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A MORERA THEOREM FOR THE BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS IN THE UNIT BALL IN $\mathbf{C}^N$

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1. Introduction and the result. Let $B_N \subset \mathbf{C}^N$, $N \geq 2$, be the open unit ball. Suppose that an affine complex subspace Λ of complex dimension k intersects bB_N transversely. We say that $f \in C(bB_N)$ has the Morera property with respect to Λ if the integral $\int_{\Lambda \cap bB_N} f\alpha$ vanishes for each $(k, k-1)$-form α on $\mathbf{C}^N$ with constant coefficients [**3**].

Functions that typically satisfy the Morera conditions are the ones that belong to $A(B_N)$, that is, are continuous on $\overline{B}_n$ and holomorphic on B_N. If f is such a function, then $f|bB_N$ has the Morera property with respect to every affine linear subspace of complex dimension k, $1 \leq k \leq N-1$, which intersects bB_N transversely.

A function $f \in C(bB_N)$ is said to be a CR function if it satisfies the weak tangential Cauchy-Riemann equations on bB_N, that is, if $\int_{bB_N} f\bar{\partial}\alpha = 0$ for every smooth $(N, N-2)$-form α on $\mathbf{C}^N$. A function $f \in C(bB_N)$ extends through B_N as a member of $A(B_N)$ if and only if f is a CR function [**9**].

Several Morera theorems are known [**3**], [**6**], [**2**], [**4**], [**5**]. These theorems specify various open sets $\mathcal{S}$ of affine complex planes of complex dimension k such that if $f \in C(bB_N)$ has the Morera property with respect to every $\lambda \in \mathcal{S}$ which intersects bB_N transversely, then f is a CR function on bB_N.

The following theorem [**1**] shows that the Morera property of $f \in C(bB_N)$ with respect to certain families of affine complex planes of complex dimension k at a fixed distance from the origin is sufficient to guarantee that f extends through B_N as a member of $A(B_N)$.

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Theorem 1.1 [**1**, Theorem 2.4]. *Let $N \geq 2$, $1 \leq k \leq N-1$ and $0 < r < 1$. Assume that $f \in C(bB_N)$ satisfies the Morera condition with respect to every affine complex plane Λ of complex dimension k at a fixed distance r from the origin.*

If $k < N-1$, then f extends through B_N as a member of $A(B_N)$.

In the case when $k = N-1$, let E be the set of all r's, $0 < r < 1$, such that $r^2/(1-r^2)$ is a root of a polynomial of the form

$$\beta_{p,q}(t) = \sum_{l=\max(p+1-q,0)}^{p} \frac{(-1)^l t^l}{l!(p-l)!(l+q-p-1)!(N+p-l-1)!}, \qquad p \geq 0,\ q \geq 1.$$

If $r \notin E$, then f extends through B_N as a member of $A(B_N)$. Moreover, if $r \in E$ then there exists a function $f \in C(bB_N)$ which does not continue through B_N as a member of $A(B_N)$ but satisfies the Morera condition with respect to every affine complex hyperplane Λ at a distance r from the origin.

For $N = 2$, Theorem 1.1 is a result of Globevnik and Stout [**3**, Theorem 2.5.1]. The hypothesis in Theorem 1.1 is the weakest when $k = N-1$. The space $A(bB_N)$, the restriction of the ball algebra $A(B_N)$ to bB_N, is described in terms of the conditions on integrals over intersections of the boundary bB_N with complex affine hyperplanes in $\mathbf{C}^N$ at a fixed distance r from the origin, where r does not belong to an exceptional set E. These intersections are submanifolds of bB_N of real codimension 2. Integral conditions over submanifolds of bB_N of bigger codimension are stronger. Conditions on integrals over submanifolds of bB_N of minimal real codimension 1 are in some sense the weakest. In our context the natural submanifolds of bB_N of minimal codimension 1 are the intersections of bB_N with affine real hyperplanes in $\mathbf{C}^N$ at a fixed distance r from the origin.

The Morera property with respect to arbitrary affine real hyperplane in $\mathbf{C}^N$ was defined in a natural way in [**4**]. If an affine real hyperplane H meets bB_N transversely, then $f \in C(bB_N)$ is said to have the Morera property with respect to H, if $\int_{H \cap bB_N} f\alpha = 0$ for every $(N, N-2)$-form α with constant coefficients.

