NEW EXAMPLES OF TAME FAMILIES OF STEIN MANIFOLDS

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ABSTRACT. Let X be a smooth open manifold of even dimension, T be a topological space, and $\{J_t\}_{t\in T}$ be a continuous family of smooth integrable Stein structures on X. The family is said to be tame if the J_t-convex hulls of any compact set in X are upper semicontinuous with respect to $t \in T$. It was recently shown that tame Stein families have good function-theoretic properties; in particular, they satisfy the Oka principle for families of holomorphic maps to any Oka manifold. We analyse some constructions and operations on Stein manifolds that preserve tameness of families and we exhibit several conceptually new examples of tame Stein families.

1. INTRODUCTION

In this paper, we study the notion of tameness of families of Stein manifolds, which was recently introduced by the first author and Álfheiður Edda Sigurðardóttir in [10]. Tame Stein families have good function-theoretic properties, analogous to those of individual Stein manifolds. In particular, under suitable regularity assumptions on a parameter space T, every tame family of smooth integrable Stein structures $\mathscr{J} = \{J_t\}_{t\in T}$ on a smooth manifold X, depending continuously on $t \in T$, enjoys the Oka–Weil approximation theorem for families of holomorphic functions on X [10, Theorem 6.3], as well as the Oka principle with approximation for families of maps to any Oka manifold [10, Theorem 6.1]. (See also [9] for results on families of open Riemann surfaces, in which case every family of complex structures is tame.) It is therefore of interest to understand which operations on Stein manifolds preserve tameness in families.

We begin by recalling the definition of a tame Stein family; see [10, Definition 5.1].

Definition 1.1. A continuous family $\mathscr{J} = \{J_t\}_{t\in T}$ of integrable Stein structures on a smooth manifold X is *tame* at a point $t_0 \in T$ if for every compact set $K \subset X$ and open set $U \subset X$ containing the J_{t_0}-convex hull $\widehat{K}_{J_{t_0}}$ of K there is a neighbourhood $T_0 \subset T$ of t_0 such that $\widehat{K}_{J_t} \subset U$ holds for all $t \in T_0$. The family $\mathscr{J}$ is tame if it is tame at every point $t_0 \in T$.

In other words, tameness means that the hulls $\widehat{K}_{J_t}$ of any compact set K in X form an upper semicontinuous set-valued function of the parameter t.

Assuming that the parameter space T is locally compact Hausdorff, [10, Proposition 5.2] shows that tameness is equivalent to asking that for every compact subset $K \subset X$ its $\mathscr{J}$-convex hull

$$\widehat{K}_{\mathscr{J}} = \bigcup_{t\in T} \{t\} \times \widehat{K}_{J_t} \subset T \times X \tag{1.1}$$

is such that the projection $\pi : \widehat{K}_{\mathscr{J}} \to T$ is proper. Here, $\pi : T \times X \to T$ is the projection on the first factor. Assuming in addition that the Stein structures J_t are sufficiently smooth, tameness of $\mathscr{J}$ is equivalent to each of the following conditions.

Date: 20 September 2026.

2020 *Mathematics Subject Classification.* Primary 32Q56; Secondary 32E10, 32H02.

Key words and phrases. Stein manifold, Oka principle, Oka manifold, fibre bundle, vector bundle, principal bundle.

(a) For any compact set $K \subset X$ and $t_0 \in T$ there is a neighbourhood $T_0 \subset T$ such that the set $\bigcup_{t \in T_0} \widehat{K}_{J_t}$ is relatively compact in X (see [10, Proposition 5.3] and note that this holds for an arbitrary topological space T).
(b) For every $t_0 \in T$ there exist a neighbourhood $T_0 \subset T$ and a continuous function $\rho : T_0 \times X \to \mathbb{R}$ such that $\rho(t, \cdot\,)$ is a strongly J_t-plurisubharmonic exhaustion function on X for every $t \in T_0$ (see [10, Theorem 5.5]).
(c) If $t_0 \in T$ and $K \subset X$ is a compact J_{t_0}-convex set (that is, $K = \widehat{K}_{J_{t_0}}$) which is the closure of a strongly J_{t_0}-pseudoconvex domain, then there is a neighbourhood $T_0 \subset T$ of t_0 such that K is J_t-convex for every $t \in T_0$ (see [10, Proposition 5.7]).
(d) For every $t_0 \in T$ and $f \in \mathcal{O}_{J_{t_0}}(X)$ there are a neighbourhood $T_0 \subset T$ of t_0 and a $\mathscr{J}$-holomorphic function $F : T_0 \times X \to \mathbb{C}$ such that $F(t_0, \cdot\,) = f$ (see [10, Corollary 6.4]).

Here, $\mathcal{O}_J(X)$ denotes the space of holomorphic functions on the complex manifold (X, J). If $\mathscr{J} = \{J_t\}_{t \in T}$ is a continuous family of complex structures on X, a continuous function $F : T \times X \to \mathbb{C}$ is said to be $\mathscr{J}$-holomorphic if the function $F(t, \cdot\,) : X \to \mathbb{C}$ is J_t-holomorphic for every $t \in T$. We denote by $\mathcal{O}_{\mathscr{J}}(T \times X)$ the space of continuous $\mathscr{J}$-holomorphic functions on $T \times X$. The analogous notion applies to maps $T \times X \to Y$ to any complex manifold Y, and we use the notation $\mathcal{O}_{\mathscr{J}}(T \times X, Y)$ for the space of continuous $\mathscr{J}$-holomorphic maps from $T \times X$ to Y.

Every family of complex structures on a smooth surface X is tame, since in this case holomorphic convexity is a topological property by the Runge approximation theorem and hence independent of the complex structure. Families of open Riemann surfaces were treated in [9]. An example of a nontame family $\{J_t\}_{t \in \mathbb{R}}$ of smooth Stein structures on $\mathbb{R}^4$, depending smoothly on the parameter $t \in \mathbb{R}$, is given in [10, Theorem 4.1]. In that example, $(\mathbb{R}^4, J_t)$ is biholomorphic to $\mathbb{C}^2$ for every $t \in \mathbb{R}$ but the J_t-convex hull of the closed ball explodes as $t \to 0$. The construction is based on the existence of an injective holomorphic map $F : \mathbb{C}^2 \to \mathbb{C}^2$ with non-Runge image $F(\mathbb{C}^2) \subsetneq \mathbb{C}^2$, due to Wold [31]. A similar construction applies on bounded convex domains $X \subset \mathbb{C}^n$ for $n > 1$ [10, Remark 4.3].

