

ON AN EXTREMAL PROBLEM FOR HARMONIC MAPS CONFORMAL AT A POINT

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ABSTRACT. Let \mathbb{D} denote the unit disc in \mathbb{C} . For a domain $D \subset \mathbb{C}$ and a point $p \in D$, let $M_D(p)$ denote the supremum of $\|df_0\|$ over all harmonic maps $f : \mathbb{D} \rightarrow D$ with $f(0) = p$ whose differential df_0 at $0 \in \mathbb{D}$ is conformal. If $f : \mathbb{D} \rightarrow D$ is a conformal diffeomorphism onto D with $f(0) = p$, then $\|df_0\| \leq M_D(p)$. In a recent paper, the authors proved that equality holds when $D = \mathbb{D}$, and they asked whether equality can hold only when D is a round disc. We give a negative answer by proving that, among bounded convex pointed domains $p \in D \subset \mathbb{C}$ and up to translations, rotations, and reflections, equality holds if and only if, after moving p to the origin, $D = F(\mathbb{D})$ where $F : \mathbb{D} \rightarrow \mathbb{C}$ is a holomorphic map with $F(0) = 0$ and

$$F'(z) = \frac{c}{1 + az + \lambda z^2},$$

where $c > 0$, $|\lambda| < 1$, and $|a - \bar{a}\lambda| < 1 - |\lambda|^2$. This family contains strongly convex examples which are not round discs.

RÉSUMÉ. Soit \mathbb{D} le disque unité dans \mathbb{C} . Pour un domaine $D \subset \mathbb{C}$ et un point $p \in D$, soit $M_D(p)$ le supremum de $\|df_0\|$ sur l'ensemble des applications harmoniques $f : \mathbb{D} \rightarrow D$ avec $f(0) = p$ dont la différentielle df_0 en $0 \in \mathbb{D}$ est conforme. Si $f : \mathbb{D} \rightarrow D$ est un difféomorphisme conforme sur D avec $f(0) = p$, alors $\|df_0\| \leq M_D(p)$. Dans un article récent, les auteurs ont prouvé que l'égalité est vérifiée lorsque $D = \mathbb{D}$, et ils se sont demandé si l'égalité ne pouvait être vérifiée que lorsque D est un disque circulaire. Nous donnons une réponse négative en prouvant que, parmi les domaines convexes pointus et bornés $p \in D \subset \mathbb{C}$ et à l'exception des translations, rotations et réflexions, l'égalité est vérifiée si et seulement si, après avoir déplacé p à l'origine, $D = F(\mathbb{D})$ où $F : \mathbb{D} \rightarrow \mathbb{C}$ est une application holomorphe avec $F(0) = 0$ et

$$F'(z) = \frac{c}{1 + az + \lambda z^2},$$

où $c > 0$, $|\lambda| < 1$ et $|a - \bar{a}\lambda| < 1 - |\lambda|^2$. Cette famille contient des exemples fortement convexes qui ne sont pas des disques ronds.

1. INTRODUCTION

The classical Schwarz lemma and its Schwarz–Pick refinement control the derivative of a holomorphic self-map of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and characterise the extremal maps as automorphisms of the disc. The analogous extremal problems for harmonic maps are more subtle, especially when one imposes conformality only at a single point. We refer to Duren [6], Forstnerič and Kalaj [8], and Kalaj [9, 10] for related results and background.

The motivation for the present paper is [8, Problem 4.3] posed by the authors. Let D be a connected and simply connected domain in the complex plane \mathbb{C} . Given a point $p \in D$, set

$$(1.1) \quad M_D(p) = \sup\{\|df_0\| : f : \mathbb{D} \rightarrow D \text{ harmonic, } f(0) = p, df_0 \text{ conformal}\}.$$

If $f : \mathbb{D} \rightarrow D$ is a holomorphic map with $f(0) = p$, then clearly

$$(1.2) \quad |f'(0)| \leq M_D(p).$$

Assuming that D is a bounded convex domain, we prove in Theorem 2.1 that there exists an extremal harmonic map $f : \mathbb{D} \rightarrow D$ with $f(0) = p$ such that $\|df_0\| = M_D(p)$.