Since the Morera theorem above for complex hyperplanes at a fixed distance r from the origin holds only if r does not belong to an exceptional set, one would expect that this is the strongest possible result in the sense that one cannot replace the Morera conditions along complex hyperplanes with the weaker Morera conditions along real hyperplanes. However, this turns out to be possible and this is the result of the present paper:

Theorem 1.2. *Let $N \geq 2$ and $0 < r < 1$. Assume that $f \in C(bB_N)$ satisfies the Morera condition with respect to every affine real hyperplane at a fixed distance r from the origin. Let E be the set of all r's, $0 < r < 1$ such that $r/\sqrt{1-r^2}$ is a root of a polynomial of the form*

$$\beta_{p,q}(t) = \begin{cases} \displaystyle\sum_{l=0}^{p} \frac{(-1)^l}{(1+l)(2+l)\cdots(N-1+l)} \binom{p}{l}\binom{q-1}{l} \\ \displaystyle\quad \cdot \int_{-1}^{1} (1-x^2)^{N-1+l}(x^2+t^2)^{p-l}(x-it)^{q-1-p}\,dx \\ \qquad\qquad \text{if } p \leq q-1 \\ \displaystyle\sum_{l=0}^{q-1} \frac{(-1)^l}{(1+l)(2+l)\cdots(N-1+l)} \binom{p}{l}\binom{q-1}{l} \\ \displaystyle\quad \cdot \int_{-1}^{1} (1-x^2)^{N-1+l}(x^2+t^2)^{q-1-l}(x+it)^{p-q+1}\,dx \\ \qquad\qquad \text{if } p \geq q-1 \end{cases}$$

where p is a nonnegative integer and q is a positive integer.

Suppose that $r \notin E$. Then f extends through B_N as a member of $A(B_N)$.

Moreover, if $r \in E$, then there exists a function $f \in C(bB_N)$ which does not continue through B_N as a member of $A(B_N)$, yet f satisfies the Morera condition with respect to every affine real hyperplane at a distance r from the origin.

2. Proof of Theorem 1.2. Let Y be the subspace of all functions $f \in C(bB_N)$ satisfying the Morera condition with respect to every affine real hyperplane at a distance r from the origin, that is, the subspace of all functions $f \in C(bB_N)$ satisfying the condition $\int_{H\cap bB_N} f\alpha = 0$ for each $(N, N-2)$-form α with constant coefficients and for each real hyperplane H at a distance r from the origin. Indeed, Y is a closed $\mathcal{U}$-invariant subspace of $C(bB_N)$, where $\mathcal{U}$ is the unitary group on $\mathbf{C}^N$: if $f \in Y$, then $f \circ U \in Y$ for each $U \in \mathcal{U}$ since we have

$$\begin{aligned}\int_{H\cap bB_N} (f\circ U)\alpha &= \int_{U(H)\cap bB_N} (U^{-1})^*((f\circ U)\alpha)\\ &= \int_{U(H)\cap bB_N} f[(U^{-1})^*\alpha] = 0\end{aligned}$$

for each $(N, N-2)$-form α with constant coefficients, for each real hyperplane H at a distance r from the origin and for each $U \in \mathcal{U}$. Given $p \geq 0$, $q \geq 0$, let $H(p,q)$ be the space of all harmonic homogeneous polynomials of total degree p in the variables $z_1, \dots, z_N$ and of total degree q in the variables $\bar{z}_1, \dots, \bar{z}_N$. By a result of Nagel and Rudin [**7**, Theorem 4.4], every function in Y extends through B_N as a member of $A(B_N)$ if and only if Y contains no $H(p,q)$ with $p \geq 0$ and $q \geq 1$. In fact, either $H(p,q) \subset Y$ or $H(p,q) \cap Y = \{0\}$ [**7**].