In this paper, we present several constructions which preserve tameness of Stein families.

Recall that a domain Ω in a complex manifold X is said to be *Runge* in X if $\{f|_\Omega : f \in \mathcal{O}(X)\}$ is a dense subset of the space $\mathcal{O}(\Omega)$ of all holomorphic functions on Ω in the compact-open topology. For later reference, we recall the following results from [10, Sect. 5]. We assume that $\mathscr{J} = \{J_t\}_{t \in T}$ is a continuous family of smooth Stein structures on a smooth manifold X.

(i) If every point $t_0 \in T$ has a neighbourhood $T_0 \subset T$ and a family of biholomorphic maps $\Phi_t : (X, J_{t_0}) \to (X, J_t)$ depending continuously on $t \in T_0$, then the family $\mathscr{J}$ is tame. Indeed, for any compact set $K \subset X$ we have $\widehat{K}_{J_t} = \Phi_t\big(\widehat{K}_{J_{t_0}}\big)$ for $t \in T_0$, and these sets depend continuously on t.
(ii) [10, Proposition 5.9]: If for every $t_0 \in T$ there are a compact set $L \subset X$ and a neighbourhood $T_0 \subset T$ of t_0 such that $J_t = J_{t_0}$ holds on $X \setminus L$ for all $t \in T_0$, then $\mathscr{J}$ is tame.
(iii) [10, Proposition 5.11]: If X is a smooth manifold, (Y, J) is a Stein manifold, and $\Phi_t : X \to \Phi_t(X) \subset Y$ is a continuous family of diffeomorphisms onto Stein Runge domains in Y, then the family of Stein structures $\Phi_t^* J$ on X is tame.
(iv) [10, Example 5.12]: Assume that (Y, J_Y) is a Stein manifold and $F : T \times X \to Y$ is a continuous map such that $F_t = F(t, \cdot\,) : X \to Y$ is a smooth proper immersion whose image $F_t(X)$ is an immersed complex submanifold of Y for each $t \in T$. Let J_t denote the unique complex structure on X such that the map F_t is (J_t, J_Y)-holomorphic. (We shall say that $J_t = F_t^* J_Y$ is the pullback of J_Y by F_t.) Then, the family $\mathscr{J} = \{J_t\}_{t \in T}$ is tame.

We now describe the main new results of the present paper.

In Section 2, we establish the tameness of families of pullbacks of a Stein fibre bundle by a continuous family of maps from a tame Stein family; see Propositions 2.1 and 2.2.

In Section 3, we show that for a complex Lie group G that is either connected or discrete, and in some other cases as well, a family of holomorphic G-bundles with a Stein fibre on a tame Stein family has a tame family of Stein total spaces; see Theorem 3.6. This includes, for example, covering spaces, principal bundles, and vector bundles.

In Section 4, we show that, in a tame family of Stein manifolds, the family of complements of a continuous family of smooth complex hypersurfaces is a tame Stein family; see Theorem 4.2. The analogous result holds for continuous families of smooth complex hypersurfaces in complex projective spaces; see Theorem 4.7.

We have already mentioned the example of a nontame smooth family of Stein structures on Euclidean 4-space $\mathbb{R}^4$, given in [10, Theorem 4.1]. We pose the following open problems, the first of which has already been mentioned in [10, Problem 5.13].

Problem 1.2. (a) Does there exist a nontame *holomorphic* family of Stein structures on some smooth manifold X? More precisely, is there a holomorphic submersion $\pi : Z \to T$ which is smoothly trivial, with fibre X, and whose fibres $\pi^{-1}(t)$ determine a nontame family of Stein structures on X?

(b) Does every Stein manifold of complex dimension > 1 admit a nontame deformation of its Stein structure? Does this hold for every smoothly bounded strongly pseudoconvex domain in a Stein manifold?

Remark 1.3. In the recent paper [15], Huang, Kutzschebauch, and Rong gave sufficient conditions on a Stein manifold X with Varolin's density property for the existence of an injective holomorphic map $F : X \to X$ whose image $F(X)$ is not Runge in X. (Such a domain $F(X)$ is called a Fatou–Bieberbach domain of the second kind.) If in addition X is contractible, then the construction in [10, proof of Theorem 4.1] applies and gives a nontame deformation of the Stein structure on X depending smoothly on a parameter $t \in \mathbb{R}$. A particularly interesting example which fits in this context is the Koras–Russell cubic threefold

$$\mathrm{KR} = \left\{(x, y, z_0, z_1) \in \mathbb{C}^4 : x^2 y + x + z_0^2 + z_1^3 = 0\right\}. \tag{1.2}$$

Although KR is diffeomorphic to $\mathbb{R}^6$, it is not algebraically isomorphic to $\mathbb{C}^3$. Makar-Limanov [18] proved this by introducing and computing the invariant that now bears his name. Furthermore, the algebraic automorphism group of KR does not act transitively (see Moser-Jauslin [20] and Petitjean [21]). On the other hand, it is still an open problem whether KR is biholomorphic to $\mathbb{C}^3$. Leuenberger [17] proved that KR, regarded as a Stein manifold, has the density property, so its holomorphic automorphism group acts transitively. By [15, Theorem 5.4], KR admits a Fatou–Bieberbach domain of the second kind. Since KR is smoothly contractible, the proof of [10, Theorem 4.1] gives the following corollary.

Corollary 1.4. *The natural Stein structure on the Koras–Russell cubic* (1.2) *admits a smooth nontame deformation.*

The construction of a nontame deformation of a given Stein structure J on X in [10, proof of Theorem 4.1] applies more generally if X is a Stein manifold with a smooth strongly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$ without critical points near infinity which admits a biholomorphic map $F : X \to F(X) \subsetneq X$ whose non-Runge image $F(X)$ contains the nontrivial topology of X. This implies that for every sufficiently large $t > 0$, there is a diffeomorphism $\Psi_t : X \to X$ with $\Psi_t = F$ on $X_t = \{x \in X : \rho(x) \leq t\}$, depending continuously on t. In this situation, [10, proof of Theorem 4.1] shows that the continuous family of pullback Stein structures $J_t = \Psi_t^* J$ on X fails to be tame. It is likely that some of the examples in [15], besides the Koras–Russell cubic, are of this kind.