In the special case when D is a round disc, the extremal maps for which equality holds in (1.2) are conformal diffeomorphisms of \mathbb{D} onto D ; see [8, Theorem 1.1] due to the authors. In

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particular, conformality of a harmonic map $f : \mathbb{D} \rightarrow \mathbb{D}$ at the point 0, together with extremality of $\|df_0\|$, implies conformality of f at all points. This is a nontrivial generalisation of the classical Schwarz–Pick lemma, whose proof in [8] is inspired by the seminal work of Lempert [11] on geodesics of the Kobayashi metric on convex domains in \mathbb{C}^n . The analogous result holds for harmonic maps $\mathbb{D} \rightarrow \mathbb{B}^n$ to the ball in the real Euclidean space \mathbb{R}^n for any $n \geq 3$, and the extremal maps are conformal harmonic diffeomorphisms from \mathbb{D} onto proper affine linear discs in \mathbb{B}^n ; see [8, Theorem 2.1]. This led to the introduction of a Kobayashi-type pseudometric on domains in \mathbb{R}^n , $n \geq 3$, called the *minimal metric* since it pertains to conformal minimal surfaces; see [8, Sect. 6] and [5]. By [8, Theorem 6.2] the minimal metric on the ball \mathbb{B}^n is the Cayley–Klein metric, one of the classical models of hyperbolic geometry.

By [8, Example 4.2], conformal diffeomorphisms $\mathbb{D} \rightarrow D$ onto more general domains $D \subset \mathbb{C}$ fail to reach the extremal value in (1.2) even if D is convex. The question raised in [8, Problem 4.3] is whether this holds only if D is a round disc. In this paper, we give a negative answer. We identify a family of bounded strongly convex pointed domains $p \in D \subset \mathbb{C}$ for which a conformal diffeomorphism $\mathbb{D} \rightarrow D$ satisfies equality in (1.2). Such domains will be called exceptional.

Theorem 1.1. *Let $a, \lambda \in \mathbb{C}$ be such that the quadratic polynomial $q(z) := 1 + az + \lambda z^2$ has no zeros on \mathbb{D} . Equivalently, by the Schur–Cohn criterion (see Lemma 3.1),*

$$(1.3) \quad |\lambda| < 1 \quad \text{and} \quad |a - \bar{a}\lambda| < 1 - |\lambda|^2.$$

Given $c > 0$, let $F : \mathbb{D} \rightarrow \mathbb{C}$ be the holomorphic map determined by the conditions

$$(1.4) \quad F(0) = 0 \quad \text{and} \quad F'(z) = \frac{c}{1 + az + \lambda z^2}.$$

Then F is a biholomorphic map onto the bounded strongly convex domain $D = F(\mathbb{D})$, and $M_D(0) = |F'(0)| = c$. The harmonic extremal maps attaining $M_D(0)$ are

$$z \mapsto F(e^{i\theta}z) \quad \text{and} \quad z \mapsto F(e^{i\theta}\bar{z}), \quad \theta \in \mathbb{R}.$$

We get a bigger family of exceptional domains by applying translations, rotations, and reflections to the maps F in the above theorem. However, postcomposition with a nonrigid map does not preserve the class of harmonic maps used in the definition of $M_D(p)$. In particular, we are not allowed to use holomorphic automorphisms of D in general. Hence, exceptional domains should be understood as pointed domains $p \in D$.

The simplest noncircular example in the above family of domains is obtained by taking $a = 0$, $\lambda = \frac{1}{2}$, and $c = 1$. This gives

$$F'(z) = \frac{1}{1 + z^2/2}, \quad F(z) = \sqrt{2} \arctan\left(\frac{z}{\sqrt{2}}\right);$$

see Corollary 3.3. The image $F(\mathbb{D})$ is not a disc, which gives a negative answer to [8, Problem 4.3].

Our next result is a converse to Theorem 1.1 in the class of bounded convex domains. The two results together classify exceptional pointed domains among bounded convex domains.

Theorem 1.2. *Let $\Phi : \mathbb{D} \rightarrow D$ be a biholomorphic map onto a bounded convex domain with $\Phi(0) = 0$ and $|\Phi'(0)| = M_D(0)$. Then, up to a rotation, Φ equals a map F from Theorem 1.1.*

Theorem 1.2 is proved in Sect. 4. The proof is based on the boundary value formulation of the harmonic extremal problem.

It would be interesting to look at the analogous problem for harmonic maps $\mathbb{D} \rightarrow D$ to domains $D \subset \mathbb{R}^n$ for $n \geq 3$. Let $z = x + iy$ be the coordinate on \mathbb{C} . Given a point $p \in D$ and a tangent vector $v \in \mathbb{R}^n \setminus \{0\}$, one defines $M_D(p, v)$ as the supremum of the numbers $\|df_0\|$ over all harmonic maps $f : \mathbb{D} \rightarrow D$ with $f(0) = p$, $f_x(0) = \lambda v$ for some $\lambda > 0$, and $df_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ a conformal linear map.

