ON THE FORNÆSS–HENKIN EMBEDDING THEOREM

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ABSTRACT. We prove that under suitable conditions on pseudoconvex domains $D \Subset X$ and $\Omega \Subset Y$ in Stein manifolds the following holds. Given a continuous map $f : X \to Y$ which is an injective holomorphic immersion on a neighbourhood of $\bar{D}$ and satisfies $f(D) = f(X) \cap \Omega$, there are a proper holomorphic embedding $F : X \hookrightarrow Y$ which approximates f on $\bar{D}$ and a small biholomorphic deformation $\Omega' \subset Y$ of Ω such that $F(D) = F(X) \cap \Omega'$.

A domain D with $\mathscr{C}^2$ boundary in a complex manifold X is said to be *strongly pseudoconvex* if every point $p \in bD$ has a coordinate neighbourhood in X in which bD is strongly geometrically convex near p. Equivalently, there are a neighbourhood $U \subset X$ of $\bar{D}$ and a $\mathscr{C}^2$ defining function $\rho : U \to \mathbb{R}$ such that $D = \{x \in U : \rho(x) < 0\}$, $d\rho_x \neq 0$ for all $x \in bD$, and ρ is strongly plurisubharmonic on a neighbourhood of bD. (This means that $dd^c\rho > 0$, where $d^c = -d \circ J$ is the conjugate differential and $J : TX \to TX$ is the almost complex structure operator.) If X is Stein then ρ can be chosen strongly plurisubharmonic on a neighbourhood of $\bar{D}$. For such D, the closure $\bar{D}$ is holomorphically convex in X (also called $\mathscr{O}(X)$-convex) if and only if there is a strongly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$ for which the above hold. Holomorphic convexity of the closure is a $\mathscr{C}^2$-stable property for relatively compact strongly pseudoconvex domains in Stein manifolds [16, Corollary 1.4].

The following classical embedding theorem is due to Fornæss [9, Theorem 9, p. 543] and Henkin [23, p. 668]; see also [15, Theorem 1.2] for domains with real analytic ($\mathscr{C}^\omega$) boundaries.

Theorem 1. *Let D be a relatively compact, strongly pseudoconvex domain in a Stein manifold X whose boundary bD is of class $\mathscr{C}^k$ $(k \in \{2, 3, \dots, \infty, \omega\})$ and whose closure $\bar{D}$ is holomorphically convex in X. There exist an integer N, a proper holomorphic embedding $F : X \hookrightarrow \mathbb{C}^N$, and a bounded strongly convex domain $\Omega \subset \mathbb{C}^N$ with $\mathscr{C}^k$ boundary such that*

$$F(D) = F(X) \cap \Omega \ \text{ and } F \text{ intersects } b\Omega \text{ transversely along } bD. \tag{1}$$

The same holds if X is a Stein space with bounded embedding dimension and $bD \cap X_{\mathrm{sing}} = \varnothing$.

The last statement in (1) follows from the Hopf–Oleĭnik lemma [24, 28]. The domain Ω in this theorem necessarily depends on D since a generic smoothly bounded strongly pseudoconvex domain does not embed properly holomorphically and smoothly up to the boundary in any fixed domain of the same kind; see [15, Theorem 1.1] and [11]. There are many constructions of proper holomorphic maps and embeddings of (strongly) pseudoconvex domains into model domains in Euclidean spaces and in other complex manifolds; see [27, 15, 21, 29, 22, 5, 6, 7, 8, 12], among others. In some of these results, the map extends continuously up to the boundary.

The following question appeared to me. Suppose that $D \Subset X$ and $\Omega \Subset \mathbb{C}^N$ are as in Theorem 1, $U \subset X$ is an open neighbourhood of $\bar{D}$, and $f : U \to \mathbb{C}^N$ is an injective holomorphic immersion satisfying $f(D) = f(U) \cap \Omega$. Since $\bar{D}$ is $\mathscr{O}(X)$-convex, the Oka–Weil theorem shows that f can be approximated uniformly on $\bar{D}$ by entire maps $F : X \to \mathbb{C}^N$. Can we choose F such that (1) holds?

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It turns out that this is false in general; here is an example. Recall that a proper holomorphic map $f : \mathbb{B}^n \to \mathbb{B}^N$ $(1 < n \le N)$ between balls in respective Euclidean spaces, which extends as a map of class $\mathscr{C}^{N-n+1}$ to a neighbourhood of a point $p \in b\mathbb{B}^n$ in $\overline{\mathbb{B}}^n$, is a rational map [10, Theorem 1.3]. (The case $n = N$ is due to Alexander [1].) Since a bounded Stein domain $X \subset \mathbb{C}^n$ cannot be mapped properly to $\mathbb{C}^N$ by a rational map, the answer to the question is negative for any such pair $\mathbb{B}^n \subset X$.