2. TAMENESS OF FAMILIES OF PULLBACKS

In this section, we prove the following result on tameness of families of pullbacks of a holomorphic fibre bundle with Stein total space by a continuous family of holomorphic maps from a tame family of Stein manifolds. See also Proposition 2.2.

Proposition 2.1. *Assume that $\pi : E \to Y$ is a holomorphic fibre bundle whose total space E is a Stein manifold, T is a locally compact Hausdorff space, $X_t = (X, J_t)$ $(t \in T)$ is a tame family of Stein manifolds, and $f_t : X_t \to Y$ is a continuous family of holomorphic maps. Assume that the total spaces of the pullback bundles $E_t := f_t^* E \to X_t$ $(t \in T)$ are smoothly diffeomorphic to one another, with the diffeomorphisms depending continuously on t (in the compact-open topology). Then, the induced family of Stein structures on E_t is continuous in t and tame.*

Proof. Given a holomorphic map $f : X \to Y$, the pullback of $\pi : E \to Y$ has the total space

$$f^*E = \{(x, e) \in X \times E : f(x) = \pi(e)\}.$$

We have the holomorphic fibre bundle projection $f^*\pi : f^*E \to X$, given by $(f^*\pi)(x, e) = x$, and the holomorphic map $\iota : f^*E \to E$ given by $\iota(x, e) = e$. Note that ι maps the fibre of f^*E over $x \in X$ isomorphically onto the fibre of E over $f(x) \in Y$ and the following diagram commutes:

$$\begin{tikzcd} f^*E \arrow[r, "\iota"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By trivially extending the bundle $\pi : E \to Y$ to $\tilde{\pi} : \widetilde{E} \to X \times Y$, with $\widetilde{E} \cong X \times E$ the pullback of E by the projection $X \times Y \to Y$, the bundle $f^*E \to X$ is equivalent to the restriction of $\widetilde{E} \to X \times Y$ to the graph

$$\Gamma_f = \{(x, f(x)) : x \in X\} \subset X \times Y, \tag{2.1}$$

with the map $x \mapsto (x, f(x))$ serving as the change of the base. In this picture, the total space $f^*E = \tilde{\pi}^{-1}(\Gamma_f)$ is a closed complex submanifold of $\widetilde{E}$, hence Stein if both X and E are Stein.

Assume now that $X_t = (X, J_t)$ $(t \in T)$, $f_t : X_t \to Y$, and $\pi : E \to Y$ are as in the proposition. Note that the family of Stein structures $\widetilde{J}_t$ on $X \times E \cong \widetilde{E}$, given by J_t on X and by the Stein structure on E, is tame. Since $E_t := f_t^*E \subset X \times E$ is a continuous family of closed $\widetilde{J}_t$-complex submanifolds of the tame Stein family $(X \times E, \widetilde{J}_t)$ for $t \in T$, the family E_t is also tame. (At this point, we tacitly use the hypothesis that the pullback bundles $E_t := f_t^*E \to X_t$ $(t \in T)$ are smoothly diffeomorphic to one another, with a continuous dependence of the diffeomorphism on t.) To see this, fix a point $t_0 \in T$ and a compact neighbourhood $T_0 \subset T$ of $t_0 \in T$ such that there is a continuous family of strongly $\widetilde{J}_t$-plurisubharmonic exhaustion functions $\rho_t : X_t \times E \to \mathbb{R}$ $(t \in T_0)$; see condition (b) in the introduction or [10, Theorem 5.5]. Then the restrictions $\rho_t|_{E_t} : E_t \to \mathbb{R}$ for $t \in T_0$ form a continuous family of strongly $\widetilde{J}_t$-plurisubharmonic exhaustion functions on E_t for $t \in T_0$. Since this holds for every $t_0 \in T$, the family $\{(E_t, \widetilde{J}_t)\}_{t \in T}$ is tame by [10, Theorem 5.5]. (A similar argument can be found in [10, Example 5.12].) □

The condition on the bundles E_t in Proposition 2.1 to be smoothly diffeomorphic to one another, with the diffeomorphisms depending continuously on the parameter t, is not satisfied in general. In particular, non-homotopic maps $X \to Y$ may in general give non-diffeomorphic pullbacks. However, in the applications of this result in the proof of Theorem 3.6 we shall only need the following local case.

Proposition 2.2. *Assume that $E \to Y$ is a smooth fibre bundle, X is a smooth manifold, T is a topological space, and $f_t : X \to Y$ $(t \in T)$ is a continuous family of smooth maps. Set $E_t := f_t^* E \to X$ for $t \in T$. Given a point $t_0 \in T$ and a relatively compact domain $U \Subset X$, there are a neighbourhood $T_0 \subset T$ of t_0 and a continuous family of smooth fibre bundle diffeomorphisms $\Theta_t : E_t|_U \to E_{t_0}|_U$ for $t \in T_0$.*

Proof. As before, we consider E as a fibre bundle on $X \times Y$. For every $t \in T$, let $F_t : X \to X \times Y$ be the smooth map

$$F_t(x) = (x, f_t(x)), \qquad x \in X.$$

The pullback $f_t^* E$ is then identified with the restriction of E to the closed smooth submanifold

$$\Gamma_t = F_t(X) \subset X \times Y,$$

the graph of f_t, with the map F_t serving as the change of the base. Let $p : X \times Y \to X$ be the projection $p(x, y) = x$. Fix a point $t_0 \in T$. By the tubular neighbourhood theorem, $\Gamma_{t_0} = F_{t_0}(X)$ has an open tubular neighbourhood $O \subset X \times Y$ with a smooth homotopy $h : O \times I \to O$ $(s \in I = [0, 1])$ satisfying the following conditions.

