Packings in bipartite prisms and hypercubes

Boštjan Brešar [a,b,*], Sandi Klavžar [a,b,c], Douglas F. Rall [d]

[a] *Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia*
[b] *Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*
[c] *Faculty of Mathematics and Physics, University of Ljubljana, Slovenia*
[d] *Department of Mathematics, Furman University, Greenville, SC, USA*

A B S T R A C T

The 2-packing number $\rho_2(G)$ of a graph G is the cardinality of a largest 2-packing of G and the open packing number $\rho^o(G)$ is the cardinality of a largest open packing of G, where an open packing (resp. 2-packing) is a set of vertices in G no two (closed) neighborhoods of which intersect. It is proved that if G is bipartite, then $\rho^o(G \square K_2) = 2\rho_2(G)$. For hypercubes, the lower bounds $\rho_2(Q_n) \geq 2^{n-\lfloor \log n \rfloor - 1}$ and $\rho^o(Q_n) \geq 2^{n-\lfloor \log(n-1) \rfloor - 1}$ are established. These findings are applied to injective colorings of hypercubes. In particular, it is demonstrated that Q_9 is the smallest hypercube which is not perfect injectively colorable. It is also proved that $\gamma_t(Q_{2^k} \times H) = 2^{2^k - k}\gamma_t(H)$, where H is an arbitrary graph with no isolated vertices.

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1. Introduction

For many reasons, hypercubes are ubiquitous in theoretical computer science and in combinatorics. Understanding their structure is therefore a fundamental problem. Although hypercubes have a seemingly simple structure, we quickly encounter very complex problems. For instance, one of them was the middle levels problem, which was successfully dismissed [15]. On the other hand, the problem of determining the domination number of hypercubes is beyond the reach of existing methods. To date, exact values of $\gamma(Q_n)$ are only known for $n \leq 9$, where the value $\gamma(Q_9) = 62$ was obtained in [16], and for the following two infinite families.

Theorem 1.1. ([7,19]) *If $k \geq 1$, then $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$ and $\gamma(Q_{2^k}) = 2^{2^k-k}$.*

The values $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$ can be obtained from the fact that hypercubes Q_{2^k-1} admit 1-perfect codes, in which case the domination number coincides with the cardinality of a 1-perfect code.

The most important variation of the domination number is the total domination number; see a recent monograph [8] surveying domination theory with the two invariants in the central role. Roughly speaking, domination operates with closed neighborhoods while total domination with open neighborhoods, which often causes a different behavior of the invariants. However, as proved in [1] by using hypergraph transversals, $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$ for all n, which makes the domination

* Corresponding author.
E-mail addresses: bostjan.bresar@um.si (B. Brešar), sandi.klavzar@fmf.uni-lj.si (S. Klavžar), Doug.Rall@furman.edu (D.F. Rall).

number and the total domination number in hypercubes tightly connected. More generally, the authors of [1] proved that $\gamma_t(G \square K_2) = 2\gamma(G)$ as soon as G is a bipartite graph.

The concepts of packing number and open packing number of a graph are often used in domination theory, since they present natural lower bounds on the domination number and the total domination number, respectively, of the graph. The concept of packing was used back in 1975 by Meir and Moon in their classical theorem stating that in a tree the domination number equals the packing number [11]. On the other hand, open packing was introduced by Henning and Slater [9], and was later used in [18] to prove a canonical formula for the total domination number of the direct product of two graphs, which holds if one of the factors has the total domination number equal to its open packing number. Similarly as total domination is related to domination, open packing can be regarded as a version of packing in which closed neighborhoods are replaced with open neighborhoods. See [12–14] for some recent studies of (open) packings as well as [5] for their application.

Open packings are also related to the so-called injective colorings of graphs, cf. [17]. More precisely, an injective coloring of a graph is exactly a partition of its vertex set into open packings. In a recent paper [3], graphs that admit injective colorings such that each of the color classes is a maximum open packing were considered. While proving this property for hypercubes of some small dimensions, it was also proved for those whose dimension is a power of 2. Yet, nothing else was known, including whether there exists a hypercube that does not satisfy this property. One of the reasons for the difficulty of this question is that the open packing number (i.e., the cardinality of a maximum open packing) has not been known.

