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CROSSING GRAPHS AS JOINS OF GRAPHS AND CARTESIAN PRODUCTS OF MEDIAN GRAPHS*

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Abstract. For a partial cube G its crossing graph $G^{\#}$ is the graph whose vertices are the Θ-classes of G, two classes being adjacent if they cross on some cycle in G. The following problem posed in [S. Klavžar and H. M. Mulder, *SIAM J. Discrete Math.*, 15 (2002), pp. 235–251, Problem 7.1] is considered: What can be said about the partial cube G if $G^{\#}$ is the join $A \oplus B$ of graphs A and B with at least one edge? It is proved that for arbitrary graphs A and B, where at least one of them contains an edge, there exists a Cartesian prime partial cube G such that $G^{\#} = A \oplus B$. On the other hand, if G is a median graph, then $G^{\#} = A \oplus B$ if and only if $G = H \,\square\, K$, where $H^{\#} = A$ and $K^{\#} = B$. Along the way some new facts about partial cubes are obtained; for instance, a bipartite graph of radius 2 is a partial cube if and only if it is $K_{2,3}$-free.

Key words. intersection graph, partial cube, median graph, Cartesian product of graphs, join of graphs

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1. Introduction.

Intersection concepts in graph theory have been extensively studied [16]. Although some of the intersection operations yield all graphs (for instance, every graph is the intersection graph of some set system), their importance is due to their usefulness in the characterization of particular classes of graphs, thus leading to a deeper structural understanding. Here we study a nonstandard intersection operation where vertices of the intersection graph (called *crossing graph*) are equivalence classes of a certain equivalence relation Θ defined on the edge-set of a graph. Hence the edges of the crossing graph are not defined in the standard way (by intersections of subsets). The graphs that we are interested in are isometric subgraphs of hypercubes, and the relation Θ is of great importance for understanding the structure of these graphs. So before presenting the preliminary work on these graphs and the crossing graph operation, let us recall necessary definitions.

The distance $d_G(u,v)$ between vertices u and v of a graph G is the length of a shortest u,v-path in G. A subgraph U of G is *isometric* if $d_U(u,v) = d_G(u,v)$ for all $u,v \in U$. The *interval* $I_G(u,v)$ is the set of vertices that lie on shortest paths between u and v in G. A subgraph U is *convex* if $I_G(u,v) \subseteq U$ for all $u,v \in U$. (Indices in the above definitions are omitted when the graph is understood from the context.) Recall that the *hypercube* Q_k, or *k-cube*, is the graph with the vertex set $\{0,1\}^k$, where two vertices are adjacent whenever they differ in exactly one position.

Partial cubes are isometric subgraphs of hypercubes. This class of graphs has been extensively investigated; see, for instance, [3, 5, 6, 7, 8, 21]. A well-known

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characterization of partial cubes is by the relation Θ on the edge-set of a graph. Two edges $e = xy$ and $f = uv$ of a graph G are in the Djoković–Winkler [7, 21] relation Θ_G, Θ for short, if $d_G(x,u)+d_G(y,v) \neq d_G(x,v)+d_G(y,u)$. Winkler [21] proved that a bipartite graph is a partial cube if and only if Θ is transitive. Letting R^* denote the transitive closure of a relation R, Winkler's result reads as follows: A connected bipartite graph G is a partial cube if and only if $\Theta = \Theta^*$. Hence in partial cubes the relation Θ is an equivalence relation on $E(G)$, and the classes of the corresponding partition will be called *Θ-classes*.

For a partial cube G its *crossing graph* $G^\#$ was introduced in [15] as follows. The vertices of $G^\#$ are the Θ-classes of G, two vertices being adjacent if the respective Θ-classes meet (or cross) on some cycle (that is, there is a cycle C that contains edges of both Θ-classes). In fact, in the class of median graphs the same concept was introduced earlier by Bandelt and Chepoi under the name *incompatibility graph* [1].

