THE OKA PRINCIPLE FOR HOLOMORPHIC FIBRE BUNDLES OF HÖLDER–ZYGMUND CLASSES ON STRONGLY PSEUDOCONVEX DOMAINS

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ABSTRACT. Let $\overline{\Omega}$ be a compact strongly pseudoconvex domain with smooth boundary in a Stein manifold, and let $h : Z \to \overline{\Omega}$ be a fibre bundle of Hölder–Zygmund class Λ^r, $r > 0$, which is holomorphic over Ω. Assuming that the fibre is an Oka manifold, we prove that every continuous section $f_0 : \overline{\Omega} \to Z$ is homotopic to a section $f_1 : \overline{\Omega} \to Z$ of class $\Lambda^r(\overline{\Omega})$ which is holomorphic on Ω, and we establish a parametric version of the same result. As an application, we obtain the Oka principle for the classification of vector bundles and principal bundles of Hölder–Zygmund classes.

In Memory of the 100th Birthday of Professor Lu Qikeng

CONTENTS

1. INTRODUCTION

A complex manifold Y is said to be an *Oka manifold* (see [14] and [15, Chap. 5]) if maps $X \to Y$ from any Stein manifold X satisfy the h-principle, also called the Oka principle. This means in particular that any continuous map $f_0 : X \to Y$ is homotopic to a holomorphic map $f_1 : X \to Y$; if f_0 is holomorphic on a neighbourhood of a compact holomorphically convex subset K of X and on a closed complex subvariety X' of X then a homotopy $f_t : X \to Y$ $(t \in [0,1])$ from f_0 to a holomorphic map f_1 can be chosen to consist of maps which are holomorphic on a neighbourhood of K, uniformly close to f_0 on K, and such that the homotopy is fixed on X'. The analogous results hold for sections of holomorphic fibre bundles with Oka fibres, and for sections of elliptic holomorphic submersions $Z \to X$ onto a Stein space. For the theory of Oka manifolds and Oka maps, see [11, 12, 15, 16]. Classical examples of Oka manifolds include complex homogeneous manifolds (see Grauert [24, 25] and [15, Proposition 5.6.1]) and, more generally, Gromov elliptic manifolds (see [27] and [15, Corollary 5.6.14]). Oka manifolds Y are characterised by the approximation property for holomorphic maps $K \to Y$ from (neighbourhoods of) compact convex sets $K \subset \mathbb{C}^n$ by entire maps

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$\mathbb{C}^n \to Y$ (the *convex approximation property*, CAP); see [14] and [15, Theorem 5.4.4]), and by *convex relative ellipticity*, CRE (see Kusakabe [35, Theorem 1.3] and [12, Definition 1.5 and Theorem 1.6]). It has recently been shown that every projective Oka manifold is elliptic [17], but there exist noncompact Oka manifolds which fail to be elliptic [36, 11]. Modern Oka theory has diverse applications.

In the present paper, we establish the following 1-parametric Oka principle for sections of fibre bundles of Hölder–Zygmund classes $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ $(r > 0)$ with Oka fibres on compact strongly pseudoconvex domains $\overline{\Omega}$ with Stein interior. The definition of the Banach space $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ is given in Subsect. 2.1, and fibre bundles of this class are introduced in Subsect. 2.2.

Theorem 1.1. *Assume that Ω is a relatively compact, strongly pseudoconvex domain with smooth boundary $b\Omega$ in a Stein manifold X, $r > 0$, and $h : Z \to \overline{\Omega}$ is a fibre bundle of Hölder–Zygmund class $\Lambda^r(\overline{\Omega})$ which is holomorphic on Ω. Assuming that the fibre of h is an Oka manifold, every continuous section $f_0 \in \Gamma(\overline{\Omega}, Z)$ of h is homotopic to a section $f \in \Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$ of class $\Lambda^r(\overline{\Omega})$ which is holomorphic on Ω. Furthermore, every homotopy of continuous sections $\{f_t\}_{t\in[0,1]} \in \Gamma(\overline{\Omega}, Z)$ with $f_0, f_1 \in \Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$ can be deformed with fixed ends to a homotopy in $\Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$.*

Theorem 1.1 is proved in Sect. 6 where we also give a fully parametric version, Theorem 6.1. The proof will also give an approximation theorem common in Oka theory: if the given section f_0 in the theorem is holomorphic on a neighbourhood of a compact holomorphically convex subset $K \subset \Omega$, then the homotopy from f_0 to a holomorphic section f_1 can be chosen to consist of section that are holomorphic on a neighbourhood of K and approximate f_0 uniformly on K. A stronger approximation theorem, with $K \subset \overline{\Omega}$ holomorphically convex in $\overline{\Omega}$, is also possible; see [8, Theorem 6.1] where the analogue of this result is established for fibre bundles $Z \to \overline{\Omega}$ of class $\mathscr{A}^r(\overline{\Omega}) = \{f \in \mathscr{C}^r(\overline{\Omega}) : f|_\Omega \in \mathscr{O}(\Omega)\}$, $r \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Here, $\mathscr{O}(\Omega)$ is the space of holomorphic functions on Ω.

The motivation for the present generalisation of the results from [8] is that the spaces Λ^r behave better than the $\mathscr{C}^k$ or Lipschitz spaces for many operators considered in analysis, as was already noticed by Zygmund in 1945. More explicitly, I was asked by Andrei Teleman in a private communication (January 2026) whether Theorem 1.1 holds. He intends to apply this result in a project of his.

Although Theorem 1.1 and the other results of the paper are stated for domains in Stein manifolds, they hold for any compact complex manifold $\overline{\Omega}$ with Stein interior and smooth strongly pseudoconvex boundary. Indeed, by Ohsawa [45], Heunemann [31], and Catlin [5] such a manifold holomorphically embeds as a smoothly bounded strongly pseudoconvex domain in a Stein manifold.

We refer to Subsect. 2.1 for the definition and basic properties of Hölder–Zygmund spaces $\Lambda^r(\overline{\Omega})$. For $r = k + \alpha$ with $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, $\Lambda^r(\overline{\Omega})$ coincides with the Hölder space $\mathscr{C}^{k,\alpha}(\overline{\Omega})$. For integer values $r = k + 1 \in \{1, 2, \ldots\}$, the Zygmund space $\Lambda^{k+1}(\overline{\Omega})$ properly contains the Lipschitz space $\mathscr{C}^{k,1}(\overline{\Omega})$ of functions whose derivatives up to order k are Lipschitz continuous on $\overline{\Omega}$. The main difference is that, in the definition of the Λ^1 norm of a function f, one replaces the first difference $\Delta_h f(x)$ (see (2.1)) by the second difference $\Delta_h^2 f(x) = \Delta_h \circ \Delta_h f(x)$ (see (2.2)). The Zygmund class Λ^1 is the natural substitute for the Lipschitz class $\mathscr{C}^{0,1} = \mathrm{Lip}^1$ in many contexts. There are continuous strict embeddings among the classical and the Hölder–Zygmund scales on any Lipschitz domain; see (2.10). An important fact used in the proof of Theorem 1.1 is that the canonical (Kohn) solution operator for the $\overline{\partial}$-equation is bounded on $\Lambda^r(\overline{\Omega})$ when Ω is a smoothly bounded strongly pseudoconvex domain; see Beals et al. [1]. The result that we shall use is stated as Theorem 2.2.

On the way to Theorem 1.1, we obtain several other results of independent interest. Given a Lipschitz domain $\Omega \Subset X$, we denote by $\mathscr{O}(\Omega)$ the space of holomorphic functions on Ω, and for any $r > 0$ we let

$$\Lambda^r_{\mathscr{O}}(\overline{\Omega}) = \{f \in \Lambda^r(\overline{\Omega}) : f|_\Omega \in \mathscr{O}(\Omega)\}.$$

Similarly we define the mapping spaces $\mathscr{O}(\Omega, Y)$ and $\Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y) \subset \Lambda^r(\overline{\Omega}, Y)$ for any complex manifold Y. Given a compact set $K \subset X$, we denote by $\mathscr{O}(K)$ the space of restrictions to K of holomorphic functions in open neighbourhoods of K, endowed with the inverse limit topology. The notation $\mathscr{O}(K, Y)$ is used for the space of maps of this kind to a complex manifold Y.

We have the following approximation theorem.

Theorem 1.2. *Let Ω be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold X, and let Y be a complex manifold. Every map in $\Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$, $r > 0$, can be approximated in the $\Lambda^r(\overline{\Omega}, Y)$ topology by functions in $\mathscr{O}(\overline{\Omega}, Y)$.*

Theorem 1.2 is proved in Sect. 3; see also the parametric version in Theorem 3.2. The analogue of Theorem 1.2 is known for spaces $\mathscr{A}^r(\overline{\Omega})$ with $r \in \mathbb{Z}_+$; see [29, Theorem 2.9.2, p. 87] and [10, Theorem 24, p. 165]. For approximation of manifold-valued maps in $\mathscr{A}^r(\overline{\Omega}, Y)$, where Y is a complex manifold, by holomorphic maps from open neighbourhoods of $\overline{\Omega}$ in X, see [8, Theorem 1.2], [15, Theorem 8.11.4], and [10, Corollary 9, p. 178]. The last mentioned result in [10] is a more general Mergelyan-type approximation theorem on strongly admissible sets in Stein manifolds.

We also have the following result generalising [13, Theorem 1.1 (i)]. The proof in [13, Sect. 2] also applies to spaces $\Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$.

Theorem 1.3. *Assume that Ω is a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold and Y is a complex manifold. For every $r > 0$ the space $\Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$ is a complex Banach manifold. The tangent space $T_f \Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$ at a point $f \in \Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$ is the space of sections of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ of the complex vector bundle $f^*TY \to \overline{\Omega}$.*

In the proof of Theorem 1.1, we shall need several results concerning vector bundles $E \to \overline{\Omega}$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ on smoothly bounded strongly pseudoconvex Stein domains Ω; see Sect. 4. In particular, we obtain Theorem A for such bundles; see Theorems 4.1 and 4.10. Their proof is based on Cartan's lemma for maps of Hölder–Zygmund classes to a complex Lie group; see Lemmas 4.3, 4.5 and Remark 4.6. The main technical results used in the proof of Theorem 1.1 are a splitting lemma (see Lemma 5.1) and a gluing lemma (see Lemma 5.3) for sprays of sections of class $\Lambda^r_{\mathscr{O}}$.

An application of Theorem 1.1 is the following Oka principle for vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, proved in Sect. 7. The analogous result for principal fibre bundles is Theorem 7.2.

Theorem 1.4. *Let Ω be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold. The following hold for every real number $r > 0$.*

- (i) *Every topological complex vector bundle on $\overline{\Omega}$ is isomorphic to a vector bundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$.*
- (ii) *If a pair of vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ are isomorphic as topological complex vector bundles, then they are also isomorphic as $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ vector bundles.*

This result is classical for vector bundles on Stein spaces; see Grauert [26], with the special case of line bundles due to Oka [46]. (See also Leiterer [39] and [15, Theorem 5.3.1].) For vector bundles of class $\mathscr{A}^k(\overline{\Omega})$, $k \in \mathbb{Z}_+$, see Leiterer [37, 38] and Heunemann [30, Theorem 2].

Remark 1.5. The results of this paper apply to a wider class of mapping spaces. Assume that $\mathcal{F}$ is a contravariant functor from the category of compact smooth manifolds M with boundary to the category of Banach algebras of $\mathbb{C}$-valued functions on them, satisfying the following conditions:

- (a) $\mathscr{C}^\infty(M) \subset \mathcal{F}(M) \subset \mathscr{C}(M)$ and both inclusions are continuous.
- (b) A smooth map $\Phi : M \to M'$ induces a homomorphism $\Phi^* : \mathcal{F}(M') \to \mathcal{F}(M)$ of Banach algebras by $f \mapsto f \circ \Phi$ for $f \in \mathcal{F}(M')$.
- (c) Postcomposition by a smooth function $\mathbb{C} \to \mathbb{C}$ induces a continuous selfmap of $\mathcal{F}(M)$.

These properties imply that the topology on $\mathcal{F}(M)$ can be defined via local charts, and hence the definition of these classes extends to differential forms and other tensor fields on M. Furthermore, we can introduce vector bundles and more general fibre bundles of class $\mathcal{F}$; see Palais [47, 48] and the references in [13]. When $M = \overline{\Omega}$ is a compact complex manifold with smoothly boundary, we define

$$\mathcal{F}_{\mathscr{O}}(\overline{\Omega}) = \{f \in \mathcal{F}(\overline{\Omega}) : f|_{\Omega} \in \mathscr{O}(\Omega)\}.$$

The analogous definition yields the mapping space $\mathcal{F}_{\mathscr{O}}(\overline{\Omega}, Y)$ for any complex manifold Y. Conditions (a)–(c) on the functor $\mathcal{F}$ clearly imply the following properties:

- A smooth map $\Phi : \overline{\Omega} \to \overline{\Omega}'$ which is holomorphic on Ω induces a homomorphism $\Phi^* : \mathcal{F}_{\mathscr{O}}(\overline{\Omega}') \to \mathcal{F}(\overline{\Omega})$ by $f \mapsto f \circ \Phi$, $f \in \mathcal{F}_{\mathscr{O}}(\overline{\Omega}')$.
- Postcomposition by an entire function $g : \mathbb{C} \to \mathbb{C}$ induces a continuous selfmap of $\mathcal{F}_{\mathscr{O}}(\overline{\Omega})$.