On the other hand, the answer is affirmative if we allow a small biholomorphic deformation of the target domain $\Omega \subset \mathbb{C}^N$. The following is a corollary to Theorem 3.

Corollary 2. *Suppose that $D \Subset X$ and $\Omega \Subset \mathbb{C}^N$ are as in Theorem 1, $U \subset X$ is an open neighbourhood of $\bar{D}$, and $f : U \to \mathbb{C}^N$ is an injective holomorphic immersion satisfying $f(D) = f(U) \cap \Omega$. Assuming that $N > 2 \dim X$, there are a proper holomorphic embedding $F : X \hookrightarrow \mathbb{C}^N$ approximating f on $\bar{D}$, a neighbourhood $\Omega' \subset \mathbb{C}^N$ of $\overline{\Omega}$, and a biholomorphic map $\Phi : \Omega' \to \Phi(\Omega') \subset \mathbb{C}^N$ close to the identity on Ω' such that $F(D) = F(X) \cap \Phi(\Omega)$.*

We now explain the setup for our main theorem. Recall that a Stein manifold Y satisfies the holomorphic *density property* if the Lie algebra generated by all the $\mathbb{C}$-complete holomorphic vector fields on Y is dense in the Lie algebra of all holomorphic vector fields on X in the compact-open topology. (See Varolin [32, 31] or [17, Sect. 4.10].) There is an analogous notion of the algebraic density property of an affine algebraic manifold, asking that the Lie algebra generated by the $\mathbb{C}$-complete algebraic vector fields agrees with the Lie algebra of all algebraic vector fields. Clearly, the algebraic density property of an affine manifold implies the holomorphic density property. Similarly one defines the *volume density property* of a Stein manifold, endowed with a holomorphic volume form ω, by considering the Lie algebra of all holomorphic vector fields on Y with vanishing ω-divergence; their flows can be approximated by ω-preserving holomorphic automorphisms of Y. These properties were first discovered on Euclidean spaces of dimension > 1 by Andersén and Lempert [2, 3], and they holds for many other Stein manifolds and volume forms; see the recent survey [14]. They imply that the holomorphic automorphism group of Y is very large; in particular, it acts infinitely transitively on Y. One of the main results is that isotopies of pseudoconvex Runge domains in Y can be approximated by isotopies of holomorphic automorphisms of Y; see [20] and [17, Theorem 4.10.5].

We denote by $\mathscr{O}(X)$ the algebra of holomorphic functions on a complex manifold X. A compact set K in a Stein manifold X is said to be holomorphically convex, or $\mathscr{O}(X)$-convex, if for every point $x_0 \in X \setminus K$ there exists $g \in \mathscr{O}(X)$ such that $|g(x_0)| > \sup_{x \in K} |g(x)|$. (See Stout [30, Sect. 6.2].) The interior of such a set is pseudoconvex (equivalently, Stein); see [30, p. 27] or [25, Corollary 2.5.7], and for any open neighbourhood $U \subset X$ of K there is a strongly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$ such that $\rho < 0$ on K and $\rho > 0$ on $X \setminus U$ (see [25, Theorem 5.1.6]). It follows that K admits a basis of strongly pseudoconvex neighbourhoods with compact $\mathscr{O}(X)$-convex closures.

We prove the following result.

Theorem 3. *Assume that X and Y are Stein manifolds such that $2 \dim X < \dim Y$ and Y has the density property, $D \Subset X$ and $\Omega \Subset Y$ are domains with $\mathscr{C}^1$ boundaries and holomorphically convex closures, and $f : X \to Y$ is a continuous map which is an injective holomorphic immersion on a neighbourhood $U \subset X$ of $\bar{D}$ such that $f(D) = f(X) \cap \Omega$ and f is transverse to $b\Omega$ along bD. Then, there are a proper holomorphic embedding $F : X \hookrightarrow Y$ which approximates f on $\bar{D}$, a neighbourhood $\Omega' \subset Y$ of $\overline{\Omega}$, and a biholomorphic map $\Phi : \Omega' \to \Phi(\Omega') \subset Y$ close to the identity on Ω' such that*

(a) *$F(D) = F(X) \cap \Phi(\Omega)$,*
(b) *F intersects $\Phi(b\Omega)$ transversely along bD, and*
(c) *the domain $\Phi(\overline{\Omega})$ is $\mathscr{O}(Y)$-convex.*

As pointed out above, the conclusion is false in general with Φ the identity map.