- $p \circ h_s = p$ for all $s \in I$, where $h_s = h(\cdot\,, s)$.
- $h_s|_{\Gamma_{t_0}}$ is the identity on Γ_{t_0} for all $s \in I$.
- h_0 is the identity on O.
- $h_1(O) = \Gamma_{t_0}$.

By the invariance of pullbacks of smooth fibre bundles under homotopies (see Proposition 5.1), the restricted bundle $E|_O$ is isomorphic to the bundle $h_1^*(E|_{\Gamma_{t_0}})$. In particular, we have a smooth fibre-preserving map $\Theta : E|_O \to E|_{\Gamma_{t_0}} \cong E_{t_0}$, covering the retraction $h_1 : O \to \Gamma_{t_0} = F_{t_0}(X)$, which maps any fibre $E_{(x,y)}$ with $(x, y) \in O$ diffeomorphically onto the fibre $E_{F_{t_0}(x))}$. For any $t \in T$ sufficiently close to t_0 we have $F_t(U) \subset O$. For such t, the restriction of Θ to $E|_{F_t(U)}$ is a fibre bundle diffeomorphism

$$\Theta_t : E|_{F_t(U)} \to E|_{F_{t_0}(U)}$$

depending continuously on t. Since $E|_{F_t(U)}$ is isomorphic to $E_t|_U$, this completes the proof. ☐

Remark 2.3. (a) If T is a connected smooth manifold, then any two points $t_0, t \in T$ are connected by a smooth path. By the invariance of pullbacks of smooth fibre bundles under homotopies (see Proposition 5.1), the corresponding pullback bundles are isomorphic. Locally in the parameter t this gives a family of isomorphisms $E_t \cong E_{t_0}$ depending continuously on t. However, it is not clear how to get a well-defined global family of such isomorphisms for paths in the same homotopy class.

(b) If X is a Stein manifold, G is a complex Lie group and $\pi : E \to Y$ is a holomorphic G-bundle, the proof of Proposition 2.2, together with the Oka principle of Grauert [11] and the existence of holomorphic tubular neighbourhoods of Stein submanifolds [8, Theorem 3.3.3, p. 74], shows that the family of pullbacks $f_t^* E$ by a continuous family of holomorphic maps $f_t : X \to Y$ is locally stable (independent of t up to a holomorphic equivalence of holomorphic G-bundles) on any relatively compact Stein domain $U \Subset X$. A related stability result is due to Leiterer [16, Theorem 2.7].

3. Tameness of holomorphic G-bundles on tame Stein families

In this section, we address the following problem.

Problem 3.1. Let X be a smooth manifold and $\{J_t\}_{t \in T}$ be a tame family of Stein structures on X depending continuously on the parameter $t \in T$. Assume that $(E, \widetilde{J}_t) \to (X, J_t)$ is a continuous family of holomorphic fibre bundles with a given fibre F and structure group G, and that $(E, \widetilde{J}_t)$ is Stein for every $t \in T$. Under which conditions is the family $\{\widetilde{J}_t\}_{t \in T}$ also tame?

First, we must ask when the manifolds $(E, \widetilde{J}_t)$ are Stein. An obvious necessary condition is that the fibre F is Stein. J-P. Serre famously asked in 1953 [25] whether every holomorphic fibre bundle with Stein base and Stein fibre has Stein total space: *un espace fibré holomorphe dont la base et la fibre sont des variétés de Stein, est-il toujours une variété de Stein?* The following definition is convenient.

Definition 3.2. An effective action of a complex Lie group G on a Stein manifold F is *good* if, whenever E is a holomorphic G-bundle with fibre F over a Stein manifold X, the total space E is Stein. (By an action, we mean a continuous and hence real-analytic action by biholomorphisms.)

Some of the many affirmative answers to Serre's question may be summarised as follows; we do not attempt a comprehensive survey.

Proposition 3.3. *The action of a complex Lie group G on a Stein manifold F is good if one of the following holds.*

(a) *G is a linear group (Serre* [25]*). This includes the case of vector bundles.*
(b) *G is connected (Matsushima and Morimoto* [19, Theorem 6]*).*
(c) *G has finitely many connected components.*
(d) *G acts freely and transitively on F* [19, Theorem 4]*. This is the case of principal G-bundles.*
(e) *F is discrete (Stein* [29]*). This is the case of covering spaces, since G can be taken to be the monodromy group. This group is countable, being a quotient of the fundamental group of X, and can therefore be regarded as a zero-dimensional Lie group.*

Proof. We only need to justify part (c). Let $P \to X$ be the principal G-bundle associated to the given holomorphic G-bundle $E = P \times_G F \to X$, and let G^0 denote the identity component of G. Since G/G^0 is finite, the quotient $X' = P/G^0 \to X$ is a finite unbranched holomorphic covering. Hence X' is Stein by [29]. The pullback of E to X' is naturally isomorphic to $E' = P \times_{G^0} F \to X'$, so it has structure group G^0. Since G^0 is connected, E' is Stein by [19, Theorem 6] (see (b)). The natural identification $E' = P \times_{G^0} F \cong X' \times_X E$ shows that $E' \to E$ is the pullback of the finite covering $X' \to X$. Hence $E' \to E$ is a finite unbranched holomorphic covering. Since E' is Stein, it follows that E is Stein. □

Several other papers in the literature provide positive answers to Serre's question for fibres that are bounded domains of various kinds in Stein manifolds and therefore do not admit an effective action by a complex Lie group. Such an action is essential for us in order to be able to apply Grauert's theorem and get uniquely determined complex structures on our total spaces; see Proposition 3.5. On the other hand, negative answers to Serre's question with infinite-dimensional structure groups have been given by Skoda [26, 27], Demailly [5, 6], Coeure and Loeb [3], Rosay [23], and others.

Let us further recall that if a complex Lie group G acts effectively on a Stein manifold, then G is itself Stein (see Matsushima and Morimoto [19, Theorem 3]). The question of which complex Lie groups are Stein is answered by the following result.