To conditions (a)–(c) we add the following condition:

(d) There is a bounded linear operator $T : \mathcal{F}_{0,1}(\overline{\Omega}) \to \mathcal{F}(\overline{\Omega})$ satisfying $\overline{\partial}\, T(\alpha) = \alpha$ for any $(0,1)$-form $\alpha \in \mathcal{F}_{0,1}(\overline{\Omega})$ with $\overline{\partial}\alpha = 0$.

In the proofs of our results, the operator $\overline{\partial}$ is only applied to functions of the form χf with $\chi \in \mathscr{C}^\infty(\overline{\Omega})$ and $f \in \mathcal{F}_{\mathscr{O}}(\overline{\Omega})$. In this case, the derivative $\overline{\partial}(\chi f) = f\overline{\partial}\chi \in \mathcal{F}_{0,1}(\overline{\Omega})$ is of the same class as f.

Inspections of proofs of our results show that they hold on Banach algebras $\mathcal{F}_{\mathscr{O}}(\overline{\Omega})$, and for vector and fibre bundles of this class, when $\mathcal{F}$ satisfies condition (a)–(d). Examples include Hölder spaces $\mathscr{C}^{k,\alpha}(\overline{\Omega})$ $(k \in \mathbb{Z}_+,\ 0 < \alpha < 1)$, Hölder–Zygmund spaces $\Lambda^r(\overline{\Omega})$ $(r > 0)$, and Sobolev spaces $W^{k,p}(\overline{\Omega})$ $(k \in \mathbb{N},\ 1 \le p < \infty,\ kp > \dim_{\mathbb{R}} \Omega)$, among others.

2. Preliminaries on Hölder–Zygmund spaces Λ^r

In the first subsection, we recall the definition and basic properties of Hölder spaces $\mathscr{C}^{k,\alpha}$ $(k \in \mathbb{Z}_+,\ 0 < \alpha \le 1)$ and Hölder–Zygmund spaces Λ^r for any real $r > 0$. We refer to the papers by Gong [21, Sect. 5], Wallin [53], and the monographs by Gilbarg and Trudinger [19, Sect. 4.1] and Stein [52, Sect. V.4] for more information. In the second subsection, we introduce vector bundles and more general fibre bundles of these classes. In the third subsection, we recall the results on the canonical solution to the $\overline{\partial}$-equation in the spaces Λ^r, which will be used in the paper.

2.1. **Hölder–Zygmund spaces.** Let Ω be a domain in $\mathbb{R}^n$. Given a function $f : \Omega \to \mathbb{C}$ and $h \in \mathbb{R}^n \setminus \{0\}$, the first and the second difference of f at $x \in \Omega$ with step h are defined by

$$\Delta_h(x) = f(x+h) - f(x), \qquad x + h \in \Omega; \tag{2.1}$$

$$\Delta_h^2 f(x) = f(x+2h) + f(x) - 2f(x+h), \qquad x+h,\ x+2h \in \Omega. \tag{2.2}$$

The function f belongs to $\mathscr{C}^{0,\alpha}(\Omega, x)$ $(x \in \Omega,\ 0 < \alpha \le 1)$ if

$$[f]_{\alpha,x} := \sup_{h \neq 0,\ x+h \in \Omega} |h|^{-\alpha} |\Delta_h f(x)| = \sup_{y \in \Omega \setminus \{x\}} \frac{|f(y) - f(x)|}{|y - x|^\alpha} < \infty.$$

Such f is said to be Hölder class α at x. For $\alpha = 1$, $\mathscr{C}^{0,\alpha}(\Omega, x) = \mathrm{Lip}^1(\Omega, x)$ is the Lipschitz class. The Hölder-α space on Ω is

$$\mathscr{C}^{0,\alpha}(\Omega) = \big\{ f : \Omega \to \mathbb{C} : \|f\|_{\mathscr{C}^{(0,\alpha)}(\Omega)} = \sup_{x \in \Omega} |f(x)| + \sup_{x \in \Omega} [f]_{\alpha,x} < \infty \big\}. \tag{2.3}$$

For $\alpha = 1$ we have the Lipschitz space $\mathscr{C}^{0,1}(\Omega) = \mathrm{Lip}^1(\Omega)$.

Let $x = (x_1, \ldots, x_n)$ be the coordinates on $\mathbb{R}^n$. Given $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$, set $|\beta| = \beta_1 + \cdots + \beta_n$ and $D^\beta f = \partial^{|\beta|} f / \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}$. For $k \in \mathbb{Z}_+$ we denote by $\mathscr{C}^k(\Omega)$ the space of k-times continuously differentiable functions f on Ω with

$$\|f\|_{\mathscr{C}^k(\Omega)} = \sum_{|\beta| \le k} \sup_{x \in \Omega} |D^\beta f(x)| < \infty.$$

The Hölder space $\mathscr{C}^{k,\alpha}(\Omega)$ for $k \in \mathbb{Z}_+$ and $0 < \alpha \le 1$ is defined by

$$\mathscr{C}^{k,\alpha}(\Omega) = \Big\{ f \in \mathscr{C}^k(\Omega) : D^\beta f \in \mathscr{C}^{0,\alpha}(\Omega) \text{ for all } \beta \in \mathbb{Z}_+^n \text{ with } |\beta| \le k, \tag{2.4}$$

$$\|f\|_{\mathscr{C}^{k,\alpha}(\Omega)} := \|f\|_{\mathscr{C}^k(\Omega)} + \sum_{|\beta| = k} \|D^\beta f\|_{\mathscr{C}^{0,\alpha}(\Omega)} < \infty \Big\}. \tag{2.5}$$

Assume now that $\Omega \Subset \mathbb{R}^n$ is a bounded Lipschitz domain. This means that its boundary $b\Omega$ is locally at each point a Lipschitz graph over an affine hyperplane in $\mathbb{R}^n$, with the domain lying on one side of the graph. A function $f : \overline{\Omega} \to \mathbb{C}$ belongs to $\mathscr{C}^k(\overline{\Omega})$ for some $k \in \mathbb{Z}_+$ if it is the restriction to $\overline{\Omega}$ of a function $\tilde{f} \in \mathscr{C}^k(\mathbb{R}^n)$. (For $k = 0$, this coincides with the usual definition of continuous functions by Tietze's extension theorem.) Given $\beta \in \mathbb{Z}_+^n$ with $|\beta| \le k$, we denote by $D^\beta f$ the restriction of $D^\beta \tilde{f}$ to $\overline{\Omega}$; note that $D^\beta f$ is independent of the choice of the extension $\tilde{f}$ if Ω is a Lipschitz domain. Given $0 < \alpha \le 1$, the Hölder space $\mathscr{C}^{k,\alpha}(\overline{\Omega})$ on the closed domain $\overline{\Omega}$ is defined by

$$\mathscr{C}^{k,\alpha}(\overline{\Omega}) = \big\{ f \in \mathscr{C}^k(\overline{\Omega}) : D^\beta f \in \mathscr{C}^{0,\alpha}(\overline{\Omega}) \text{ for all } \beta \in \mathbb{Z}_+^n \text{ with } |\beta| \le k \big\},$$

endowed with the norm $\|f\|_{\mathscr{C}^{k,\alpha}(\overline{\Omega})}$ (2.5). We observe the following.

Proposition 2.1. *If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ then every function in $\mathscr{C}^{k,\alpha}(\Omega)$ $(k \in \mathbb{Z}_+,\ 0 < \alpha \le 1)$ extends to a unique function $\tilde{f} \in \mathscr{C}^{k,\alpha}(\overline{\Omega})$.*

Proof. For $k = 0$ this holds by McShane's extension theorem [44, Corollary 1, p. 840]. If $k \ge 1$ and $f \in \mathscr{C}^{k,\alpha}(\Omega)$, then its partial derivatives $D^\beta f$ of order $|\beta| = k$ belong to $\mathscr{C}^{0,\alpha}(\Omega)$, extend to functions $g_\beta = \mathscr{C}^{0,\alpha}(\overline{\Omega})$ by McShane's theorem. By Whitney's theorem [54] (see also Malgrange [43, Theorem 3.2 and Comp. 3.5]), it follows that f extends to a function $\tilde{f} \in \mathscr{C}^k(\mathbb{R}^n)$ satisfying $D^\beta \tilde{f} = g_\beta$ on $\overline{\Omega}$ for all $|\beta| = k$. Thus, $\tilde{f}|_{\overline{\Omega}} \in \mathscr{C}^{k,\alpha}(\overline{\Omega})$. Clearly, the extension $\tilde{f}|_{\overline{\Omega}}$ of f is unique. □

We now recall the definition of Hölder–Zygmund spaces $\Lambda^r(\Omega)$ and $\Lambda^r(\overline{\Omega})$ for any real number $r > 0$. Write $r = k + \alpha$ with $k \in \mathbb{Z}_+$ and $0 < \alpha \le 1$. If r is not an integer (that is, $\alpha \ne 1$), set $\Lambda^r(\Omega) = \mathscr{C}^{k,\alpha}(\Omega)$ and $\Lambda^r(\overline{\Omega}) = \mathscr{C}^{k,\alpha}(\overline{\Omega})$; both spaces are endowed with the norm $\|f\|_{\Lambda^{k+\alpha}(\Omega)} := \|f\|_{\mathscr{C}^{k,\alpha}(\Omega)}$ (2.5). Assume now that $r = k+1 \in \mathbb{N}$ is an integer. Recall that the second difference $\Delta_h^2 f$ is given by (2.2). We define

$$\Lambda^1(\Omega) = \big\{ f \in \mathscr{C}(\Omega) : \|f\|_{\Lambda^1(\Omega)} = \sup_{x \in \Omega} |f(x)| + \sup_{x, x+h \in \Omega,\, h \ne 0} |h|^{-1} |\Delta_h^2 f(x)| < \infty \big\}, \tag{2.6}$$

$$\Lambda^r(\Omega) = \big\{ f \in \mathscr{C}^{r-1}(\Omega) : \|f\|_{\Lambda^r(\Omega)} = \|f\|_{\mathscr{C}^{r-1}(\Omega)} + \sum_{|\beta| = r-1} \|D^\beta f\|_{\Lambda^1(\Omega)} < \infty \big\}, \tag{2.7}$$

where $r > 1$ in (2.7). The space $\Lambda^1(\mathbb{R}^n)$ is the classical (nonhomogeneous) Zygmund space on $\mathbb{R}^n$; omitting the term $\|f\|_\infty = \sup_{x \in \Omega} |f(x)|$ in (2.6) gives the homogeneous Zygmund space.

For $r \in \mathbb{N}$ the Zygmund space $\Lambda^r(\overline{\Omega})$ on a closed domain $\overline{\Omega}$ cannot be defined by considering only values of functions in the points of $\overline{\Omega}$. Indeed, if Ω is strictly convex and $x, x+h \in b\Omega$ with $h \ne 0$, then $x + 2h \notin \overline{\Omega}$, so $\Delta_h^2 f(x)$ is not defined. One possible definition is the following; see Wallin [53, Definition 8, p. 105]. It can be used for any $r > 0$, not only for integers.

$$\Lambda^r(\overline{\Omega}) = \big\{f = \tilde f|_{\overline{\Omega}} : \tilde f \in \Lambda^r(\mathbb{R}^n)\big\}, \tag{2.8}$$

$$\|f\|_{\Lambda^r(\overline{\Omega})} = \inf\big\{\|\tilde f\|_{\Lambda^r(\mathbb{R}^n)} : \tilde f \in \Lambda^r(\mathbb{R}^n),\ \tilde f|_{\overline{\Omega}} = f\big\}. \tag{2.9}$$

One can also use extensions of f to any domain $D \subset \mathbb{R}^n$ containing $\overline{\Omega}$; it is easily seen that the resulting norms are comparable. However, it is not clear whether these norms decrease to the interior norm $\|f\|_{\Lambda^r(\Omega)}$ (2.7) as D shrinks to $\overline{\Omega}$. Stein [52, Sect. VI.2] constructed a linear extension operator $E : \mathscr{C}(\overline{\Omega}) \to \mathscr{C}_0(\mathbb{R}^n)$ on any Lipschitz domain $\Omega \Subset \mathbb{R}^n$ such that $E : \Lambda^r(\overline{\Omega}) \to \Lambda^r(\mathbb{R}^n)$ is bounded for every $r > 0$. (Stein proved that his extension operator is bounded on Sobolev spaces, and for Hölder–Zygmund spaces the same was shown by Gong [22].)

It turns out that for any bounded Lipschitz domain Ω in $\mathbb{R}^n$, the interior and the exterior definition of the spaces $\Lambda^r(\overline{\Omega})$ are equivalent. Indeed, Shi and Yao [51, Theorem 1.1] have recently shown that $\Lambda^r(\Omega) = \{f|_\Omega : f \in \Lambda^r(\mathbb{R}^n)\}$, and the interior norm $\|f\|_{\Lambda^r(\Omega)}$ (2.7) is comparable to the exterior norm $\|f\|_{\Lambda^r(\overline{\Omega})}$ (2.9). Their proof uses Rychkov's universal extension for Besov spaces $B^r_{p,q}(\Omega)$; see [49] and note that $\Lambda^r(\Omega) = B^r_{\infty,\infty}(\Omega)$. For noninteger values $r > 0$ and domains Ω with smooth boundaries, see also [19, Lemma 6.37] whose proof is based on Seeley's extension theorem [50].

