Projective Manifolds Covered by Stein Domains with the Holomorphic Density Property

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The AI (Chat GPT 5.6 think deeper) was asked the following question.

In the paper [5] it is proved that the unit disc bundle in a semipositively curved ample line bundle over a flag manifold X is an Oka domain. The result is based on an important property of flag manifolds that they are covered by Zariski open domains, each of which is a Stein manifold with the holomorphic density property. I would like to know whether there is a bigger class of projective manifolds X having the latter property. Let me know if you find something useful on this topic.

Here is unedited reply by the AI. I am posting this here, should anyone be interested.

A natural and genuinely larger class for which the required hypothesis holds is the class of smooth projective varieties covered by affine spaces. More generally, one can replace affine spaces by suitable smooth affine varieties with the algebraic density property. The main caveat is that the substantially larger classes of uniformly rational or spherical varieties are not presently known, in general, to possess such a covering. The discussion below concerns the geometric hypothesis used in the paper [5], namely that the compact base manifold is covered by Zariski-open subsets which are Stein manifolds with the holomorphic density property.

1 The immediate enlargement: projective varieties covered by affine spaces

Let X be a smooth projective n-fold, with $n \geq 2$, such that every point $x \in X$ admits a Zariski-open neighbourhood

$$x \in U_x \subset X, \qquad U_x \cong \mathbb{A}^n_{\mathbb{C}}.$$

Such varieties are often called $\mathbb{A}^n$*-covered varieties*. They are also described as varieties of class $\mathcal{A}_0$ in some of the literature on uniformly rational varieties. Since

$$\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$$

has the holomorphic density property for $n \geq 2$, every chart U_x is Stein and has the density property. Consequently, every smooth projective $\mathbb{A}^n$-covered variety has precisely the covering property used in the proof of the Oka-tube theorem. Thus one obtains the following immediate observation.

Proposition 1. *Let X be a smooth projective manifold of dimension $n \geq 2$. Assume that X is covered by Zariski-open subsets biholomorphic to $\mathbb{C}^n$. Then X is covered by Zariski-open Stein domains with the holomorphic density property.*

This is not merely a formal generalization. There is now a substantial algebraic-geometric literature on varieties covered by affine spaces. A particularly relevant reference is the paper of Arzhantsev, Kaliman, and Zaidenberg [1]. Their work studies projective varieties covered by affine spaces as a subclass of uniformly rational projective varieties and establishes algebraic ellipticity results for them and for their affine cones. It also discusses smooth projective spherical varieties, including toric varieties. Although the principal results of [1] concern algebraic ellipticity rather than the density property of the affine charts, the paper provides useful terminology, examples, and permanence properties for the class which is relevant here.

2 Examples beyond flag manifolds

2.1 Smooth complete toric varieties

Smooth complete toric varieties are covered by their torus-invariant affine charts. If X is a smooth n-dimensional toric variety, then the affine chart corresponding to a smooth maximal cone is isomorphic to $\mathbb{A}^n$. Hence every smooth projective toric variety is $\mathbb{A}^n$-covered. It follows that every smooth projective toric variety of dimension at least two has the desired covering by Stein domains with the holomorphic density property. This gives a substantial enlargement beyond flag manifolds. It includes, for example,

- all smooth projective toric surfaces;
- arbitrary smooth projective toric blowups;
- products of projective spaces;
- toric projective bundles;
- many higher-dimensional toric Fano varieties.

The paper [1] explicitly singles out toric varieties as examples within the broader setting of smooth projective spherical varieties.

2.2 Products

If X and Y are covered by affine spaces, then $X \times Y$ is also covered by affine spaces. Indeed, if

$$U \cong \mathbb{A}^{\dim X} \quad \text{and} \quad V \cong \mathbb{A}^{\dim Y},$$

then

$$U \times V \cong \mathbb{A}^{\dim X} \times \mathbb{A}^{\dim Y} \cong \mathbb{A}^{\dim X + \dim Y}.$$

Thus products provide many additional examples.

2.3 Projectivized vector bundles

Projectivized vector bundles can also be $\mathbb{A}^n$-covered when the vector bundle is suitably trivial over the affine charts of the base. More precisely, suppose that X is covered by open subsets $U_\alpha \cong \mathbb{A}^m$, and that a vector bundle $E \to X$ is algebraically trivial over every U_α. Then

$$\mathbb{P}(E)|_{U_\alpha} \cong U_\alpha \times \mathbb{P}^{r-1},$$

where $r = \operatorname{rank} E$. Covering $\mathbb{P}^{r-1}$ by its standard affine charts gives a covering of $\mathbb{P}(E)|_{U_\alpha}$ by affine spaces of dimension $m + r - 1$. Such constructions yield many nonhomogeneous examples.