Problem 1.3. *For which bounded convex pointed domains $p \in D \subset \mathbb{R}^n$, $n \geq 3$, and vectors $v \in \mathbb{R}^n$ is the supremum $M_D(p, v)$ reached by a conformal harmonic map $f : \mathbb{D} \rightarrow D$?*

By [8, Theorem 2.1], the answer is affirmative when $p \in D = \mathbb{B}^n$ for any $v \in \mathbb{R}^n \setminus \{0\}$.

2. EXISTENCE OF EXTREMAL MAPS

Recall that for a domain $D \subset \mathbb{C}$ and a point $p \in D$, $M_D(p)$ is defined by (1.1). By $\text{Per}(D)$ we denote the perimeter of a bounded domain D . In this section we prove the following result.

Theorem 2.1. *Let $D \subset \mathbb{C}$ be a bounded convex domain and $p \in D$. There exists a harmonic map $f : \mathbb{D} \rightarrow D$ such that $f(0) = p$ and $\|df_0\| = M_D(p)$. Furthermore,*

$$M_D(p) \leq \frac{\text{Per}(D)}{2\pi}.$$

Proof. Let $z = x + iy$ be the complex coordinate on \mathbb{C} . Given a differentiable function f on a domain in \mathbb{C} , we denote

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Then, $df = f_z dz + f_{\bar{z}} d\bar{z}$. The condition $f_{\bar{z}}(z) = 0$ means that f is conformal orientation preserving at z , and in this case, $f_z(z) = f'(z)$ is the complex derivative of f at z .

By a translation of D we may assume that $p = 0$. Let \mathcal{A}_D denote the set of all functions $\psi \in L^\infty([0, 2\pi])$ such that

$$\psi(t) \in \bar{D} \quad \text{for a.e. } t$$

and

$$\int_0^{2\pi} \psi(t) dt = 0, \quad \int_0^{2\pi} e^{it} \psi(t) dt = 0.$$

We may think of $\psi \in \mathcal{A}_D$ as a function on the circle $\partial\mathbb{D} = \{e^{it} : t \in [0, 2\pi]\}$. Let $f = P[\psi] : \mathbb{D} \rightarrow \mathbb{C}$ denote its Poisson extension. Then f is harmonic, and the above conditions on ψ imply

$$f(0) = 0, \quad f_{\bar{z}}(0) = 0.$$

Hence df_0 is conformal and orientation preserving. Conversely, replacing $f(z)$ by $f(\bar{z})$ if necessary, every admissible harmonic map $\mathbb{D} \rightarrow D$ arises in this way.

After precomposing with a rotation of \mathbb{D} , we may assume that $f_z(0) > 0$, so

$$\|df_0\| = f_z(0).$$

Thus

$$M_D(0) = \sup_{\psi \in \mathcal{A}_D} \Re L(\psi), \quad L(\psi) := f_z(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} \psi(t) dt.$$

Let h_D be the support function of D ,

$$h_D(t) := \sup_{w \in D} \Re(e^{-it} w), \quad t \in [0, 2\pi].$$

For every $\psi \in \mathcal{A}_D$ we have $\psi(t) \in \bar{D}$ for almost every t , and hence

$$\Re(e^{-it} \psi(t)) \leq h_D(t) \quad \text{for a.e. } t.$$

Therefore,

$$\Re L(\psi) = \frac{1}{2\pi} \int_0^{2\pi} \Re(e^{-it} \psi(t)) dt \leq \frac{1}{2\pi} \int_0^{2\pi} h_D(t) dt.$$

