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Partial domination in supercubic graphs

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ABSTRACT

For some α with $0 < \alpha \le 1$, a subset X of vertices in a graph G of order n is an α-partial dominating set of G if the set X dominates at least $\alpha \times n$ vertices in G. The α-partial domination number $\mathrm{pd}_\alpha(G)$ of G is the minimum cardinality of an α-partial dominating set of G. In this paper partial domination of graphs with minimum degree at least 3 is studied. It is proved that if G is a graph of order n and with $\delta(G) \ge 3$, then $\mathrm{pd}_{\frac{7}{8}}(G) \le \frac{1}{3}n$. If in addition $n \ge 60$, then $\mathrm{pd}_{\frac{9}{10}}(G) \le \frac{1}{3}n$, and if G is a connected cubic graph of order $n \ge 28$, then $\mathrm{pd}_{\frac{13}{14}}(G) \le \frac{1}{3}n$. Along the way it is shown that there are exactly four connected cubic graphs of order 14 with domination number 5.

1. Introduction

One of the central themes of the theory of graph domination is establishing upper bounds for graphs with a prescribed minimum degree as a function of graph order. The topic is in depth surveyed in the paper [14] as well as in the 2023 book [13]. Special attention has been paid to cubic graphs and graphs of minimum degree at least 3. For the latter graphs, Reed [25] in 1996 established a best possible upper bound.

Theorem 1.1. ([25]) *If G is a graph of order n with $\delta(G) \ge 3$, then $\gamma(G) \le \frac{3}{8}n$.*

In 2009, Kostochka and Stocker sharpened Reed's bound for connected cubic graphs as follows.

Theorem 1.2. ([17]) *If G is a connected cubic graph of order n, then $\gamma(G) \le \frac{5}{14}n$, unless G is one of the two graphs A_1 and A_2 shown in Fig. 1.*

Kostochka and Stocker further proved that the graphs A_1 and A_2 are the only connected, cubic graphs that achieve the $\frac{3}{8}$-bound of Theorem 1.1. On the other hand, Reed [25] conjectured that $\gamma(G) \le \lceil \frac{1}{3}n \rceil$ whenever G is a connected cubic graph of order n. Kostochka and Stodolsky [18] disproved this conjecture by constructing an infinite sequence $\{G_k\}_{k=1}^{\infty}$ of connected, cubic graphs with

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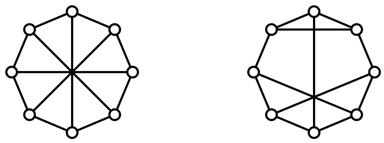


(a) A_1 (b) A_2

Fig. 1. The (non-planar) cubic graphs A_1 and A_2 of order $n = 8$ with $\gamma(A_1) = \gamma(A_2) = 3$

$$\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{69}.$$

Subsequently, Kelmans [16] constructed an infinite series of 2-connected, cubic graphs H_k with

$$\lim_{k\to\infty} \frac{\gamma(H_k)}{|V(H_k)|} \geq \frac{1}{3} + \frac{1}{60}.$$

Thus, there exist connected cubic graphs G of arbitrarily large order n satisfying

$$\gamma(G) \geq \left(\frac{1}{3} + \frac{1}{60}\right)n.$$

So, $\gamma(G) \leq \lceil \frac{1}{3}n \rceil$ does not hold for all connected cubic graphs. On the other hand, in 2010 Verstraëte conjectured that if G is a cubic graph of order n and girth at least 6, then $\gamma(G) \leq \frac{1}{3}n$, see [13, Conjecture 10.23]. In [21] the conjecture has been verified for cubic graphs with girth at least 83. Further upper bounds on the domination number of a cubic graph in terms of its order and girth were proved in [19,20,24].

The following concepts were independently introduced in [4,5]. Let $G = (V(G), E(G))$ be a graph of order n. For some α with $0 < \alpha \leq 1$, a set $S \subseteq V(G)$ is an *α-partial dominating set* of G if

$$|N_G[S]| \geq \alpha \times n,$$

that is, the set S dominates at least αn vertices in G. The *α-partial domination number* of G, denoted by $\mathrm{pd}_\alpha(G)$ (also by $\gamma_\alpha(G)$ in the literature), is the minimum cardinality of an α-partial dominating set of G. Investigations on the concept of partial domination in graphs can be found in [3–6,22,23]. At this point, it should be pointed out that the term “partial domination” is also used to refer to a concept that is different from ours [2]. We also remark that the concept of an α-dominating set [7,8,15] is different from our concept of an α-partial dominating set.

In light of the above, this paper addresses the following natural question: What is the largest possible value on α such that the α-partial domination number of a connected cubic graph is at most one-third the order of the graph? We further consider the same question in the more general setting of graphs with minimum degree at least 3.

We proceed as follows. In Section 2, we present the graph theory terminology we adopt in this paper, and state preliminary results. In Section 3, we prove that the $\frac{7}{8}$-partial domination number of a connected cubic graph G of order 14 is at most 4. Thereafter in Section 4, we prove that the $\frac{7}{8}$-partial domination number of a graph with minimum degree at least 3 is at most one-third its order, and prove a stronger statement if the order of the graph is large enough. In Section 5 we show that there are exactly four connected cubic graphs of order 14 with domination number 5, and conjecture that these are the only graphs achieving equality in the upper bound $\gamma(G) \leq \frac{5}{14}n$ given by Kostochka and Stocker in Theorem 1.2.

2. Preliminaries

In this section, we call up the definitions, concepts and known results that we need for what follows. Let $G = (V(G), E(G))$ be a graph. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of vertices adjacent to v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. For a set $D \subseteq V(G)$, its *open neighborhood* is the set $N_G(D) = \cup_{v\in D} N_G(v)$, and its *closed neighborhood* is the set $N_G[D] = N_G(D) \cup D$. The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The graph G is *r-regular* if every vertex in G has degree r. A 3-regular graph is called a *cubic graph*, and a graph G with $\Delta(G) \leq 3$ a *subcubic graph*. To these established terms we add the term *supercubic graph* which refers to graphs G with $\delta(G) \geq 3$.

A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A γ-set of G is a dominating set of G of minimum cardinality $\gamma(G)$. Let X and Y be subsets of vertices in G. The set X *dominates* the set Y if every vertex in Y is in the set X or has a neighbor in the set X, that is, if $Y \subseteq N_G[X]$. If X is a set of vertices in a graph G, then we denote by $\mathrm{dom}_G(X)$ the number of vertices dominated by the set X, and so $\mathrm{dom}_G(X) = |N_G[X]|$. A thorough treatise on domination in graphs can be found in [11–13].

For a set of vertices S in a graph G and a vertex $v \in S$, the *S-private neighborhood* of v is defined by $\mathrm{pn}[v, S] = \{w \in V(G) \colon N_G[w] \cap S = \{v\}\}$. The *$S$-external private neighborhood* of v is the set $\mathrm{epn}[v, S] = \mathrm{pn}[v, S] \setminus S$. (The set $\mathrm{epn}[v, S]$ is also denoted $\text{epn}(v, S)$ in the literature.) An *S-external private neighbor* of v is a vertex in $\text{epn}[v, S]$. In 1979, Bollobás and Cockayne [1] established the following property of minimum dominating sets in graphs to be used later on.