Note that Theorem 3 includes Corollary 2 as a special case. Indeed, since $\overline{\Omega} \subset \mathbb{C}^N$ is a compact convex set, it is $\mathscr{O}(\mathbb{C}^N)$-convex, and a map f in Corollary 2 clearly extends from a neighbourhood of $\bar{D}$ to a continuous map $f : X \to \mathbb{C}^N$ satisfying $f(D) = f(X) \cap \Omega$, so Theorem 3 applies.

The proof of Theorem 3 relies on a nontrivial result from the theory of Stein manifolds with the density property; see Theorem 4 due to Andrist, Ritter, Wold and the author [4]. The second tool used in the proof is an extension lemma for deformations of holomorphic embeddings into a Stein manifold to ambient holomorphic deformations; see Lemma 6.

Proof of Theorem 3. Let $f : X \to Y$ and $U \supset \bar{D}$ be as in the theorem. Since $\bar{D}$ is $\mathscr{O}(X)$-convex, there is a compact $\mathscr{O}(X)$-convex subset $K \subset U$ containing $\bar{D}$ in its interior, and hence $f(bK) \cap \overline{\Omega} = \varnothing$. (We can choose K to be the closure of a strongly pseudoconvex domain.) Pick a compact $\mathscr{O}(Y)$-convex subset $L \subset Y$ satisfying

$$\overline{\Omega} \subset \mathring{L} = L \setminus bL \ \text{ and } \ f(bK) \cap L = \varnothing. \tag{2}$$

We shall use the following result, which is a special case of [4, Theorem 1.2]. (For $Y = \mathbb{C}^N$ see also [19, Theorem 15].)

Theorem 4. *Assume that Y is a Stein manifold satisfying the density property or the volume density property, $L \subset Y$ is a compact $\mathscr{O}(Y)$-convex set, X is a Stein manifold with $2\dim X < \dim Y$, $K \subset X$ is a compact $\mathscr{O}(X)$-convex subset, and $f : X \to Y$ is a continuous map which is holomorphic on a neighbourhood of K and satisfies $f(\overline{X \setminus K}) \subset Y \setminus L$. Let dist_Y be a distance function on Y inducing the manifold topology. Given $\epsilon > 0$ there is a proper holomorphic embedding $F : X \hookrightarrow Y$ such that*

$$\sup_{x \in K} \mathrm{dist}_Y(F(x), f(x)) < \epsilon \ \ \text{and} \ \ F(\overline{X \setminus K}) \subset Y \setminus L. \tag{3}$$

Remark 5. This is an Oka-theoretic result whose proof could now be somewhat simplified by using the recent result of Kusakabe [26, Theorem 1.2] saying that, if Y is a Stein manifold with the density property and $L \subset Y$ is a compact holomorphically convex subset then the complement $Y \setminus L$ is an Oka manifold. This means in particular that any map $f : X \to Y$ as in Theorem 3, mapping $\overline{X \setminus K}$ to $Y \setminus L$, can be approximated on K by holomorphic maps $X \to Y$ with the same property (see [13, Theorem 1.3]). The Oka property of $Y \setminus L$ does not suffice to construct proper maps $X \to Y$, and one must use that Y has the density or volume density property as explained in [4, proof of Theorem 1.2].

With K and L as in (2), pick strongly pseudoconvex domains

$$\Omega_1 \Subset \Omega_2 \Subset Y \tag{4}$$

such that

$$L \subset \Omega_1 \quad \text{and} \quad f(bK) \cap \overline{\Omega}_2 = \varnothing. \tag{5}$$

Shrinking U around K we may assume that $f(U \setminus \mathring{K}) \cap \overline{\Omega}_2 = \varnothing$. It follows that

$$V := f(U) \cap \Omega_2 \tag{6}$$

is a closed complex submanifold of Ω_2 whose closure $\overline{V}$ is contained in the relative interior of $f(K)$ in $f(U)$, and we have $V = f(U')$ where $U' \subset X$ is an open neighbourhood of $\bar{D}$ and $\overline{U}' \subset \mathring{K}$.