Taking the supremum over $\psi \in \mathcal{A}_D$ gives

$$M_D(0) \leq \frac{1}{2\pi} \int_0^{2\pi} h_D(t) dt.$$

By Cauchy's perimeter formula for bounded convex planar domains (see [13]),

$$\int_0^{2\pi} h_D(t) dt = \text{Per}(D),$$

and hence

$$M_D(0) \leq \frac{\text{Per}(D)}{2\pi}.$$

Since D is bounded, \mathcal{A}_D is contained in a weak-* compact ball of L^∞ . Because \bar{D} is convex and closed, it is an intersection of closed halfplanes, so \mathcal{A}_D is weak-* closed and hence weak-* compact. Since $\Re L$ is weak-* continuous, there exists $\psi_0 \in \mathcal{A}_D$ such that

$$M_D(0) = \Re L(\psi_0).$$

(The existence of such ψ_0 can also be seen by a normal families argument.) Let $f_0 = P[\psi_0]$ denote its Poisson integral. We claim that $f_0(\mathbb{D}) \subset D$. Let

$$H_{\beta,c} := \{w \in \mathbb{C} : \Re(e^{-i\beta}w) < c\}$$

be any open supporting halfplane of D . Since $0 \in D$, we have $c > 0$. Since the a.e. boundary values of f_0 lie in $\bar{D} \subset \overline{H_{\beta,c}}$, the harmonic function

$$u(z) := \Re(e^{-i\beta}f_0(z)) - c$$

satisfies $u \leq 0$ on \mathbb{D} , while

$$u(0) = \Re(e^{-i\beta} \cdot 0) - c = -c < 0.$$

By the maximum principle, $u < 0$ on \mathbb{D} . Since this holds for every supporting halfplane of D , we conclude that

$$f_0(\mathbb{D}) \subset D.$$

Therefore f_0 is admissible and $\|df_0\| = M_D(0)$. \square

3. EXCEPTIONAL DOMAINS AND A NONCIRCULAR EXAMPLE

In this section we prove Theorem 1.1. We begin with preliminaries. We shall use the following form of the classical Schur–Cohn criterion [14, 3].

Lemma 3.1. *The quadratic holomorphic map*

$$(3.1) \quad q(z) = 1 + az + \lambda z^2, \quad a, \lambda \in \mathbb{C}$$

has no zeros on $\bar{\mathbb{D}}$ if and only if

$$(3.2) \quad |\lambda| < 1 \quad \text{and} \quad |a - \bar{a}\lambda| < 1 - |\lambda|^2.$$

Proof. Put

$$P(w) = w^2 + aw + \lambda.$$

If $z \neq 0$ and $w = 1/z$, then

$$q(z) = 0 \iff P(w) = 0.$$

Thus q has no zeros on $\bar{\mathbb{D}}$ if and only if both zeros of P lie in \mathbb{D} . For the monic quadratic polynomial $P(w) = w^2 + aw + \lambda$, the Schur–Cohn criterion is precisely (3.2). \square

Lemma 3.2. *Assume that the numbers $a, \lambda \in \mathbb{C}$ satisfy*

$$|\lambda| < 1 \quad \text{and} \quad |a - \bar{a}\lambda| \leq 1 - |\lambda|^2,$$

and let $F : \mathbb{D} \rightarrow \mathbb{C}$ be such that

$$F'(z) = \frac{c}{1 + az + \lambda z^2}, \quad c > 0.$$

Then F maps \mathbb{D} biholomorphically onto the convex domain $D = F(\mathbb{D})$. If in addition

$$|a - \bar{a}\lambda| < 1 - |\lambda|^2,$$

then $D = F(\mathbb{D})$ is a bounded strongly convex domain.

Proof. By the non-strict form of the Schur–Cohn criterion, the assumptions on a and λ imply that the polynomial $1 + az + \lambda z^2$ has no zeros in \mathbb{D} . Hence, F' is holomorphic and nonzero in \mathbb{D} . By the standard convexity criterion for holomorphic functions, it suffices to prove that

$$(3.3) \quad \Re\left(1 + z \frac{F''(z)}{F'(z)}\right) \geq 0, \quad z \in \mathbb{D}.$$

A computation gives

$$(3.4) \quad p(z) := 1 + z \frac{F''(z)}{F'(z)} = \frac{1 - \lambda z^2}{1 + az + \lambda z^2}.$$

Set

$$\omega(z) := \frac{p(z) - 1}{p(z) + 1} = -\frac{z(a + 2\lambda z)}{2 + az}.$$

The assumption $|a - \bar{a}\lambda| \leq 1 - |\lambda|^2$ implies

$$|a|(1 - |\lambda|) \leq |a - \bar{a}\lambda| \leq 1 - |\lambda|^2.$$

It follows that $|a| \leq 1 + |\lambda| < 2$ and hence $2 + az \neq 0$ for $|z| \leq 1$, thereby showing that ω is holomorphic in a neighbourhood of the closed disc $\bar{\mathbb{D}}$. We claim that

$$(3.5) \quad |\omega(z)| < 1, \quad z \in \mathbb{D}.$$

For $|z| = 1$ we have

$$|\omega(z)| \leq 1 \iff |a + 2\lambda z| \leq |2 + az|.$$

After squaring and expanding, we have

$$|2 + \bar{a}z|^2 - |a + 2\lambda z|^2 = 4(1 - |\lambda|^2) + 4\Re((a - \bar{a}\lambda)z).$$

