, there are numbers $0 < r_0 < r_1 < 1$ and a smoothly bounded compact G-invariant domain $K_0 \subset X$ that is a strong deformation retract of X such that

$$F_0(\overline{X \setminus K_0}) \subset r_1\mathbb{B} \setminus r_0\overline{\mathbb{B}}. \tag{5.2}$$

Choose any increasing sequence $r_1 < r_2 < r_3 < \cdots$ such that

$$\lim_{j\to\infty} r_j = 1 \quad \text{and} \quad \sum_{j\geq 2} \sqrt{r_{j+1}^2 - r_{j-1}^2} < \infty. \tag{5.3}$$

Fix a point $x_0 \in \mathring{K}_0$ and a number $\epsilon_0 > 0$. To complete the proof we shall inductively construct a sequence of nondegenerate G-equivariant conformal minimal immersions $F_j : \overline{X} \to \mathbb{R}^n$, smoothly bounded compact G-invariant domains $K_j \subset \mathring{X}$ that are strong deformation retracts of X, and positive numbers $\epsilon_j > 0$, satisfying the following conditions for every $j = 1, 2, \ldots$

- (a_j) $K_{j-1} \subset \mathring{K}_j$.
- (b_j) $F_j(\overline{X \setminus K_j}) \subset r_{j+1}\mathbb{B} \setminus r_j\overline{\mathbb{B}}$.
- (c_j) $F_j(\overline{X \setminus K_{j-1}}) \subset r_{j+1}\mathbb{B} \setminus r_{j-1}\overline{\mathbb{B}}$.
- (d_j) $\|F_j - F_{j-1}\|_{1,K_{j-1}} < \epsilon_{j-1}$. (Here, $\|\cdot\|_{1,K} = \|\cdot\|_{\mathscr{C}^1(K)}$.)
- (e_j) $\|F_j - F_{j-1}\|_{0,\overline{X}} < \sqrt{r_{j+1}^2 - r_{j-1}^2}$.
- (f_j) $\mathrm{dist}_{F_j}(x_0, bK_j) > j$.
- (g_j) $\epsilon_j < \epsilon_{j-1}/2$ and if $F : X \to \mathbb{R}^n$ is a conformal harmonic map such that $\|F - F_j\|_{1,K_j} < 2\epsilon_j$, then $F|_{K_j}$ is a nondegenerate immersion, $F(\overline{K_j \setminus K_{j-1}}) \subset r_{j+1}\mathbb{B} \setminus r_{j-1}\overline{\mathbb{B}}$, and $\mathrm{dist}_F(x_0, bK_j) > j$.

In addition, we will ensure that

$$X = \bigcup_{j\geq 1} K_j. \tag{5.4}$$

With such sequences in hand, conditions (d_j), (g_j), and (5.4) ensure that there is a limit nondegenerate G-equivariant conformal minimal immersion $F = \lim_{j\to\infty} F_j : X \to \mathbb{R}^n$ such that

$$\|F - F_j\|_{1,K_j} < 2\epsilon_j \quad \text{for every } j = 0, 1, 2, \ldots \tag{5.5}$$

By (5.3) and (e_j) the sequence F_j also converges uniformly on $\overline{X}$ to a continuous G-equivariant map $F : \overline{X} \to \overline{\mathbb{B}}$, while (5.3), (5.5), and (g_j) imply that the restriction $F|_X : X \to \mathbb{B}$ is proper and complete. Properness also gives $F(bX) \subset b\mathbb{B}$.

It therefore remains to explain the induction. The basis is given by the already fixed F_0, K_0, and ϵ_0. Condition (b_0) holds by (5.2), while (f_0) is implied by the fact that $x_0 \in \mathring{K}_0$. All other conditions are vacuous for $j = 0$. For the inductive step, assume that for some $j \geq 1$ we have F_{j-1}, K_{j-1}, and ϵ_{j-1} satisfying (b_{j-1}). This condition enables us to apply Lemma 4.1 to the data

$$F = F_{j-1}, \quad K = K_{j-1}, \quad r = r_{j-1}, \quad s = r_{j+1}, \quad \delta = r_{j+1} - r_j,$$

obtaining a nondegenerate G-equivariant conformal minimal immersion $F_j : \overline{X} \to \mathbb{R}^n$ satisfying (c_j), (d_j), (e_j), and

$$F_j(bX) \subset r_{j+1}\mathbb{B} \setminus r_j\overline{\mathbb{B}}. \tag{5.6}$$