Theorem 3.4 ([19, Theorems 1 and 2])**.** *A complex Lie group G is Stein if and only if G does not contain any connected complex subgroup of positive dimension contained in a compact subgroup of G. A connected complex Lie group is Stein if and only if its connected centre subgroup is isomorphic to the group $\mathbb{C}^n \times (\mathbb{C}^*)^m$ for some $n, m \geq 0$.*

Assume now that X and F are complex manifolds and G is a complex Lie group acting effectively on F. A fibre bundle $E \to X$ with structure group G is called a *G-bundle*. If X is a Stein manifold, Grauert's Oka principle [11] says that every topological G-bundle E is topologically isomorphic to a holomorphic G-bundle, and the holomorphic G-bundle structure on E compatible with the given topological structure is unique up to holomorphic automorphisms of G-bundles. It follows that the complex structure on E, which makes it a holomorphic G-bundle in a given topological isomorphism class of G-bundles, is uniquely determined. We summarise these observations as follows.

Proposition 3.5. *Let X be a smooth manifold, F be a complex manifold, G be a complex Lie group acting effectively on F, and $E \to X$ be a smooth G-bundle with fibre F. A family $\{J_t\}_{t\in T}$ of smooth Stein structures on X determines on $E \to X$ a family of J_t-holomorphic G-bundle structures, which are unique up to holomorphic isomorphisms. Hence, there is a unique family $\{\widetilde{J}_t\}_{t\in T}$ of associated complex structures on E determined by these conditions. If J_t is continuous in $t \in T$, then so is $\widetilde{J}_t$. If F is Stein and the action of G on F is good, then $(E, \widetilde{J}_t)$ is a Stein manifold for every $t \in T$.*

The statement in the above proposition regarding continuity of the complex structures $\widetilde{J}_t$ with respect to the parameter $t \in T$ follows from Remark 3.7.

The following is the main result of this section.

Theorem 3.6. *Assume that X is a smooth manifold and $\pi : E \to X$ is a G-bundle whose fibre F is a Stein manifold with a good action (see Definition 3.2) of the structure group G. Let T be a locally compact Hausdorff space and $\{J_t\}_{t\in T}$ be a tame family of smooth Stein structures on X. Then the associated family $\{\widetilde{J}_t\}_{t\in T}$ of Stein structures on E (see Proposition 3.5) is also tame.*

This result applies in all cases mentioned in Proposition 3.3.

Proof. We begin by showing that for any $t_0 \in T$ and smoothly bounded strongly J_{t_0}-pseudoconvex domain $U \Subset X$, the family of Stein structures $\widetilde{J}_t$ on the restricted bundle $E|_U$ is continuous and tame at t_0. Set $E_t = (E, \widetilde{J}_t)$ for $t \in T$. Fix a smoothly bounded, strongly J_{t_0}-pseudoconvex domain $\Omega \Subset X$ with $\overline{U} \subset \Omega$. By [10, Theorem 3.1], there are a neighbourhood $T_0 \subset T$ of t_0 and a family of (J_t, J_{t_0})-biholomorphisms

$$\Phi_t : \Omega \to \Phi_t(\Omega) \subset X \quad (t \in T_0), \quad \Phi_{t_0} = \mathrm{Id}_\Omega, \tag{3.1}$$

depending continuously on $t \in T_0$. For every $t \in T_0$, the pullback $(\Phi_t^* E_{t_0})|_\Omega \to \Omega$ is a J_t-holomorphic G-bundle with fibre F. Assuming that the neighbourhood T_0 of t_0 is small enough, $(\Phi_t^* E_{t_0})|_U$ is isomorphic to $E_t|_U$ as a topological G-bundle (see Proposition 2.2). By Grauert's Oka principle [11], the bundle $(\Phi_t^* E_{t_0})|_U$ is isomorphic to $E_t|_U$ as a J_t-holomorphic G-bundle. Under the above holomorphic bundle isomorphism, the complex structure $\widetilde{J}_t$ on $E_t|_U$ corresponds to the complex structure on the pullback bundle $(\Phi_t^* E_{t_0})|_U$. Equivalently, it is obtained from $\widetilde{J}_{t_0}$ by a bundle diffeomorphism covering $\Phi_t|_U : U \to \Phi_t(U) \subset X$ (see (3.1)). Hence, Propositions 2.1 and 2.2 imply that the family of complex structures $\{\widetilde{J}_t\}_{t\in T_0}$ on $E|_U$ is continuous and tame if the neighbourhood T_0 is small enough. Since this argument applies at every point $t_0 \in T$, we conclude that the family of complex structures $\{\widetilde{J}_t\}_{t\in T}$ on E is continuous.

Remark 3.7. So far we have not used the hypothesis that the action of the structure group G on the fibre F is good. This shows that if the family of complex structures $\{J_t\}_{t\in T}$ on X is continuous in t, then the associated family $\{\widetilde{J}_t\}_{t\in T}$ of complex structures on E is also continuous in t.

To complete the proof, it remains to show that the family of Stein structures $\widetilde{J}_t$ on E is tame. We need the following observation. Lacking a reference, we include a proof.

Proposition 3.8. *Let $p : Y \to X$ be a holomorphic map between Stein manifolds and let $U \subset X$ be a Runge open subset. Assume that $D := p^{-1}(U)$ is Stein. Then D is Runge in Y.*

The proposition follows immediately from the following lemma by choosing exhaustions of U by compact $\mathcal{O}(X)$-convex sets L_j and of Y by compact $\mathcal{O}(Y)$-convex sets B_j. Recall that a Stein domain

D in a Stein manifold Y is Runge if and only if it admits an exhaustion by compact $\mathcal{O}(Y)$-convex sets [14, Theorem 4.3.3].