In this paper we address the problem of what can be said about the partial cube G if $G^\# = A \oplus B$, where A and B have at least one edge. Here $A \oplus B$ denotes the *join* of graphs A and B, that is, the graph obtained from the disjoint union of A and B by joining every vertex of A with every vertex of B by an edge. In the next section we state important properties of the Cartesian product of graphs and median graphs that are needed later. In section 3 we prove that for arbitrary graphs A and B, where at least one of them contains an edge, there exists a Cartesian prime partial cube G such that $G^\# = A \oplus B$. Then we restrict our attention to median graphs and prove that the crossing graph of a median graph G is the join of two graphs A and B if and only if G is a Cartesian product graph. In due course we also characterize partial cubes of radius 2 and observe that a partial cube contains no nontrivial convex subgraph that meets all of its Θ-classes.

2. Cartesian products and median graphs. The *Cartesian product* $G \,\square\, H$ of the graphs G and H is the graph with the vertex set $V(G) \times V(H)$ in which two vertices (a,x) and (b,y) are adjacent whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product is associative and commutative with K_1 as its unit. It is easy to see that the Cartesian product of k copies of K_2 is the *hypercube* Q_k. A graph G is called *prime* (with respect to the Cartesian product) if it cannot be represented as the product of two nontrivial graphs; that is, $G = G_1 \,\square\, G_2$ implies that G_1 or G_2 is the one-vertex graph K_1.

The well-known prime factorization theorem, proved by Sabidussi [19] and independently by Vizing [20], states that every connected graph has a unique prime factor decomposition with respect to the Cartesian product. This decomposition can be made explicit in the following way: Edges uv and uw are said to be in relation τ_G, or τ for short, if u is the unique common neighbor of v and w. Feder [9] proved (cf. also [11, Theorem 4.8] and [13]) that $(\Theta \cup \tau)^*$ is the Cartesian product relation of a connected graph. This actually means that the equivalence classes of the relation $(\Theta \cup \tau)^*$ determine the prime factor decomposition of a graph—every equivalence class yields one factor of the decomposition. The following consequence of this theorem will be useful for us.

Corollary 1. *A connected graph G is prime if and only if $(\Theta_G \cup \tau_G)^* = E(G)$.*

We will also need the following result (in a way part of the folklore) on the Cartesian product; see [4].

Lemma 2. *A subgraph C of the Cartesian product $G_1 \,\square \cdots \square\, G_m$ of connected graphs is convex if and only if $C = p_1(C) \,\square \cdots \square\, p_m(C)$, where $p_i(C)$ is convex in G_i, $1 \leq i \leq m$. (Here p_i is the projection map from G onto G_i.)*

The most important subclass of partial cubes are median graphs. They have been rediscovered several times, and a rich theory of these graphs and related structures has been developed; cf. the survey [14]. The most common definition is the following: G is a *median graph* if for every triple of vertices $u, v, w \in V(G) : I(u, v) \cap I(u, w) \cap I(v, w)$ consists of precisely one vertex (which is called the *median* of the triple u, v, w). One of the most well-known characterizations of median graphs involves a certain expansion procedure, a result due to Mulder [17]. (By the way, it inspired Chepoi [5] to prove a similar characterization of partial cubes.) In this note we will make use of a variation of the expansion procedure that involves peripheral subgraphs of a median graph [18]; see also [2].

Let G be a connected graph and G_0 a convex subgraph. Then the *peripheral expansion* of G is the graph G' obtained as follows. Take the disjoint union of a copy of G and a copy of G_0. Join each vertex u in the copy of G_0 with the vertex that corresponds to u in the copy of G (actually in the subgraph G_0 of G). We say that the resulting graph G' is *obtained by a (peripheral) expansion from G along G_0*. We also say that we *expand* G_0 in G to obtain G'. Note that in a peripheral expansion one new Θ-class appears. It is easy to prove that expanding a convex subgraph of a median graph yields again a median graph. It is more surprising that the converse is also true, as proved by Mulder in [18].