The space $\Lambda^r(\overline{\Omega})$, $r > 0$, on a Lipschitz domain is a commutative unital Banach algebra under pointwise multiplication, with Moser-type estimates for products, compositions, and inverses; see [22, Lemmas 3.1–3.3], [21, Lemma 6.3], and [2]. In particular, precompositions and postcompositions by smooth maps preserve the Hölder–Zygmund classes. There are continuous strict embeddings among the classical and the Hölder–Zygmund scales on any Lipschitz domain:

$$\mathscr{C}^{k+1} \subset \mathscr{C}^{k,1} \subset \Lambda^{k+1} \subset \Lambda^{k+\alpha} \subset \Lambda^{k+\beta} \subset \mathscr{C}^k, \quad k \in \mathbb{Z}_+,\ 0 < \beta < \alpha < 1. \tag{2.10}$$

An example showing that $\mathrm{Lip}^1(\mathbb{R}) = \mathscr{C}^{0,1}(\mathbb{R}) \subsetneq \Lambda^1(\mathbb{R})$ can be found in [52, Example 4.3.1, p. 148].

If X is a smooth manifold and Ω is a Lipschitz domain in X with compact closure, one defines the spaces $\Lambda^r(\Omega)$ and $\Lambda^r(\overline{\Omega})$ by using a finite system of smooth coordinate charts on X covering $\overline{\Omega}$. The norms obtained in this way are comparable to one another. For the details, see Palais [47, 48] and the discussion and references in [2], [13, Sect. 2] and [23, Subsect. 3.2].

2.2. **Mapping spaces and fibre bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$.** Let Ω be a relatively compact domain with Lipschitz boundary in a smooth manifold X. Although the function spaces $\Lambda^r(\overline{\Omega})$ introduced above are given by global conditions, the definition is local in the sense that $f \in \Lambda^r(\overline{\Omega})$ if and only if every point $x \in \overline{\Omega}$ has a compact Lipschitz neighbourhood $O_x \subset \overline{\Omega}$ such that $f|_{O_x} \in \Lambda^r(O_x)$. Since postcompositions of functions in $\Lambda^r(\overline{\Omega})$ with smooth functions on $\mathbb{C}$ are again in $\Lambda^r(\overline{\Omega})$ (and such a postcomposition is a smooth operator, see [2]), we can define for any smooth manifold Y the space $\Lambda^r(\overline{\Omega}, Y)$ of maps $f : \overline{\Omega} \to Y$ of class Λ^r. Assume now that X and Y are complex manifolds. Set

$$\Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y) = \{f \in \Lambda^r(\overline{\Omega}, Y) : f \text{ is holomorphic on } \Omega\}, \quad r > 0.$$

In particular, $\Lambda^r_{\mathscr{O}}(\overline{\Omega}, \mathbb{C}) = \Lambda^r_{\mathscr{O}}(\overline{\Omega})$.

A holomorphic fibre bundle $h : Z \to \overline{\Omega}$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ has total space of the form $Z = \bigsqcup_{i=1}^m \overline{U}_i \times Y/\sim$ where the fibre Y is a complex manifold, the sets $\overline{U}_i \subset \overline{\Omega}$ and $\overline{U}_{i,j} = \overline{U}_i \cap \overline{U}_j$ are compact and have Lipschitz boundaries, $\bigcup_{i=1}^m \overline{U}_i = \overline{\Omega}$, and a point $(x, y) \in \overline{U}_j \times Y$ ($x \in \overline{U}_{i,j}$) is identified by the equivalence relation $\sim$ with the point $(x, \phi_{i,j}(x, y)) \in \overline{U}_i \times Y$, where the map $\overline{U}_{i,j} \times Y \ni (x, y) \mapsto \phi_{i,j}(x, y) \in Y$ is of class $\Lambda^r_{\mathscr{O}}(\overline{U}_{i,j})$ in x, holomorphic in y, and $\phi_{i,j}(x, \cdot\,) \in \mathrm{Aut}(Y)$ for every $x \in \overline{U}_{i,j}$. A section $f : \overline{\Omega} \to Z$ of $h : Z \to \overline{\Omega}$ is given by a collection of maps $f_i : \overline{U}_i \to Y$ ($i = 1, \ldots, m$) satisfying the compatibility conditions

$$f_i(x) = \phi_{i,j}(x, f_j(x)), \quad x \in \overline{U}_{i,j},\ i, j = 1, \ldots, m.$$

The section f is of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ if and only if $f_i \in \Lambda^r_{\mathscr{O}}(\overline{U}_i, Y)$ for $i = 1, \ldots, m$. We denote by $\Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$ the space of all sections of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. If Y is a smooth manifold, the analogous definition gives fibre bundles $Z \to \overline{\Omega}$ of class $\Lambda^r(\overline{\Omega})$.

A vector bundle $\pi : E \to \overline{\Omega}$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ and rank n has fibre $\mathbb{C}^n$ and transitions maps

$$\phi_{i,j}(x, v) = A_{i,j}(x)v, \quad v \in \mathbb{C}^n,\ A_{i,j} \in \Lambda^r_{\mathscr{O}}(\overline{U}_{i,j}, GL_n(\mathbb{C}))$$

satisfying the 1-cocycle conditions (with $I \in GL_n(\mathbb{C})$ the identity matrix):

$$A_{i,i} = I, \quad A_{i,j}A_{j,i} = I, \quad A_{i,j}A_{j,k}A_{k,i} = I \quad \text{for all } i, j, k.$$

A section of E over $\overline{\Omega}$ is given by a collection of maps $f_i \in \Gamma^r_{\mathscr{O}}(\overline{U}_i, \mathbb{C}^n)$ satisfying

$$f_i(x) = A_{i,j}(x)f_j(x), \quad x \in \overline{U}_{i,j},\ i, j = 1, \ldots, m.$$

Similarly one defines vector bundle morphisms of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, subbundles, quotient bundles, etc.

2.3. **Regularity of the canonical solution operator for the $\overline{\partial}$-equation in Hölder–Zygmund spaces.** Let $p \geq 0$, $q \geq 1$ be integers, and let Ω be a domain in a complex manifold X. The $\overline{\partial}$-problem on Ω asks for the existence and regularity properties of solutions of the equation $\overline{\partial}u = f$ for a differential (p, q)-form f on Ω (or on its closure $\overline{\Omega}$) satisfying the necessary condition $\overline{\partial}f = 0$. One of the most successful techniques in this field is the $\overline{\partial}$–Neumann method introduced in the pioneering works of Kohn [33, 34]; see the books by Folland and Kohn [9] and Chen and Shaw [6], among others.

Let $\overline{\Omega}$ be a compact complex Hermitian manifold of dimension $n + 1$ with $\mathscr{C}^\infty$ boundary $b\Omega$ and Stein interior Ω such that at each point of $b\Omega$ the Levi form has at least $n + 1 - q$ positive eigenvalues. Let N_q denote the Neumann operator for the complex Laplacian

$$\Box_q = \overline{\partial}^*_q\overline{\partial}_q + \overline{\partial}_{q-1}\overline{\partial}^*_{q-1}, \quad q \geq 1$$

acting on (p, q)-forms which satisfy the $\overline{\partial}$-Neumann boundary conditions on $b\Omega$. This means that $N_q\Box_q = \Box_qN_q$ is the L^2 orthogonal projection onto the range of $\Box_q$. Then, $T_q = \overline{\partial}^*_{q-1}N_q$ is Kohn's canonical solution operator for the $\overline{\partial}$-equation $\overline{\partial}_{q-1}u = f$ with f a (p, q)-form with $\overline{\partial}_qf = 0$. Denote by $\Lambda^r_{p,q}(\overline{\Omega})$ the space of (p, q)-forms on $\overline{\Omega}$ of class $\Lambda^r(\overline{\Omega})$ for $r > 0$. (Note that (p, q)-forms are sections of certain vector bundles on $\overline{\Omega}$ derived from the tangent bundle $TX|_{\overline{O}mega}$, so the notion $\Lambda^r(\overline{\Omega})$ classes makes sense.) The following is a special case of [1, Theorem 2] by Beals, Greiner and Stanton.

Theorem 2.2. *Let Ω be as above. The canonical solution $T_q = \overline{\partial}^*_{q-1}N_q$ to $\overline{\partial}u = f$ for $f \in \Lambda^r_{p,q}(\overline{\Omega})$ satisfying $\overline{\partial}f = 0$ maps $\Lambda^r_{p,q}(\overline{\Omega}) \cap \ker\overline{\partial}$ boundedly to $\Lambda^{r+1/2}_{p,q-1}(\overline{\Omega})$ for every $p \geq 0$, $q \geq 1$, and $r > 0$.*

We shall use this result for $p = 0$. Although the gain of regularity for $1/2$ will not be used, it implies that the operator $T_q : \Lambda^r_{p,q}(\overline{\Omega}) \cap \ker\overline{\partial} \to \Lambda^r_{p,q-1}(\overline{\Omega})$ is bounded and compact, a fact which might be useful in other applications.

3. Approximation of maps of class $\Lambda^r_{\mathscr{O}}$ on strongly pseudoconvex domains

In this section, we prove Theorem 1.2 and its parametric version, Theorem 3.2. We shall use the following notion of a Cartan pair; see [15, Definitions 5.7.1 and 5.10.2].

Definition 3.1. A pair (A, B) of compact sets in a complex manifold X is a *Cartan pair* if

(i) A, B, $D = A \cup B$ and $C = A \cap B$ are closures of smoothly bounded, strongly pseudoconvex domains with Stein interior, and
(ii) $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$.

A Cartan pair (A, B) is *special* if there is a coordinate neigbourhood of B in X in which the sets B and C are strongly convex. In this case, B is said to be a convex bump attached to A.

A more general notion of a Cartan pair in [15, Definition 5.7.1 (I)] is suitable when considering holomorphic maps on neighbourhoods of the respective sets. We shall not need it in this paper since our focus in on mapping spaces on compact domains.

Proof of Theorem 1.2. Since the domain $\Omega \Subset X$ is smoothly bounded and strongly pseudoconvex, there is a smooth strongly plusubharmonic function ρ on a neighbourhood $U \subset X$ of $\overline{\Omega}$ such that $\Omega = \{x \in U : \rho(x) < 0\}$ and $d\rho_x \neq 0$ for every point $x \in b\Omega = \{\rho = 0\}$. Pick $c > 0$ such that ρ has no critical values on the interval $[0, c]$. Set $A = \overline{\Omega} = \{\rho \leq 0\}$ and $A' = \{\rho \leq c\}$. Given an open cover $\mathcal{U} = \{U_j\}$ of $\overline{A' \setminus A}$ consisting of holomorphic coordinate charts $U_j \subset X$, [15, Lemma 5.10.3] gives compact, smoothly bounded, strongly pseudoconvex domains

$$A_0 := A \subset A_1 \subset \cdots \subset A_m = A' \tag{3.1}$$

for some $m \in \mathbb{N}$ such that for every $k = 0, 1, \ldots, m-1$ we have $A_{k+1} = A_k \cup B_k$, where (A_k, B_k) is a special Cartan pair (see Definition 3.1) and $B_k \subset U_j$ for some $j = j(k)$.

It therefore suffices to show that for every special Cartan pair (A, B) we can approximate any map $f \in \Lambda^r_{\mathscr{O}}(A, Y)$ as closely as desired by maps $\tilde{f} \in \Lambda^r_{\mathscr{O}}(D, Y)$ where $D = A \cup B$; the theorem then follows by a finite induction using the sequence (3.1). We first consider the case of functions ($Y = \mathbb{C}$). Fix such a pair (A, B) and $f \in \Lambda^r_{\mathscr{O}}(A)$. Set $C = A \cap B$. We can find a holomorphic function g on a neighbourhood of B which approximates $f|_C$ as closely as desired in $\Lambda^r(C)$. To do this, we proceed as follows. By the assumption, there is a neighbourhood $W \subset X$ of B and a holomorphic coordinate map $\psi : W \to \widetilde{W} \subset \mathbb{C}^n$ ($n = \dim X$) such that the sets $\widetilde{C} = \psi(C)$ and $\widetilde{B} = \psi(B)$ are strongly convex. Choose a point p in the interior of $\widetilde{C}$; we may assume that $p = 0 \in \mathbb{C}^n$. For $t \in (0, 1)$ the holomorphic map $\phi_t : W \to W$ defined by $\phi_t(x) = \psi^{-1}(t\psi(x))$, $x \in W$, satisfies $\phi_t(C) \subset \mathring{C}$. The function $f_t = f \circ \phi_t$ is then holomorphic on a neighbourhood $V_t \subset W$ of C and it approximates $f|_C$ for t close to 1. Fix t and pick a compact neigbourhood $C' \subset V_t$ of C such that $\psi(C')$ is convex. By the Oka–Weil theorem, we can approximate f_t uniformly on C' by a function $g \in \mathscr{O}(B)$. If the approximation is close enough in both steps then g approximates $f|_C$ to the desired precision in $\Lambda^r(C)$.