Lemma 3.9. *Let $p : Y \to X$ be a holomorphic map between Stein manifolds. If $L \subset X$ is compact and $\mathcal{O}(X)$-convex, and $B \subset Y$ is compact and $\mathcal{O}(Y)$-convex, then $K := p^{-1}(L) \cap B$ is $\mathcal{O}(Y)$-convex.*

Proof. Let $y \in Y \setminus K$. Suppose that $p(y) \notin L$. Since L is $\mathcal{O}(X)$-convex, there exists $f \in \mathcal{O}(X)$ such that $|f(p(y))| > \sup_{x \in L} |f(x)|$. The pullback $f \circ p$ belongs to $\mathcal{O}(Y)$, and

$$|(f \circ p)(y)| > \sup_{z \in K} |(f \circ p)(z)|$$

because $p(K) \subset L$. Hence y does not belong to the $\mathcal{O}(Y)$-hull of K. Suppose now that $p(y) \in L$. Since $y \notin K$, necessarily $y \notin B$. As B is $\mathcal{O}(Y)$-convex, there exists $g \in \mathcal{O}(Y)$ satisfying

$$|g(y)| > \sup_{z \in B} |g(z)| \geq \sup_{z \in K} |g(z)|.$$

Hence again y does not belong to the $\mathcal{O}(Y)$-hull of K. $\square$

We now complete the proof of Theorem 3.6. Fix $t_0 \in T$ and a compact $\mathcal{O}(E, \widetilde{J}_{t_0})$-convex set $L \subset E$. Given an open set $V \subset E$ with $L \subset V$, we must find a neighbourhood $T_0 \subset T$ of t_0 such that

$$\widehat{L}_{(E, \widetilde{J}_t)} \subset V \ \text{ for all } t \in T_0. \tag{3.2}$$

Choose an open smoothly bounded domain $U \Subset X$ which is strongly pseudoconvex and Runge in (X, J_{t_0}) such that $L \subset E|_U$. Since the family $\{J_t\}_{t \in T}$ is tame, there is a neighbourhood $T_0 \subset T$ of t_0 such that U is strongly J_t-pseudoconvex and Runge in (X, J_t) for all $t \in T_0$ [10, Proposition 5.7]. By the first part of the proof, the family $(E|_U, \widetilde{J}_t)$ is tame at t_0. Hence, shrinking T_0 if necessary, we have

$$\widehat{L}_{(E|_U, \widetilde{J}_t)} \subset V \ \text{ for all } t \in T_0.$$

Since $E|_U$ is $\widetilde{J}_t$-Runge in E for all $t \in T_0$ by Proposition 3.8, the $\widetilde{J}$-hull of L in $E|_U$ agrees with its $\widetilde{J}_t$-hull in E. This shows that (3.2) holds, thereby completing the proof. $\square$

4. Complements of families of complex hypersurfaces

If X is a Stein manifold and H is a closed complex hypersurface in X, then the complement $X \setminus H$ is also a Stein manifold (see [12, Theorem 5, p. 129]). Suppose now that T is a topological space and $\mathscr{J} = \{J_t\}_{t \in T}$ is a family of smooth Stein structures on X.

Definition 4.1. A family $H_t \subset X$ $(t \in T)$ of closed J_t-complex hypersurfaces is said to be continuous if for some (hence for any) $t_0 \in T$ there exists a continuous family of smooth diffeomorphisms $\Theta_t : X \to X$ $(t \in T)$ such that $\Theta_{t_0} = \mathrm{Id}_X$ and $H_t = \Theta_t(H_{t_0})$ for every $t \in T$.

Note that $\widetilde{J}_t = \Theta_t^* J_t$ for $t \in T$ is then a continuous family of Stein structures on the fixed smooth manifold X such that $H = H_{t_0}$ is a $\widetilde{J}_t$-complex hypersurface in X for every $t \in T$.

Theorem 4.2. *(Notation as above.) If the family of Stein structures $\mathscr{J} = \{J_t\}_{t \in T}$ on X is tame, then the family of Stein structures $\widetilde{\mathscr{J}} = \{\widetilde{J}_t\}_{t \in T}$ on $X \setminus H$ is tame.*

The conclusion of Theorem 4.2 is equivalent to the statement that the family of Stein manifolds $\{(X \setminus H_t, J_t)\}_{t \in T}$ is tame. This viewpoint will be convenient in the proof. However, since the notion of tameness formally pertains to a family of Stein structures on a fixed smooth manifold, we have transferred the Stein structures J_t on the variable family of domains $X \setminus H_t$ to the Stein structures $\widetilde{J}_t = \Theta_t^* J_t$ on the fixed domain $X \setminus H$ via the diffeomorphisms Θ_t. Here is an equivalent statement.

Theorem 4.3. *If $\{J_t\}_{t\in T}$ is a tame family of smooth Stein structures on X and $H \subset X$ is a closed J_t-holomorphic hypersurface for every $t \in T$, then the Stein family $\{(X \setminus H, J_t)\}_{t\in T}$ is also tame.*

In the proof of Theorem 4.2 we shall use the following observation.

Lemma 4.4. *Let $\pi : E \to X$ be a holomorphic line bundle on a Stein manifold X, and let $f_t : X \to E$ ($t \in T$) be a continuous family of holomorphic sections. Set $H_t = f_t(X)$. Then there exists a continuous family $\phi_t : E \setminus H_t \to \mathbb{R}_+$ ($t \in T$) of smooth strongly plurisubharmonic exhaustion functions. In particular, $\{E \setminus H_t\}_{t\in T}$ is a tame family of Stein manifolds.*

Proof. Since E is Stein, the complement $E \setminus E_0$ of its zero section E_0 is also Stein (see [12, Theorem 5, p. 129]). Hence, there is a smooth strongly plurisubharmonic exhaustion function $\phi : E\setminus E_0 \to \mathbb{R}_+$. We identify E_0 with X. For every $t \in T$, the map $F_t : E \to E$, defined by

$$F_t(e) = e - f_t(\pi(e)), \qquad e \in E,$$

is a fibre-preserving biholomorphism of E which is affine linear on each fibre and maps $H_t = f_t(X)$ onto $E_0 \cong X$. Taking $\phi_t = \phi \circ F_t$ gives a family satisfying the lemma. Note that the biholomorphisms $F_t : E \setminus H_t \to E \setminus E_0$ identify the varying complements $\{E \setminus H_t\}_{t\in T}$ with the fixed smooth manifold $E \setminus E_0$. ☐

We shall need the following version of Rossi's local maximum principle [24], due to Rosay [22, Proposition, p. 430]. It is stated in terms of plurisubharmonic functions as opposed to holomorphic functions. Note that in any Stein manifold, the plurisubharmonic hull of a compact set agrees with its holomorphic hull (see [14, Theorem 5.2.10]). We state Rosay's result for complex manifolds, although his result also applies to almost complex (not necessarily integrable) manifolds.