Lemma 2.1. ([1]) *Every isolate-free graph G contains a γ-set D such that* $\text{epn}[v, D] \neq \emptyset$ *for every vertex $v \in D$.*

A set S of vertices in G is a *packing* in G if the closed neighborhoods of vertices in S are pairwise disjoint. Equivalently, S is a packing in G if the vertices in S are pairwise at distance at least 3. A packing is sometimes called a 2-packing in the literature. The *packing number* of G, denoted by $\rho(G)$, is the maximum cardinality of a packing in G. In 1996, Favaron [9] proved the following result on the packing number of a cubic graph.

Theorem 2.2. (Favaron [9]) *If G is a connected cubic graph of order n different from the Petersen graph, then $\rho(G) \geq \frac{n}{8}$.*

For a set of vertices in G, the subgraph of G induced by S is denoted by $G[S]$. Finally, the *boundary* of a set S of vertices in G, denoted by $\partial(S)$, is the set of vertices not in S that have a neighbor in S, that is, $\partial(S) = N_G[S] \setminus S$.

3. (Partial) domination in cubic graphs of order 14

In this section, we present a preliminary result that the $\frac{7}{8}$-partial domination number of a connected cubic graph G of order 14 is at most 4. We will need this result when proving our main theorem in Section 4.

Theorem 3.1. *If G is a connected cubic graph of order $n = 14$, then*

$$\text{pd}_{\frac{7}{8}}(G) \leq 4 < \frac{1}{3}n \,.$$

Proof. Let $\alpha = \frac{7}{8}$ and let G be a connected cubic graph of order $n = 14$. Let $\gamma = \gamma(G)$. By Theorem 1.2, $\gamma \leq \lfloor \frac{5}{14}n \rfloor = 5$. If $\gamma \leq \lfloor \frac{1}{3}n \rfloor = 4$, then every γ-set of G is certainly an α-partial dominating set of G. Thus in this case, $\text{pd}_\alpha(G) \leq \gamma \leq 4$, as desired. Hence we may assume in what follows that $\gamma = 5$.

By Theorem 2.2, the graph G has packing number $\rho(G) \geq \lceil \frac{n}{8} \rceil = 2$. Let P be a maximum packing in G, and so $|P| = \rho(G) \geq 2$. Suppose that $\rho(G) > 2$, implying that $\rho(G) = 3$. In this case, $\text{dom}_G(P) = |N_G[P]| = 12$. Thus if v is any one of the two vertices in $V(G) \setminus N_G[P]$ and $S = P \cup \{v\}$, then the set S satisfies $\text{dom}_G(S) \geq 13 > \frac{7}{8}n$, and so $\text{pd}_\alpha(G) \leq |S| = 4$. Hence we may assume that $|P| = \rho(G) = 2$, for otherwise the desired result follows.

Let $X = V(G) \setminus N_G[P]$, and so $|X| = 6$. If a vertex in X has all three of its neighbors in the set X, then we can add such a vertex to the set P to produce a packing of cardinality 3, contradicting our assumption that $\rho(G) = 2$. Hence, every vertex in X has at most two neighbors that belong to X.

Suppose that a vertex $x \in X$ has two neighbors in the set X. In this case, we let $P_x = P \cup \{x\}$. The resulting set P_x satisfies $|P_x| = 3$ and $\text{dom}_G(P_x) = 4 + 4 + 3 = 11$. Let $Z = V(G) \setminus N_G[P_x]$, and so $|Z| = 3$. If there is a vertex $z \in Z$ with at least one neighbor in Z, then the set $S = P_x \cup \{z\}$ satisfies $\text{dom}_G(S) \geq 13$ and $|S| = 4$, and so as before $\text{pd}_\alpha(G) \leq |S| = 4$. Hence, we may assume that Z is an independent set in G. Thus, every vertex in Z has all three of its neighbors contained in the boundary $\partial(P_x)$ of the set P_x. Denoting by ℓ_1 the number of edges between the sets Z and $\partial(P_x)$, we obtain $\ell_1 = 3|Z| = 9$. Since $|\partial(P_x)| = \text{dom}_G(P_x) - |P_x| = 11 - 3 = 8$, by the Pigeonhole Principle at least one vertex v in the boundary $\partial(P_x)$ of P_x has at least two neighbors in Z. Thus the set $S = P_x \cup \{v\}$ satisfies $\text{dom}_G(S) \geq 13$ and $|S| = 4$, and so as before $\text{pd}_\alpha(G) \leq |S| = 4$.

Hence, we may assume that every vertex in X has at most one neighbor that belongs to X, and therefore at least two neighbors that belong to the boundary $\partial(P)$ of P. Denoting by ℓ_2 the number of edges between the sets X and $\partial(P)$, we obtain $\ell_2 \geq 2|X| = 2 \times 6 = 12$. However every vertex in $\partial(P)$ has one neighbor in P and therefore at most two neighbors in X, and so $\ell_2 \leq 2|\partial(P)| = 2 \times 6 = 12$. Consequently, $\ell_2 = 12$, implying that $\partial(P)$ is an independent set and each vertex in $\partial(P)$ has exactly two neighbors in X. Furthermore, each vertex in X has exactly two neighbors in $\partial(P)$ and one neighbor in X. In particular, the subgraph induced by the set X consists of three disjoint copies of P_2, that is, $G[X] = 3P_2$.

Let $Y = \partial(P)$, and let H be the graph with vertex set $X \cup Y$ and with edge set consisting of all edges in G between X and Y. By our earlier observations, $|X| = |Y| = 6$. The resulting bipartite graph H has partite sets X and Y and is a 2-regular graph of order 12. Thus, either $H = 2C_6$, or $H = 3C_4$, or $H = C_4 \cup C_8$, or $H = C_{12}$. Let $P = \{v_1, v_2\}$. Let $X = \{x_1, x_2, \ldots, x_6\}$ and $Y = \{y_1, y_2, \ldots, y_6\}$.