To ensure that F_j is nondegenerate, use that F_{j-1} is nondegenerate and choose the approximation in (d_j) close enough. In turn, by applying Lemma 4.2 to F_j we can assume in addition that

$$\mathrm{dist}_{F_j}(x_0, bX) > j. \tag{5.7}$$

The additional perturbation is chosen sufficiently small uniformly on $\overline{X}$ and in the $\mathscr{C}^1$ topology on K_{j-1} so that conditions (c_j)–(e_j) and (5.6) remain valid. By continuity of F_j and compactness of $\overline{X}$, we can choose in view of (5.6) and (5.7) a smoothly bounded compact G-invariant domain $K_j \subset X$ which is a strong deformation retract of X and satisfies conditions (a_j), (b_j), and (f_j). Finally, choose $\epsilon_j > 0$ so small that (g_j) holds. It is clear that in this process we can choose the compact domain K_j in each step large enough to ensure (5.4). This closes the induction and completes the proof. □

Proof of Theorem 1.3. Reasoning as in the proof of Theorem 1.8, we can assume that we have objects X, G, R, X_0, F_0, r_0, and K_0 as in that proof such that (5.1) holds and

$$F_0(\overline{X} \setminus \mathring{K}_0) \subset \mathbb{B} \setminus r_0\overline{\mathbb{B}}. \tag{5.8}$$

Cf. (5.2). Fix $\epsilon_0 > 0$. A recursive application of Lemma 4.1 then gives a sequence of nondegenerate G-equivariant conformal minimal immersions $F_j : \overline{X} \to \mathbb{R}^n$, smoothly bounded compact G-invariant domains $K_j \subset \mathring{X}$ that are strong deformation retracts of X, and positive numbers $\epsilon_j > 0$, such that the following conditions are satisfied for every $j = 1, 2, \ldots$

- (a'_j) $K_{j-1} \subset \mathring{K}_j$.
- (b'_j) $F_j(\overline{X \setminus K_j}) \subset (j+1)\mathbb{B} \setminus j\overline{\mathbb{B}}$.
- (c'_j) $F_j(\overline{X \setminus K_{j-1}}) \subset (j+1)\mathbb{B} \setminus (j-1)\overline{\mathbb{B}}$.
- (d'_j) $\|F_j - F_{j-1}\|_{1,K_{j-1}} < \epsilon_{j-1}$.
- (g'_j) $\epsilon_j < \epsilon_{j-1}/2$ and if $F : X \to \mathbb{R}^n$ is a conformal harmonic map such that $\|F - F_j\|_{1,K_j} < 2\epsilon_j$, then $F|_{K_j}$ is a nondegenerate immersion and $F(\overline{K_j \setminus K_{j-1}}) \subset (j+1)\mathbb{B} \setminus (j-1)\overline{\mathbb{B}}$.

In addition, we ensure (5.4). This is done exactly as in the proof of Theorem 1.8 but replacing r_j by j for $j = 2, 3, \ldots$; cf. conditions (a_j)–(d_j) and (g_j). For the initial step $j = 1$, we apply Lemma 4.1 with $r = r_0$, $s = 2$, and $\delta = 1$. (We may also guarantee for every $j \geq 2$ that $\|F_j - F_{j-1}\|_{0,\overline{X}} < 2\sqrt{j}$ as in (e_j), but this is not needed for the current aim.) By (d'_j), (5.4), and (g'_j), there is a limit nondegenerate G-equivariant conformal minimal immersion $F = \lim_{j\to\infty} F_j : X \to \mathbb{R}^n$ satisfying (5.5). This condition and the second part of (g'_j) ensure that $F : X \to \mathbb{R}^n$ is a proper map. □

Proof of Theorem 1.7. As in the proof of Theorem 1.8, we may assume that X is a relatively compact G-invariant domain with real analytic boundary in an open Riemann surface R, that G acts on R by holomorphic automorphisms, and that the stabiliser locus of the action of G on R equals the finite set X_0 in (5.1). For the existence assertion, Theorem 1.1 supplies an initial nondegenerate G-equivariant conformal minimal immersion on R. For the approximation assertion, we start instead with the prescribed nonflat conformal G-equivariant minimal immersion $\overline{X} \to \mathbb{R}^n$ and apply Theorem 2.4 to approximate it uniformly on $\overline{X}$ by a nonflat G-equivariant conformal minimal immersion F_0 defined on a neighbourhood of $\overline{X}$. Nondegeneracy is ensured by additional value interpolation at finitely many points outside $\overline{X}$ in the chosen neighbourhood.

It remains to show that F_0 obtained above can be approximated as closely as desired uniformly on $\overline{X}$ by a continuous G-equivariant map $F : \overline{X} \to \mathbb{R}^n$ whose restriction $F|_X : X \to \mathbb{R}^n$ is a complete, nondegenerate, conformal minimal immersion.