Proposition 4.5. *Let K be a compact set in a complex manifold X, and let $\widehat{K}$ be its plurisubharmonic hull. Let $x \in \widehat{K} \setminus K$. For any relatively compact neighbourhood $V \Subset X$ of x that does not intersect K and any continuous plurisubharmonic function u defined on a neighbourhood of $\overline{V}$, we have*

$$u(x) \le \sup\big\{u(y) : y \in \widehat{K} \cap bV\big\}.$$

Proof of Theorem 4.2. Using the notation in Definition 4.1, it suffices to prove tameness of the family $\widetilde{\mathscr{J}}$ at $t = t_0$, for the situation is symmetric with respect to the choice of $t_0 \in T$. Fix a compact set $K \subset X \setminus H_{t_0}$. Since the family H_t depends continuously on t, there is a neighbourhood $T_0 \subset T$ of t_0 such that $K \subset X\setminus H_t$ for all $t \in T_0$. Pick a smooth strongly J_{t_0}-plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$ such that the domain $U = \{\rho < 0\}$ is smoothly bounded and contains $\widehat{K}_{J_{t_0}}$. Since the family of Stein structures $\mathscr{J} = \{J_t\}_{t\in T}$ is tame, after shrinking the neighbourhood $T_0 \subset T$ around t_0 if necessary, the set $\bigcup_{t\in T_0} \widehat{K}_{J_t}$ is relatively compact in U. Let K_t denote the J_t-convex hull of K in the Stein manifold $(X \setminus H_t, J_t)$ for $t \in T_0$. Note that

$$K_t \subset \widehat{K}_{J_t} \setminus H_t \subset U \ \text{ for } t \in T_0.$$

To complete the proof of the theorem, we show the following.

Lemma 4.6. *(Notation as above.) After shrinking T_0 around t_0, there is a continuous family of strongly J_t-plurisubharmonic exhaustion functions $\phi_t : U \setminus H_t \to \mathbb{R}$ for $t \in T_0$.*

From Lemma 4.6 and Proposition 4.5 it follows that the hulls K_t of K in $X \setminus H_t$ are bounded away from $H_t \cup bU$ for $t \in T_0$. Since the point $t_0 \in T$ and the compact set $K \subset X \setminus H_{t_0}$ were arbitrary, property (a) in the introduction (see [10, Proposition 5.3]) implies that the family $\{(X \setminus H_t, J_t)\}_{t\in T}$ is tame. This is equivalent to saying that the family $\{(X \setminus H_{t_0}, \widetilde{J}_t)\}_{t\in T}$ is tame.

Proof of Lemma 4.6. Let $\rho : X \to \mathbb{R}$ be as above with $U = \{\rho < 0\}$. If T_0 is a small enough neighbourhood of t_0 then $\rho|_U$ is strongly J_t-plurisubharmonic for every $t \in T_0$ (see [10, Lemma 5.6]). Pick a smooth, convex, strongly increasing function $\xi : (-\infty, 0) \to \mathbb{R}$ with $\lim_{t\to 0} \xi(t) = +\infty$. It follows that $\xi \circ \rho : U \to \mathbb{R}$ is a strongly J_t-plurisubharmonic exhaustion function for every $t \in T_0$. This is an immediate consequence of the formulas

$$dd^c(\xi \circ \rho) = (\xi' \circ \rho)dd^c\rho + (\xi'' \circ \rho)d\rho \wedge d^c\rho \tag{4.1}$$

where $d^c\rho(Jv) = d\rho(-Jv)$ for $v \in TX$, $dd^c\rho$ is the Levi form of ρ, and

$$(d\rho \wedge d^c\rho)(v \wedge Jv) = (d\rho(v))^2 + (d^c\rho(v))^2, \tag{4.2}$$

which hold for any complex structure J on X and tangent vector $v \in TX$. (This also follows from the formula [1, Eq. (8.9), p. 343] for the Hessian of $\xi \circ \rho$, noting that $dd^c\rho$ on a complex line $L = \mathrm{Span}(v, Jv) \subset TX$ $(0 \neq v \in TX)$ is the trace of the Hessian of ρ restricted to L.)

We shall find a continuous family of strongly J_t-plurisubharmonic functions τ_t on a neighbourhood of $H_t \cap \overline{U}$ in X such that τ_t tends to $+\infty$ along the hypersurface H_t for every $t \in T_0$. For a suitable constant $c > 0$ and up to shrinking T_0 around t_0,

$$\phi_t := \max\{\xi \circ \rho, c\,\tau_t\}$$

is then a well-defined strongly J_t-plurisubharmonic exhaustion function on $U \setminus H_t$ depending continuously on $t \in T_0$. (To make ϕ_t smooth, we replace $\max$ by the regularised maximum, see [8, p. 69]. However, this is inessential for the application.)

It remains to construct the functions τ_t with the stated properties. In the sequel, we will shrink T_0 around t_0 when necessary without mentioning it again. By the Docquier–Grauert tubular neighbourhood theorem (see [13, p. 257, Theorem 8] or [8, Theorem 3.3.3, p. 74]), the hypersurface H_{t_0} has an open J_{t_0}-Stein neighbourhood $V \subset X$ which is J_{t_0}-biholomorphic to a neighbourhood $\widetilde{V} \subset E$ of the zero section E_0 of E. Denote by $\Psi : V \to \widetilde{V}$ this biholomorphism and identify H_{t_0} with the zero section $\Psi(H_{t_0}) = E_0$ of E. Choose a pair of strongly J_{t_0}-pseudoconvex domains $\Omega \Subset \Omega_0 \Subset X$ with $\overline{U} \subset \Omega$. By [10, Theorem 3.1] there is a continuous family of (J_t, J_{t_0})-biholomorphic maps $\Phi_t : \Omega_0 \to \Phi_t(\Omega_0) \subset X$, with $\Phi_{t_0} = \mathrm{Id}_{\Omega_0}$, such that $\Omega \subset \Phi_t(\Omega_0)$ for every $t \in T_0$. Let $\pi : E \to H_{t_0}$ denote the J_{t_0}-holomorphic normal line bundle of H_{t_0} in X. Recall that H_t is a J_t-complex hypersurface in X for all $t \in T$. For t close to t_0 the set