Fix a distance function dist_Y on Y inducing the manifold topology and a number $\epsilon > 0$. Let $F = f_1 : X \to Y$ be a proper holomorphic embedding given by Theorem 4. Since $f|_U : U \to Y$ is an injective holomorphic immersion, the map

$$\phi = F \circ (f|_{U'})^{-1} : f(U') = V \to Y \tag{7}$$

is well defined, holomorphic, and it satisfies

$$F = \phi \circ f \ \text{ on } U' \quad \text{and} \quad \mathrm{dist}_Y(\phi(y), y) < \epsilon \ \text{ for all } y \in V. \tag{8}$$

The estimate follows from (3).

To complete the proof of the theorem, we will apply the following lemma with the map ϕ in (7) and the domain Ω_1 in (4).

Lemma 6. *Let Y be a Stein manifold, $\Omega \Subset Y$ be a domain with $\mathscr{C}^1$ boundary whose closure $\overline{\Omega}$ has a basis of Stein neighbourhoods, and $V \subset Y$ be a locally closed complex submanifold which intersects $b\Omega$ transversely and such that $V \cap \overline{\Omega}$ is compact. There are constants $C > 0$ and $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ and for any holomorphic map $\phi : V \to Y$ satisfying $\mathrm{dist}_Y(\phi(y), y) < \epsilon$ for all $y \in V$ there exists an injective holomorphic map $\Phi : \Omega \to \Phi(\Omega) \subset Y$ satisfying*

$$\Phi = \phi \text{ on } V \cap \Omega \quad \text{and} \quad \mathrm{dist}_Y(\Phi(y), y) < C\epsilon \text{ for all } y \in \Omega.$$

Proof. We begin by explaining the case when $Y = \mathbb{C}^N$. Denote by $H^\infty(\Omega)$ the Banach space of bounded holomorphic functions on a complex manifold Ω with the sup norm. Pick pseudoconvex domains $\Omega_1 \Subset \Omega_2 \Subset \mathbb{C}^N$ such that $\overline{\Omega} \subset \Omega_1$ and $V \cap \overline{\Omega}_2$ is compact. By [18, Lemma 3.1] there is a bounded linear extension operator $S : H^\infty(V) \to H^\infty(\Omega_1)$ satisfying

$$(Sg)(x) = g(x) \text{ for all } g \in H^\infty(V) \text{ and } x \in V \cap \Omega_1.$$

For a vector valued function $g = (g_1, \ldots, g_N) : V \to \mathbb{C}^N$ we set $Sg = (Sg_1, \ldots, Sg_N)$. Denote by $c > 0$ the norm of S. Let $\phi : V \to \mathbb{C}^N$ be as in the lemma. The map

$$\Phi = S(\phi|_V) : \Omega_1 \to \mathbb{C}^N$$

then satisfies $\|\Phi - \mathrm{Id}_{\Omega_1}\|_\infty < c\epsilon$. By Cauchy estimates, there is a constant $C > 0$ such that $\|\Phi - \mathrm{Id}_\Omega\|_{\mathscr{C}^1(\Omega)} < C\epsilon$. Assuming that $\epsilon > 0$ is small enough, it follows that $\Phi : \Omega \to Y$ is a biholomorphic map onto its image.

Consider now the general case. Pick a relatively compact Stein domain $\Omega_2 \Subset Y$ with $\overline{\Omega} \subset \Omega_2$ such that $V \cap \overline{\Omega}_2$ is compact. By Cartan's Theorem A there exists finitely many holomorphic vector fields $V_1, \ldots, V_N$ on Y which generate the tangent space to Y at every point $y \in \overline{\Omega}_2$. Let $\theta_{i,t}$ denote the flow of V_i for time $t \in \mathbb{C}$. There is a number $\delta > 0$ such that the flow $\theta_{i,t}(y)$ exists for all $y \in \Omega_2$ and $|t| < \delta$. Hence, there is a ball $0 \in B \subset \mathbb{C}^N$ such that the map $\Theta : \Omega_2 \times B \to Y$ given by

$$\Theta(y, t_1, \ldots, t_N) = \theta_{1,t_1} \circ \theta_{2,t_2} \circ \cdots \circ \theta_{N,t_N}(y), \quad y \in \Omega_2,\ t = (t_1, \ldots, t_N) \in B$$

is well defined, and it is obviously holomorphic. Note that

$$\frac{\partial}{\partial t_i}\Theta(y,t)\Big|_{t=0} = V_i(y), \quad i = 1, \ldots, N.$$