2.4 Blowups

The class of varieties covered by affine spaces is stable under several suitable blowup operations. The paper [1] studies smooth projective spherical varieties successively blown up along smooth subvarieties and also treats uniformly rational projective varieties more generally. One must, however, distinguish carefully between two statements:

(i) preservation of algebraic ellipticity under blowups;

(ii) preservation of an actual covering by affine spaces.

The blowup results for algebraic ellipticity are broader. Preservation of an $\mathbb{A}^n$-covering generally requires additional hypotheses on the center and on its position relative to the affine charts. Therefore the blowup theorems in [1] should not automatically be interpreted as asserting that all the resulting varieties admit a covering by affine spaces with the density property.

3 A broader formulation using affine varieties with the algebraic density property

The condition that every chart be isomorphic to $\mathbb{A}^n$ is stronger than necessary. One may instead consider a smooth projective variety X admitting a Zariski-open covering

$$X = \bigcup_{\alpha} U_\alpha$$

such that each U_α is a smooth affine algebraic variety with the algebraic density property. Recall that the algebraic density property means that the Lie algebra of all algebraic vector fields on U_α is generated by complete algebraic vector fields. The algebraic density property implies the holomorphic density property after analytification. Hence any such covering also has the property required in the Oka-tube argument. A particularly useful source of examples is the theorem of Kaliman and Kutzschebauch [6]. In a standard formulation, their result says that a connected affine homogeneous space of a linear algebraic group has the algebraic density property, apart from certain low-dimensional exceptional cases involving the affine line or an algebraic torus. Consequently, one can consider projective varieties admitting a Zariski-open covering by affine homogeneous spaces having the algebraic density property. Flag manifolds fit this mechanism in the simplest possible form: their translates of the big Bruhat cell are affine spaces. This suggests looking more generally at projective equivariant compactifications whose appropriate boundary complements are affine homogeneous spaces with the algebraic density property. I am not aware of a general classification of projective manifolds admitting such a covering.

4 What about smooth projective spherical varieties?

Smooth projective spherical varieties form a particularly tempting larger class. Recall that a normal G-variety is spherical if a Borel subgroup $B \subset G$ has an open dense orbit. Important examples include

- flag varieties;
- toric varieties;
- symmetric varieties;
- wonderful compactifications and related equivariant compactifications.

The paper [1] establishes strong algebraic ellipticity results for smooth projective spherical varieties and places many of them in the uniformly rational framework. These results do not, by themselves, imply that every smooth projective spherical variety is covered by Zariski-open Stein manifolds with the holomorphic density property. Indeed, the local structure theorem for spherical varieties typically produces charts involving torus factors, schematically of the form

$$\mathbb{A}^r \times (\mathbb{C}^*)^s,$$

or more general affine spherical varieties. Such charts need not be affine spaces. The occurrence of torus factors is relevant because algebraic tori do not behave like affine spaces from the point of view of the ordinary density property. Thus one cannot simply replace "flag manifold" by "smooth projective spherical manifold" in the argument and expect all of the required properties to remain valid. There is nevertheless an important connection with the theory of flexible algebraic varieties. An affine algebraic variety is called *flexible* if at every smooth point its tangent space is spanned by the tangent vectors to orbits of algebraic $\mathbb{G}_a$-actions. Flexibility therefore expresses the existence of many complete algebraic vector fields. Recent work on affine spherical varieties gives strong flexibility and transitivity results under suitable assumptions, for example when all invertible regular functions are constant. Such results are highly suggestive in the present context. Nevertheless, flexibility and the algebraic density property are not identical conditions. Flexibility alone should therefore not be quoted as establishing the precise chartwise density property required by the Oka-tube argument.

Remark 1. *Smooth projective toric varieties cause no difficulty because smooth toric affine charts corresponding to maximal cones are affine spaces. The difficulty concerns general spherical varieties, whose natural affine charts can contain nontrivial torus factors or more complicated affine spherical pieces.*