Claim 1. *$H \neq 2C_6$.*

Proof. Suppose, to the contrary, that $H = 2C_6$. Renaming vertices in X and Y if necessary, we may assume that $Q_1 : x_1y_1x_2y_2x_3y_3x_1$ and $Q_2 : x_4y_4x_5y_5x_6y_6x_4$ are the two 6-cycles in H, and so $H = Q_1 \cup Q_2$. Renaming vertices if necessary, we may assume that v_1y_1 is an edge of G. Since v_1 is adjacent to at most two vertices from the cycle Q_2, we may assume, renaming vertices of Q_2 if necessary, that v_2y_4 is an edge of G. Thus the graph F shown in Fig. 2 is a spanning

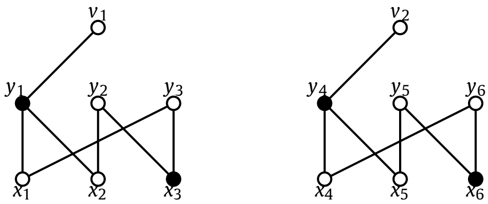


Fig. 2. A spanning subgraph F of G in the proof of Claim 1

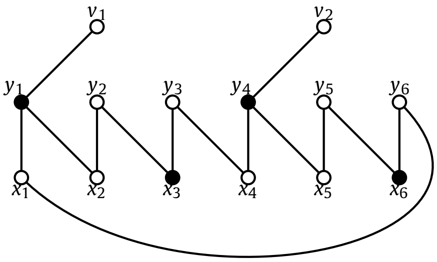


Fig. 3. A spanning subgraph F of G in the proof of Claim 2

subgraph of G. In this case, the set $S = \{y_1, x_3, y_4, x_6\}$ is a dominating set of F (where the vertices in S are shaded in Fig. 2), and so $\gamma \le \gamma(F) = 4$, a contradiction. (□)

Claim 2. $H \neq C_{12}$.

Proof. Suppose, to the contrary, that $H = C_{12}$. Renaming vertices in X and Y if necessary, we may assume that H is the cycle $x_1y_1x_2y_2 \ldots x_6y_6x_1$. The vertex v_1 has three edges to Y, implying that v_1 has exactly one edge to at least one of the three sets $\{y_1, y_4\}$, $\{y_2, y_5\}$ and $\{y_3, y_6\}$. Renaming vertices if necessary, we may assume that v_1 has exactly one edge to the set $\{y_1, y_4\}$. Further, we may assume that v_1y_1 is an edge of G, and so v_1y_4 is not an edge of G. Since every vertex in Y is adjacent to exactly one of v_1 and v_2, this implies that v_2y_4 is an edge. Thus the graph F shown in Fig. 3 is a spanning subgraph of G. In this case, the set $S = \{y_1, x_3, y_4, x_6\}$ is a dominating set of F (see Fig. 3), and so $\gamma \le \gamma(F) \le 4$, a contradiction. (□)

Claim 3. *If* $H = 3C_4$, *then* $\mathrm{pd}_\alpha(G) \le 4$.

Proof. Suppose that $H = 3C_4$. Renaming vertices in X and Y if necessary, we may assume that $Q_1: x_1y_1x_2y_2x_1$, $Q_2: x_3y_3x_4y_4x_3$ and $Q_3: x_5y_5x_6y_6x_5$ are the three 4-cycles in H, and so $H = Q_1 \cup Q_2 \cup Q_3$.

Suppose that v_1 is adjacent in G to a vertex from each of the three 4-cycles of H. Renaming vertices if necessary, we may assume that $N_G(v_1) = \{y_1, y_3, y_5\}$. Since every vertex in Y is adjacent to exactly one of v_1 and v_2, this implies that $N_G(v_2) = \{y_2, y_4, y_6\}$. Thus the graph F shown in Fig. 4(a) is a spanning subgraph of G. In this case, the set $S = \{v_1, y_2, y_4, y_6\}$ is a dominating set of F (see Fig. 4(a)), and so $\gamma \le \gamma(F) = 4$, a contradiction.

Hence, neither v_1 nor v_2 is adjacent in G to a vertex from each of the three 4-cycles of H. Renaming vertices if necessary, we may assume that $N_G(v_1) = \{y_1, y_2, y_3\}$ and $N_G(v_2) = \{y_4, y_5, y_6\}$. By our earlier observations, $G[X] = 3P_2$. If x_1x_2 is an edge, then the graph F shown in Fig. 4(b) is a spanning subgraph of G. In this case, the set $S = \{v_2, x_2, y_3, y_5\}$ is a dominating set of F (see Fig. 4(b)), and so $\gamma \le \gamma(F) = 4$, a contradiction. Hence, $x_1x_2 \notin E(G)$. By symmetry, $x_5x_6 \notin E(G)$.

Suppose that x_3x_4 is an edge. Renaming vertices in necessary, we may assume in this case that x_1x_6 and x_2x_5 are edges. Thus the graph G is determined and is shown in Fig. 4(c). In this case, the set $S = \{x_2, x_6, y_3, y_4\}$ is a dominating set of G (see Fig. 4(c)), and so $\gamma \le 4$, a contradiction. Hence, $x_3x_4 \notin E(G)$.

The graph G is therefore determined. Renaming vertices if necessary, we may assume that $G = G_{14.1}$, where $G_{14.1}$ is the graph shown in Fig. 4(d). We note that $\gamma = 5$. In this case, the set $S = \{y_1, y_3, y_5, v_2\}$ satisfies $\mathrm{dom}_G(S) = 13$ (the vertex y_2 represented by the square in Fig. 4(d) is the only vertex not dominated by S) and $|S| = 4$, implying that $\mathrm{pd}_\alpha(G) \le |S| \le 4$. This completes the proof of Claim 3. (□)

Claim 4. *If* $H = C_4 \cup C_8$, *then* $\mathrm{pd}_\alpha(G) \le 4$.

Proof. Suppose that $H = C_4 \cup C_8$. Renaming vertices in X and Y if necessary, we may assume that $Q_1: x_1y_1x_2y_2x_1$ is the 4-cycle in H and $Q_2: x_3y_3x_4y_4x_5y_5x_6y_6x_3$ is the 6-cycle in H.