By the assumption $|a - \bar{a}\lambda| \leq 1 - |\lambda|^2$, the right-hand side is nonnegative, and hence $|\omega(z)| \leq 1$ for $|z| = 1$. By the maximum principle, $|\omega(z)| \leq 1$ for all $z \in \bar{\mathbb{D}}$. Since $\omega(0) = 0$, the function ω is not identically equal to a unimodular constant, which implies (3.5). Therefore, the function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)}, \quad z \in \mathbb{D},$$

has positive real part in \mathbb{D} . In view of (3.4), this establishes (3.3), so F is convex in \mathbb{D} . If

$$|a - \bar{a}\lambda| < 1 - |\lambda|^2,$$

then the boundary inequality above is strict:

$$|\omega(z)| < 1, \quad |z| = 1.$$

By the standard strict convexity criterion for analytic functions, F maps \mathbb{D} conformally onto a bounded strongly convex domain. \square

Proof of Theorem 1.1. By Lemma 3.1, the assumption that $q(z) = 1 + az + \lambda z^2$ has no zeros on $\bar{\mathbb{D}}$ is equivalent to

$$|\lambda| < 1, \quad |a - \bar{a}\lambda| < 1 - |\lambda|^2.$$

In particular, F' , and hence F , extends holomorphically to a neighbourhood of $\bar{\mathbb{D}}$. By Lemma 3.2, F maps \mathbb{D} conformally onto a bounded strongly convex domain $D = F(\mathbb{D})$.

We now prove the extremal property $|F'(0)| = M_D(0)$. To this end, we must show that if $f : \mathbb{D} \rightarrow D$ is a harmonic map with $f(0) = 0$ such that df_0 is conformal, then $\|df_0\| \leq c = |F'(0)|$. If $df_0 = 0$, there is nothing to prove. If df_0 is orientation reversing, we replace $f(z)$ by $f(\bar{z})$; this does not change $\|df_0\|$. Thus, we may assume that $f_{\bar{z}}(0) = 0$. Precomposing with a rotation of \mathbb{D} , we may also assume that $f_z(0) > 0$. Then

$$\|df_0\| = f_z(0).$$

Since D is bounded, f has radial boundary values $\psi : \partial\mathbb{D} \rightarrow \bar{D}$ almost everywhere, and $f = P[\psi]$ is the Poisson integral of ψ . By the Poisson formula,

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{it}) dt = 0$$

and

$$f_z(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} \psi(e^{it}) dt, \quad f_{\bar{z}}(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{it} \psi(e^{it}) dt.$$

Therefore

$$\|df_0\| = f_z(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} \psi(e^{it}) dt.$$

Using the constraints

$$\int_0^{2\pi} \psi(e^{it}) dt = 0, \quad \int_0^{2\pi} e^{it} \psi(e^{it}) dt = 0,$$

we may add the corresponding zero terms and obtain

$$(3.6) \quad \|df_0\| = f_z(0) = \frac{1}{2\pi} \int_0^{2\pi} (e^{-it} + a + \lambda e^{it}) \psi(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} (V(t) \psi(e^{it})) dt$$

where

$$(3.7) \quad V(t) := e^{-it} + a + \lambda e^{it} = e^{-it} q(e^{it}) \neq 0, \quad t \in \mathbb{R}.$$

Here, $q(z) = 1 + az + \lambda z^2$ is as in (3.1). Recall that q has no zeros on $\bar{\mathbb{D}}$. We claim that, for every $t \in \mathbb{R}$, the real linear functional

$$w \mapsto \Re(V(t)w)$$

is uniquely maximized over \bar{D} at $w = F(e^{it})$. Indeed, (1.4) gives $q(e^{it})F'(e^{it}) = c$, and hence

$$e^{it} F'(e^{it}) = \frac{c e^{it}}{q(e^{it})}.$$

From this and (3.7) we obtain

$$\frac{\overline{V(t)}}{e^{it} F'(e^{it})} = e^{it} \overline{q(e^{it})} \frac{q(e^{it})}{c e^{it}} = \frac{1}{c} |q(e^{it})|^2 > 0.$$