Condition (ii) in Definition 3.1 ensures the existence of a smooth function $\chi : X \to [0, 1]$ which equals 1 on a neighbourhood of $\overline{A \setminus B}$ and equals 0 on a neighbourhood of $\overline{B \setminus A}$. Set

$$u = \chi f + (1 - \chi) g \in \Lambda^r(D).$$

Note that $u = f$ on $\overline{A \setminus B}$, $u = g$ on $\overline{B \setminus A}$, $f - u = (1 - \chi)(f - g)$ on A, and hence

$$\|f - u\|_{\Lambda^r(A)} = \|(1 - \chi)(f - g)\|_{\Lambda^r(A)} \leq c_0 \|f - g\|_{\Lambda^r(C)} \tag{3.2}$$

for some $c_0 > 0$ depending only on χ. Furthermore, $\overline{\partial} u = (f - g)\overline{\partial}\chi \in \Lambda^r_{0,1}(D)$ is a $\overline{\partial}$-closed $(0, 1)$-form whose support is disjoint from $\overline{A \setminus B} \cup \overline{B \setminus A}$. It follows that

$$\|\overline{\partial} u\|_{\Lambda^r_{0,1}(D)} \leq c_1 \|f - g\|_{\Lambda^r(C)}$$

for some $c_1 > 0$ depending on χ. By Theorem 2.2 there exists $\tilde{u} \in \Lambda^r(D)$ satisfying $\overline{\partial}\tilde{u} = \overline{\partial} u$ and

$$\|\tilde{u}\|_{\Lambda^r(D)} \leq c_2 \|\overline{\partial} u\|_{\Lambda^r(D)} \leq c_1 c_2 \|f - g\|_{\Lambda^r(C)} \tag{3.3}$$

for some $c_2 > 0$ depending only on D. The function $\tilde{f} = u - \tilde{u} \in \Lambda^r(D)$ satisfies $\overline{\partial}\tilde{f} = 0$, so it is holomorphic on $\mathring{D}$. On A we have $f - \tilde{f} = (f - u) + \tilde{u}$, and it follows from (3.2)–(3.3) that $\|f - \tilde{f}\|_{\Lambda^r(A)} \leq (c_0 + c_1 c_2)\|f - g\|_{\Lambda^r(C)}$. This proves the theorem for functions. Assume now that Y is a complex manifold. Every map $f \in \Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$ for $r > 0$ is continuous on $\overline{\Omega}$ and holomorphic on Ω, so its graph has a Stein neigbourhood in $X \times Y$ (see [13, Theorem 1.2] or [15, Corollary 8.11.2]). The proof is then reduced to the case of functions as in [15, Theorem 8.11.4]. $\square$

Theorem 1.2 has the following extension to the parametric case.

Theorem 3.2. *Assume that Ω is a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold X, Y is a complex manifold, and P is a compact Hausdorff space. Every continuous map $f : P \to \Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$, $r > 0$, can be approximated by continuous maps $\tilde{f} : P \to \mathscr{O}(\overline{\Omega}, Y)$. If in addition Q is a closed subspace of P which is a strong neighbourhood deformation retract and $f|_Q : Q \to \mathscr{O}(\overline{\Omega}, Y)$, we can choose $\tilde{f}$ to agree with f on Q.*

Proof. Denote by $pr_X : X \times Y \to X$ the projection on the first factor. Fix a point $p_0 \in P$. The map $f(p_0) \in \Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$ is continuous on $\overline{\Omega}$, so its graph over $\overline{\Omega}$ has an open Stein neighbourhood $V_0 \subset X \times Y$. Furthermore, V_0 can be chosen fibrewise biholomorphic (with respect to the projection pr_X) to a domain with convex fibres in a holomorphic vector bundle $E_0 \to U_0$ over an open neighbourhood $U_0 \subset X$ of $\overline{\Omega}$ (see [13, Theorem 1.2] or [15, Theorem 8.11.1]). Hence, the notion of a convex combination of points in the fibres of $pr_X : V_0 \to X$ is well defined. By Theorem 1.2 there is a holomorphic map $\tilde{f}(p_0) \in \mathscr{O}(U_0, Y)$ on a neighbourhood $U_0 \subset X$ of $\overline{\Omega}$ which approximates $f(p_0)$ in $\Lambda^r(\overline{\Omega}, Y)$ to a desired precision and whose graph is contained in V_0. For $p \in P$ close to p_0, the graph of $f(p) \in \Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$ lies in V_0 and $\tilde{f}(p_0)|_{\overline{\Omega}}$ is an approximant to $f(p)$. Repeating the same procedure at other points of P gives a finite open covering $\mathcal{P} = \{P_i\}_{i=0}^m$ of P and for each $i = 0, \ldots, m$ open Stein domains $U_i \subset X$, $V_i \subset X \times Y$ and a map $\tilde{f}_i \in \mathscr{O}(U_i, Y)$ which is close to all maps $f(p)$, $p \in U_i$, to a desired precision in $\Lambda^r(\overline{\Omega}, Y)$ such that the graphs of these maps belong to V_i. In view of the fibrewise convex structure of every Stein domain V_i, we can use the *method of successive patching* (see [15, p. 78, p. 282] for the details) to obtain a map $\tilde{f} : P \to \mathscr{O}(\overline{\Omega}, Y)$ approximating f to a desired precision. For the last statement in the theorem, we precompose f by a continuous map $\psi : P \to P$ which maps a small neighbourhood $Q_0' \subset P$ of Q to itself, it retracts a neighbourhood $Q_0 \subset Q_0'$ of Q onto Q, and it equals the identity map on $P \setminus Q_0'$. (Such ψ exists since Q is a strong deformation neighbourhood retract in P.) This yields a continuous map $f_0 = f \circ \psi : P \to \Lambda^r_{\mathscr{O}}(\overline{\Omega}, Y)$ which approximates f, it agrees with f on Q, and such that $f_0|_{Q_0} : Q_0 \to \mathscr{O}(\overline{\Omega}, Y)$. Applying the above procedure to f_0 yields an approximating map $\tilde{f} : P \to \mathscr{O}(\overline{\Omega}, Y)$ which agrees with f on Q. ☐

4. Cartan's lemma and Theorem A for vector bundles of class $\Lambda^r_{\mathscr{O}}$

The notion of a complex vector bundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, $r > 0$, was introduced in Subsect. 2.2. In this section, we prove the following version of Cartan's Theorem A for such vector bundles.

Theorem 4.1. *Let $\overline{\Omega}$ be a compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold X, and let $\pi : E \to \overline{\Omega}$ be a vector bundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ for some $r > 0$. There exist finitely many sections in $\Gamma^r_{\mathscr{O}}(\overline{\Omega}, E)$ spanning every fibre $E_x := \pi^{-1}(x)$, $x \in \overline{\Omega}$.*

The classical Theorem A of Cartan gives such a statement for holomorphic vector bundles on finite dimensional Stein spaces. For vector bundles of class $\mathscr{A}^r(\overline{\Omega})$ with $r \in \mathbb{Z}_+$, and for more general coherent analytic sheaves of this class, the analogue of Theorem 4.1 is due to Leiterer [38] for domains in Euclidean spaces and Heumenann [32, Theorems 2, 6] in general.

An equivalent statement is that every vector bundle $E \to \overline{\Omega}$ as in the theorem admits a vector bundle epimorphism $\Phi : \overline{\Omega} \times \mathbb{C}^m \to E$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. Indeed, if $\xi_1, \ldots, \xi_m : \overline{\Omega} \to E$ are sections of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ spanning every fibre of E then the map $\Phi : \overline{\Omega} \times \mathbb{C}^m \to E$ given by

$$\Phi(x, z_1, \ldots, z_m) = \sum_{i=1}^m z_i \xi_i(x) \in E_x = \pi^{-1}(x), \quad x \in \overline{\Omega},\ z = (z_, \ldots, z_m) \in \mathbb{C}^m \tag{4.1}$$

is a vector bundle epimorphism of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. Conversely, given such an epimorphism, the images of standard basis sections of the trivial bundle generate each fibre of E.

We also have the following embedding result for vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$.

Corollary 4.2. *Given a vector bundle $\pi : E \to \overline{\Omega}$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ as in Theorem 4.1, there exists a vector bundle embedding $E \hookrightarrow \overline{\Omega} \times \mathbb{C}^m$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ for some $m \in \mathbb{N}$.*

Proof. Theorem 4.1 gives a vector bundle epimorphism $\Phi : \overline{\Omega} \times \mathbb{C}^m \to E^*$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. Its dual $\Phi^* : (E^*)^* = E \to (\overline{\Omega} \times \mathbb{C}^m)^* \cong \overline{\Omega} \times \mathbb{C}^m$ is a vector bundle embedding of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. □

Denote by $M_n = M_n(\mathbb{C})$ the space of complex $n \times n$ matrices and by $GL_n = GL_n(\mathbb{C})$ the general linear group of rank n over $\mathbb{C}$. By $I \in GL_n$ we denote the identity matrix. In the proof of Theorem 4.1 we shall use the following version of Cartan's lemma. See also the parametric version, Lemma 4.5, and [3] or [28, Section VI. E] for the classical Cartan lemma.

Lemma 4.3. *Let (A, B) be a Cartan pair in a complex manifold X (see Definition 3.1) such that $C := A \cap B$ is holomorphically convex in B. Given a map $\gamma \in \Lambda^r_{\mathscr{O}}(C, GL_n)$ which is homotopic to the constant map $C \in x \mapsto I \in GL_n$ by a path in $\mathscr{C}(C, GL_n)$ and $\epsilon > 0$, there are maps $\alpha \in \Lambda^r_{\mathscr{O}}(A, GL_n)$ and $\beta \in \Lambda^r_{\mathscr{O}}(B, GL_n)$ such that $\|\alpha - I\|_{\Lambda^r(A)} < \epsilon$ and*

$$\gamma = \alpha^{-1} \cdot \beta \quad \textit{holds on } C. \tag{4.2}$$

Proof. We first construct the product splitting (4.2) for γ close to $I \in GL_n$. Write $\gamma = I + c$ with $c \in \Lambda^r_{\mathscr{O}}(C, M_n)$. We have the following solution to the Cousin problem in $\Lambda^r_{\mathscr{O}}$.

Lemma 4.4. *There are bounded linear operators $\mathcal{A} : \Lambda^r_{\mathscr{O}}(C) \to \Lambda^r_{\mathscr{O}}(A)$, $\mathcal{B} : \Lambda^r_{\mathscr{O}}(C) \to \Lambda^r_{\mathscr{O}}(B)$ with*

$$\mathcal{A}c + \mathcal{B}c = c \quad \textit{for all } c \in \Lambda^r_{\mathscr{O}}(C). \tag{4.3}$$

Proof. Set $D = A \cup B$. Condition (ii) in Definition 3.1 implies that there is a smooth function $\chi : X \to [0, 1]$ which equals 1 on a neighbourhood of $\overline{A \setminus B}$ and equals 0 on a neighbourhood of $\overline{B \setminus A}$. Given $c \in \Lambda^r_{\mathscr{O}}(C)$, we have $c\chi \in \Lambda^r(A)$, $c(1-\chi) \in \Lambda^r(B)$, and $\overline{\partial}(c\chi) = c\,\overline{\partial}\chi \in \Lambda^r_{0,1}(D)$ is a closed $(0, 1)$-form on D of class $\Lambda^r(D)$ supported on C. Let

$$T = \overline{\partial}^* N : \{\omega \in \Lambda^r_{0,1}(D) : \overline{\partial}\omega = 0\} \to \Lambda^r(D)$$

denote the (bounded, linear) canonical solution operator for the $\overline{\partial}$-equation for $(0, 1)$-forms of class $\Lambda^r(D)$; see Theorem 2.2. The linear operators on $\Lambda^r_{\mathscr{O}}(C)$ defined by

$$\mathcal{A}c = c\,\chi - T(c\,\overline{\partial}\chi) \in \Lambda^r_{\mathscr{O}}(A), \quad \mathcal{B}c = c(1-\chi) + T(c\,\overline{\partial}\chi) \in \Lambda^r_{\mathscr{O}}(B)$$

for all $c \in \Lambda^r_{\mathscr{O}}(C)$ then satisfy the lemma. □

Applying Lemma 4.4 componentwise gives bounded linear operators

$$\mathcal{A} : \Lambda^r_{\mathscr{O}}(C, M_n) \to \Lambda^r_{\mathscr{O}}(A, M_n), \quad \mathcal{B} : \Lambda^r_{\mathscr{O}}(C, M_n) \to \Lambda^r_{\mathscr{O}}(B, M_n)$$

satisfying $\mathcal{A} + \mathcal{B} = \mathrm{Id}$ on $\Lambda^r_{\mathscr{O}}(C, M_n)$; see (4.3). We define the map $\Phi : U \to \Lambda^r_{\mathscr{O}}(C, GL_n)$ on a small neighbourhood $U \subset \Lambda^r_{\mathscr{O}}(C, M_n)$ of $c = 0$ by

$$\Phi(c) = (I - \mathcal{A}c)^{-1}(I + \mathcal{B}c) \in \Lambda^r_{\mathscr{O}}(C, GL_n), \quad c \in U.$$

Clearly, Φ is smooth and satisfies $\Phi(0) = I$ and $d\Phi_0 = \mathcal{A} + \mathcal{B} = \mathrm{Id}$. If follows that Φ has a smooth inverse Ψ on a neigbourhood $V \subset \Lambda^r_{\mathscr{O}}(C, GL_n)$ of I such that $\Psi(I) = 0$ and

$$(\Phi \circ \Psi)(\gamma) = (I - \mathcal{A} \circ \Psi(\gamma))^{-1}\,(I + \mathcal{B} \circ \Psi(\gamma)) = \gamma \quad \text{for all } \gamma \in V.$$

The smooth operators $\widetilde{\mathcal{A}} : V \to \Lambda^r_{\mathscr{O}}(A, GL_n)$, $\widetilde{\mathcal{B}} : V \to \Lambda^r_{\mathscr{O}}(B, GL_n)$ defined by

$$\widetilde{\mathcal{A}} = I - \mathcal{A} \circ \Psi, \qquad \widetilde{\mathcal{B}} = I + \mathcal{B} \circ \Psi$$

then provide a splitting $\gamma = (\widetilde{\mathcal{A}}\gamma)^{-1}(\widetilde{\mathcal{B}}\gamma)$ for $\gamma \in V$ (see (4.2)) satisfying

$$\|\widetilde{\mathcal{A}}\gamma - I\|_{\Lambda^r(A)} \le const\|\gamma - I\|_{\Lambda^r(C)}, \quad \|\widetilde{\mathcal{B}}\gamma - I\|_{\Lambda^r(B)} \le const\|\gamma - I\|_{\Lambda^r(C)} \tag{4.4}$$

for some $const > 0$ depending only on (A, B) and r.