We proceed as follows. In the remainder of this introduction, we provide the definitions and concepts we need for the following. In Section 2 we prove that the open packing number of a prism over a bipartite graph G is twice the 2-packing number of G. This result nicely complements [1, Theorem 1] which states that the total domination number of a prism over a bipartite graph G is twice the domination number of G. We also demonstrate that in general, the open packing number of a prism over a graph G can be arbitrary larger that the 2-packing number of G. In Section 3 we prove lower bounds on the 2-packing number and the open packing number of hypercubes. The bounds are sharp for small dimensions and for two infinite families, but are not sharp in general. In the subsequent section we apply these findings to injective colorings of hypercubes. In particular we demonstrate that Q_9 is the smallest hypercube which is not perfect injectively colorable. In the concluding remarks, we give an overview of the known values for the hypercube invariants considered here and also derive the total domination number of the direct product of Q_{2^k} and an arbitrary graph.

1.1. Preliminaries

Let $G = (V(G), E(G))$ be a graph and $x \in V(G)$. The *open neighborhood* $N(x)$ is the set of vertices adjacent to x and the *closed neighborhood* is $N[x] = N(x) \cup \{x\}$. A set $D \subseteq V(G)$ is a *dominating set* of G if each vertex of $V(G) \setminus D$ has a neighbor in D. The cardinality of a smallest dominating set of G is the *domination number* $\gamma(G)$ of G. Similarly, $D \subseteq V(G)$ is a *total dominating set* of G if each vertex of $V(G)$ has a neighbor in D. The cardinality of a smallest dominating set of G is the *total domination number* $\gamma_t(G)$ of G.

Let G be a connected graph and let x and y be vertices in G. The *distance*, $d_G(x, y)$, between x and y is the length of a shortest path between x and y in G. Let $X \subseteq V(G)$. Then X is a 2-*packing* of G if every two distinct vertices in X are at distance at least 3; alternatively, $N[x] \cap N[y] = \emptyset$ for every pair of distinct vertices $x, y \in X$. Similarly, if $N(x) \cap N(y) = \emptyset$ for every pair of distinct vertices $x, y \in X$, then X is an *open packing* of G. The cardinality of a largest 2-packing of G is the 2-*packing number* $\rho_2(G)$ of G and the cardinality of a largest open packing of G is the *open packing number* $\rho^o(G)$ of G. By a ρ_2-*set* of G we mean a 2-packing of G of cardinality $\rho_2(G)$. A ρ^o-*set* of G is defined analogously.

If X is a 2-packing such that $V(G) = \cup_{x \in X} N[x]$, then we say that X is a 1-*perfect code* of G. In domination theory, 1-perfect codes are known as *efficient dominating sets*, see [8, Chapter 9] and [10]. Since $\gamma(G) \geq \rho_2(G)$ for every graph G, if X is a 1-perfect code of G, then X is also a dominating set of G. This observation leads to the following well known fact.

Proposition 1.2. *If G admits a 1-perfect code, then $\gamma(G) = \rho_2(G)$. If in addition G is r-regular, then $\gamma(G) = \rho_2(G) = \frac{n(G)}{r+1}$.*

An *injective coloring* of a graph G is a partition of the vertex set of G into open packings. The *injective chromatic number*, $\chi_i(G)$, of G is the minimum cardinality of an injective coloring in G.