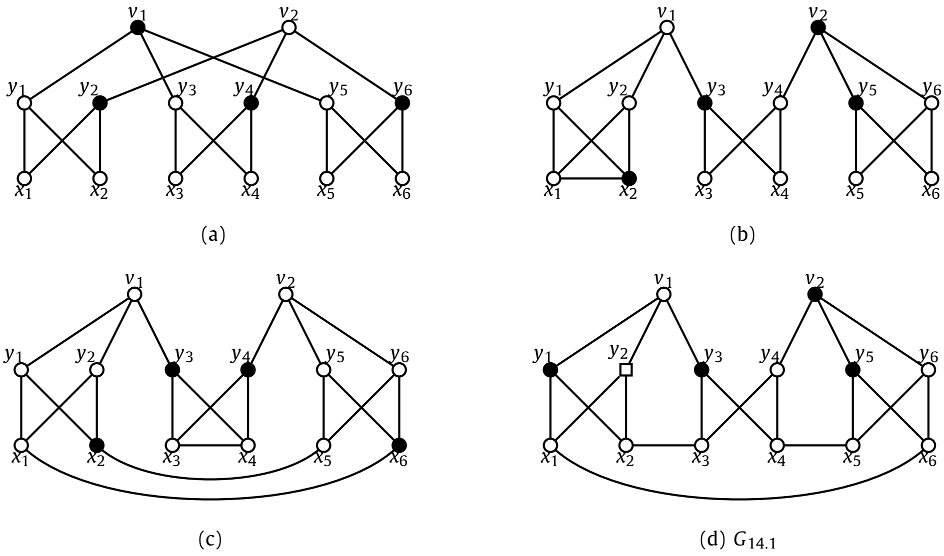


Fig. 4. Spanning subgraphs F of G in the proof of Claim 3

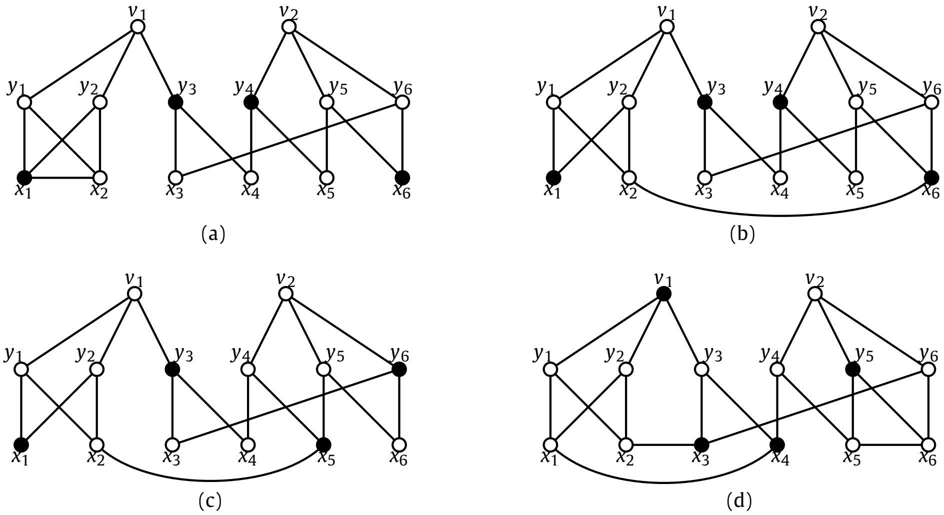


Fig. 5. Spanning subgraphs F of G in the proof of Claim 4.1

Claim 4.1. *Both v_1 and v_2 are adjacent to exactly one vertex from the cycle Q_1.*

Proof. Suppose, to the contrary, that v_1 or v_2, say v_1, is adjacent in G to two vertices in the cycle Q_1. Renaming vertices if necessary, we may assume that $N_G(v_1) = \{y_1, y_2, y_3\}$, and so $N_G(v_2) = \{y_4, y_5, y_6\}$. Recall that $G[X] = 3P_2$.

If x_1x_2 is an edge, then the graph F shown in Fig. 5(a) is a spanning subgraph of G. In this case, the set $S = \{x_1, x_6, y_3, y_4\}$ is a dominating set of F (see Fig. 5(a)), and so $\gamma \le \gamma(F) = 4$, a contradiction. Hence, $x_1x_2 \notin E(G)$. If x_2x_6 is an edge, then the graph F shown in Fig. 5(b) is a spanning subgraph of G. In this case, the set $S = \{x_1, x_6, y_3, y_4\}$ is a dominating set of F (see Fig. 5(b)), and so $\gamma \le \gamma(F) = 4$, a contradiction. Hence, $x_2x_6 \notin E(G)$. By symmetry, $x_1x_6 \notin E(G)$. If x_2x_5 is an edge, then the graph F shown in Fig. 5(c) is a spanning subgraph of G. In this case, the set $S = \{x_1, x_5, y_3, y_6\}$ is a dominating set of F (see Fig. 5(c)), and so $\gamma \le \gamma(F) = 4$, a contradiction. Hence, $x_2x_5 \notin E(G)$. By symmetry, $x_1x_5 \notin E(G)$.

Renaming x_1 and x_2 if necessary, we may assume that x_1x_4 and x_2x_3 are edges. The remaining edge in $G[X]$ is therefore the edge x_5x_6. Thus, the graph G is determined and is shown in Fig. 5(d). In this case, the set $S = \{v_1, x_3, x_4, y_5\}$ is a dominating set of G (see Fig. 5(d)), and so $\gamma \le 4$, a contradiction. This completes the proof of Claim 4.1. (□)

By Claim 4.1, both v_1 and v_2 are adjacent to exactly one vertex from the cycle Q_1. Renaming y_1 and y_2 if necessary, we may assume that v_1y_1 and v_2y_2 are edges.

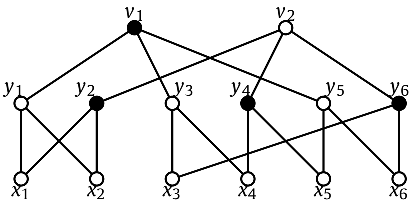


Fig. 6. A spanning subgraph F of G in the proof of Claim 4.2

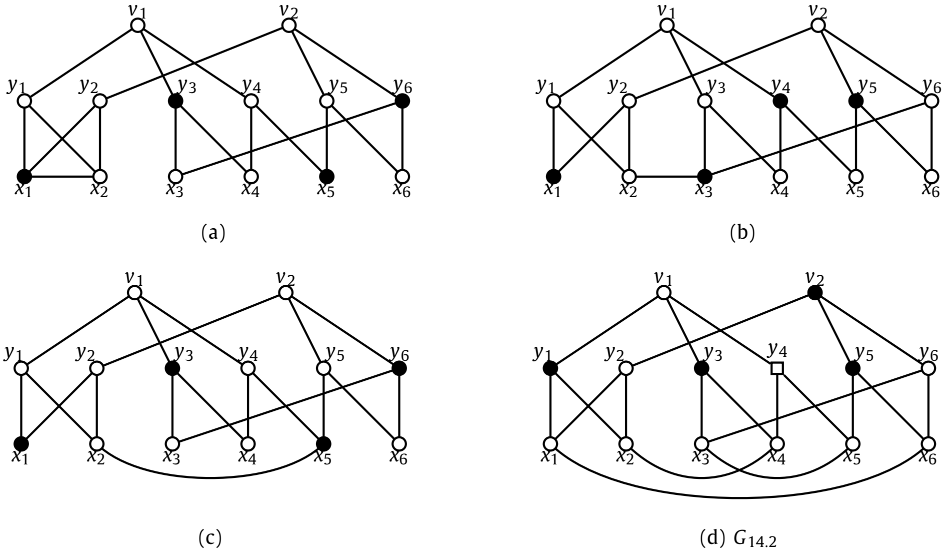


Fig. 7. Spanning subgraphs F of G in the proof of Claim 4

Claim 4.2. *The vertex v_1 is adjacent to two vertices in Q_2 at distance 2 in Q_2.*

Proof. Suppose, to the contrary, that v_1 is adjacent to two vertices in Q_2 at distance 4. Renaming vertices if necessary, we may assume that v_1y_3 and v_1y_5 are edges. Thus, $N_G(v_1) = \{y_1, y_3, y_5\}$ and $N_G(v_2) = \{y_2, y_4, y_6\}$. Thus the graph F shown in Fig. 6 is a spanning subgraph of G. In this case, the set $S = \{v_1, y_2, y_4, y_6\}$ is a dominating set of F (see Fig. 6), and so $\gamma \le \gamma(F) = 4$, a contradiction. (□)

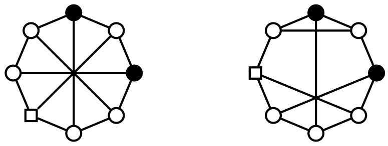
By Claim 4.2, the vertex v_1 is adjacent to two vertices in Q_2 at distance 2 in Q_2. Renaming vertices if necessary, we may assume that $N_G(v_1) = \{y_1, y_3, y_4\}$ and $N_G(v_2) = \{y_2, y_5, y_6\}$. If x_1x_2 is an edge, then the graph F shown in Fig. 7(a) is a spanning subgraph of G. In this case, the set $S = \{x_1, x_5, y_3, y_6\}$ is a dominating set of F (see Fig. 7(a)), and so $\gamma \le \gamma(F) = 4$, a contradiction. Hence, $x_1x_2 \notin E(G)$.