Thus for every t , $\overline{V(t)}$ is a positive real multiple of $e^{it} F'(e^{it})$, which is the outward normal direction to the positively oriented boundary curve $t \mapsto F(e^{it}) \in bD$. Since D is strongly convex, the supporting functional $w \mapsto \Re(V(t)w)$ has the unique maximizer $w = F(e^{it})$ on \bar{D} as claimed. Therefore, for almost every t ,

$$\Re(V(t)\psi(e^{it})) \leq \Re(V(t)F(e^{it})).$$

In view of (3.6), it follows that

$$(3.8) \quad \|df_0\| \leq \frac{1}{2\pi} \Re \int_0^{2\pi} V(t) F(e^{it}) dt.$$

Note that

$$\int_0^{2\pi} F(e^{it}) dt = 2\pi F(0) = 0.$$

Also, since F is holomorphic in a neighbourhood of $\bar{\mathbb{D}}$,

$$\int_0^{2\pi} e^{it} F(e^{it}) dt = 0.$$

Hence

$$\frac{1}{2\pi} \Re \int_0^{2\pi} V(t) F(e^{it}) dt = \frac{1}{2\pi} \Re \int_0^{2\pi} e^{-it} F(e^{it}) dt = F'(0) = c.$$

From this and (3.8) we conclude that

$$\|df_0\| \leq c.$$

Since F itself is admissible and $\|dF_0\| = F'(0) = c$, it follows that

$$M_D(0) = c.$$

Equality $\|df_0\| = c$ can hold only if the boundary value function ψ of f satisfies

$$\psi(e^{it}) = F(e^{it}) \quad \text{for a.e. } t,$$

because, for every t , the functional $w \mapsto \Re(V(t)w)$ has a unique maximizer $w = F(e^{it})$ over \overline{D} . Hence, by uniqueness of the Poisson extension, the only extremal satisfying

$$f_{\bar{z}}(0) = 0, \quad f_z(0) > 0$$

is $f = F$. Undoing rotations of the source, the orientation preserving extremals are precisely

$$F \circ R_\theta, \quad R_\theta(z) = e^{i\theta}z, \quad \theta \in \mathbb{R}.$$

If orientation reversing conformal differentials are also allowed, then applying the preceding argument to $f(\bar{z})$ gives the additional extremals $z \mapsto F(e^{i\theta}\bar{z})$. \square

As an immediate consequence, we obtain a negative answer to [8, Problem 4.3].

Corollary 3.3. *There exists a bounded strongly convex domain $0 \in D \subset \mathbb{C}$, which is not a disc, such that the supremum of $\|df_0\|$ over all harmonic maps*

$$f : \mathbb{D} \rightarrow D, \quad f(0) = 0,$$

whose differential is conformal at 0, is attained by a biholomorphic map $F : \mathbb{D} \rightarrow D$ with $F(0) = 0$. Moreover, among orientation preserving extremals, F is unique up to precomposition by rotations of \mathbb{D} ; after the normalization $F_z(0) > 0$, it is unique.

Proof. Choose

$$\lambda = \frac{1}{2}, \quad a = 0, \quad c = 1$$

in Theorem 1.1. Then

$$F(0) = 0, \quad F'(z) = \frac{1}{1 + z^2/2},$$

and hence

$$(3.9) \quad F(z) = \sqrt{2} \arctan\left(\frac{z}{\sqrt{2}}\right).$$

By Theorem 1.1, the image $D = F(\mathbb{D})$ is a bounded strongly convex domain, and $M_D(0) = |F'(0)| = 1$. Moreover, the extremals are precisely those described in Theorem 1.1. If D were a disc, then the map $F : \mathbb{D} \rightarrow D$ would be a Möbius transformation. Since the map (3.9) is not a Möbius transformation, D is not a disc. \square

4. BOUNDED CONVEX DOMAINS: CLASSIFICATION OF EQUALITY

In this section we prove Theorem 1.2.

Proof. Let $\Phi : \mathbb{D} \rightarrow D$ be a biholomorphic map onto a bounded convex domain $D \subset \mathbb{C}$ with $\Phi(0) = 0$. Rotating the target $D = \Phi(\mathbb{D})$, we may assume that $\Phi'(0) > 0$. Our goal is to prove that Φ equals a map F in (1.4) from Theorem 1.1.