To prove that every function in Y extends through B_N as a member of $A(B_N)$, it is enough to show that for every $p \geq 0$ and $q \geq 1$ the function $f(z) = z_{N-1}^p \bar{z}_N^q$ does not belong to Y. We will show that $f \in Y$ if and only if $\beta_{p,q}(r/\sqrt{1-r^2}\,) = 0$. Consider the $(N, N-2)$-form $\alpha_J = dz_1 \wedge \cdots \wedge dz_N \wedge d\bar{z}_1 \cdots \wedge \widehat{d\bar{z}_{j_1}} \wedge \cdots \wedge \widehat{d\bar{z}_{j_2}} \wedge \cdots \wedge d\bar{z}_N$, where $J = (j_1, j_2)$, $1 \leq j_1 < j_2 \leq N$. Write $\zeta_j = x_j + iy_j$, $1 \leq j \leq N$, and consider the real hyperplane $\Lambda = U(\Lambda_0)$ where $\Lambda_0 = \{\zeta \in \mathbf{C}^N, y_N = r\}$ and $U \in \mathcal{U}$. Then

$$\begin{aligned}\int_{\Lambda\cap bB_N} f\alpha_J &= \int_{\Lambda_0\cap bB_N} (f\circ U)U^*\alpha_J\\ &= \int_{\Lambda_0\cap bB_N} (U_{N-1}(\zeta))^p\overline{(U_N(\zeta))}^q\, dU_1(\zeta)\wedge\cdots\wedge dU_N(\zeta)\wedge\\ &\qquad \wedge d\overline{U_1(\zeta)}\wedge\cdots\wedge \widehat{d\overline{U_{j_1}(\zeta)}}\wedge\cdots\wedge\widehat{d\overline{U_{j_2}(\zeta)}}\wedge\cdots\wedge d\overline{U_N(\zeta)}.\end{aligned}$$

If $(u_{j,l})_{j,l=1,\dots,N}$ is the matrix of U in the canonical basis of $\mathbf{C}^N$, then we denote by $\Delta(U)$ the determinant of the matrix $(u_{j,l})_{j,l=1,\dots,N}$ and we denote by $\Delta_{((j_1,j_2);(l_1,l_2))}(U)$, $1 \leq l_1 < l_2 \leq N$, the determinant of the matrix obtained from the above matrix $(u_{j,l})_{j,l=1,\dots,N}$ deleting the j_1th, j_2th rows and the l_1th, l_2th columns. Then on Λ_0 we have

$$\begin{aligned} U_j(\zeta) &= u_{j,1}\zeta_1 + \cdots + u_{j,N-1}\zeta_{N-1} + u_{j,N}(x_N + ir), \\ dU_j(\zeta) &= u_{j,1}d\zeta_1 + \cdots + u_{j,N-1}d\zeta_{N-1} + u_{j,N}dx_N, \end{aligned}$$

$$\begin{aligned} &dU_1(\zeta) \wedge \cdots \wedge dU_N(\zeta) \wedge d\overline{U_1(\zeta)} \wedge \cdots \wedge \widehat{d\overline{U_{j_1}(\zeta)}} \wedge \cdots \wedge \widehat{d\overline{U_{j_2}(\zeta)}} \wedge \cdots \wedge d\overline{U_N(\zeta)} \\ &\qquad = \Delta(U) \sum_{l_1=1}^{N-1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)}\, d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge \\ &\qquad\quad \wedge dx_N \wedge d\bar\zeta_1 \wedge \cdots \wedge \widehat{d\bar\zeta_{l_1}} \wedge \cdots \wedge d\bar\zeta_{N-1}. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\Lambda_0 \cap bB_N} f\alpha_J \\ &\quad = \Delta(U) \Bigg[\sum_{l_1=1}^{N-1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)} \Bigg] \int_{\Lambda_0 \cap bB_N} (U_{N-1}(\zeta))^p \overline{(U_N(\zeta))}^q \\ &\qquad \cdot d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \wedge d\bar\zeta_1 \wedge \cdots \wedge \widehat{d\bar\zeta_{l_1}} \wedge \cdots \wedge d\bar\zeta_{N-1}. \end{aligned}$$