Theorem 3. *A graph G is a median graph if and only if it can be obtained from K_1 by a sequence of peripheral expansions.*

Hence each median graph contains a *peripheral subgraph*, that is, a subgraph H whose vertices are all incident with a particular Θ-class F in G, such that H is a connected component of $G - F$ (the graph obtained from G by removal of edges from F). Even more is known [18], as stated in the following proposition.

Proposition 4. *Let G be a median graph and F any Θ-class in G. Then both connected components of $G - F$ contain a peripheral subgraph of G.*

It is easy to see that median graphs are closed under Cartesian multiplication and that, conversely, if a median graph is not prime, all of the factors also must be median graphs.

3. Partial cubes whose crossing graphs are joins. Crossing graphs of Cartesian products have a simple structure [15, Proposition 6.1].

Proposition 5. *Let H and K be partial cubes. Then $(H \,\square\, K)^{\#} = H^{\#} \oplus K^{\#}$.*

Let A and B be graphs. Clearly, $A \oplus B$ is a complete bipartite graph if and only if both A and B have no edges. In [15] it has also been proved that $G^{\#}$ is a complete bipartite graph if and only if G is the Cartesian product of two trees. In this section we show, a bit surprisingly, that any other join of graphs can be realized as the crossing graph of a partial cube that is prime with respect to the Cartesian product.

Recall that the *radius* of a connected graph G is $\min_{u \in V(G)} \max_{v \in V(G)} d_G(u, v)$ and that G is called $K_{2,3}$-free if it contains no induced subgraph isomorphic to $K_{2,3}$. Note that partial cubes are $K_{2,3}$-free, as follows readily from the fact that Θ is not transitive on $K_{2,3}$.

For the main result of this section we first state the following lemma, which might be of independent interest.

Lemma 6. *Let G be a bipartite graph of radius 2. Then G is a partial cube if and only if G is $K_{2,3}$-free.*

Proof. We only need to show that if G is bipartite of radius 2 and $K_{2,3}$-free, then G is a partial cube. Let u be a vertex that realizes the radius of G and let $v_1, \ldots, v_k$ be its neighbors. As G is bipartite, $v_1, \dots, v_k$ is an independent set of G. Let $w_1, \dots, w_r$ be the remaining vertices of G; then they are all at distance 2 from u. Again, there is no edge between w_i and w_j.

Note that a graph is a partial cube if and only if the graph obtained from it by removing a pendant vertex is a partial cube. Hence we may without loss of generality assume that G has no pendant vertex. Since G is $K_{2,3}$-free, it follows that every vertex w_i is of degree 2. Moreover, no two vertices w_i and w_j, $i \neq j$, have the same pair of neighbors. Therefore every edge of the form $w_i v_j$ lies in precisely one square.

No two edges uv_i and uv_j, $i \neq j$, are in relation Θ. We claim that G isometrically embeds into Q_k and construct edge-subsets $E_1, \dots, E_k$ of $E(G)$ as follows. For $i = 1, \dots, k$ put uv_i in E_i. Consider an edge $w_i v_j$ and let $w_i v_j u v_\ell$ be the unique square containing this edge. Then $w_i v_j$ is in relation Θ with uv_ℓ. Put $w_i v_j \in E_\ell$. We claim that $E_1, \dots, E_k$ form the $\Theta = \Theta^*$-classes of G.

Clearly, $E_1, \dots, E_k$ is a partition of $E(G)$. Suppose $w_i v_j$ and $w_{i'} v_{j'}$ are two distinct edges of E_ℓ. Note first that $i \neq i'$, for otherwise w_i would have three neighbors at distance 1 from u; see Figure 1(i). The case $j = j'$ leads to another $K_{2,3}$; see Figure 1(ii). Hence $i \neq i'$ and $j \neq j'$ and we have the situation as shown in Figure 1(iii).