This proves the lemma for maps $\gamma \in \Lambda^r_{\mathscr{O}}(C, GL_n)$ near the constant map $C \ni x \mapsto I$. The general case is obtained as follows. By Theorem 1.2 we can approximate γ as closely as desired in $\Lambda^r_{\mathscr{O}}(C)$ by a holomorphic map $\gamma' : U \to GL_n$ from an open neighbourhood $U \subset X$ of C. Since GL_n is an Oka manifold (every complex homogeneous manifold is an Oka manifold by Grauert [24, 25]; see also [15, Proposition 5.6.1]), γ is homotopic to the constant map, and C if holomorphically convex in B, the Oka principle [15, Corollary 5.4.5] shows that γ' can be approximated uniformly on a compact neighbourhood $C' \subset U$ of C by holomorphic maps $\tilde\gamma \in \mathscr{O}(B, GL_n)$. Then, $\gamma = (\gamma\tilde\gamma^{-1})\tilde\gamma$ on C, and $\gamma\tilde\gamma^{-1}$ is close to I in $\Lambda^r(C, GL_n)$. By the first part we have $\gamma\tilde\gamma^{-1} = \alpha^{-1}\tilde\beta$ with $\alpha \in \Lambda^r_{\mathscr{O}}(A, GL_n)$ close to I and $\tilde\beta \in \Lambda^r_{\mathscr{O}}(B, GL_n)$. Setting $\beta = \tilde\beta\tilde\gamma \in \Lambda^r_{\mathscr{O}}(B, GL_n)$ gives $\gamma = \alpha^{-1}\beta$ as in (4.2). □

Lemma 4.3 has the following generalisation to the parametric case.

Lemma 4.5. *Assume that X and (A, B) are as in Lemma 4.3. Given a compact Hausdorff space P, a closed subspace $Q \subset P$ which is a strong neighbourhood deformation retract, and a continuous map $\gamma : P \to \Lambda^r_{\mathscr{O}}(C, GL_n)$ such that $\gamma(p) = I$ for $p \in Q$ and there is a homotopy $\gamma_t : P \to \mathscr{C}(C, GL_n)$ ($t \in [0, 1]$) which is fixed on Q such that $\gamma_0 = I$ and $\gamma_1 = \gamma$, there are continuous maps $\alpha : P \to \Lambda^r_{\mathscr{O}}(A, GL_n)$, $\beta : P \to \Lambda^r_{\mathscr{O}}(B, GL_n)$ such that $\|\alpha - I\|_{\Lambda^r(A)}$ is arbitrarily small, $\alpha|_Q = \beta|_Q = I$, and $\gamma = \alpha^{-1} \cdot \beta$ holds on C.*

Proof. By Theorem 3.2 and the argument in the last part of the proof of Lemma 4.3, we can reduce to the case when γ is close to the constant map $P \to I \in GL_n$. Since the splitting (4.3) in Lemma 4.4 is given by bounded linear operators, it also applies to the parametric case. The proof of the first part of Lemma 4.3 then carries over verbatim. □

Remark 4.6. Lemmas 4.3 and 4.5 also hold, with the same proofs, for maps with values in any complex Lie group G in place of $GL_n(\mathbb{C})$. In this case, we replace the matrix algebra M_n (which is the Lie algebra of $GL_n(\mathbb{C})$) by the Lie algebra $\mathfrak{g} \cong \mathbb{C}^d$ of G. Furthermore, the analogous results hold with the spaces $\Lambda^r_{\mathscr{O}}(\overline{\Omega}, G)$ replaced by any space $\mathcal{F}_{\mathscr{O}}(\overline{\Omega}, G)$ described in Remark 1.5.

The main step in the proof of Theorem 4.1 is given by the following lemma.

Lemma 4.7. *Assume that (A, B) is a special Cartan pair (see Definition 3.1). Let $D = A \cup B$ and $\pi : E \to D$ be a vector bundle of class $\Lambda^r_{\mathscr{O}}(D)$, $r > 0$, such that $E|_B$ is $\Lambda^r_{\mathscr{O}}(B)$-isomorphic to a trivial bundle. Then, every vector bundle epimorphism $\Phi : A \times \mathbb{C}^n \to E|_A$ of class $\Lambda^r_{\mathscr{O}}(A)$ can be approximated in $\Lambda^r(A)$ by vector bundle epimorphisms $\widetilde\Phi : D \times \mathbb{C}^n \to E$ of class $\Lambda^r_{\mathscr{O}}(D)$.*

Proof. Let $e_1, \ldots, e_n$ be sections of $E|_A$ of class $\Lambda^r_{\mathscr{O}}(A)$ which are Φ-images of the standard basis sections of $A \times \mathbb{C}^n$. Also, let $g_1, \ldots, g_m$ with $m = \mathrm{rank} E$ be basis sections of the trivial bundle $E|_B \cong B \times \mathbb{C}^m$. On C, we have $e_i = \sum_{j=1}^m \gamma_{i,j} b_j$ ($i = 1, \ldots, n$) with $\gamma_{i,j} \in \Lambda^r_{\mathscr{O}}(C)$ and the $n \times m$ matrix $\Gamma'(x) = (\gamma_{i,j}(x))$ has rank m at every $x \in C$. We claim that there is an $n \times n$ matrix function

$$\Gamma = (\gamma_{i,j})_{i,j=1}^n = (\Gamma', \Gamma'') \in \Lambda^r_{\mathscr{O}}(C, GL_n)$$

whose left hand side $n \times m$ submatrix equals Γ'. This is equivalent to saying that the subbundle E' of $C \times \mathbb{C}^n$, spanned by the columns of Γ', has a trivial complementary subbundle $E'' \subset C \times \mathbb{C}^n$ of class $\Lambda^r_{\mathscr{O}}(C)$. The existence of a complementary subbundle $\widetilde{E}''$ of class $\mathscr{A}(C)$ (continuous on C and holomorphic on $\mathring{C}$) follows from [30, Theorem 3]. Since C is contractible, $\widetilde{E}''$ is trivial as a bundle of class $\mathscr{A}(C)$ by [30, Theorem 2]. Let $\widetilde\Gamma''$ be an $n \times (n - m)$ matrix of class $\mathscr{A}(C)$ spanned by the basis sections of $\widetilde{E}''$. Since C is convex, we can approximate $\widetilde\Gamma''$ uniformly on C by a matrix Γ'' of class $\Lambda^r_{\mathscr{O}}(C)$. If the approximation is close enough then $\Gamma = (\Gamma', \Gamma'') \in \Lambda^r_{\mathscr{O}}(C, GL_n)$. Set $g_{m+1} = 0, \ldots, g_n = 0$. We have $(e_1, e_2, \ldots, e_n)^t = \Gamma(g_1, g_2, \ldots, g_n)^t$ where the superscript t

denotes the transpose. By Lemma 4.3 we have $\Gamma = \alpha^{-1}\beta$ where $\alpha \in \Lambda^r_{\mathscr{O}}(A, GL_n)$ is close to I and $\beta \in \Lambda^r_{\mathscr{O}}(B, GL_n)$. Then, $\alpha(e_1, e_2, \ldots, e_n)^t = \beta(g_1, g_2, \ldots, g_n)^t$ holds on C. The resulting collection of sections of $E \to D$ determines a vector bundle epimorphism $\widetilde{\Phi} : D \times \mathbb{C}^n \to E$ of class $\Lambda^r_{\mathscr{O}}(D)$ (see (4.1)) which approximates Φ in $\Lambda^r_{\mathscr{O}}(A)$. $\square$

Proof of Theorem 4.1. Pick a smooth strongly plurisubharmonic function ρ on a neighbourhood U of $\overline{\Omega}$ such that $\Omega = \{x \in U : \rho(x) < 0\}$ and $d\rho_x \neq 0$ for every point $x \in b\Omega = \{\rho = 0\}$. Choose $c < 0$ such that ρ has no critical values on $[c, 0]$ and set $A_0 = \{\rho \le c\}$. By [15, Lemma 5.10.3], given an open cover $\mathcal{U} = \{U_j\}$ of $\overline{\Omega \setminus A_0} = \{c \le \rho \le 0\}$ consisting of holomorphic coordinate charts $U_j \subset X$, there are compact, smoothly bounded, strongly pseudoconvex domains $A_0 \subset A_1 \subset \cdots \subset A_{k_0} = \overline{\Omega}$ for some $k_0 \in \mathbb{N}$ such that for every $k = 0, 1, \ldots, k_0 - 1$ we have $A_{k+1} = A_k \cup B_k$, where (A_k, B_k) is a special Cartan pair (see Definition 3.1) and $B_k \subset U_j$ for some $j = j(k)$. We choose the sets U_j small enough so that the restricted bundle $E|_{B_k}$ is trivial for $k = 0, 1, \ldots, k_0 - 1$. Since E is holomorphic over Ω, there is a holomorphic vector bundle epimorphism $\Phi_0 : A_0 \times \mathbb{C}^n \to E|_{A_0}$ for some $n \in \mathbb{N}$. We inductively apply Lemma 4.7 to find vector bundle epimorphisms $\Phi_k : A_k \times \mathbb{C}^n \to E|_{A_k}$ of class $\Lambda^r_{\mathscr{O}}(A_k)$ for $k = 1, \ldots, k_0$, starting with $\Phi_0 = \Phi$ and ensuring that Φ_{k+1} approximates Φ_k on A_k for $k = 0, 1, \ldots, k_0 - 1$. Then, $\Phi = \Phi_{k_0} : \overline{\Omega} \times \mathbb{C}^n \to E$ satisfies the conclusion of the theorem. $\square$

Remark 4.8. In the sequel, we shall be using the notion of a continuous family of holomorphic vector bundles on a smoothly bounded domain $\Omega \Subset X$ which are continuous or better on $\overline{\Omega}$; see Leiterer [40, Definition 2.14, p. 70] (where the parameter space is $[0, 1]$) or the statement of his stability theorem [40, Theorem 2.7] for the general case. This means that, locally in the parameter, the family is defined by a continuous family of 1-cocycles on the same finite open covering of $\overline{\Omega}$. The stability theorem says that, under suitable cohomological conditions (i), (ii) on the endomorphism bundle $\mathrm{Ad}(E)$ of a vector bundle $E \to \overline{\Omega}$ of class $\mathscr{A}(\overline{\Omega})$, all nearby vector bundles of the same class are isomorphic to E with continuous dependence on the parameter. This follows from an implicit function theorem in Banach spaces [40, Theorem 2.9]. The aforementioned cohomological conditions in [40, Theorem 2.7] are satisfied if for every continuous E-valued $(0, q)$-form ϕ on $\overline{\Omega}$ $(q > 0)$ the equation $\overline{\partial}\psi = \phi$ can be solved with a continuous ψ on $\overline{\Omega}$. This holds in particular if Ω is strongly pseudoconvex. By Theorems 2.2 and 4.11, the same conclusions hold in Hölder–Zygmund spaces, so [40, Theorem 2.7] also holds for vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. For later reference we state this explicitly.

Theorem 4.9. *Assume that $\overline{\Omega}$ is a compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold X, P is a topological space, and $E_p \to \overline{\Omega}$ $(p \in P)$ is a continuous family of vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, $r > 0$. Then for each $p_0 \in P$ there are a neighbourhood $P_0 \subset P$ and a continuous family of vector bundle isomorphisms $\phi_p : E_p \to E_{p_0}$ $(p \in P_0)$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$.*

We have the following parametric version of Theorem 4.1.