Fix $\epsilon > 0$. Choose a number $0 < \epsilon_0 < \epsilon/2$, a smoothly bounded compact G-invariant domain $K_0 \subset X$ that is a strong deformation retract of X, and a point $x_0 \in \mathring{K}_0$. To complete the proof, it suffices to construct a sequence of nondegenerate G-equivariant conformal minimal immersions $F_j : \overline{X} \to \mathbb{R}^n$, smoothly bounded compact G-invariant domains $K_j \subset \mathring{X}$ that are strong deformation retracts of X, and positive numbers $\epsilon_j > 0$ satisfying (5.4) and the following conditions for every $j = 1, 2, \ldots$:

- (A_j) $K_{j-1} \subset \mathring{K}_j$.
- (B_j) $\|F_j - F_{j-1}\|_{0,\overline{X}} < \epsilon_{j-1}$.
- (C_j) $\|F_j - F_{j-1}\|_{1,K_{j-1}} < \epsilon_{j-1}$.
- (D_j) $\mathrm{dist}_{F_j}(x_0, bK_j) > j$.
- (E_j) $\epsilon_j < \epsilon_{j-1}/2$, and if $F : X \to \mathbb{R}^n$ is a conformal harmonic map such that $\|F - F_j\|_{1,K_j} < 2\epsilon_j$ then $F|_{K_j}$ is a nondegenerate immersion and $\mathrm{dist}_F(x_0, bK_j) > j$.

Indeed, by (B_j) and (E_j) there is a limit continuous G-equivariant map $F = \lim_{j\to\infty} F_j : \overline{X} \to \mathbb{R}^n$ with

$$\|F - F_j\|_{0,\overline{X}} < 2\epsilon_j \quad \text{for every } j = 0, 1, 2, \ldots \tag{5.9}$$

(Cf. (5.5)). In particular, $\|F - F_0\|_{0,\overline{X}} < 2\epsilon_0 < \epsilon$. Moreover, (5.9) together with (C_j), (D_j) and (E_j) imply that $F|_X : X \to \mathbb{R}^n$ is a nondegenerate, complete, conformal minimal immersion with $\|F - F_j\|_{1,K_j} < 2\epsilon_j$ for $j = 0, 1, 2, \ldots$.

The induction is done as follows. For the base case, note that F_0, K_0, and ϵ_0 meet (D_0) while all other conditions are vacuous. Fix $j \geq 1$ and assume that we have objects F_{j-1}, K_{j-1}, and ϵ_{j-1} as above. (It is not needed that (D_{j-1}) holds.) Lemma 4.2 then provides a nondegenerate G-equivariant conformal minimal immersion $F_j : \overline{X} \to \mathbb{R}^n$ satisfying (B_j), (C_j) and $\mathrm{dist}_{F_j}(x_0, bX) > j$. (To ensure that F_j is nondegenerate, use that F_{j-1} is such and choose the approximation in (B_j) close enough.) This enables us to choose first K_j so large that (A_j) and (D_j) hold (and to ensure (5.4) in the process), and then $\epsilon_j > 0$ so small that (E_j) holds. This closes the induction and completes the proof. □

Proof of Theorem 1.11. We use the null-curve version of Theorem 3.3 from Corollary 3.4. Fix a G-invariant Hermitian inner product on $\mathbb{C}^n$, for example

$$\langle u, v\rangle_G = \sum_{g\in G} \langle gu, gv\rangle_{\mathrm{st}},$$

where $\langle \cdot\,, \cdot\,\rangle_{\mathrm{st}}$ denotes the standard Hermitian inner product. The action of G is unitary with respect to this Hermitian inner product. Its associated norm and Riemannian metric are equivalent to the standard Euclidean ones. Consequently, properness, boundedness, and completeness do not depend on which of these metrics is used. For null curves one works directly with holomorphic null immersions rather than with their real parts. The exposing construction on the source is unchanged. The differential data on the attached discs and the period-dominating sprays take values in the punctured null quadric, and all local data are chosen on one representative of each G-orbit and transported by the complex orthogonal action of G. Using the G-invariant Hermitian norm in the boundary-lifting argument, the proof of Theorem 1.3 gives part (a). Using the Riemannian metric associated to the same Hermitian inner product, the distance-increasing construction and the proof of Theorem 1.7 give part (b). Equivalence with the standard Euclidean metric completes the proof. □

Acknowledgements. Alarcón is partially supported by the State Research Agency (AEI) via the grant no. PID2023-150727NB-I00, funded by MICIU/AEI/10.13039/501100011033 and ERDF/EU, Spain. Forstnerič is supported by the European Union (ERC Advanced grant HPDR, 101053085) and grant P1-0291 from ARIS, Republic of Slovenia.

Declaration on the use of AI. The mathematical ideas, statements, and proofs in this article are due to the authors. Generative AI tools (mainly, GPT 6.1 Sol) were used to verify mathematical correctness and completeness, and to assist with language editing. The authors take full responsibility for the correctness and presentation of the article.