The *Cartesian product* $G \square H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$, and two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$ if either $g_1 = g_2$ and $h_1 h_2$ is an edge in H or $h_1 = h_2$ and $g_1 g_2$ is an edge in G. For a vertex g of G, the subgraph of $G \square H$ induced by the set $\{(g, h) : h \in V(H)\}$ is an *H-fiber* and is denoted by ${}^g H$. Similarly, for $h \in H$, the *G-fiber*, G^h, is the subgraph induced by $\{(g, h) : g \in V(G)\}$. Cartesian product is commutative and associative. The *hypercube* of dimension n, or the *n-cube*, is isomorphic to $K_2 \square \cdots \square K_2$, where there are n factors K_2, and is denoted by Q_n. The equality $Q_n = Q_{n-1} \square K_2$ will be used (at least implicitly) several times in the paper. Note that the vertices of the n-cube enjoy a natural representation as binary strings of length n, where two vertices are adjacent precisely when the corresponding strings differ in exactly one position. Finally, the *direct product* $G \times H$ of graphs G and H has the vertex set $V(G) \times V(H)$, and two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \times H$ if $g_1 g_2$ is an edge in G and $h_1 h_2$ is an edge in H.
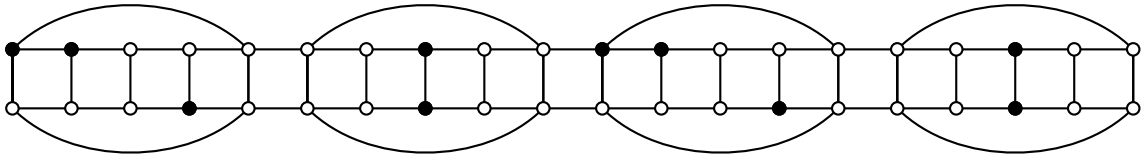
Fig. 1. An open packing in $G_2 \square K_2$.

2. Packing vs. open packing in bipartite prisms

In [1] it was proved that if G is a bipartite graph, then $\gamma_t(G \square K_2) = 2\gamma(G)$. In this section we prove an analogous result that connects the open packing number and the packing number.

We begin with the following simple lemma, which holds in all graphs (see also [13, Theorem 3.3] for a more general bound).

Lemma 2.1. *If G is a graph, then $\rho^o(G \square K_2) \geq 2\rho_2(G)$.*

Proof. Let G be a graph, and let P be a ρ_2-set of G. Then $P \times V(K_2)$ is an open packing of $G \square K_2$, hence the result. □

In general, $\rho^o(G \square K_2)$ can be arbitrary larger than $2\rho_2(G)$. For an example consider the family of graphs G_k, $k \geq 1$, defined as follows. G_k contains $2k$ disjoint cycles C_5 connected in a row by an edge between two consecutive 5-cycles. This informal definition of G_k should be clear from Fig. 1 where $G_2 \square K_2$ is drawn. As an arbitrary packing of G_k contains at most one vertex of each C_5 we infer that $\rho_2(G_k) = 2k$. On the other hand, repeating the pattern as shown in Fig. 1 for $k = 2$, we get $\rho^o(G_k \square K_2) \geq 5k$.

For bipartite graphs, however, the above phenomena cannot occur as the main result of this section asserts.

Theorem 2.2. *If G is a bipartite graph, then $\rho^o(G \square K_2) = 2\rho_2(G)$.*

Proof. Let G be a bipartite graph with parts A and B forming a natural partition of $V(G)$. By Lemma 2.1, we have $\rho^o(G \square K_2) \geq 2\rho_2(G)$. To prove the reversed inequality, consider an open packing O in $G \square K_2$ such that $|O| = \rho^o(G \square K_2)$. We will show that O can be transformed into an open packing O' of the form $P' \times V(K_2)$, where P' is a subset of $V(G)$. (Clearly, the latter also implies that P' is a 2-packing.) Note that O can be presented as the disjoint union $I \cup R$, where I is the set of vertices that are isolated in the subgraph of $G \square K_2$ induced by O, while R is the set of vertices that have exactly one neighbor in O. Clearly, at least one of the sets I or R is non-empty. Set $V(K_2) = \{1, 2\}$, and let $I_i = I \cap V(G^i)$ and $R_i = R \cap V(G^i)$ for all $i \in \{1, 2\}$. In addition, let $I_i^A = \{(u, i) \in I_i : u \in A\}$, $I_i^B = \{(u, i) \in I_i : u \in B\}$ for $i \in \{1, 2\}$, and similarly let $R_i^A = \{(u, i) \in R_i : u \in A\}$, $R_i^B = \{(u, i) \in R_i : u \in B\}$ for $i \in \{1, 2\}$. Next, we compare the two quantities $|I_1^A| + |I_2^B|$ and $|I_2^A| + |I_1^B|$. We may assume with no loss of generality that $|I_1^A| + |I_2^B| \geq |I_2^A| + |I_1^B|$. Now, the announced transformation of O to O' is defined as follows:

- if $(u, t) \in I_1^A \cup I_2^B$, then let $\{u\} \times V(K_2) \subseteq O'$;
- if $(u, t) \in I_2^A \cup I_1^B$, then let $(\{u\} \times V(K_2)) \cap O' = \emptyset$;
- if $(u, 1) \in R_1$ and $(u, 2) \in R_2$, then let $\{u\} \times V(K_2) \subseteq O'$;
- if $(u, 1) \in R_1^A$ and $(v, 1) \in R_1^B$, where $uv \in E(G)$, then let $\{u\} \times V(K_2) \subseteq O'$ and $(\{v\} \times V(K_2)) \cap O' = \emptyset$;
- if $(u, 2) \in R_2^A$ and $(v, 2) \in R_2^B$, where $uv \in E(G)$, then let $\{v\} \times V(K_2) \subseteq O'$ and $(\{u\} \times V(K_2)) \cap O' = \emptyset$.

We claim that $|O'| \geq |O|$. Indeed, the first two rows in the above transformation show that for every vertex $(u, t) \in I_1^A \cup I_2^B$ we get two vertices in O', while for every vertex $(u, t) \in I_2^A \cup I_1^B$ we get no vertices in O', yet $|I_1^A \cup I_2^B| > |I_2^A \cup I_1^B|$ by the earlier assumption. By the last three rows of the above transformation, every pair of vertices in R is replaced by two vertices in O'. This altogether implies that $|O'| \geq |O|$, so it remains to prove that O' is an open packing in $G \square K_2$.

If $(u, 1) \in I_1^A$ and $(v, 1) \in I_1^A$, then $d_G(u, v) \geq 4$, because the vertices belong to O, which is an open packing, and u and v are both in A. Thus vertices in $\{u\} \times V(K_2)$ will be at distance at least 4 from the vertices in $\{v\} \times V(K_2)$. By symmetry, we get the same conclusion for vertices $(u, 2) \in I_2^B$ and $(v, 2) \in I_2^B$. If $(u, 1) \in I_1^A$ and $(v, 2) \in I_2^B$, then $d_G(u, v) \geq 3$, because u and v belong to different parts, A and B, respectively, of the bipartition of $V(G)$ and they belong to O, which is an open packing. Thus, vertices in $\{u\} \times V(K_2)$ will be at distance at least 3 from the vertices in $\{v\} \times V(K_2)$, as desired. Clearly, if $(u, t) \in I_1^A \cup I_2^B$, then $d_G(u, v) \geq 3$ for any $v \in V(G)$ such that $\{(v, 1), (v, 2)\} \subset R$. This yields that vertices in $\{u\} \times V(K_2)$ will be at distance at least 3 from the vertices in $\{v\} \times V(K_2)$. If $(u, 1) \in I_1^A$ and $(v, 1) \in R_1^A$, we have $d_G(u, v) \geq 4$. On the other hand, if $(u, 1) \in I_1^A$ and $(v, 2) \in R_2^B$ we have $d_G(u, v) \geq 3$. In either case, the corresponding vertices in O' are at least three apart. By symmetry, we can find that for vertices in I_2^B and vertices in $R_1^A \cup R_2^B$ their distances are sufficiently large

so that the corresponding K_2-fibers that are in O' will be at distance at least 3. This completes the proof that the distance between the vertices in O' that appear in the first row of the above transformation to all other vertices in O' will be at least 3, except of course for two vertices in O' that belong to the same K_2-fiber and are adjacent.