Then, by Stokes' theorem,

$$\begin{aligned} &\int_{\Lambda_0 \cap bB_N} f\alpha_J \\ &\quad = \Delta(U) \Bigg[\sum_{l_1=1}^{N-1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)} \Bigg] \int_{\Lambda_0 \cap B_N} d[(U_{N-1}(\zeta))^p \overline{(U_N(\zeta))}^q] \\ &\qquad \cdot d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \wedge d\bar\zeta_1 \wedge \cdots \wedge \widehat{d\bar\zeta_{l_1}} \wedge \cdots \wedge d\bar\zeta_{N-1} \\ &\quad = \Delta(U) \Bigg[\sum_{l_1=1}^{N-1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)} \Bigg] \\ &\qquad \cdot \int_{\Lambda_0 \cap B_N} \sum_{i=1}^{N} \Bigg(\frac{\partial}{\partial \zeta_i} [(U_{N-1}(\zeta))^p \overline{(U_N(\zeta))}^q]\, d\zeta_i \\ &\qquad\qquad + \frac{\partial}{\partial \bar\zeta_i} [(U_{N-1}(\zeta))^p \overline{(U_N(\zeta))}^q]\, d\bar\zeta_i \Bigg) \wedge \end{aligned}$$

$$
\begin{aligned}
&\wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \wedge d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_{l_1}} \wedge \cdots \wedge d\bar{\zeta}_{N-1} \\
&= \Delta(U)\Big[\sum_{l_1=1}^{N-1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)}\Big] \\
&\quad \cdot \int_{\Lambda_0\cap B_N} \left(\frac{\partial}{\partial\bar{\zeta}_{l_1}}\,[(U_{N-1}(\zeta))^p\overline{(U_N(\zeta))}^q]\,d\bar{\zeta}_{l_1}\right)\wedge \\
&\wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \wedge d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_{l_1}} \wedge \cdots \wedge d\bar{\zeta}_{N-1} \\
&= \Delta(U)\Big[\sum_{l_1=1}^{N-1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)}\Big] \\
&\quad \cdot \int_{\Lambda_0\cap B_N} q(U_{N-1}(\zeta))^p\overline{(U_N(\zeta))}^{q-1}\bar{u}_{N,l_1}\,d\bar{\zeta}_{l_1}\wedge \\
&\wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \wedge d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_{l_1}} \wedge \cdots \wedge d\bar{\zeta}_{N-1} \\
&= \Delta(U)\Big[\sum_{l_1=1}^{N-1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)}\Big] \\
&\quad \cdot \int_{\Lambda_0\cap B_N} q(U_{N-1}(\zeta))^p\overline{(U_N(\zeta))}^{q-1}\bar{u}_{N,l_1}(-1)^{N+l_1-1} \\
&\quad \cdot d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{l_1} \wedge \cdots \wedge d\bar{\zeta}_{N-1}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\int_{\Lambda_0\cap bB_N} f\alpha_J = \Delta(U)\Big[&\sum_{l_1=1}^{N-1}(-1)^{N+l_1-1}\overline{\Delta_{((j_1,j_2);(l_1,N))}(U)}\,\bar{u}_{N,l_1}\Big] \\
&\cdot \int_{\Lambda_0\cap B_N} q(U_{N-1}(\zeta))^p\overline{(U_N(\zeta))}^{q-1} d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \\
&\wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{N-1}.
\end{aligned}
$$

If $j_2 < N$, we have

$$
\sum_{l_1=1}^{N-1}(-1)^{N+l_1-1}\bar{u}_{N,l_1}\overline{\Delta_{((j_1,j_2);(l_1,N))}(U)} = \begin{vmatrix}
\overline{u}_{1,1} & \cdots & \overline{u}_{1,N-1} \\
\cdots & \cdots & \cdots \\
\widehat{\overline{u}_{j_1,1}} & \cdots & \widehat{\overline{u}_{j_1,N-1}} \\
\cdots & \cdots & \cdots \\
\widehat{\overline{u}_{j_2,1}} & \cdots & \widehat{\overline{u}_{j_2,N-1}} \\
\cdots & \cdots & \cdots \\
\overline{u}_{N,1} & \cdots & \overline{u}_{N,N-1} \\
\overline{u}_{N,1} & \cdots & \overline{u}_{N,N-1}
\end{vmatrix} = 0.
$$