If x_2x_3 is an edge, then the graph F shown in Fig. 7(b) is a spanning subgraph of G. In this case, the set $S = \{x_1, x_3, y_4, y_5\}$ is a dominating set of F (see Fig. 7(b)), and so $\gamma \le \gamma(F) = 4$, a contradiction. Hence, $x_2x_3 \notin E(G)$. By symmetry, $x_1x_3 \notin E(G)$.

If x_2x_5 is an edge, then the graph F shown in Fig. 7(c) is a spanning subgraph of G. In this case, the set $S = \{x_1, x_5, y_3, y_6\}$ is a dominating set of F (see Fig. 7(c)), and so $\gamma \le \gamma(F) = 4$, a contradiction. Hence, $x_2x_5 \notin E(G)$. By symmetry, $x_1x_5 \notin E(G)$.

Renaming x_1 and x_2 if necessary, we may assume that x_1x_6 and x_2x_4 are edges. The remaining edge in $G[X]$ is therefore the edge x_3x_5. Thus, the graph G is determined. Renaming vertices if necessary, we may assume that $G = G_{14.2}$, where $G_{14.2}$ is the graph shown in Fig. 7(d). We note that $\gamma = 5$. In this case, the set $S = \{y_1, y_3, y_5, v_2\}$ satisfies $\mathrm{dom}_G(S) = 13$ (the vertex y_4 represented by the square in Fig. 7(d) is the only vertex not dominated by S) and $|S| = 4$, implying that $\mathrm{pd}_\alpha(G) \le |S| = 4$. This completes the proof of Claim 4. (□)

We now return to the proof of Theorem 3.1 one final time. As observed earlier, there are four possibilities for the graph H, namely $H = 2C_6$ or $H = 3C_4$ or $H = C_4 \cup C_8$ or $H = C_{12}$. By Claim 1, $H \ne 2C_6$. By Claim 2, $H \ne C_{12}$. By Claim 3, if $H = 3C_4$, then $\mathrm{pd}_\alpha(G) \le 4$. By Claim 4, if $H = C_4 \cup C_8$, then $\mathrm{pd}_\alpha(G) \le 4$. This completes the proof of Theorem 3.1. □



(a) A_1 (b) A_2

Fig. 8. $\frac{7}{8}$-partial dominating sets in A_1 and A_2. In each, the vertex represented by the square is the only vertex not dominated by the two shaded vertices.

4. Partial domination in supercubic graphs

We start this section by proving a useful lemma. We first present a key lemma, which allows us to grow a given set of vertices to a larger set that dominates more vertices. Recall that we refer to graphs G with $\delta(G) \geq 3$ as supercubic graphs.

Lemma 4.1. *Let k be a positive integer and G a supercubic graph of order n. If $S \subseteq V(G)$, $U_S = V(G) \setminus N_G[S]$, and*

$$4|U_S| > k(n - |S|)\,,$$

then there exists a vertex in $\partial(S) \cup U_S$ that dominates at least $k+1$ vertices from U_S.

Proof. Consider the 'useful' vertex pairs (x, y) such that $y \in U_S$ and x dominates y (allowing $x = y$). Denote by p the number of useful pairs. As all vertices from U_S can be dominated by itself or one of its at least three neighbors, $p \geq 4|U_S|$. Since $y \in U_S = V(G) \setminus N_G[S]$, we have $N_G[y] \cap S = \emptyset$. It follows that $x \in \partial(S) \cup U_S$.

To prove the statement, we suppose that there is no vertex in G which dominates more than k vertices from U_S. Equivalently, every vertex $x \in \partial(S) \cup U_S$ belongs to at most k different useful pairs (x, y) (s.t. x is the first entry). We then conclude

$$k(|\partial(S)| + |U_S|) = k(n - |S|) \geq p \geq 4|U_S|$$

that contradicts the given condition and therefore proves the statement. □

We are now in a position to prove that the $\frac{7}{8}$-partial domination number of a cubic graph G is at most one-third of the order of G. In fact, our result states that the same is true for every supercubic graph.

Theorem 4.2. *If G is a supercubic graph of order n, then*

$$\mathrm{pd}_{\frac{7}{8}}(G) \leq \frac{1}{3}n\,.$$

Proof. First suppose that G is the disjoint union of the components $G_1, \ldots, G_k$. It was already observed in [5] that $\mathrm{pd}_\alpha(G) \leq \sum_{i=1}^{k} \mathrm{pd}_\alpha(G_i)$ holds for each α. Therefore, it suffices to prove the statement for connected graphs.

Let G be a connected graph of order n and of minimum degree $\delta(G) \geq 3$. Let $\alpha = \frac{7}{8}$ and $\gamma = \gamma(G)$. We proceed further with two claims.

Claim 5. *If $n \leq 14$, then $\mathrm{pd}_\alpha(G) \leq \frac{1}{3}n$.*

Proof. By Theorem 1.1, $\gamma \leq \left\lfloor \frac{3}{8}n \right\rfloor$ holds, so $\left\lfloor \frac{3}{8}n \right\rfloor$ vertices are enough to dominate the entire vertex set. Since $\left\lfloor \frac{3}{8}n \right\rfloor = \left\lfloor \frac{1}{3}n \right\rfloor$ holds whenever $n \leq 14$ and $n \notin \{8, 11, 14\}$, it suffices to consider graphs of order 8, 11 and 14.

Suppose first that G is cubic. Then only $n \in \{8, 14\}$ must be considered. If $n = 14$, then by Theorem 3.1 we have $\mathrm{pd}_\alpha(G) \leq \frac{1}{3}n$. Hence we may assume that $n = 8$. If G is isomorphic to A_1 or A_2, then as illustrated in Fig. 8, there exists a set S of two (shaded) vertices in G that dominates seven vertices. For any other cubic graph G of order 8 we have $\gamma(G) \leq 2$ by Theorem 1.2. Hence $\mathrm{pd}_\alpha(G) \leq 2 = \frac{1}{4}n < \frac{1}{3}n$ for each cubic graph G of order 8.

Assume in the rest that G is supercubic but not cubic. Hence there exists a vertex u of degree at least 4.