Vertices of O' that appear in the third row of the transformation remain at distance at least 3 from all other vertices in O' (with the clear exception of two adjacent such vertices). Therefore, it remains to consider the vertices in O' that appear in the last two rows of the above transformation. Suppose there are two vertices in R_1^A (and a similar argument can be applied if they are in R_2^B), say, $(u, 1)$ and $(v, 1)$, which are not adjacent. Then $d_G(u, v) \geq 4$, and so $\{u\} \times V(K_2)$ will be at distance at least 4 from the vertices in $\{v\} \times V(K_2)$ (by symmetry, the same conclusion applies if $(u, 2)$ and $(v, 2)$ are in R_2^B). Finally, let $(u, 1) \in R_1^A$ and $(v, 2) \in R_2^B$. Since O is an open packing, we have $d_G(u, v) > 1$, and since they are in different parts of the bipartition, we get $d_G(u, v) \geq 3$. We derive that $\{u\} \times V(K_2)$ will be at distance at least 3 from the vertices in $\{v\} \times V(K_2)$, which concludes the proof that O' is an open packing. Since $|O| = \rho^o(G \square K_2)$ and $|O'| \geq |O|$, we derive $|O'| = |O| = \rho^o(G \square K_2)$. In addition, there exists a set $P' \subset V(G)$ such that $O' = P' \times \{1, 2\}$, where P' is a 2-packing of G. Hence, $|P'| \leq \rho_2(G)$, and so $|O'| = 2|P'| \leq 2\rho_2(G)$, implying $\rho^o(G \square K_2) \leq 2\rho_2(G)$. $\square$

3. (Open) packings in hypercubes

The following lemma follows by observing that the restriction of a 2-packing in $G \square K_2$ to a G-layer is a 2-packing of that layer (see also [13, Proposition 3.2] for a more general bound).

Lemma 3.1. *If G is a graph, then $\rho_2(G \square K_2) \leq 2\rho_2(G)$.*

We can now bound ρ_2 and ρ^o of hypercubes as follows.

Theorem 3.2. *If $n \geq 2$, then*

(i) $\rho_2(Q_n) \geq 2^{n-\lfloor \log n \rfloor - 1}$ *and*
(ii) $\rho^o(Q_n) \geq 2^{n-\lfloor \log(n-1) \rfloor - 1}$.

Proof. (i) Suppose first that $n = 2^k - 1$, where $k \geq 2$. As already mentioned, in these cases Q_n admits a 1-perfect code, say S. Then $|S| = 2^{2^k - 1}/2^k = 2^{2^k - k - 1}$ and consequently

$$\rho_2(Q_n) = |S| = 2^{2^k - k - 1} = 2^{2^k - 1 - (k-1) - 1} = 2^{n - \lfloor \log n \rfloor - 1}.$$

Consider now the hypercubes Q_n, where $k \geq 3$ and $2^{k-1} - 1 < n < 2^k - 1$. In particular, if $n = 2^k - 2$, then since $Q_{2^k - 1} = Q_{2^k - 2} \square K_2$, Lemma 3.1 implies that

$$\rho_2(Q_n) = \rho_2(Q_{2^k - 2}) \geq \frac{1}{2}\rho_2(Q_{2^k - 1}) = 2^{2^k - k - 2} = 2^{2^k - 2 - (k-1) - 1} = 2^{n - \lfloor \log n \rfloor - 1}.$$

Inductively applying the lemma, the result holds for all n such that $2^{k-1} - 1 < n < 2^k - 1$. Therefore, (i) holds for all $n \geq 2$.

(ii) Applying Theorem 2.2 and (i), we have

$$\rho^o(Q_n) = 2\rho_2(Q_{n-1}) \geq 2 \cdot 2^{(n-1) - \lfloor \log(n-1) \rfloor - 1} = 2^{n - \lfloor \log(n-1) \rfloor - 1}$$

for all $n \geq 2$ and we are done. $\square$

If $n \leq 7$, then equality holds in Theorem 3.2(i). The cases when $n \in \{2, 3, 4\}$ can be easily argued by case analysis. The equality in cases when $n \in \{5, 6\}$ then follows by combining Lemma 3.1 and Theorem 3.2(i). For $n = 7$, the equality holds because Q_7 has a 1-perfect code. One is thus tempted to conjecture that the lower bound in Theorem 3.2(i) holds as equality for all n. However, with the help of a computer, we found the set