References

[1] A. Alarcón. The Yang problem for complete bounded complex submanifolds: a survey. In *Cutting-edge mathematics. RSME 2022, Ciudad Real, Spain, January 17–21, 2022*, pages 1–25. Cham: Springer, 2024.

[2] A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, and F. J. López. Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves. *Proc. Lond. Math. Soc. (3)*, 111(4):851–886, 2015.

[3] A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, and F. J. López. Minimal surfaces in minimally convex domains. *Trans. Amer. Math. Soc.*, 371(3):1735–1770, 2019.

[4] A. Alarcón and F. Forstnerič. Every bordered Riemann surface is a complete proper curve in a ball. *Math. Ann.*, 357(3):1049–1070, 2013.

[5] A. Alarcón and F. Forstnerič. The Calabi–Yau problem, null curves, and Bryant surfaces. *Math. Ann.*, 363(3-4):913–951, 2015.

[6] A. Alarcón, F. Forstnerič, and F. J. López. Embedded minimal surfaces in $\mathbb{R}^n$. *Math. Z.*, 283(1-2):1–24, 2016.

[7] A. Alarcón, F. Forstnerič, and F. J. López. *Minimal surfaces from a complex analytic viewpoint*. Springer Monographs in Mathematics. Springer, Cham, 2021.
[8] A. Alarcón and F. J. López. Minimal surfaces in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$. *J. Differential Geom.*, 90(3):351–381, 2012.
[9] M. Černe. Nonlinear Riemann-Hilbert problem for bordered Riemann surfaces. *Am. J. Math.*, 126(1):65–87, 2004.
[10] J. P. D'Angelo. *Rational sphere maps*, volume 341 of *Prog. Math.* Cham: Birkhäuser, 2021.
[11] B. Drinovec Drnovšek and F. Forstnerič. Holomorphic curves in complex spaces. *Duke Math. J.*, 139(2):203–253, 2007.
[12] F. Forstnerič. Recent developments on Oka manifolds. *Indag. Math., New Ser.*, 34(2):367–417, 2023.
[13] F. Forstnerič. Minimal surfaces with symmetries. *Proc. Lond. Math. Soc. (3)*, 128(3):32, 2024. Id/No e12590.
[14] F. Forstnerič. The Oka principle for multivalued sections of ramified mappings. *Forum Math.*, 15(2):309–328, 2003.
[15] F. Forstnerič. *Stein manifolds and holomorphic mappings (The homotopy principle in complex analysis)*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Cham, second edition, 2017.
[16] F. Forstnerič and E. F. Wold. Bordered Riemann surfaces in $\mathbb{C}^2$. *J. Math. Pures Appl. (9)*, 91(1):100–114, 2009.
[17] H. Grauert. Analytische Faserungen über holomorph-vollständigen Räumen. *Math. Ann.*, 135:263–273, 1958.
[18] R. C. Gunning and R. Narasimhan. Immersion of open Riemann surfaces. *Math. Ann.*, 174:103–108, 1967.
[19] L. P. d. M. Jorge and F. Xavier. A complete minimal surface in $\mathbb{R}^3$ between two parallel planes. *Ann. of Math. (2)*, 112(1):203–206, 1980.
[20] F. Kutzschebauch, F. Lárusson, and G. W. Schwarz. Equivariant Oka theory: survey of recent progress. *Complex Anal. Synerg.*, 8(3):13, 2022. Id/No 15.
[21] B. Maskit. The conformal group of a plane domain. *Am. J. Math.*, 90:718–722, 1968.
[22] R. Miranda. *Algebraic curves and Riemann surfaces*, volume 5 of *Grad. Stud. Math.* Providence, RI: AMS, American Mathematical Society, 1995.
[23] S. Morales. On the existence of a proper minimal surface in $\mathbb{R}^3$ with a conformal type of disk. *Geom. Funct. Anal.*, 13(6):1281–1301, 2003.
[24] N. Nadirashvili. Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces. *Invent. Math.*, 126(3):457–465, 1996.
[25] R. Schoen and S. T. Yau. *Lectures on harmonic maps*. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.
[26] E. L. Stout. Bounded holomorphic functions on finite Riemann surfaces. *Trans. Amer. Math. Soc.*, 120:255–285, 1965.

Antonio Alarcón, Departamento de Geometría y Topología e Instituto de Matemáticas (IMAG), Universidad de Granada, Campus de Fuentenueva s/n, E–18071 Granada, Spain.

Email address: alarcon@ugr.es

Franc Forstnerič, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Franc Forstnerič, Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

Email address: franc.forstneric@fmf.uni-lj.si