Since the vector fields V_i span the tangent bundle of Y at every point of Ω_2, it follows that the partial differential

$$\frac{\partial}{\partial t}\Theta(y,t)\Big|_{t=0} : \mathbb{C}^N \to T_yY$$

is surjective for every $y \in \Omega_2$. By Cartan's Theorem B, we have $\Omega_2 \times \mathbb{C}^N = E \oplus E'$ where E and E' are holomorphic vector subbundles of the trivial bundle $\Omega_2 \times \mathbb{C}^N$ such that E' is the kernel of the map $\frac{\partial}{\partial t}\Theta(\cdot\,, t)\Big|_{t=0}$. It follows that there is a neighbourhood $E_0 \subset E$ of the zero section of E such that the restriction $\Theta|_{E_0} : E_0 \to Y$ is well defined and fibrewise biholomorphic onto its image. Therefore, for any holomorphic map $\phi : V \to Y$ satisfying $\mathrm{dist}_Y(\phi(y), y) < \epsilon$ for $y \in V$ and a small enough $\epsilon > 0$ there is a unique holomorphic section $\tilde{\phi}$ of $E|_{V \cap \Omega_1}$ such that

$$\Theta(y, \tilde{\phi}(y)) = \phi(y) \quad \text{for all } y \in V \cap \Omega_1. \tag{9}$$

Considering E as a subbundle of the trivial bundle $\Omega_2 \times \mathbb{C}^N$, we may view $\tilde{\phi}$ as a map $\tilde{\phi} : V \cap \Omega_1 \to \mathbb{C}^N$ satisfying $|\tilde{\phi}| < c\epsilon$ for some constant $c > 0$. This reduces the proof to the special case considered above. Indeed, taking a smaller domain $\Omega_0 \Subset \Omega_1$ with $\overline{\Omega} \subset \Omega_0$, [18, Lemma 3.1] gives a holomorphic extension $\widetilde{\Phi} : \Omega_0 \to \mathbb{C}^N$ of $\tilde{\phi}|_{V\cap\Omega_1}$ which is close to the zero map. If the approximation is close enough then the map $\Phi : \Omega \to Y$ given by $\Phi(y) = \Theta(y, \widetilde{\Phi}(y)) \in Y$ $(y \in \Omega)$ is injective holomorphic and close to the identity. Note that $\Phi|_{V\cap\Omega} = \phi$ in view of (9). □

The proof is now completed as follows. Let $F : X \to Y$ be a proper holomorphic embedding as above, given by Theorem 4, which approximates f on K and satisfies (3). Also, let $\Omega_1 \Subset \Omega_2$ be domains as in (4) such that (5) holds, V be defined by (6), and ϕ be defined by (7) so that (8) holds. Assuming that F approximates f sufficiently closely on K, Lemma 6 applied to ϕ furnishes a biholomorphic map $\Phi : \Omega_1 \to \Phi(\Omega_1) \subset Y$, close to the identity on Ω_1, such that $\Phi = \phi$ on $V \cap \Omega_1$ and $\Phi(\overline{\Omega}) \subset L$ in view of (2). Since V intersects $b\Omega$ transversely, we have $\phi(V) \cap \Phi(\Omega) = \phi(V \cap \Omega)$ provided that the approximations are close enough. From this and (3) we see that the pair (F, Φ) satisfies conditions (a) and (b) in the theorem. Finally, condition (c) holds provided that the map Φ is close enough to the identity on Ω_1. Indeed, since $\overline{\Omega}$ is $\mathscr{O}(Y)$-convex, it has a smoothly bounded strongly pseudoconvex neighbourhood $\Omega' \Subset \Omega_1$ with $\mathscr{O}(Y)$-convex closure. If Φ is close enough to Id_{Ω_1} then $\Phi(\overline{\Omega}')$ is $\mathscr{O}(Y)$-convex by [16, Corollary 1.4]. Since $\overline{\Omega}$ is $\mathscr{O}(Y)$-convex (and hence also $\mathscr{O}(\Omega_1)$-convex) and the map $\Phi : \Omega_1 \to \Phi(\Omega_1)$ is biholomorphic, $\Phi(\overline{\Omega})$ is $\mathscr{O}(\Phi(\Omega_1))$-convex. Finally, since $\Phi(\overline{\Omega}) \subset \Phi(\overline{\Omega}') \subset \Phi(\Omega_1)$ and $\Phi(\overline{\Omega}')$ is $\mathscr{O}(Y)$-convex, it follows that $\Phi(\overline{\Omega})$ is $\mathscr{O}(Y)$-convex. □

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