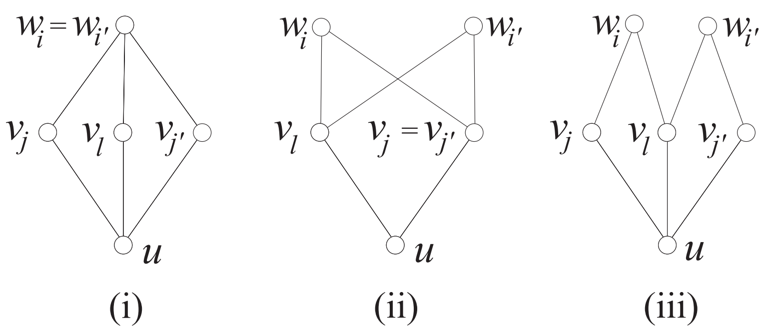

FIG. 1. *Cases in the proof of Lemma* 6.

Then $w_i v_\ell \in E(G)$ and $w_{i'} v_\ell \in E(G)$, which implies that $w_i v_j$ is in relation Θ with $w_{i'} v_{j'}$. Thus all pairs of edges from E_ℓ are in relation Θ. Now assume $w_i v_j \in E_\ell$ and $w_{i'} v_{j'} \in E_{\ell'}$, where $\ell \neq \ell'$. If $i = i'$ or $j = j'$, then clearly $w_i v_j$ and $w_{i'} v_{j'}$ are not in relation Θ. Next, if $\ell = j'$, then $d(w_i, w_{i'}) + d(v_j, v_{j'}) = 2 + 2$ is equal to $d(w_i, v_{j'}) + d(w_{i'}, v_j) = 1 + 3$; hence they are again not in relation Θ (the case $\ell' = j$ is analogous). Otherwise we get $d(w_i, w_{i'}) + d(v_j, v_{j'}) = 4 + 2 = 3 + 3 = d(w_i, v_{j'}) + d(w_{i'}, v_j)$. Hence we conclude that $\Theta = \Theta^*$ and thus G is a partial cube by Winkler's theorem. ☐

THEOREM 7. *Let A and B be arbitrary graphs, where at least one of them contains an edge. Then there exists a Cartesian prime partial cube G such that $G^\# = A \oplus B$.*

Proof. For a graph H let $\widetilde{H}$ be the graph obtained from H by subdividing all edges of H and adding a new vertex u joined to all the original vertices of H. (This construction has been introduced in [12] to establish a connection between median graphs and triangle-free graphs.) We claim that $G = \widetilde{A \oplus B}$ does the job.

Let $V(A) = \{a_1, \dots, a_n\}$ and $V(B) = \{b_1, \dots, b_m\}$, so that in G the vertex u is adjacent to $a_1, \dots, a_n$ and to $b_1, \dots, b_m$. Let x_{ij} be the vertex of G obtained by subdividing the edge $a_i b_j$, $1 \leq i \leq n$, $1 \leq j \leq m$.

We first observe that G is a partial cube by Lemma 6. Let E_i be the Θ-classes of G with the representative ua_i, $1 \leq i \leq n$, and let F_i be the Θ-classes of G with the representative ub_i, $1 \leq i \leq m$. Consider the square $ua_ix_{ij}b_j$ to infer that E_i and F_j cross. Similarly, E_i and E_j (resp., F_i and F_j) cross if and only if $a_ia_j \in E(A)$ (resp., $b_ib_j \in E(B)$). Hence $G^{\#} = A \oplus B$.

It remains to show that G is prime with respect to the Cartesian product. Assume without loss of generality that $n \geq 2$ and that $a_1a_2 \in E(A)$. Let a_i, a_j, $i \neq j$, be arbitrary vertices of A and b_k a vertex of B. Then we have $x_{ik}b_k \in E_i$ and $x_{jk}b_k \in E_j$. By the construction of G (recall that x_{ik} and x_{jk} are of degree 2) we infer that the edges $x_{ik}b_k$ and $x_{jk}b_k$ are in relation τ. As i and j were arbitrary, it follows that $E_1, \ldots, E_n$ belong to the same equivalence class of $(\Theta_G \cup \tau_G)^*$. Analogously, $F_1, \ldots, F_m$ belong to the same equivalence class of $(\Theta_G \cup \tau_G)^*$. Let y be the vertex of G obtained by subdividing the edge a_1a_2. Then we have $a_1y \in E_2$ and $a_1x_{11} \in F_1$. Moreover, a_1y is in relation τ with a_1x_{11}, which implies that $(\Theta_G \cup \tau_G)^*$ consists of a single equivalence class. By Corollary 1 we conclude that G is a Cartesian prime graph. $\square$