If $j_2 = N$, we have

$$\sum_{l_1=1}^{N-1} (-1)^{N+l_1-1} \bar{u}_{N,l_1} \overline{\Delta_{((j_1,j_2);(l_1,N))}(U)} = \begin{vmatrix} \overline{u}_{1,1} & \cdots & \overline{u}_{1,N-1} \\ \cdots & \cdots & \cdots \\ \widehat{\overline{u}_{j_1,1}} & \cdots & \widehat{\overline{u}_{j_1,N-1}} \\ \cdots & \cdots & \cdots \\ \overline{u}_{N,1} & \cdots & \overline{u}_{N,N-1} \end{vmatrix}$$
$$= \overline{\Delta_{(j_1;N)}(U)},$$

where we denote by $\Delta_{(j_1;N)}(U)$ the determinant obtained from the matrix $(u_{j,l})_{j,l=1,\ldots,N}$ deleting the j_1th row and the Nth column. It remains to consider the case $J = (j_1, N)$ for $1 \leq j_1 < N$. Now

$$\int_{\Lambda \cap bB_N} f\alpha_J = q\Delta(U)\overline{\Delta_{(j_1;N)}(U)}$$
$$\cdot \int_{\Lambda_0 \cap B_N} (U_{N-1}(\zeta))^p \overline{U_N(\zeta)}^{q-1}$$
$$\cdot\, d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge dx_N \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{N-1}.$$

Computing the powers and using the Fubini's theorem we obtain that the last integral is

(1)

$$A_J p!(q-1)!$$
$$\cdot \sum \frac{(u_{N-1,1})^{p_1} \cdots (u_{N-1,N-1})^{p_{N-1}} (u_{N-1,N})^{p_N} (\overline{u}_{N,1})^{q_1} \cdots (\overline{u}_{N,N-1})^{q_{N-1}} (\overline{u}_{N,N})^{q_N}}{p_1! \cdots p_N! q_1! \cdots q_N!}$$
$$\cdot \int_{-\sqrt{(1-r^2)}}^{\sqrt{(1-r^2)}} (x_N + ir)^{p_N} (x_N - ir)^{q_N}\, dx_N$$
$$\cdot \int_{\sqrt{(1-r^2-x_N^2)}\, B_{N-1}} \zeta_1^{p_1} \overline{\zeta}_1^{q_1} \zeta_2^{p_2} \overline{\zeta}_2^{q_2} \cdots \zeta_{N-1}^{p_{N-1}} \overline{\zeta}_{N-1}^{q_{N-1}}$$
$$\cdot\, d\zeta_1 \wedge \cdots \wedge d\zeta_{N-1} \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{N-1},$$

where the summation is over all $(p_1, \ldots, p_N)$, $(q_1, \ldots, q_N)$ such that $0 \leq p_i \leq p$, $0 \leq q_i \leq q-1$ $(1 \leq i \leq N)$ and $p_1 + \cdots + p_N = p$, $q_1 + \cdots + q_N = q-1$ and where A_J is a nonzero constant. Since the last integral in (1) vanishes when $(p_1, \dots p_{N-1}) \neq (q_1, \dots, q_{N-1})$ [**8**, pp.15–16], (1) equals

$$
\begin{aligned}
&(2)\\
&A_J p!(q-1)!\\
&\quad \cdot \sum \tfrac{(u_{N-1,1})^{p_1} \cdots (u_{N-1,N-1})^{p_{N-1}} (u_{N-1,N})^{p_N} (\overline{u}_{N,1})^{p_1} \cdots (\overline{u}_{N,N-1})^{p_{N-1}} (\overline{u}_{N,N})^{q_N}}{p_1! \cdots p_N! p_1! \cdots p_{N-1}! q_N!}\\
&\quad \cdot \int_{-\sqrt{(1-r^2)}}^{\sqrt{(1-r^2)}} (x_N + ir)^{p_N} (x_N - ir)^{q_N} \, dx_N\\
&\quad \cdot \int_{\sqrt{(1-r^2-x_N^2)}\, B_{N-1}} |\zeta_1|^{2p_1} |\zeta_2|^{2p_2} \cdots |\zeta_{N-1}|^{2p_{N-1}} \, d\zeta_1 \wedge \cdots\\
&\quad \wedge d\zeta_{N-1} \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{N-1},
\end{aligned}
$$