Theorem 4.10. *Assume that $\overline{\Omega}$ is a compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold X, P is a compact Hausdorff space, and $E_p \to \overline{\Omega}$ $(p \in P)$ is a continuous family of vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, $r > 0$. There exist an integer $N \in \mathbb{N}$ and a continuous family of vector bundle epimorphisms $\Phi_p : \overline{\Omega} \times \mathbb{C}^N \to E_p$, $p \in P$, of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$.*

Proof. By Theorem 4.9, every point $p_0 \in P$ has a neighbourhood $P_0 \subset P$ and a continuous family of vector bundle isomorphisms $\phi_p : E_p \to E_{p_0}$ $(p \in P_0)$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. A vector bundle epimorphism $\Psi_{p_0} : \overline{\Omega} \times \mathbb{C}^{n_0} \to E_{p_0}$, given by Theorem 4.1, extends to a continuous family of such epimorphisms $\Psi_p = \phi_p^{-1} \circ \Psi_{p_0} : \overline{\Omega} \times \mathbb{C}^{n_0} \to E_p$ for $p \in P_0$. Since P is compact, we obtain finitely many pairs of open subsets $P_i' \Subset P_i \subset P$ $(i = 1, \ldots, m)$ such that $\bigcup_{i=1}^m P_i' = P$ and continuous families of vector bundle epimorphism $\Psi^i_p : \overline{\Omega} \times \mathbb{C}^{n_i} \to E_p$, $p \in P_i$. For every $i = 1, \ldots, m$ let $\chi_i : P \to [0, 1]$ be a continuous function such that $\chi_i = 1$ on P_i' and $\mathrm{supp}\chi_i \subset P_i$. Let $\Phi^i_p : \overline{\Omega} \times \mathbb{C}^{n_i} \to E_p$ be defined by $\Phi_p^i(x,z) = \Psi_p^i(x,\chi_i(p)z)$. Take $N = \sum_{i=1}^m n_i$. The family $\Phi_p = \bigoplus_{i=1}^m \Phi_p^i : \overline{\Omega} \times \mathbb{C}^N \to E_p$ for $p \in P$ clearly satisfies the theorem. □

Next, we show that every complex vector subbundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ is complemented.

Theorem 4.11. *Let $\pi : E \to \overline{\Omega}$ be a complex vector bundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. For every complex vector subbundle $E' \subset E$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ there is a complementary to E' complex vector subbundle $E'' \subset E$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ such that $E \cong E' \oplus E''$.*

Proof. We first consider the case when $E = \overline{\Omega} \times \mathbb{C}^n$ is a trivial bundle. By [32, Theorem 3] there is a vector subbundle $\widetilde{E} \subset E$ of class $\mathscr{A}(\overline{\Omega})$ complementary to E', i.e., $\widetilde{E}$ is holomorphic on Ω and continuous on $\overline{\Omega}$. To complete the proof, it suffices to approximate $\widetilde{E}$ sufficiently closely by a subbundle $E'' \subset E = \overline{\Omega} \times \mathbb{C}^n$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. We follow the idea in [18, Proof of Lemma 2.2]. Let $\widetilde{L} : \overline{\Omega} \to M_n = \mathrm{Lin}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$ be the unique map such that $\widetilde{L}(x) : \mathbb{C}^n \to \mathbb{C}^n$ is the projection onto $\widetilde{E}_x$ with kernel E'_x for every $x \in \overline{\Omega}$. Clearly, $\widetilde{L}$ is of class $\mathscr{A}(\overline{\Omega})$, and the eigenvalues of $\widetilde{L}(x)$ are 0 and 1 for every $x \in \overline{\Omega}$. Let $C = \{z \in \mathbb{C} : |z-1| = 1/2\}$. We can approximate $\widetilde{L}$ uniformly on $\overline{\Omega}$ by a holomorphic map $L : U \to M_n$ on a neighbourhood U of $\overline{\Omega}$. Assuming that the approximation is close enough, $L(x)$ has no eigenvalues on the curve C for $x \in \overline{\Omega}$, we have $\mathbb{C}^n = V_{x,+} \oplus V_{x,-}$ where $V_{x,+}$ resp. $V_{x,-}$ are the $L(x)$-invariant subspaces of $\mathbb{C}^n$ spanned by the generalised eigenvectors of $L(x)$ inside resp. outside of C, and $V_{x,+}$ is close to the subspace $\widetilde{E}_x = \widetilde{L}(x)(\mathbb{C}^n) \subset \mathbb{C}^n$. The map

$$P(L(x)) = \frac{1}{2\pi \mathrm{i}} \int_{\zeta \in C} (\zeta I - L(x))^{-1} \, d\zeta \in M_n$$

is the projection of $\mathbb{C}^n$ onto $V_{x,+}$ with kernel $V_{x,-}$ (see [20]), it depends holomorphically on x in a neighbourhood of $\overline{\Omega}$, and it approximates $L(x)$ uniformly on $x \in \overline{\Omega}$. Assuming that the approximations are close enough, the subbundle $E'' \subset E$ with fibres $E''_x = V_{x,+} = P(L(x))(\mathbb{C}^n)$, $x \in \overline{\Omega}$, is complementary to E', and E'' is holomorphic on a neighbourhood of $\overline{\Omega}$.

For a general vector bundle $E \to \overline{\Omega}$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, we apply Theorem 4.1 to find a vector bundle epimorphism $\Phi : \overline{\Omega} \times \mathbb{C}^N \to E$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. The preimage $\Phi^{-1}(E')$ is a vector subbundle of $\overline{\Omega} \times \mathbb{C}^N$ class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, so it has a complementary subbundle $\widetilde{E}'' \subset \overline{\Omega} \times \mathbb{C}^N$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. Its image $E'' = \Phi(\widetilde{E}'')$ is then a subbundle of E of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ which is complementary to E'. □

We also have the following parametric version of Theorem 4.11.

Theorem 4.12. *Let Ω be as Theorem 4.11, P a compact Hausdorff space, and $\pi_p : E_p \to \overline{\Omega}$ a continuous family of vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. (See Remark 4.8.) Given a continuous family $E'_p \subset E_p$ $(p \in P)$ of vector subbundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, there is a continuous family $E''_p \subset E_p$ $(p \in P)$ of complementary vector subbundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ such that*

$$E_p \cong E'_p \oplus E''_p \text{ holds for all } p \in P. \tag{4.5}$$

If in addition Q is a closed subspace of P which is a strong neighbourhood deformation retract and $E''_p \subset E_p$ $(p \in Q)$ is a continuous family of complementary to E'_p vector subbundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$, then the family E''_p can be extended to all values $p \in P$ so that (4.5) *holds.*

Proof. Assume first that the family of vector bundles $E_p = E \to \overline{\Omega}$ is constant. Consider the short exact sequence of vector bundle homomorphisms

$$0 \longrightarrow E'_p \longrightarrow E \xrightarrow{\phi_p} E/E'_p \longrightarrow 0,$$

where ϕ_p is the quotient projection. A complementary to E'_p subbundle of E is the image of a vector bundle homomorphism $\sigma_p : E/E'_p \to E$ satisfying $\phi_p \circ \sigma_p = \mathrm{Id}$. Such σ_p is called a splitting of the above sequence. Clearly, a splitting at $p_0 \in P$ also gives a splitting at nearby values of $p \in P$, and a convex combination of splittings is again a splitting. Hence, the proof follows from Theorem 4.11 by using a continuous partition of unity on the parameter space P.

The case of a continuously variable family $E_p \to \overline{\Omega}$, $p \in P$, can be reduced to the special case as follows. By Theorem 4.9, every point $p_0 \in P$ has a neighbourhood $P_0 \subset P$ and a continuous family of vector bundle isomorphisms $\phi_p : E_p \to E_{p_0}$ $(p \in P_0)$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. Hence, the family E_p is locally constant and our proof applies locally on P. It remains to patch the locally defined families of splittings by a continuous partition of unity on P. $\square$

5. A GLUING LEMMA FOR SPRAYS OF CLASS $\Lambda^r_{\mathscr{O}}$

Let X be a complex manifold. Given a compact smoothly bounded subset K of X and an open ball $0 \in W \subset \mathbb{C}^N$, we shall consider maps $\gamma : K \times W \to K \times \mathbb{C}^N$ of the form

$$\gamma(x, w) = \big(x, \psi(x, w)\big), \quad x \in K,\ w \in W \tag{5.1}$$

and of class $\Lambda^r_{\mathscr{O}}(K \times W)$. The ball W will have the role of a parameter space and will be allowed to shrink during the construction. Let $\mathrm{Id}(x, w) = (x, w)$ denote the identity map on $X \times \mathbb{C}^N$.

The following splitting lemma is an analogue of [15, Proposition 5.8.1, p. 235], which pertains to function spaces $\mathscr{A}^r(K)$ with $r \in \mathbb{Z}_+$. It was first proved in [8, Theorem 3.2] and [13, Lemma 3.2].

Lemma 5.1. *Let (A, B) be a Cartain pair in a Stein manifold X (see Def. 3.1) and set $C = A \cap B$, $D = A \cup B$. Given a ball $0 \in W \subset \mathbb{C}^N$ and a number $\epsilon \in (0, 1)$, there is a number $\delta > 0$ satisfying the following. For every map $\gamma : C \times W \to C \times \mathbb{C}^N$ of the form* (5.1) *and class $\Lambda^r_{\mathscr{O}}(C \times W)$ satisfying $\|\gamma - \mathrm{Id}\|_{\Lambda^r(C\times W)} < \delta$ there exist maps*

$$\alpha_\gamma : A \times \epsilon W \to A \times \mathbb{C}^N, \qquad \beta_\gamma : B \times \epsilon W \to B \times \mathbb{C}^N$$

of the form (5.1) *and class $\Lambda^r_{\mathscr{O}}(A \times \epsilon W)$ and $\Lambda^r_{\mathscr{O}}(B \times \epsilon W)$, respectively, depending smoothly on γ, such that $\alpha_{\mathrm{Id}} = \mathrm{Id}$, $\beta_{\mathrm{Id}} = \mathrm{Id}$, and*

$$\gamma \circ \alpha_\gamma = \beta_\gamma \quad \textit{holds on } C \times \epsilon W. \tag{5.2}$$

If γ agrees with Id *to order $m \in \mathbb{N}$ along $w = 0$ then so do α_γ and β_γ. The analogous result holds with a continuous dependence of maps on a parameter $p \in P$ in a compact Hausdorff space.*

Proof. One follows the proof of [15, Proposition 5.8.1], taking into account [15, Remark 5.8.3 (B), p. 238] and Theorem 2.2 in the present paper. In particular, the use of [15, Lemma 5.8.2, p. 236] is replaced by Lemma 4.4. We leave the obvious details to the reader. $\square$

Assume that D is a compact strongly pseudoconvex domain with smooth boundary in a Stein manifold X and $h : Z \to D$ is a topological fibre bundle which is holomorphic on $\mathring{D} = D \setminus bD$. Let Y the denote the fibre of h, a complex manifold. For $x \in D$ write $Z_x = h^{-1}(x) \cong Y$. The set

$$VT_z(Z) = T_z Z_{\pi(z)}, \quad z \in Z$$

is called the vertical tangent space to Z at z, and the vector bundle $VT(Z) \to Z$ with fibres $VT_z(Z)$ is the vertical tangent bundle of (Z, h). When h is differentiable in the variable $x \in D$, we have $VT_z(Z) = \ker dh_z,\ z \in Z$. If the bundle $h : Z \to D$ is of Hölder–Zygmund class $\Lambda^r_{\mathscr{O}}(D)$ (see Subsect. 2.2) then $VT(Z)$ is of local class $\Lambda^r_{\mathscr{O}}(Z)$, and for every section $f : D \to Z$ of class $\Lambda^r_{\mathscr{O}}(D)$ the pullback bundle $f^*VT(Z) \to D$ is a vector bundle of class $\Lambda^r_{\mathscr{O}}(D)$.

We recall the notion of (local) dominating sprays of sections; see [15, Definition 5.9.1, p. 239] for sprays over open domains and [15, Sect. 8.10] for sprays over compact domains in a complex manifold.

Definition 5.2. A holomorphic spray of sections of $h : Z \to D$ is a continuous map $f : D \times W \to Z$ which is holomorphic on $\mathring{D} \times W$, where $0 \in W \subset \mathbb{C}^N$ is an open convex set, such that

$$h(f(x,w)) = x \quad \text{for } x \in D \text{ and } w \in W. \tag{5.3}$$

The section $f_0 = f(\cdot\,,0) : D \to Z$ is called the core of f. The spray f is dominating on a subset $K \subset D$ if the vertical derivative of f at $w = 0$, given by

$$\partial_w|_{w=0} f(x,w) : T_0\mathbb{C}^N \cong \mathbb{C}^N \longrightarrow VT_{f(x,0)}Z,$$

is surjective for all $x \in K$. We say that f is dominating if this holds on $K = D$.

We shall consider fibre bundles $h : Z \to D$ and sprays $f : D \times W \to Z$ of classes $\Lambda^r_{\mathscr{O}}(D)$ for $r > 0$. The main ingredient in the proof of Theorem 1.1 is the following gluing lemma for such sprays. Its analogue in spaces $\mathscr{A}^r(D)$, $r \in \mathbb{Z}_+$, was first proved in [7, Proposition 4.3]. The shorter proofs in [8, Proposition 2.4] and [15, Proposition 5.9.2] use the implicit function theorem in Banach spaces.