Other constructions that yield joins of graphs as crossing graphs can also be obtained. Let A be a graph and let G be the graph that is obtained from $\widetilde{A}$ by the Chepoi expansion (cf. [5]) with covering sets A and the star induced by u and its neighbors. Then G is a partial cube with $G^{\#} = K_1 \oplus A$. This construction is illustrated in Figure 2 for the case when A is the graph on four vertices and five edges. The new Θ-class of G that yields the K_1 in the join decomposition is denoted with thick lines.

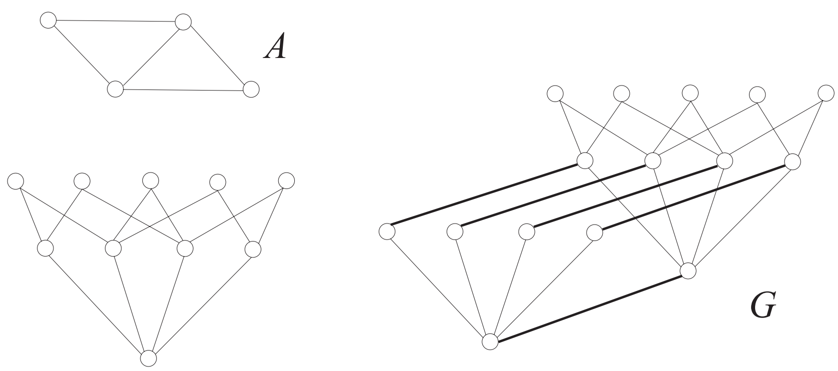

FIG. 2. *Expanding $\widetilde{A}$ into G, so that $G^{\#} = K_1 \oplus A$.*

4. The case of median graphs. Crossing graphs of median graphs are easier to study than those of general partial cubes, since if two Θ-classes of a median graph cross on some cycle, then there exists a square in which they cross. This fact can be easily seen by using the expansion procedure and induction.

In [15] it is proved that every graph is the crossing graph of some median graph. However, it was erroneously mentioned that there are prime median graphs whose crossing graphs are joins of two graphs. The graph presented in Figure 7.2 of [15] is a Cartesian product graph, namely $P_3 \,\square\, G$, where G is the graph obtained from C_4 and another vertex joined to one of the vertices of C_4. In this section we prove that the above remark is indeed wrong by proving that a median graph whose crossing graph is the join of two graphs is necessarily the Cartesian product of two graphs. Note that this is in surprising contrast to the situation from the previous section. We will need the following lemma that might be of independent interest. It follows from the

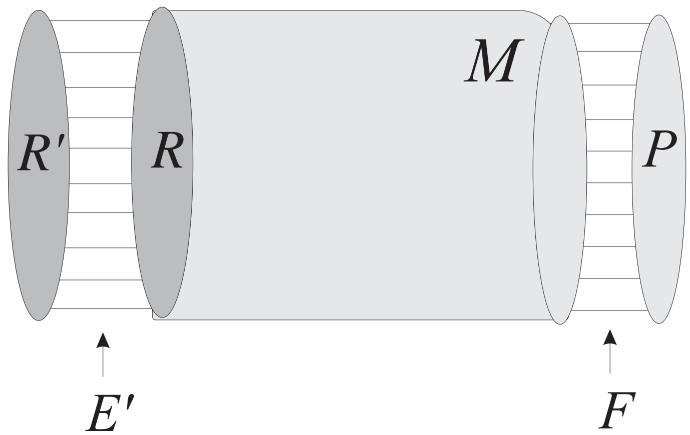

FIG. 3. *Case* $|A| = 1$ *in the proof of Theorem* 9.