where the summation is over all $(p_1, \dots, p_N)$, q_N such that $0 \leq q_N \leq q-1$, $0 \leq p_i \leq p$, $1 \leq i \leq N$, and $p_1 + \cdots + p_{N-1} = p - p_N = q - 1 - q_N$. Computing the last integral we get

$$
\begin{aligned}
&\int_{\sqrt{(1-r^2-x_N^2)}\, B_{N-1}} |\zeta_1|^{2p_1} |\zeta_2|^{2p_2} \cdots |\zeta_{N-1}|^{2p_{N-1}} \, d\zeta_1 \wedge \cdots\\
&\quad \wedge d\zeta_{N-1} \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{N-1}\\
&\qquad = c_{N-1} \Big(\sqrt{(1 - r^2 - x_N^2)} \Big)^{2(N-1+p_1+\cdots+p_{N-1})}\\
&\qquad\quad \cdot \int_{B_{N-1}} |\zeta_1|^{2p_1} |\zeta_2|^{2p_2} \cdots |\zeta_{N-1}|^{2p_{N-1}} \, d\nu_{N-1}\\
&\qquad = c_{N-1} \Big(\sqrt{(1 - r^2 - x_N^2)} \Big)^{2(N-1+p-p_N)} \frac{(N-1)! p_1! \cdots p_{N-1}!}{(N-1+p-p_N)!}
\end{aligned}
$$

[**8**, p. 17].

Here ν_{N-1} is the Lebesgue measure on $\mathbf{C}^{N-1} = \mathbf{R}^{2N-2}$ and c_{N-1} is a nonzero constant.

We have shown that $\int_{\Lambda\cap bB_N} f\alpha_J$ equals $A_J c_{N-1}\Delta(U)\overline{\Delta_{(j_1;N)}(U)}q \times (N-1)!F_r(U)$, where

$$
\begin{aligned}
&F_r(U) = p!(q-1)! \\
&\cdot \sum \frac{(u_{N-1,1})^{p_1}\cdots(u_{N-1,N-1})^{p_{N-1}}(u_{N-1,N})^{p_N}(\overline{u}_{N,1})^{p_1}\cdots(\overline{u}_{N,N-1})^{p_{N-1}}(\overline{u}_{N,N})^{q_N}}{p_1!\cdots p_N!q_N!(N-1+p-p_N)!} \\
&\cdot \int_{-\sqrt{(1-r^2)}}^{\sqrt{(1-r^2)}} (x_N+ir)^{p_N}(x_N-ir)^{q_N} \\
&\cdot \left(\sqrt{1-r^2-x_N^2}\,\right)^{2(N-1+p-p_N)} dx_N,
\end{aligned}
$$

where the summation is over all $(p_1,\dots,p_N)$, q_N such that $0\le q_N\le q-1$, $0\le p_i\le p$, $1\le i\le N$, and $p_1+\cdots+p_{N-1}=p-p_N=q-1-q_N$.

We now simplify the expression for $F_r(U)$. For each $U\in\mathcal{U}$ we have

$$
u_{N-1,1}\bar{u}_{N,1}+\cdots+u_{N-1,N-1}\bar{u}_{N,N-1} = -\,u_{N-1,N}\,\bar{u}_{N,N}.
$$

Putting this into (3), we obtain

$$
\begin{aligned}
F_r(U) = \sum p!(q-1)! \,&\frac{(-u_{N-1,N}\overline{u}_{N,N})^{p-p_N}(u_{N-1,N})^{p_N}(\overline{u}_{N,N})^{q_N}}{(p-p_N)!(N-1+p-p_N)!p_N!q_N!} \\
&\cdot \int_{-\sqrt{(1-r^2)}}^{\sqrt{(1-r^2)}} (x_N+ir)^{p_N}(x_N-ir)^{q_N} \\
&\cdot \left(\sqrt{1-r^2-x_N^2}\,\right)^{2(N-1+p-p_N)} dx_N,
\end{aligned}
$$

where the summation is over all p_N, q_N such that $0\le p_N\le p$, $0\le q_N\le q-1$ and $p-p_N=q-1-q_N$. Now