If $n = 8$, a vertex u of maximum degree dominates $|N_G[u]| = \mathrm{dom}_G(\{u\}) \geq 5$ vertices. If $\mathrm{dom}_G(\{u\}) \geq 6$, then any undominated vertex $u' \notin N_G[u]$ can be added to the set and we have $\mathrm{dom}_G(\{u, u'\}) \geq 7$. If $\mathrm{dom}_G(\{u\}) = 5$, we apply Lemma 4.1 with $k = 1$ and $S = \{u\}$. Since $4|U_S| = 4 \times 3 > 8 - 1$, there exists a vertex u' such that $\mathrm{dom}_G(\{u, u'\}) \geq 7$. In both cases, $\mathrm{dom}_G(\{u, u'\}) \geq 7$ implies $\mathrm{pd}_\alpha(G) \leq 2 = \left\lfloor \frac{1}{3}n \right\rfloor$.

If $n = 11$, we want to prove that there are three vertices v_1, v_2, v_3 that dominate at least 10 vertices in G. Then, $\mathrm{pd}_\alpha(G) \leq 3 = \left\lfloor \frac{1}{3}n \right\rfloor$ will follow. Let v_1 be a vertex of maximum degree. We have $\mathrm{dom}_G(\{v_1\}) \geq 5$. If $\mathrm{dom}_G(\{v_1\}) = 5$ then, for $S = \{v_1\}$, the inequality $4|U_S| = 24 > 2(n - |S|) = 20$ holds and Lemma 4.1 implies the existence of a vertex v_2 that dominates at least three vertices from U_S. It follows that $\mathrm{dom}_G(\{v_1, v_2\}) \geq 8$. If $\mathrm{dom}_G(\{v_1\}) = 6$ then, by setting $S = \{v_1\}$, we get $4|U_S| = 20 > n - |S| = 10$ that shows, by Lemma 4.1, the existence of a vertex v_2 which dominates at least two new vertices. Again, we have that $\mathrm{dom}_G(\{v_1, v_2\}) \geq 8$. If $\mathrm{dom}_G(\{v_1\}) \geq 7$, then v_2 can be chosen as an arbitrary undominated vertex and $\mathrm{dom}_G(\{v_1, v_2\}) \geq 8$ is achieved. For the choice of the last vertex, we consider two cases. If $\mathrm{dom}_G(\{v_1, v_2\}) = 8$, the set $S = \{v_1, v_2\}$ satisfies the condition in Lemma 4.1 with $k = 1$ and the existence of a vertex v_3 which dominates at least two vertices from U_S follows. It means $\mathrm{dom}_G(\{v_1, v_2, v_3\}) \geq 10$ as required. If $\mathrm{dom}_G(\{v_1, v_2\}) \geq 9$, then any undominated vertex can be chosen as v_3 and we have $\mathrm{dom}_G(\{v_1, v_2, v_3\}) \geq 10$ again.

If $n = 14$, we want to prove that there exist four vertices v_1, v_2, v_3, v_4 which together dominate at least 13 vertices. Then, $\mathrm{pd}_\alpha(G) \leq 4 = \lfloor \frac{1}{3}n \rfloor$ will follow. Let v_1 be a vertex of maximum degree. If $\mathrm{dom}_G(\{v_1\}) = 5$ and $N(v_1)$ is a dominating set in G, then $\mathrm{dom}_G(N(v_1)) = 14$ and we are ready. In the other case, $\mathrm{dom}_G(\{v_1\}) = 5$ and $N(v_1)$ is not a dominating set in G. Then there exists a vertex v_2 such that $\{v_1, v_2\}$ is a packing in G. If v_2 is a vertex of degree 3, then $\mathrm{dom}_G(\{v_1, v_2\}) = 9$ and for $S = \{v_1, v_2\}$ and $k = 1$, the condition $4 \times 5 > 14 - 2$ holds and Lemma 4.1 implies the existence of a vertex v_3 with $\mathrm{dom}_G(\{v_1, v_2, v_3\}) \geq 11$. If v_2 is a vertex of degree at least 4, then $\mathrm{dom}_G(\{v_1, v_2\}) \geq 10$ and $\mathrm{dom}_G(\{v_1, v_2, v_3\}) \geq 11$ can be easily achieved. For the choice of the last vertex, we consider two further subcases. If $\mathrm{dom}_G(\{v_1, v_2, v_3\}) = 11$, we have $4 \times 3 > 14 - 3$ and Lemma 4.1 implies the existence of a vertex v_4 with $\mathrm{dom}_G(\{v_1, v_2, v_3, v_4\}) \geq 13$. If $\mathrm{dom}_G(\{v_1, v_2, v_3\}) \geq 12$ and there are undominated vertices, then we may choose such a vertex v_4 and get $\mathrm{dom}_G(\{v_1, v_2, v_3, v_4\}) \geq 13$. (□)

By Claim 5, we may assume that $n \geq 15$, for otherwise the desired result follows. Let $D = \{v_1, v_2, \ldots, v_\gamma\}$ be a γ-set of G satisfying the Bollobás-Cockayne Lemma 2.1, and so $\mathrm{epn}[v, D] \neq \emptyset$ for every vertex $v \in D$. By Theorem 1.1, we have $\gamma \leq \lfloor \frac{3}{8}n \rfloor$. If $\gamma \leq \frac{1}{3}n$, then the set D is certainly an α-partial dominating set of G of cardinality at most $\frac{1}{3}n$. Thus in this case, $\mathrm{pd}_\alpha(G) \leq |D| \leq \frac{1}{3}n$. Hence we may assume that $\gamma > \frac{1}{3}n$, for otherwise the desired result is immediate.

Using the vertices $v_1, \ldots, v_\gamma$ from D, let $(V_1, V_2, \ldots, V_\gamma)$ be a partition of the vertex set $V(G)$ such that for all $i \in [\gamma]$, the following properties hold: (i) $v_i \in V_i$, (ii) $\mathrm{epn}[v_i, D] \subset V_i$, and (iii) $V_i \subseteq N_G[v_i]$. As observed earlier, $|\mathrm{epn}[v_i, D]| \geq 1$, and so $|V_i| \geq |\{v_i\}| + |\mathrm{epn}[v_i, D]| \geq 2$ for all $i \in [\gamma]$. Renaming the vertices $v_1, v_2, \ldots, v_\gamma$ if necessary, we may assume that $|V_i| \geq |V_{i+1}|$ for all $i \in [\gamma - 1]$, that is,

$$|V_1| \geq |V_2| \geq \cdots \geq |V_\gamma| \geq 2. \tag{1}$$

Let $k_1 = \lfloor \frac{1}{3}n \rfloor$ and let $k_2 = \gamma - k_1$. By assumption, $\frac{1}{3}n < \gamma$. By Theorem 1.1, $\gamma \leq \lfloor \frac{3}{8}n \rfloor$. Hence, $\frac{1}{3}n < \gamma \leq \lfloor \frac{3}{8}n \rfloor$. By definition of k_1 and k_2 and by our earlier observations and assumptions,

$$1 \leq k_2 = \gamma - k_1 \leq \left\lfloor \frac{3}{8}n \right\rfloor - \left\lfloor \frac{1}{3}n \right\rfloor. \tag{2}$$

Let $S = \{v_1, v_2, \ldots, v_{k_1}\}$, and so $|S| = k_1 = \lfloor \frac{1}{3}n \rfloor$. Since $(V_1, V_2, \ldots, V_\gamma)$ is a partition of the vertex set $V(G)$, we note that the number of vertices dominated by the set S is at least the number of vertices in the sets $V_1 \cup \cdots \cup V_{k_1}$, that is,

$$\mathrm{dom}_G(S) \geq \sum_{i=1}^{k_1} |V_i|. \tag{3}$$

We proceed further with the following claim.