$$T = \{00000000, 00001110, 00110010, 00111100, 01010110, 01011000,$$

$$01100100, 01101001, 01111111, 10010100, 10100101, 10101011,$$

$$11000111, 11001100, 11011011, 11100010, 11110001\}$$

which is a 2-packing in Q_8 with $|T| = 17$, hence $\rho_2(Q_8) \geq 17$. By Theorem 2.2, this in turn implies that $\rho^o(Q_9) \geq 34$. Hence also the lower bound in Theorem 3.2(ii) is not sharp in general. It is sharp however for all $n \leq 8$ because the lower bound in Theorem 3.2(i) is sharp for $n \leq 7$ and because of Theorem 2.2. Furthermore, by using Theorem 2.2 and the fact that the lower bound in Theorem 3.2(i) is sharp when $n = 2^k - 1$, it follows that the lower bound in Theorem 3.2(ii) is sharp for each value of n that is a power of 2.
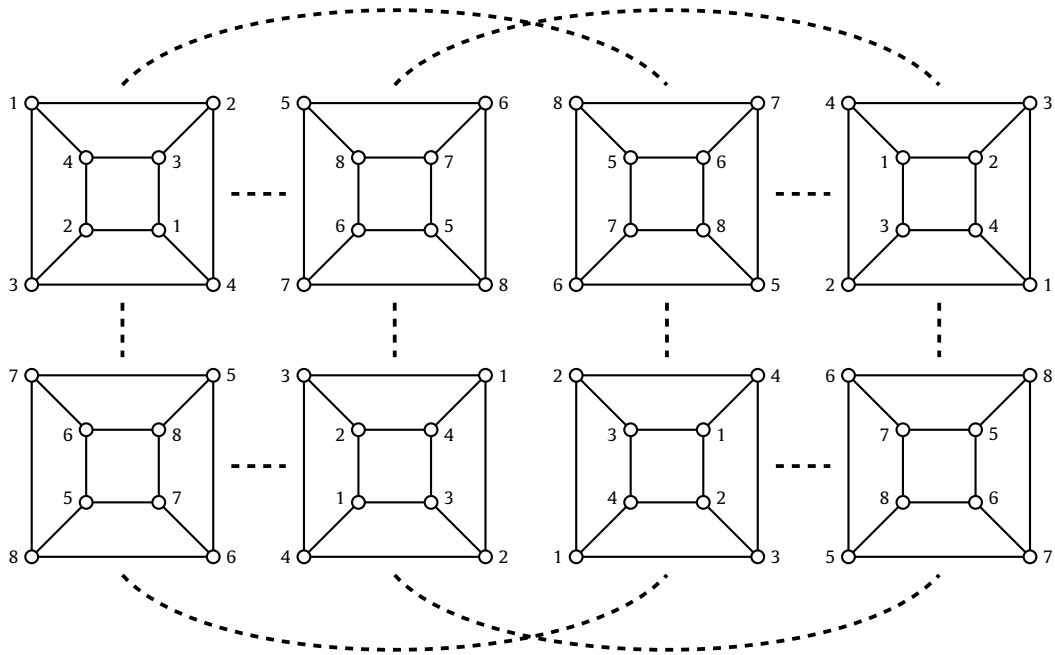
Fig. 2. Partition of $V(Q_6)$ into (maximum) 2-packings of Q_6.

4. Application to injective colorings

The concept of injective coloring was introduced by Hahn, Kratochvíl, Siráň and Sotteau [6] back in 2002, and has been considered by a number of authors, cf. [2,4]. In the recent paper [3], graphs that admit special types of injective colorings were considered: a graph G is a *perfect injectively colorable graph* if it has an injective coloring in which every color class forms a ρ^o-set of G. The authors of [3] considered hypercubes that are perfect injectively colorable. They noticed that such are the hypercubes Q_n, where $n \in [5]$, and proved that for all $k \in \mathbb{N}$, the hypercube Q_{2^k} is a perfect injectively colorable graph. Apart from the mentioned cases, it was asked in [3, Problem 1] in which other dimensions the hypercube is perfect injectively colorable. Since an answer to the question is closely related to computing the value of the open packing number of hypercubes, it was also asked in [3, Problem 2] what is the value of $\rho^o(Q_n)$ for $n \ge 6$.