Lemma 5.3 (Gluing sprays of class $\Lambda^r_{\mathscr{O}}$)**.** *Let (A,B) be a Cartain pair in a Stein manifold (see Def. 3.1) and set $C = A \cap B$, $D = A \cup B$. Assume that $h : Z \to D$ is a fibre bundle of class $\Lambda^r_{\mathscr{O}}(D)$. Given a ball $0 \in W_0 \subset \mathbb{C}^N$ and a spray of sections $f : A \times W_0 \to Z$ of class $\Lambda^r_{\mathscr{O}}(A \times W_0)$ which is dominating on C, there is a ball $W \subset \mathbb{C}^N$ with $0 \in W \subset W_0$ satisfying the following conditions.*

(a) *For every holomorphic spray of sections $g : B \times W_0 \to Z$ of class $\Lambda^r_{\mathscr{O}}(B \times W_0)$ which is sufficiently close to f in $\Lambda^r(C \times W_0)$ there exists a spray of sections $f' : D \times W \to Z$ of class $\Lambda^r_{\mathscr{O}}(D \times W)$, close to f in $\Lambda^r(A \times W)$ (depending on the Λ^r distance between f and g on $C \times W_0$), whose core f'_0 is homotopic to $f_0 = f(\cdot\,,0)$ on A and is homotopic to $g_0 = g(\cdot\,,0)$ on B.*
(b) *If f and g agree to order $m \in \mathbb{Z}_+$ along $C \times \{0\}$ then f' can be chosen to agree to order m with f along $A \times \{0\}$ and with g along $B \times \{0\}$.*

The analogous result holds for families of sprays depending continuously on a parameter $p \in P$ in a compact Hausdorff space. If in addition the sprays f, g agree over C for values of p in a closed subset $Q \subset P$ which is a strong neighbourhood deformation retract, then f' can be chosen to agree with f and g on A and B, respectively, for $p \in Q$.

Sketch of proof. The proof is analogous to that of [15, Proposition 5.9.2, p. 240], which applies to function spaces $\mathscr{A}(D)$. As noted in [15, Remark 5.8.3 (B), p. 238], the same proof applies in any Banach spaces on which there is a linear bounded solution operator for the $\overline{\partial}$-equation on the level of $(0,1)$-forms. In view of Theorem 2.2, this holds for the spaces Λ^r with $r > 0$. Nevertheless, some steps must be adjusted by using results in Sections 3 and 4. We indicate the necessary changes.

The first step is to find a smaller ball $0 \in W \subset W_0$ and a map $\gamma : C \times W \to C \times \mathbb{C}^N$ of the form

$$\gamma(x,w) = (x, w + c(x,w)), \quad x \in C,\ w \in W,$$

with $c \in \Lambda^r_{\mathscr{O}}(C \times W)$ close to 0 (depending on the $\Lambda^r(C \times W_0)$ distance between f and g), such that

$$f = g \circ \gamma \quad \text{holds on } C \times W. \tag{5.4}$$

(For maps of class $\mathscr{A}^r$ with $r \in \mathbb{Z}_+$, a solution is given by [15, Lemma 5.9.3, p. 240].) In the process of constructing γ, we must split the trivial bundle $C \times \mathbb{C}^N$ by the subbundle $E' = \ker \partial_w f|_{w=0}$, the kernel of the vertical derivative of the spray f at $w = 0$. In our case, E' is of class $\Lambda^r_{\mathscr{O}}(C)$, and a splitting is furnished by Theorem 4.11 in the basic case and Theorem 4.12 in the parametric case. The proof of [15, Lemma 5.9.3] then applies without further changes.

In the second step, we apply Lemma 5.1 to split γ in the form (5.2):

$$\gamma \circ \alpha = \beta \ \text{ on } C \times \epsilon W \text{ for some } 0 < \epsilon < 1.$$

From this and (5.4) it follows that

$$f \circ \alpha = g \circ \beta \ \text{ holds on } C \times \epsilon W. \tag{5.5}$$

Hence, the two sides of the above equation amalgamate to a spray $f' : D \times \epsilon W \to Z|_D$ satisfying the lemma. The proof in the parametric case follows the same scheme. □

6. Proof of Theorem 1.1

We begin by explaining the proof of the basic (nonparametric) case of Theorem 1.1. The parametric version in Theorem 6.1 includes the 1-parametric case stated in Theorem 1.1.

Thus, our task is to prove that any continuous section $f_0 \in \Gamma(\overline{\Omega}, Z)$ of the fibre bundle $h : Z \to \overline{\Omega}$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ is homotopic to a section $f \in \Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$.

Pick a smooth strongly plurisubharmonic function ρ on a neighbourhood $U \subset X$ of $\overline{\Omega}$ such that $\Omega = \{x \in U : \rho(x) < 0\}$ and $d\rho_x \neq 0$ for every point $x \in b\Omega = \{\rho = 0\}$. Pick $c < 0$ such that ρ has no critical values on the interval $[c, 0]$. Set $A_0 = \{\rho \leq c\}$ and $A' = \{\rho \leq 0\} = \overline{\Omega}$. Given an open cover $\mathcal{U} = \{U_j\}$ of $\overline{A' \setminus A}$ consisting of holomorphic coordinate charts $U_j \subset X$, [15, Lemma 5.10.3] gives compact, smoothly bounded, strongly pseudoconvex domains

$$A_0 \subset A_1 \subset \cdots \subset A_m = A' = \overline{\Omega} \tag{6.1}$$

for some $m \in \mathbb{N}$ such that for every $k = 0, 1, \ldots, m-1$ we have $A_{k+1} = A_k \cup B_k$, where (A_k, B_k) is a special Cartan pair (see Definition 3.1) and $B_k \subset U_j$ for some $j = j(k)$. Hence, we may assume that the bundle $Z \to \overline{\Omega}$ is trivial over B_k for all $k = 0, 1, \ldots, m-1$.

By the Oka principle on open Stein manifolds [15, Theorem 5.4.4], we may assume that f_0 is holomorphic on a neighbourhood of A_0. It suffices to show that for every special Cartan pair (A, B) in $\overline{\Omega}$ such that the bundle $Z \to \overline{\Omega}$ is trivial over B, we can approximate a section $f \in \Gamma^r_{\mathscr{O}}(A, Z)$ as closely as desired by sections $\tilde{f} \in \Gamma^r_{\mathscr{O}}(D, Z)$, where $D = A \cup B$. The theorem then follows by a finite induction using the sequence (6.1) and starting with f_0 on A_0. The existence of a homotopy from f_0 to $f = f_m$ is obvious since there is no change of topology from A_0 to $A' = \overline{\Omega}$.

Fix a special Cartan pair (A, B) and a section $f_0 \in \Gamma^r_{\mathscr{O}}(A, Z)$. The proof proceeds in three steps.

(1) We embed f_0 as the core of a dominating spray of sections $f : A \times W \to Z$ of class $\Lambda^r_{\mathscr{O}}(A \times W, Z)$ (see Definition 5.2). The construction of f is explained in the sequel.
(2) Set $C = A \cap B$ and fix a number $\delta \in (0, 1)$. Since the sets $C \subset B$ are convex in a local holomorphic coordinate on a neighbourhood of B, the bundle h is trivial over B, and the fibre Y of h is an Oka manifold, we can approximate f as closely as desired in $\Lambda^r(C \times \delta W)$ by holomorphic sprays of sections $g : B \times \delta W \to Z$. (See the proof of Theorem 1.2 for the details.)
(3) Assuming that the approximation is close enough, we apply Lemma 5.3 to glue f and g into a spray $\tilde{f} \in \Gamma^r_{\mathscr{O}}(D \times \delta' W)$ for some $\delta' \in (0, \delta)$ which approximates f in $\Lambda^r(A \times \delta' W)$.

This complete the proof modulo the construction of a dominating spray in step (1). For this, we follow the proof of [15, Proposition 8.10.2, p. 388] (the original reference is [8, Proposition 4.1]) where such a result is proved for classes $\mathscr{A}^r$ with $r \in \mathbb{Z}_+$. Here is a sketch. By Theorem 4.1 there is a surjective vector bundle morphism $L : A \times \mathbb{C}^N \to f_0^* VT(Z)$ for some $N \in \mathbb{N}$. We claim that there is a dominating spray $f : A \times W \to Z$ of class $\Lambda^r_{\mathscr{O}}(A \times W, Z)$, where $0 \in W \subset \mathbb{C}^N$ is a ball, such that

$$\partial_w|_{w=0} f(x, w) = L_x := L(x, \cdot\,) : \mathbb{C}^N \to VT_{f(x,0)} Z \ \text{ holds for all } x \in A. \tag{6.2}$$

Choose a sequence of compact, smoothly bounded, strongly pseudoconvex domains

$$A_0 \subset A_1 \subset \cdots \subset A_m = A$$

as in (6.1) such A_0 is contained in the interior of A and for every $k = 0, 1, \dots, m-1$ we have $A_{k+1} = A_k \cup B_k$, where (A_k, B_k) is a special Cartan pair and the bundle Z is trivial over B_k. The existence of a holomorphic spray $f^0 : A_0 \times W_0 \to Z|_{A_0}$ satisfying (6.2) is standard; see [15, Proposition 8.10.2]. By a finite induction we find sprays $f^k : A_k \times W_k \to Z|_{A_k}$ of classes $\Lambda^r_{\mathscr{O}}(A_k \times W_k, Z)$ satisfying satisfying (6.2) for $x \in A_k$ $(k = 1, \dots, m)$, where $W_0 \supset W_1 \supset \cdots \supset W_m$ are balls in $\mathbb{C}^n$ centred at 0. Every induction step is of the same kind and proceeds as follows.

We first approximate f^k over $C_k = A_k \cap B_k$ by a spray $g^k : B_k \times V_k \to Z|_{B_k}$ of class $\Lambda^r_{\mathscr{O}}(B_k \times V_k, Z)$ satisfying (6.2) for points $x \in B_k$, where $0 \in V_k \subset W_k$ is a smaller ball; see [15, Lemma 8.10.3] and note that its proof also applies in classes Λ^r. In particular, f^k and g^k agree along $C \times \{0\}$ to the second order. Next, we apply Lemma 5.3 to glue f^k and g^k into a spray f^{k+1} over $A_{k+1} = A_k \cup B_k$ satisfying condition (6.2) for all points $x \in A_{k+1}$. This is done by first finding a map γ^k of the form (5.1) and class $\Lambda^r_{\mathscr{O}}(C_k \times V_k)$ which agrees with the identity to the second order along $C_k \times \{0\}$ and satisfies $f^k \circ \gamma^k = g^k$ on $C_k \times V_k$; see (5.4). (The ball V_k is allowed to shrink.) Next, we apply Lemma 5.1 to find sprays α^k and β^k over A_k and B_k such that $\alpha^k \circ \gamma^k = \beta^k$ (see (5.2)), and α^k and β^k agree with the identity to the second order along $A_k \times \{0\}$ and $B_k \times \{0\}$, respectively. As in (5.5), this yields the spray f^{k+1} over $A_{k+1} = A_k \cup B_k$ defined by the identity

$$f^k \circ \alpha^k = g^k \circ \beta^k \ \text{ on } C_k \times \epsilon V_k, \quad 0 < \epsilon < 1.$$

Taking $W_{k+1} = \epsilon V_k$ completes the induction step and proves the basic case of Theorem 1.1.

We now state a general parametric version of Theorem 1.1. Recall that $\Gamma(\overline{\Omega}, Z)$ denotes the space of continuous sections of a topological fibre bundle $h : Z \to \overline{\Omega}$. If the bundle is of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ then $\Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$ denotes the space of sections of the same class.

Theorem 6.1. *Assume that Ω is as in Theorem 1.1, $r > 0$, $h : Z \to \overline{\Omega}$ is a fibre bundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ with Oka fibre, P is a compact Hausdorff space, and Q is a closed subset of P which is a strong neighbourhood deformation retract. Let $f : P \to \Gamma(\overline{\Omega}, Z)$ be a continuous map such that $f|_Q : Q \to \Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$. Then there is a homotopy $f_s : P \to \Gamma(\overline{\Omega}, Z)$, $s \in [0, 1]$, which is fixed on Q such that $f_0 = f$ and $f_1 : P \to \Gamma^r_{\mathscr{O}}(\overline{\Omega}, Z)$.*

The proof follows the same scheme as that of Theorem 1.1 using the parametric versions of the tools in Sections 4 and 5. We leave out the details.

7. The Oka principle for vector bundles and principal bundles of class $\Lambda^r_{\mathscr{O}}$

In this section, we prove Theorem 1.4. We follow [15, proof of Theorem 5.3.1], which is due to Grauert [26]. See also Cartan's exposition of Grauert's Oka principle in [4].