5 Uniformly rational varieties

A smooth algebraic n-fold X is called *uniformly rational* if every point $x \in X$ admits a Zariski-open neighbourhood U_x which is isomorphic to a Zariski-open subset of affine space:

$$U_x \cong V_x \subset \mathbb{A}^n.$$

This class contains all $\mathbb{A}^n$-covered varieties and is substantially larger. The paper [1] proves algebraic ellipticity results for complete uniformly rational varieties and also contains results for stably uniformly rational complete varieties. For the present question, however, there is an essential gap. If

$$V \subset \mathbb{A}^n$$

is an arbitrary Zariski-open subset, it does not follow in general that V has the algebraic or holomorphic density property. Deleting a hypersurface or a more complicated algebraic subset can drastically change the supply of complete algebraic and holomorphic vector fields. Thus one has the definite implication

$$X \text{ is } \mathbb{A}^n\text{-covered} \quad \Longrightarrow \quad \begin{array}{l} X \text{ is covered by Stein domains} \\ \text{with the holomorphic density property,} \end{array}$$

for $n \geq 2$, whereas no general implication

$$X \text{ is uniformly rational} \quad \Longrightarrow \quad \begin{array}{l} X \text{ is covered by Stein domains} \\ \text{with the holomorphic density property} \end{array}$$

appears to be presently available. Uniformly rational varieties may nevertheless become relevant if the proof of the Oka-tube theorem can be modified so that algebraic ellipticity, a local spray condition, or another suitable flexibility property replaces the full density property. The results of [1] make this possibility quite natural. However, it would require examining precisely where Andersén–Lempert approximation is used in the proof.

6 The resulting hierarchy

The clearest presently available hierarchy is

$$\boxed{\begin{aligned} \{\text{flag manifolds}\} &\subsetneq \{\text{smooth projective varieties covered by affine spaces}\} \\ &\subset \{\text{uniformly rational smooth projective varieties}\} \\ &\subset \{\text{rational smooth projective varieties}\}. \end{aligned}}$$

For varieties of dimension at least two, the second class definitely has the property in question, since every affine-space chart is biholomorphic to $\mathbb{C}^n$, which is Stein and has the holomorphic density property. Smooth projective toric varieties form a concrete and large subclass beyond flag manifolds. Products, suitable projective bundles, and selected blowups yield further nonhomogeneous examples. Smooth projective spherical varieties and uniformly rational projective varieties form broader and very promising classes. However, the available algebraic ellipticity or flexibility results do not automatically furnish a Zariski-open covering by Stein manifolds with the density property.

7 Conclusion

The following seems to be a useful safe formulation of the generalization.

> Let X be a smooth projective manifold of dimension at least two which is covered by Zariski-open subsets isomorphic to affine space. Then X is covered by Zariski-open Stein manifolds with the holomorphic density property. Consequently, any result whose proof for flag manifolds uses only the existence of such a covering extends to this class, provided that no other special property of flag manifolds enters the proof.

In particular, the geometric hypothesis extends immediately from flag manifolds to all smooth projective $\mathbb{A}^n$-covered varieties. This includes all smooth projective toric varieties and many nonhomogeneous examples. A further extension to all smooth projective spherical varieties or to all uniformly rational smooth projective varieties is plausible and worth investigating, but it is not a formal consequence of the presently available ellipticity results. The main unresolved point is whether these larger classes admit sufficiently many Zariski-open charts with the algebraic or holomorphic density property, or whether the density-property assumption in the Oka-tube argument can be replaced by a weaker local flexibility assumption.

Question 1. *Does every smooth projective spherical variety admit a finite Zariski-open covering by Stein manifolds with the holomorphic density property?*

A weaker and perhaps more accessible version is the following.

Question 2. *Which smooth projective uniformly rational varieties admit a Zariski-open covering by smooth affine varieties with the algebraic density property?*

These questions appear to isolate the precise issue separating the known affine-space-covered case from the broader spherical and uniformly rational settings.

References

[1] I. Arzhantsev, S. Kaliman, and M. Zaidenberg, Varieties covered by affine spaces, uniformly rational varieties and their cones, *Adv. Math.* **437** (2024), Article 109449. doi:10.1016/j.aim.2023.109449. Preprint: arXiv:2304.08608.

[2] E. Andersén and L. Lempert, On the group of holomorphic automorphisms of $\mathbb{C}^n$, *Invent. Math.* **110** (1992), 371–388.

[3] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, and M. Zaidenberg, Flexible varieties and automorphism groups, *Duke Math. J.* **162** (2013), 767–823.

[4] F. Forstnerič, *Stein Manifolds and Holomorphic Mappings: The Homotopy Principle in Complex Analysis*, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 56, Springer, Cham, 2017.

[5] F. Forstnerič and Y. Kusakabe, Oka tubes in holomorphic line bundles, *Math. Ann.*, **391:4** (2025) 5265–5292, .

[6] S. Kaliman and F. Kutzschebauch, Algebraic (volume) density property for affine homogeneous spaces, *Math. Ann.* **367** (2017), 1311–1332. arXiv:1507.07604.

[7] D. Varolin, The density property for complex manifolds and geometric structures, *J. Geom. Anal.* **11** (2001), 135–160.