Claim 6. *If* $|V_{k_1}| \geq 3$, *then* $\mathrm{pd}_\alpha(G) \leq \frac{1}{3}n$.

Proof. Suppose that $|V_{k_1}| \geq 3$. In this case, by Inequalities (1) and (3), and by our assumption that $n \geq 15$, we infer that

$$\mathrm{dom}_G(S) \geq 3k_1 = 3\left\lfloor \frac{1}{3}n \right\rfloor \geq \left\lceil \frac{7}{8}n \right\rceil. \tag{4}$$

By Inequality (4), we have $\mathrm{dom}_G(S) \geq \frac{7}{8}n$, implying that the set S is an α-partial dominating set of G, and so $\mathrm{pd}_\alpha(G) \leq |S| \leq \frac{1}{3}n$, yielding the desired result. (□)

By Claim 6, we may assume that $|V_{k_1}| = 2$, for otherwise the desired result follows. With this assumption and by inequality (1), we note that $|V_i| = 2$ for all $i \geq k_1$. Hence by Inequality (2), we have

$$\sum_{i=k_1+1}^{k_2} |V_i| = 2k_2 \leq 2\left(\left\lfloor \frac{3}{8}n \right\rfloor - \left\lfloor \frac{1}{3}n \right\rfloor \right). \tag{5}$$

Thus, by inequalities (3) and (5), and by our assumption that $n \geq 15$, we infer that

$$\mathrm{dom}_G(S) \geq \sum_{i=1}^{k_1} |V_i| = n - \sum_{i=k_1+1}^{k_2} |V_i| \geq n - 2\left(\left\lfloor \frac{3}{8}n \right\rfloor - \left\lfloor \frac{1}{3}n \right\rfloor \right) \geq \left\lceil \frac{7}{8}n \right\rceil. \tag{6}$$

By Inequality (6), we have $\mathrm{dom}_G(S) \ge \frac{7}{8}n$, implying that the set S is an α-partial dominating set of G, and so $\mathrm{pd}_\alpha(G) \le |S| \le \frac{1}{3}n$, yielding the desired result. □

The bound in Theorem 4.2 is best possible in the sense that if $\alpha > \frac{7}{8}$ and G is A_1 or A_2 (see Fig. 1), then in this case $\lceil \alpha \times n \rceil = 8 = n$, and at least three vertices are needed to dominate all vertices of G. Thus in this example, $\mathrm{pd}_\alpha(G) = 3 = \frac{3}{8}n > \frac{1}{3}n$. The same is true if every component of G is isomorphic to A_1 or A_2. Hence the value for α in the statement of Theorem 4.2 cannot be increased in general in order to guarantee that the α-partial domination number of a connected cubic graph is at most one-third its order.

However if the connected cubic graph G has sufficiently large order n, then we can improve the value $\alpha = \frac{7}{8}$ given in Theorem 4.2 to a larger value of α. For example, if $n \ge 28$, then $\alpha = \frac{13}{14}$ suffices, as the following result shows.

Theorem 4.3. *If G is a connected cubic graph of order $n \ge 28$, then*

$$\mathrm{pd}_{\frac{13}{14}}(G) \le \frac{1}{3}n\,.$$

Proof. Let G be a connected cubic graph of order $n \ge 28$ and let $\alpha = \frac{13}{14}$. We adopt exactly our notation from the proof of Theorem 4.2. In particular, D is a γ-set of G satisfying Lemma 2.1. As before, by Theorem 1.2 we have $\gamma \le \lfloor \frac{5}{14}n \rfloor$. If $\gamma \le \frac{1}{3}n$, then $\mathrm{dom}_G(D) = n$. Hence we may assume that $\gamma > \frac{1}{3}n$, for otherwise the desired result is immediate. Let k_1 and k_2 be defined exactly as in the proof Theorem 4.2. If $|V_{k_1}| \ge 3$, then

$$\mathrm{dom}_G(S) \ge 3k_1 = 3\left\lfloor \frac{1}{3}n \right\rfloor \ge \left\lceil \frac{13}{14}n \right\rceil, \tag{7}$$

implying that the set S is an α-partial dominating set of G. Thus, $\mathrm{pd}_\alpha(G) \le |S| \le \frac{1}{3}n$, yielding the desired result. Hence we may assume that $|V_{k_1}| = 2$. With this assumption, we note that $|V_i| = 2$ for all $i \ge k_1$. Thus proceeding exactly as before, we yield the inequality chain where recall that by supposition we have $n \ge 28$ and so

$$\mathrm{dom}_G(S) \ge \sum_{i=1}^{k_1} |V_i| = n - \sum_{i=k_1+1}^{k_2} |V_i| \ge n - 2\left(\left\lfloor \frac{5}{14}n \right\rfloor - \left\lfloor \frac{1}{3}n \right\rfloor\right) \ge \left\lceil \frac{13}{14}n \right\rceil. \tag{8}$$

Once again, $\mathrm{dom}_G(S) \ge \frac{13}{14}n$, implying that the set S is an α-partial dominating set of G. Thus, $\mathrm{pd}_\alpha(G) \le |S| \le \frac{1}{3}n$. □

Note that in the proof of Theorem 4.3 we used the inequality $\gamma \le \lfloor \frac{5}{14}n \rfloor$ which holds for every connected cubic graph of order at least 10. Hence we cannot avoid the assumption that G is connected. On the other hand, $\gamma \le \lfloor \frac{3}{8}n \rfloor$ holds for every supercubic graph, and we have the following result.