Convexity Lemma from [10], which asserts that an induced connected subgraph H of a bipartite graph G is convex if and only if no edge with one endvertex in H and the other not in H is in relation Θ to an edge in H.

LEMMA 8. *Let G be a partial cube and H a convex subgraph of G. If H intersects all Θ-classes of G, then $H = G$.*

Proof. Suppose H is a proper subgraph of G. Then, since H is convex and hence induced, there exists an edge uv of G such that $u \in H$ and $v \notin H$. By the Convexity Lemma, uv is in relation Θ to no edge of H. But then H does not intersect the Θ-class of uv, a contradiction. ∎

We can now state the main result of this section.

THEOREM 9. *Let G be a median graph. Then $G^{\#} = A \oplus B$ if and only if $G = H \,\Box\, K$, where $H^{\#} = A$ and $K^{\#} = B$.*

Proof. By Proposition 5 one direction is proved: The crossing graph of the Cartesian product of median graphs is the join of the crossing graphs of the factors. Hence it remains to prove the converse of this statement, for which we will use induction on the number of Θ-classes of a median graph G. Clearly the smallest graph that is the join of two graphs and the crossing graph of a median graph is K_2. It is obvious that the only median graph with exactly two Θ-classes that cross is C_4, and $C_4 = K_2 \,\Box\, K_2$, providing the basis of the induction.

Assume the statement holds for median graphs with fewer than k Θ-classes. Let G be a median graph with k Θ-classes and $G^{\#} = A \oplus B$. By Theorem 3, G can be obtained by the peripheral expansion from a median graph M along its convex subgraph R. Denote by R' the corresponding peripheral subgraph (isomorphic to R), that is, $R' = G - M$. As M has one Θ-class less than G, $M^{\#}$ is an induced subgraph of $G^{\#}$. More precisely $M^{\#} = G^{\#} - u$, where u corresponds to the peripheral Θ-class E' of G. Without loss of generality we may assume that $u \in A$.

Assume first that $|A| = 1$. By Proposition 4 both connected components of $G - E'$ contain a peripheral subgraph. One component clearly induces the peripheral subgraph R'. Let P be a peripheral subgraph in the other component of $G - E'$. Denote by F the Θ-class such that P is a component of $G - F$ and denote by v the vertex of $G^{\#}$ that corresponds to F (see Figure 3). If $F \neq E'$, then F and E' do not cross, for otherwise P would lie in both components of $G - E'$. Hence, in $G^{\#}$ vertices u and v are not adjacent, which means that they must both be in A, but this is a contradiction with $|A| = 1$. The remaining case is $E' = F$, which implies $P = R$. Hence $G = K_2 \,\Box\, R$, where $R^\# = B$.

Now, let $|A| > 1$. Then $M^\# = (A - u) \oplus B$, and by the induction hypothesis, $M = U \,\Box\, K$, where $U^\# = A - u$ and $K^\# = B$. Note that Θ-classes of M consist of Θ-classes of U and of Θ-classes of K. More precisely, if F is a Θ-class of U (resp., K), then $F \times V(K)$ (resp., $V(U) \times F$) is a Θ-class of $U \,\Box\, K$; cf. [11, Lemma 4.3]. Denote by $u_1, \ldots, u_p$ the vertices of $A - u$ that correspond to Θ-classes of U, and by $v_1, \ldots, v_r$ the vertices of B that correspond to Θ-classes of K. By Lemma 2, $R = U' \,\Box\, K'$, where U' is a convex subgraph of U and K' is a convex subgraph of K.

Suppose K' is a proper subgraph of K. By Lemma 8 there exists a Θ-class of K that does not intersect with R, and thus it does not cross with E'. This implies that there is a vertex $v_i \in B$ which is not adjacent to $u \in A$, a contradiction. Hence $K' = K$ and $R = U' \,\Box\, K$, where U' is a convex subgraph of U. We deduce that $G = H \,\Box\, K$, where H is the graph obtained from U by expanding U'. Clearly $H^\# = A$ and $K^\# = B$, which completes the proof. $\square$

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