In this note, we give some partial answers to the above two questions. One can easily find that $\rho_2(Q_5) = 4$, which by Theorem 2.2 implies that $\rho^o(Q_6) = 8$. In addition, Fig. 2 shows a maximum 2-packing of Q_6 of cardinality 8, where vertices of an arbitrary color in [8] form a maximum 2-packing. This gives, again by Theorem 2.2, that $\rho^o(Q_7) = 16$. In addition, recall that $\rho^o(Q_8) = 32$, which follows from the fact that Q_7 has a perfect code. Now, by the observation from Section 3, we have $\rho_2(Q_8) \ge 17$. On the other hand, we claim that $\rho_2(Q_8) \le 30$. Suppose to the contrary that $\rho_2(Q_8) > 30$, and let P be a ρ_2-set of Q_8. Then, partitioning $V(Q_8)$ into Q and Q', each of which induces Q_7, we infer that either $|Q \cap P|$ or $|Q' \cap P|$ is equal to 16. We may assume that $|Q \cap P| = 16$, and noting that $Q \cap P$ is a 2-packing of Q_7, this implies that $Q \cap P$ corresponds to a perfect code of Q_7, thus $Q \cap P$ is a dominating set of Q. This in turn implies that every vertex in Q' is at distance at most 2 from a vertex in $Q \cap P$, which yields that $P = Q \cap P$, and so $|P| = 16$, a contradiction proving that $\rho_2(Q_8) \le 30$. Now, using Theorem 2.2, we get $34 \le \rho^o(Q_9) \le 60$. In particular, $\rho^o(Q_9)$ is not a power of 2, which readily implies that Q_9 does not admit a partition into ρ^o-sets, and is consequently not a perfect injectively colorable graph. On the other hand, refer to Fig. 2 again, which shows a coloring of Q_6 in which each color class is a 2-packing of cardinality $\rho_2(Q_6)$. By applying Theorem 2.2 and the first part of its proof, one can construct an injective coloring of Q_7 in which each color class is an open packing of cardinality $\rho^o(Q_7)$. Therefore, Q_7 is perfect injectively colorable graph.

Summarizing the above, hypercubes Q_n, where $n \le 8$, are perfect injectively colorable graphs, and so Q_9 is the first instance of a hypercube, which is not in this class of graphs.

5. Concluding remarks

Table 1 presents values or bounds on the main domination and packing invariants in hypercubes Q_n, for all $n, n \le 9$. The values for γ and γ_t have been known earlier, while some of the values and bounds for ρ_2 and ρ^o have been obtained in this paper.

Table 1
Packing and domination invariants in hypercubes Q_n, where $n < 10$.

n	1	2	3	4	5	6	7	8	9
γ	1	2	2	4	7	12	16	32	62
γ_t	2	2	4	4	8	14	24	32	64
ρ_2	1	1	2	2	4	8	16	17-30	?
ρ^o	2	2	2	4	4	8	16	32	34-60

In addition, consider the value $\gamma_t(Q_{2^k}) = 2^{2^k-k}$, which follows from Theorem 1.1 combined with the formula $\gamma_t(G \,\square\, K_2) = 2\gamma(G)$ from [1]. Now, compare this with the bound $\rho^o(Q_{2^k}) \geq 2^{2^k-k}$, which follows from Theorem 3.2(ii) when plugging $n = 2^k$. Since $\gamma_t(G) \geq \rho^o(G)$ for every graph G with no isolated vertices, we infer that

$$\gamma_t(Q_{2^k}) = 2^{2^k-k} = \rho^o(Q_{2^k}), \text{ for all } k \in \mathbb{N}. \tag{1}$$

Recall the result from [18] stating that $\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$ whenever G is a graph with $\rho^o(G) = \gamma_t(G)$ and graphs G and H have no isolated vertices. Therefore, from the discussion above we get that

$$\gamma_t(Q_{2^k} \times H) = 2^{2^k-k}\gamma_t(H),$$

where $k \in \mathbb{N}$ and H is an arbitrary graph with no isolated vertices. An additional family of graphs with this property (that $\gamma_t = \rho^o$) can be found in [12]. It would be interesting to establish if there are any hypercubes Q_n of other dimensions than those in (1) that satisfy the equality $\gamma_t(Q_n) = \rho^o(Q_n)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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