Theorem 4.4. *If G is a supercubic graph of order $n \ge 60$, then*

$$\mathrm{pd}_{\frac{9}{10}}(G) \le \frac{1}{3}n\,.$$

Proof. We can proceed along the same lines as in the proof of Theorem 4.3. The only difference is that now we cannot apply Theorem 1.2, instead we apply Theorem 1.1. Then (7) rewrites as

$$\mathrm{dom}_G(S) \ge 3k_1 = 3\left\lfloor \frac{1}{3}n \right\rfloor \ge \left\lceil \frac{9}{10}n \right\rceil, \tag{9}$$

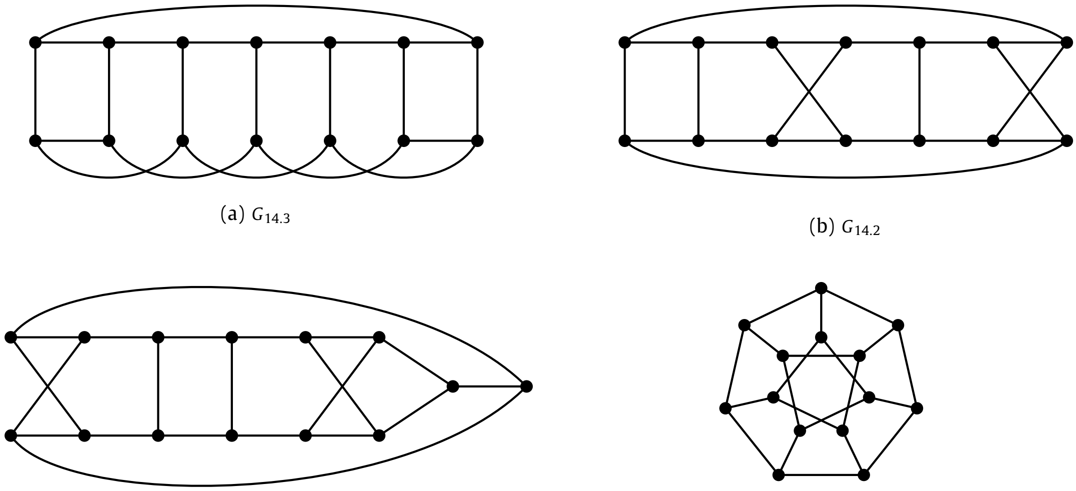
which holds for $n \ge 18$, while (8) rewrites as

$$\mathrm{dom}_G(S) \ge \sum_{i=1}^{k_1} |V_i| = n - \sum_{i=k_1+1}^{k_2} |V_i| \ge n - 2\left(\left\lfloor \frac{3}{8}n \right\rfloor - \left\lfloor \frac{1}{3}n \right\rfloor\right) \ge \left\lceil \frac{9}{10}n \right\rceil, \tag{10}$$

which holds for $n \ge 60$. The conclusion follows. □

5. Closing remarks

As a consequence of Theorems 1.1 and 1.2, we have the following result which characterizes the connected cubic graphs G of order n satisfying $\gamma(G) = \frac{3}{8}n$.



(a) $G_{14.3}$ (b) $G_{14.2}$ (c) $G_{14.1}$ (d) $P(7,2)$

Fig. 9. The four connected cubic graphs G of order $n = 14$ satisfying $\gamma(G) = 5$

Corollary 5.1. ([17,25]) *If G is a connected cubic graph of order n, then $\gamma(G) \le \frac{3}{8}n$, with equality if and only if G is one of the two graphs A_1 and A_2 shown in Fig. 1.*

A natural problem is to characterize the graphs that achieve equality in the Kostochka-Stocker Theorem 1.2, that is, to characterize the connected cubic graphs G of order n satisfying $\gamma(G) = \frac{5}{14}n$. Necessarily, for such graphs we have $n = 14k$ for some $k \ge 1$.

We show next that there are exactly four such graphs of order $n = 14$. We remark that the proof of Theorem 3.1 gave rise to two connected cubic graphs of order n satisfying $\gamma(G) = 5 = \frac{5}{14}n$, namely the graphs $G_{14.1}$ and $G_{14.2}$ shown in Figs. 4(d) and 7(d), respectively. (These two graphs are redrawn in Fig. 9(c) and 9(b), respectively.) With a bit more work, one can readily establish two additional such graphs.

In the second paragraph of the proof of Theorem 3.1, we consider the case when $\rho(G) = 3$. In this case, adding a vertex at distance 3 to a maximum packing immediately yielded an $\frac{7}{8}$-partial dominating set of G of cardinality 4, and therefore we assumed that $\rho(G) = 2$. However a more detailed analysis of the case when $\rho(G) = 3$ yields the generalized Petersen graph $P(7,2)$ shown in Fig. 9(d).

In the fourth paragraph of the proof of Theorem 3.1, we considered the case when the set $X = V(G) \setminus N_G[P]$ contains a vertex adjacent to two other vertices in X. Since this case immediately yielded an $\frac{7}{8}$-partial dominating set of G of cardinality 4, we therefore assumed that this case does not occur. However a more detailed analysis of the case when a vertex in X has two neighbors in X yields the graph $G_{14.3}$ shown in Fig. 9(a). The proof details giving rise to these two additional graphs, namely $P(7,2)$ and $G_{14.3}$, are similar to our proof of Theorem 3.1, and are not given here. Moreover, the result was also verified by computer.

Theorem 5.2. *If G is a connected cubic graph of order $n = 14$ satisfying $\gamma(G) = 5 = \frac{5}{14}n$, then $G \in \{G_{14.1}, G_{14.2}, G_{14.3}, P(7,2)\}$.*

It is not known if the $\frac{5}{14}$-upper bound on the domination number of a connected cubic graph of order n given by Kostochka and Stocker [17] is achievable when n is large. We pose the following conjecture.

Conjecture 5.3. *If G is a connected cubic graph of order n satisfying $\gamma(G) = \frac{5}{14}n$, then $G \in \{G_{14.1}, G_{14.2}, G_{14.3}, P(7,2)\}$.*

The authors in [17] remark that the bound $\gamma(G) \le \lfloor \frac{5}{14}n \rfloor$ for a connected cubic graph of order $n \ge 14$ is achievable for $n \in \{14, 16, 18\}$. It would be interesting to find graphs of orders $n \ge 20$ that achieve equality in this bound. Natural candidates are the generalized Petersen graphs $P(p,2)$ of order $n = 2p$ whose domination numbers are known (see, [10]).

Theorem 5.4. ([10]) $\gamma(P(p,2)) = p - \lfloor \frac{p}{5} \rfloor - \left\lfloor \frac{p+2}{5} \right\rfloor$ *for all $p \ge 3$.*

For $p \in \{3, 5, 6, 7, 8, 9, 11, 12\}$, we have $p - \lfloor \frac{p}{5} \rfloor - \lfloor \frac{p+2}{5} \rfloor = \lfloor \frac{5}{7}p \rfloor$. Hence as a consequence of Theorem 5.4, we have the following result.

Corollary 5.5. *For $n \in \{6, 10, 12, 14, 16, 18, 22, 24\}$, there exist connected cubic graphs G of order n satisfying $\gamma(G) = \lfloor \frac{5}{14}n \rfloor$.*

As far as we are aware, $P(12, 2)$ is the largest currently known connected cubic graph of order n satisfying $\gamma(G) = \lfloor \frac{5}{14}n \rfloor$. In addition, $\gamma(P(12, 2)) = 8 = \frac{1}{3}n$. We close with the following question, for which we suspect the answer is no.

Question 5.6. Are there infinitely many connected cubic graphs G of order n satisfying $\gamma(G) = \lfloor \frac{5}{14}n \rfloor$?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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