Since D is bounded and convex, ∂D is a rectifiable Jordan curve. Hence, Φ extends homeomorphically to $\overline{\mathbb{D}}$, $\Phi' \in H^1$, and its boundary value function

$$(4.1) \quad \varphi(t) = \psi_0(e^{it}) := \Phi(e^{it}), \quad t \in \mathbb{R},$$

is absolutely continuous. Moreover,

$$(4.2) \quad \dot{\varphi}(t) = ie^{it}\Phi'(e^{it}) \quad \text{for a.e. } t.$$

We formulate the extremal problem in terms of the boundary function. Let

$$(4.3) \quad K := \left\{ \psi \in L^\infty(\partial\mathbb{D}, \mathbb{C}) : \psi(e^{it}) \in \overline{D} \text{ for a.e. } t \right\}.$$

For $\psi \in K$, define

$$A_0(\psi) := \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{it}) dt, \quad A_1(\psi) := \frac{1}{2\pi} \int_0^{2\pi} e^{it}\psi(e^{it}) dt,$$

and

$$J(\psi) := \Re \frac{1}{2\pi} \int_0^{2\pi} e^{-it}\psi(e^{it}) dt.$$

Let

$$K_0 := \{\psi \in K : A_0(\psi) = 0, A_1(\psi) = 0\}.$$

Note that K_0 equals the set \mathcal{A}_D introduced in Sect. 2. If $\psi \in K_0$, its Poisson extension $f = P[\psi] : \mathbb{D} \rightarrow \mathbb{C}$ satisfies $f(0) = 0$, $f_{\bar{z}}(0) = 0$, and hence df_0 is an orientation preserving conformal linear map. Moreover,

$$f_z(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} \psi(e^{it}) dt.$$

By the argument in the proof of Theorem 2.1, we have $f(\mathbb{D}) \subset D$ since D is convex.

Let Φ be as above. Its boundary map $\psi_0 := \Phi|_{\partial\mathbb{D}}$ belongs to K_0 , and

$$J(\psi_0) = \Phi'(0).$$

Since $M_D(0) = \Phi'(0)$, the function ψ_0 maximizes J over K_0 .

We now apply a finite dimensional separation argument. Note that K is convex because \bar{D} is convex: if $\psi_1, \psi_2 \in K$ and $0 \leq \theta \leq 1$, then

$$\theta\psi_1(e^{it}) + (1-\theta)\psi_2(e^{it}) \in \bar{D} \quad \text{for a.e. } t.$$

Moreover, A_0, A_1 , and J are real linear functionals on K . Hence the image

$$\mathcal{S} = \{(A_0(\psi), A_1(\psi), J(\psi)) : \psi \in K\} \subset \mathbb{C}^2 \times \mathbb{R}$$

is convex in $\mathbb{C}^2 \times \mathbb{R}$. Let

$$s_0 := (0, 0, J(\psi_0)).$$

Since ψ_0 maximizes J on K_0 , the open vertical ray

$$(4.4) \quad \mathcal{R} := \{(0, 0, s) : s > J(\psi_0)\}$$

is disjoint from \mathcal{S} . Moreover, $(0, 0) \in \mathbb{C}^2$ is an interior point of the projection of \mathcal{S} to \mathbb{C}^2 . Indeed, since $0 \in D$, there is $\delta > 0$ such that $\{w \in \mathbb{C} : |w| < \delta\} \subset D$. Hence, whenever $|u| + |v| < \delta$, the function $\psi_{u,v}(e^{it}) = u + ve^{-it}$ belongs to K , and $A_0(\psi_{u,v}) = u$, $A_1(\psi_{u,v}) = v$.

Since \mathcal{S} is convex and the convex set \mathcal{R} in (4.4) is disjoint from \mathcal{S} , the finite dimensional separation theorem (see e.g. [1, Sect. 2.5]) gives a nonzero real linear functional

$$(4.5) \quad \ell(z_0, z_1, s) = \Re(\alpha z_0 + \beta z_1) + \mu s$$

on $\mathbb{C}^2 \times \mathbb{R}$ and a real number γ such that

$$\ell(x) \leq \gamma \leq \ell(y) \quad (x \in \mathcal{S}, y \in \mathcal{R}).$$

Since $s_0 = (0, 0, J(\psi_0)) \in \mathcal{S}$, we have

$$\ell(s_0) \leq \gamma.$$

On the other hand, letting $y = (0, 0, s) \in \mathcal{R}$ and then $s \downarrow J(\psi_0)$, we obtain

$$\gamma \leq \ell(s_0).$$

Therefore $\ell(s_0) = \gamma$, so the separating hyperplane supports \mathcal{S} at s_0 .