$$
\begin{aligned}
&\int_{-\sqrt{(1-r^2)}}^{\sqrt{(1-r^2)}} (x_N+ir)^{p_N}(x_N-ir)^{q_N}\left(\sqrt{1-r^2-x_N^2}\,\right)^{2(N-1+p-p_N)} dx_N \\
&\quad = (\sqrt{1-r^2}\,)^{2(N-1)+p+q}\int_{-1}^{1}(1-x^2)^{N-1+p-p_N} \\
&\qquad \cdot \left(x+i\frac{r}{\sqrt{1-r^2}}\right)^{p_N}\left(x-i\frac{r}{\sqrt{1-r^2}}\right)^{q_N} dx.
\end{aligned}
$$

We have shown that

$$F_r(U) = \big(\sqrt{1-r^2}\,\big)^{2(N-1)+p+q}\, u^p_{N-1,N}\, \overline{u}^{q-1}_{N,N}\, \beta_{p,q}\Big(\frac{r}{\sqrt{1-r^2}}\Big),$$

where

$$\beta_{p,q}(t) = \sum_{\substack{0\le p_N\le p\\ 0\le q_N\le (q-1)\\ p-p_N=q-1-q_N}} \frac{(-1)^{p-p_N}p!(q-1)!}{(p-p_N)!(N-1+p-p_N)!p_N!q_N!} \cdot \int_{-1}^{1}(1-x^2)^{N-1+p-p_N}(x+it)^{p_N}(x-it)^{q_N}\,dx,$$

that is,

$$\beta_{p,q}(t) = \begin{cases} \displaystyle\sum_{l=0}^{p} \frac{(-1)^l}{(1+l)(2+l)\cdots(N-1l)}\binom{p}{l}\binom{q-1}{l} \cdot \int_{-1}^{1}(1-x^2)^{N-1+l}(x^2+t^2)^{p-l}(x-it)^{q-1-p}\,dx & \text{if } p\le q-1\\ \displaystyle\sum_{l=0}^{q-1} \frac{(-1)^l}{(1+l)(2+l)\cdots(N-1+l)}\binom{p}{l}\binom{q-1}{l} \cdot \int_{-1}^{1}(1-x^2)^{N-1+l}(x^2+t^2)^{q-1-l}(x+it)^{p-q+1}\,dx & \text{if } p\ge q-1. \end{cases}$$

$\beta_{p,q}$ is a polynomial in t of degree $p+q-1$.

Recall that $\int_{U(\Lambda_0)\cap bB_N} f\alpha_J = A_J c_{N-1}\Delta(U)\overline{\Delta_{(j_1;N)}(U)}q(N-1)!F_r(U)$.

If $\beta_{p,q}[r/(\sqrt{1-r^2}\,)] = 0$, then $\int_{U(\Lambda_0)\cap bB_N} f\alpha_J = 0$ for each $U\in\mathcal{U}$, that is, $f\in Y$. Conversely, let $f\in Y$, that is, $\int_{U(\Lambda_0)\cap bB_N} f\alpha_J = 0$ for all $U\in\mathcal{U}$. Let $\mathcal{D}_i = \{U\in\mathcal{U}; \Delta_{(i;N)}(U)\neq 0\}$ and let $\mathcal{D} = \cup_{i=1}^{N-1}\mathcal{D}_i$. The set $\mathcal{D}$ is an open dense subset of $\mathcal{U}$. The same holds for the set of all

$U \in \mathcal{U}$ such that both $u_{N-1,N}$ and $u_{N,N}$ are different from zero. Thus, there is a $U \in \mathcal{D}$ such that $u_{N-1,N} \neq 0$ and $u_{N,N} \neq 0$. This implies that $\beta_{p,q}[r/(\sqrt{1-r^2}\,)] = 0$. This completes the proof of Theorem 1.2. $\square$

Remark 2.1. Note that the exceptional set E in Theorem 1.2 is not empty. For instance, if $p = 0$ and $q = 4$, then $\beta_{p,q}$ has a positive root and the corresponding value for r is $\sqrt{2\Gamma(1/2+N)}/\sqrt{2\Gamma(1/2+N)+3\Gamma(3/2+N)}$.

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