$$\widetilde{H}_t := \Phi_t(H_t \cap \Omega_0) \subset X$$

is then a locally closed J_{t_0}-complex hypersurface close to H_{t_0} and depending continuously on t. In particular, we have $\widetilde{H}_t \subset V$ and the image $\Psi(\widetilde{H}_t) \subset \widetilde{V} \subset E$ coincides over the J_{t_0}-Stein domain $W := E_0 \cap \Omega$ with the image of a J_{t_0}-holomorphic section $f_t : W \to E|_W$. (Recall that E_0 is identified with H_{t_0}.) Clearly, f_t depends continuously on t. By Lemma 4.4, there is a continuous family of strongly J_{t_0}-plurisubharmonic exhaustion functions $\psi_t : E|_W \setminus f_t(W) \to \mathbb{R}$, $t \in T_0$. Taking

$$\tau_t = \psi_t \circ \Psi \circ \Phi_t, \quad t \in T_0,$$

gives a continuous family of strongly J_t-plurisubharmonic functions on deleted open neighbourhoods of $H_t \cap \Omega$ which tend to $+\infty$ along $H_t \cap \Omega$. This completes the proof. □

As noted above, this proves Theorem 4.2. □

We have the following analogue of Theorem 4.2 for hypersurfaces in complex projective spaces.

Theorem 4.7. *Let $n > 1$ and T be a topological space. If $H_t \subset \mathbb{CP}^n$ $(t \in T)$ is a continuous family of smooth complex hypersurfaces, then the complements $X_t = \mathbb{CP}^n \setminus H_t$ $(t \in T)$, with the induced complex structures from $\mathbb{CP}^n$, form a tame family of Stein manifolds.*

Sketch of proof. An inspection of the construction in [7, proof of Theorem 7.2], taking into account also that every closed complex submanifold of $\mathbb{CP}^n$ has positive normal bundle (see Barth [2]), shows that there is a family of open tubular neighbourhoods $U_t \subset \mathbb{CP}^n$ of H_t and smooth strongly plurisubharmonic functions $\rho_t : U_t \setminus H_t \to \mathbb{R}$, depending continuously on $t \in T$, such that ρ_t tends to $+\infty$ along H_t for every $t \in T$. It follows that $X_t = \mathbb{CP}^n \setminus H_t$ is a holomorphically convex manifold. Since any closed complex subvariety of $\mathbb{CP}^n$ of positive dimension intersects H_t, it follows that X_t does not contain any such subvariety and hence it is Stein. By our definition of a continuous family of hypersurfaces, the Stein manifolds X_t are smoothly diffeomorphic to one another. The existence of functions ρ_t as above shows that for any $t_0 \in T$ and compact set $K \subset X_{t_0}$, the $\mathcal{O}(X_t)$-convex hull $\widehat{K}_t$ of K for t near t_0 does not enter the region in U_t where ρ_t is large positive; in particular, it does not approach H_t. This shows tameness.

An alternative argument is to show that the functions ρ_t extend from smaller deleted neighbourhoods $U'_t \setminus H_t$ of H_t to strongly plurisubharmonic exhaustion functions $\tilde{\rho}_t : X_t \to \mathbb{R}$ depending continuously on t near t_0. The conclusion then follows from condition (b) in the Introduction. $\square$

5. Appendix: Homotopy invariance of pullbacks

The following result is well known.

Proposition 5.1. *Let $\pi : E \to Y$ be a smooth fibre bundle, let X be a smooth manifold, and let $f, g : X \to Y$ be smoothly homotopic maps. Then the pullback bundles f^*E and g^*E are isomorphic as smooth fibre bundles over X.*

For principal bundles, this is proved, for example, in Steenrod's book [28, Part II, Chapter 11]. The general case can be found in several sources. However, as pointed out by del Hoyo [4] in 2016, most of these proofs contain a gap, and in the cited paper he explained how to remove it. We include the main point of his argument.

Proof. Set $I = [0, 1]$ and let $H : X \times I \to Y$ be a smooth homotopy from f to g, so that $H(\cdot\,, 0) = f$ and $H(\cdot\,, 1) = g$. Consider the pullback bundle $H^*E \longrightarrow X \times I$. Its restriction to $X \times \{0\}$ is naturally identified with f^*E, while its restriction to $X \times \{1\}$ is naturally identified with g^*E. Choose a complete Ehresmann connection on the smooth fibre bundle $H^*E \to X \times I$ (see del Hoyo [4] for the existence of such a connection). Let t be the standard coordinate on I and $\partial_t = \frac{\partial}{\partial t}$ be the corresponding vector field on $X \times I$. Its horizontal lift with respect to the complete Ehresmann connection is a complete vector field on H^*E covering ∂_t. Integrating this vector field from $t = 0$ to $t = 1$ yields a diffeomorphism

$$\Phi : (H^*E)|_{X\times\{0\}} \longrightarrow (H^*E)|_{X\times\{1\}}$$

which preserves fibres. Using the above identifications of the end fibres with the pullback bundles, we obtain a smooth bundle isomorphism $f^*E \cong g^*E$. $\square$

The above proof fails in general if the Ehresmann connection on H^*E is not complete, since the integral curves of the lifted vector field may escape to infinity before the parameter $t \in I$ reaches the value $t = 1$. (In particular, the connection obtained as the orthogonal complement to the vertical subbundle of the fibre bundle projection with respect to some Riemannian metric on H^*E need not be complete.) It was proved by del Hoyo [4] that a smooth surjective submersion $\pi : E \to X$ admits a complete Ehresmann connection if and only if it is a fibre bundle.

Acknowledgements. Forstnerič was supported by the European Union (ERC Advanced grant HPDR, 101053085) and grants P1-0291 and N1-0237 from ARIS, Republic of Slovenia. Part of this work was done during Forstnerič's visit to Adelaide University in February 2026, and he wishes to thank the university for its hospitality.

Use of artificial intelligence. AI assistants were used to search for references and weed out minor mistakes, inconsistent notation, and typographical errors.

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