We claim that $\mu > 0$ in (4.5). First, since

$$\ell(0, 0, s) = \mu s \geq \gamma \quad (s > J(\psi_0)),$$

we must have $\mu \geq 0$. If $\mu = 0$, then ℓ depends only on (z_0, z_1) . But 0 is an interior point of the projection of \mathcal{S} onto \mathbb{C}^2 , and a nonzero real linear functional cannot support a set at an interior point of its projection. Hence $\alpha = \beta = 0$, contradicting the fact that ℓ is nonzero. Thus $\mu > 0$ as claimed. Dividing by μ , we obtain constants $a, \lambda \in \mathbb{C}$ such that ψ_0 maximizes the functional

$$K \ni \psi \mapsto \Re \frac{1}{2\pi} \int_0^{2\pi} (e^{-it} + a + \lambda e^{it}) \psi(e^{it}) dt = \Re \frac{1}{2\pi} \int_0^{2\pi} V(t) \psi(e^{it}) dt$$

where

$$V(t) := e^{-it} + a + \lambda e^{it}$$

is as in (3.7). Note that

$$(4.6) \quad q(e^{it}) := e^{it} V(t) = 1 + a e^{it} + \lambda e^{2it}$$

is a nonzero quadratic polynomial in e^{it} , and hence $V(t)$ has at most two zeros on $t \in [0, 2\pi)$. Since the set K in (4.3) is defined by pointwise conditions $\psi(e^{it}) \in \overline{D}$ for a.e. t , this integral maximality of ψ_0 implies, for almost every t , the pointwise support condition

$$\Re(V(t)w) \leq \Re(V(t)\psi_0(e^{it})) \quad (w \in \overline{D}).$$

Since V , φ , and the support function of \overline{D} are continuous, this inequality holds for all t . Writing $\varphi(t) = \psi_0(e^{it})$ (see (4.1)), the support condition is equivalent to

$$\Re(V(t)(w - \varphi(t))) \leq 0 \quad (w \in \overline{D}).$$

Taking $w = \varphi(t + h)$ and using both positive and negative values of h , at every point t where φ is differentiable (which holds for almost every t , see (4.2)) we get

$$\Re(V(t)\dot{\varphi}(t)) = 0.$$

In view of (4.2), this gives

$$\Re(V(t)ie^{it}\Phi'(e^{it})) = 0 \quad \text{for a.e. } t.$$

By (4.6), this is equivalent to

$$(4.7) \quad \Im\left((1 + ae^{it} + \lambda e^{2it})\Phi'(e^{it})\right) = 0 \quad \text{for a.e. } t.$$

Set

$$G(z) := (1 + az + \lambda z^2)\Phi'(z) = q(z)\Phi'(z).$$

Since ∂D is rectifiable, we have $\Phi' \in H^1$ and hence $G \in H^1$. By (4.7), the harmonic function $\Im G$ has nontangential boundary values equal to 0 almost everywhere. By uniqueness for harmonic functions in h^1 , $\Im G \equiv 0$. Hence G is constant, $G(z) \equiv c = G(0) = \Phi'(0) > 0$, and

$$\Phi'(z) = \frac{c}{q(z)} = \frac{c}{1 + az + \lambda z^2}.$$

Since $q\Phi' = c > 0$ in \mathbb{D} , the polynomial q has no zeros in \mathbb{D} . We claim that q has no zeros on $\partial\mathbb{D}$ either. Suppose to the contrary that $q(\zeta) = 0$ for some $\zeta \in \partial\mathbb{D}$. Since q has degree at most two, ζ is either a simple or a double zero. If ζ is a simple zero, then $c/q(z)$ has a simple pole at ζ . Integrating along the radius $z = r\zeta$, $r \uparrow 1$, gives logarithmic growth of $\Phi(r\zeta)$, contradicting the boundedness of the domain $\Phi(\mathbb{D}) = D$. If ζ is a double zero, then $c/q(z)$ has a double pole at ζ . Integrating along the same radius gives a pole growth of $\Phi(r\zeta)$, again contradicting the boundedness of D . Thus, q has no zeros on $\overline{\mathbb{D}}$, and Φ is of the form (1.4) in Theorem 1.1. \square

Remark 4.1. No smoothness assumption is made in Theorem 1.2; smoothness and strict convexity are deduced from the equality assumption.

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