Proof of Theorem 1.4. A topological vector bundle $E \to \overline{\Omega}$ of rank m is the pullback $f^*\mathbb{U}$ by a continuous map f from $\overline{\Omega}$ to a suitable Grassmannian $G(m, N)$ (consisting of complex m-planes in $\mathbb{C}^N$) of the universal bundle $\mathbb{U} \to G(m, N)$ of rank m. (We take N big enough such that E embeds as a topological vector subbundle of the trivial bundle $\overline{\Omega} \times \mathbb{C}^N$.) Since $G(m, N)$ is a complex homogeneous manifold, and hence an Oka manifold by Grauert [25], Theorem 1.1 shows that f is homotopic to a map $F : \overline{\Omega} \to G(m, N)$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$. The pullback $F^*\mathbb{U} \to \overline{\Omega}$ is then a vector bundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ which is topologically isomorphic to $E \cong f^*\mathbb{U}$. This proves part (i) of the theorem.

To prove the second statement, let $E \to \overline{\Omega}$ and $E' \to \overline{\Omega}$ be vector bundles of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ and rank m. There are an open cover $\{U_j\}$ of $\overline{\Omega}$ by smoothly bounded domains and vector bundle isomorphisms $\theta_j : E|_{\overline{U}_j} \xrightarrow{\cong} \overline{U}_j \times \mathbb{C}^m$, $\theta'_j : E'|_{\overline{U}_j} \xrightarrow{\cong} \overline{U}_j \times \mathbb{C}^m$ of class $\Lambda^r_{\mathscr{O}}(\overline{U}_j)$. Set $U_{i,j} = U_i \cap U_j$. Let

$$g_{i,j} : \overline{U}_{i,j} \to GL_m(\mathbb{C}), \quad g'_{i,j} : \overline{U}_{i,j} \to GL_m(\mathbb{C})$$

denote the fibrewise holomorphic transition maps of class $\Lambda^r_{\mathscr{O}}(\overline{U}_{i,j})$ so that

$$\theta_i \circ \theta_j^{-1}(x,v) = \big(x, g_{i,j}(x)v\big), \quad x \in \overline{U}_{i,j},\ v \in \mathbb{C}^m,$$

and likewise for E'. A complex vector bundle isomorphism $\Phi : E \to E'$ is given by a collection of complex vector bundle isomorphisms $\Phi_j : \overline{U}_j \times \mathbb{C}^m \to \overline{U}_j \times \mathbb{C}^m$ of the form

$$\Phi_j(x,v) = \big(x, \phi_j(x)v\big), \quad x \in \overline{U}_j,\ v \in \mathbb{C}^m$$

with $\phi_j(x) \in GL_m(\mathbb{C})$ for $x \in \overline{U}_j$, satisfying the compatibility conditions

$$\phi_i = g'_{i,j}\phi_j g_{i,j}^{-1} = g'_{i,j}\phi_j g_{j,i} \quad \text{on } \overline{U}_{i,j}. \tag{7.1}$$

Let $h : Z \to \overline{\Omega}$ denote the fibre bundle of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ with fibre $G = GL_m(\mathbb{C})$ and transition maps (7.1). (This means that $Z|_{\overline{U}_j} \cong \overline{U}_j \times G$ for each j, an element $(x,v) \in \overline{U}_j \times G$ for $x \in \overline{U}_{i,j}$ is identified with $(x,v') \in \overline{U}_i \times G$ where $v' = g'_{i,j}(x)\, v\, g_{j,i}(x)$, and no other identifications.) A collection of maps $\phi_j : \overline{U}_j \to G$ satisfying conditions (7.1) is then a section $\overline{\Omega} \to Z$. This shows that complex vector bundle isomorphisms $E \to E'$ correspond to sections of $Z \to \overline{\Omega}$, with isomorphisms of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ corresponding to sections of the same class. Hence, part (ii) follows from Theorem 1.1. □

Similarly one can prove the following analogue of [15, Theorem 8.2.1] due to Grauert [26]. In the proof, we use the analogue of Lemma 4.3 for an arbitrary complex Lie group G; see Remark 4.6.

Theorem 7.1. *Let Ω be as in Theorem 1.4. For every complex Lie group G, the isomorphism classes of principal G bundles on $\overline{\Omega}$ of class $\Lambda^r_{\mathscr{O}}(\overline{\Omega})$ are in bijective correspondence with the topological isomorphism classes of principal G bundles.*

Remark 7.2. Grassmann manifolds $G(m,N)$ play a major role in the theory of complex vector bundles as the classifying spaces. Being projective, they are Kähler manifolds. An explicit formula for a Kähler metric on $G(m,N)$ was given by Lu Qi-Keng in 1963, see [41, 42].

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REFERENCES

[1] R. Beals, P. C. Greiner, and N. K. Stanton. L^p and Lipschitz estimates for the $\bar\partial$-equation and the $\bar\partial$-Neumann problem. *Math. Ann.*, 277:185–196, 1987.

[2] G. Bourdaud and M. Lanza de Cristoforis. Functional calculus in Hölder-Zygmund spaces. *Trans. Am. Math. Soc.*, 354(10):4109–4129, 2002.

[3] H. Cartan. Sur les matrices holomorphes de n variables complexes. *J. Math. Pures Appl.*, 19:1–26, 1940.

[4] H. Cartan. Espaces fibrés analytiques. In *Symposium internacional de topología algebraica (International symposium on algebraic topology)*, pages 97–121. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[5] D. Catlin. A Newlander-Nirenberg theorem for manifolds with boundary. *Mich. Math. J.*, 35(2):233–240, 1988.

[6] S.-C. Chen and M.-C. Shaw. *Partial differential equations in several complex variables.* Providence, RI: Amer. Math. Soc.; Somerville, MA: International Press, 2001.

[7] B. Drinovec Drnovšek and F. Forstnerič. Holomorphic curves in complex spaces. *Duke Math. J.*, 139(2):203–253, 2007.

[8] B. Drinovec Drnovšek and F. Forstnerič. Approximation of holomorphic mappings on strongly pseudoconvex domains. *Forum Math.*, 20(5):817–840, 2008.

[9] G. B. Folland and J. J. Kohn. *The Neumann problem for the Cauchy-Riemann complex*, volume 75 of *Ann. Math. Stud.* Princeton University Press, Princeton, NJ, 1972.

[10] J. E. Fornæss, F. Forstnerič, and E. Wold. Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. In *Advancements in complex analysis. From theory to practice*, pages 133–192. Cham: Springer, 2020.

[11] F. Forstnerič. Recent developments on Oka manifolds. *Indag. Math., New Ser.*, 34(2):367–417, 2023.

[12] F. Forstnerič. From stein manifolds to oka manifolds: the h-principle in complex analysis. 2025. To appear in Proc. ICM 2026. `https://arxiv.org/abs/2509.21197`.

[13] F. Forstnerič. Manifolds of holomorphic mappings from strongly pseudoconvex domains. *Asian J. Math.*, 11(1):113–126, 2007.

[14] F. Forstnerič. Oka manifolds. *C. R. Math. Acad. Sci. Paris*, 347(17-18):1017–1020, 2009.

[15] F. Forstnerič. *Stein manifolds and holomorphic mappings (The homotopy principle in complex analysis)*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Cham, second edition, 2017.

[16] F. Forstnerič and F. Lárusson. Survey of Oka theory. *New York J. Math.*, 17A:11–38, 2011.

[17] F. Forstnerič and F. Lárusson. Every projective Oka manifold is elliptic. *arXiv e-prints,* `https://arxiv.org/abs/2502.20028`, 2025. Math. Res. Lett., to appear.

[18] F. Forstnerič, E. Løw, and N. Øvrelid. Solving the d- and $\overline{\partial}$-equations in thin tubes and applications to mappings. *Michigan Math. J.*, 49(2):369–416, 2001.

[19] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order. 2nd ed*, volume 224 of *Grundlehren Math. Wiss.* Springer, Cham, 1983.

[20] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant subspaces of matrices with applications*. Can. Math. Soc. Ser. Monogr. Adv. Texts. John Wiley, New York, NY, 1986.

[21] X. Gong. On regularity of $\overline{\partial}$-solutions on a_q domains with c^2 boundary in complex manifolds. *Trans. Amer. Math. Soc.*, 378(3):1771–1829.

[22] X. Gong. Hölder estimates for homotopy operators on strictly pseudoconvex domains with C^2 boundary. *Math. Ann.*, 374(1-2):841–880, 2019.

[23] X. Gong and Z. Shi. Global Newlander–Nirenberg theorem on domains with finite smooth boundary in complex manifolds. 2024. `https://arxiv.org/abs/2410.09334`.

[24] H. Grauert. Approximationssätze für holomorphe Funktionen mit Werten in komplexen Rämen. *Math. Ann.*, 133:139–159, 1957.

[25] H. Grauert. Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. *Math. Ann.*, 133:450–472, 1957.

[26] H. Grauert. Analytische Faserungen über holomorph-vollständigen Räumen. *Math. Ann.*, 135:263–273, 1958.

[27] M. Gromov. Oka's principle for holomorphic sections of elliptic bundles. *J. Amer. Math. Soc.*, 2(4):851–897, 1989.

[28] R. C. Gunning and H. Rossi. *Analytic functions of several complex variables*. 1965.

[29] G. M. Henkin and J. Leiterer. *Theory of functions on complex manifolds*, volume 60 of *Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs]*. Akademie-Verlag, Berlin, 1984.

[30] D. Heunemann. An approximation theorem and Oka's principle for holomorphic vector bundles which are continuous on the boundary of strictly pseudoconvex domains. *Math. Nachr.*, 127:275–280, 1986.

[31] D. Heunemann. Extension of the complex structure from Stein manifolds with strictly pseudoconvex boundary. *Math. Nachr.*, 128:57–64, 1986.

[32] D. Heunemann. Theorem B for Stein manifolds with strictly pseudoconvex boundary. *Math. Nachr.*, 128:87–101, 1986.

[33] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. I. *Ann. of Math. (2)*, 78:112–148, 1963.

[34] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. II. *Ann. of Math. (2)*, 79:450–472, 1964.

[35] Y. Kusakabe. Elliptic characterization and localization of Oka manifolds. *Indiana Univ. Math. J.*, 70(3):1039–1054, 2021.

[36] Y. Kusakabe. Oka properties of complements of holomorphically convex sets. *Ann. Math. (2)*, 199(2):899–917, 2024.

[37] J. Leiterer. Analytische Faserbündel mit stetigem Rand über streng pseudokonvexen Gebieten. I. Garbentheoretische Hilfsmittel. *Math. Nachr.*, 71:329–344, 1976.

[38] J. Leiterer. Analytische Faserbündel mit stetigem Rand über streng pseudokonvexen Gebieten. II. Topologische Klassifizierung. *Math. Nachr.*, 72:201–217, 1976.

[39] J. Leiterer. Holomorphic vector bundles and the Oka-Grauert principle. Several complex variables. IV. Algebraic aspects of complex analysis, Encycl. Math. Sci. 10, 63-103 (1990); translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 10, 75-121 (1986), 1986.

[40] J. Leiterer. Holomorphic vector bundles and the Oka-Grauert principle. In *Several complex variables. IV. Algebraic aspects of complex analysis*, Encycl. Math. Sci., Vol. 10, pages 63–103. Springer-Verlag, 1990. Translation from *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya* 10, 75-121 (1986). Translated by D. N. Akhiezer.

[41] Q.-K. Lu. The elliptic geometry of extended spaces. *Acta Math. Sin.*, 13:49–62, 1963.

[42] Q.-K. Lu. A note about 'The elliptic geometry of extended spaces'. *Acta Math. Sin.*, 13:314, 1963.

[43] B. Malgrange. *Ideals of differentiable functions*, volume 3 of *Tata Inst. Fundam. Res., Stud. Math.* London: Oxford University Press, 1966.

[44] E. J. McShane. Extension of range of functions. *Bull. Am. Math. Soc.*, 40:837–842, 1934.

[45] T. Ohsawa. Holomorphic embedding of compact s.p.c. manifolds into complex manifolds as real hypersurfaces. In *Differential geometry of submanifolds (Kyoto, 1984)*, volume 1090 of *Lecture Notes in Math.*, pages 64–76. Springer, Berlin, 1984.

[46] K. Oka. Sur les fonctions analytiques de plusieurs variables. III. Deuxième problème de Cousin. *J. Sci. Hiroshima Univ., Ser. A*, 9:7–19, 1939.

[47] R. S. Palais. *Foundations of global non-linear analysis*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[48] R. S. Palais. Banach manifolds of fiber bundle sections. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 243–249. Gauthier-Villars, Paris, 1971.

[49] V. S. Rychkov. On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains. *J. Lond. Math. Soc., II. Ser.*, 60(1), 1999.

[50] R. Seeley. Extension of C^∞ functions defined in a half space. *Proc. Am. Math. Soc.*, 15:625–626, 1964.

[51] Z. Shi and L. Yao. New estimates of Rychkov's universal extension operator for Lipschitz domains and some applications. *Math. Nachr.*, 297(4):1407–1443, 2024.

[52] E. M. Stein. *Singular integrals and differentiability properties of functions*, volume 30 of *Princeton Math. Ser.* Princeton University Press, Princeton, NJ, 1970.

[53] H. Wallin. New and old function spaces. Function spaces and applications, Proc. US-Swed. Semin., Lund/Swed., Lect. Notes Math. 1302, 99-114 (1988)., 1988.

[54] H. Whitney. Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.*, 36(1):63–89, 1934.

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