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Revisiting d-distance (independent) domination in trees and in bipartite graphs

Csilla Bujtás [a,b], Vesna Iršič Chenoweth [a,b], Sandi Klavžar [a,b,c,*], Gang Zhang [b,d]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[c] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
[d] School of Mathematical Sciences, Xiamen University, China

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ABSTRACT

The d-distance p-packing domination number $\gamma_d^p(G)$ of G is the minimum size of a set of vertices of G which is both a d-distance dominating set and a p-packing. In 1994, Beineke and Henning conjectured that if $d \ge 1$ and T is a tree of order $n \ge d+1$, then $\gamma_d^1(T) \le \frac{n}{d+1}$. They supported the conjecture by proving it for $d \in \{1,2,3\}$. In this paper, it is proved that $\gamma_d^1(G) \le \frac{n}{d+1}$ holds for any bipartite graph G of order $n \ge d+1$, and any $d \ge 1$. Trees T for which $\gamma_d^1(T) = \frac{n}{d+1}$ holds are characterized. It is also proved that if T has ℓ leaves, then $\gamma_d^1(T) \le \frac{n-\ell}{d}$ (provided that $n-\ell \ge d$), and $\gamma_d^1(T) \le \frac{n+\ell}{d+2}$ (provided that $n \ge d$). The latter result extends Favaron's theorem from 1992 asserting that $\gamma_1^1(T) \le \frac{n+\ell}{3}$. In both cases, trees that attain the equality are characterized and relevant conclusions for the d-distance domination number of trees derived.

1. Introduction

Let $G = (V(G), E(G))$ be a graph, $S \subseteq V(G)$, let d and p be nonnegative integers, and let $d(\cdot,\cdot)$ denote the standard shortest-path distance. Then S is a *d-distance dominating set* of G if for every vertex $u \in V(G) \setminus S$ there exists a vertex $w \in S$ such that $d(u,w) \le d$, and S is a *p-packing* of G if $d(w,w') \ge p+1$ for every two different vertices $w, w' \in S$. The *d-distance p-packing domination number* $\gamma_d^p(G)$ of G is the minimum size of a set S which is at the same time d-distance dominating set and p-packing. (If for some parameters d and p such a set does not exist, set $\gamma_d^p(G) = \infty$.)

The d-distance p-packing domination number was introduced by Beineke and Henning [1] under the name (p,d)-*domination number* and with the notation $i_{p,d}(G)$. With the intention of placing it within the trends of contemporary graph domination theory, the notation $\gamma_d^p(G)$ was recently proposed in [2] and we follow it here. In [2] it is proved that for every two fixed integers d and p with $2 \le d$ and $0 \le p \le 2d-1$, the decision problem whether $\gamma_d^p(G) \le k$ holds is NP-complete for bipartite planar graphs. Several bounds on $\gamma_d^p(T)$, where T is a tree on n vertices with ℓ leaves and s support vertices are also proved, including $\gamma_2^0(T) \ge \frac{n-\ell-s+4}{5}$ and $\gamma_d^2(T) \le \frac{n-2\sqrt{n}+d+1}{d}$, $d \ge 2$. These results improve or extend earlier results from the literature.

* Corresponding author.
E-mail addresses: csilla.bujtas@fmf.uni-lj.si (C. Bujtás), vesna.irsic@fmf.uni-lj.si (V. Iršič Chenoweth), sandi.klavzar@fmf.uni-lj.si (S. Klavžar), gzhang@stu.xmu.edu.cn (G. Zhang).

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In this paper, our focus is on the invariants γ_d^0 and γ_d^1. For the first one we will simplify the notation to γ_d because it has been investigated under the name of *d-distance domination number* of G with the notation $\gamma_d(G)$, see the survey [8]. We also refer to [3] for algorithmic aspects. For the total version of this concept see [4]. The second invariant γ_d^1 deals with d-distance dominating sets which are 1-packings. Note that a set of vertices is a 1-packing if and only if it is an independent set, hence in this case we will say that $\gamma_d^1(G)$ is the *d-distance independent domination number* of G, cf. [5,7,8].

Meierling and Volkmann [10], and independently Raczek, Lemańska, and Cyman [12], proved that if $d \geq 1$, and T is a tree on n vertices and with ℓ leaves, then $\gamma_d(T) \geq \frac{n-d\ell+2d}{2d+1}$. On the other hand, Meir and Moon [11] proved that if $d \geq 1$ and T is a tree of order $n \geq d+1$, then $\gamma_d(T) \leq \frac{n}{d+1}$. About twenty years later, in 1991, Topp and Volkmann [13] gave a complete characterization of the graphs G with $\gamma_d(G) = \frac{n}{d+1}$. In 1994, Beineke and Henning [1] proved that if $d \in \{1,2,3\}$ and T is a tree of order $n \geq d+1$, then $\gamma_d^1(T) \leq \frac{n}{d+1}$. Moreover, they closed their paper with the following:

Conjecture 1.1. [1] *If $d \geq 1$ and T is a tree of order $n \geq d+1$, then $\gamma_d^1(T) \leq \frac{n}{d+1}$.*

We point out here that in the book's chapter [8], Conjecture 1.1 is stated as [8, Theorem 71] with the explanation that the above-mentioned bound on $\gamma_d(T)$ due to Meir and Moon [11] is proved in such a way, that the d-distance dominating set is also independent. Anyhow, in the next section we prove that the bound holds for all bipartite graphs. In Section 3 we then characterize trees T of order n for which $\gamma_d^1(T) = \frac{n}{d+1}$ holds. In Section 4, we prove that if T has ℓ leaves, then $\gamma_d^1(T) \leq \frac{n-\ell}{d}$ (provided that $n-\ell \geq d$), and $\gamma_d^1(T) \leq \frac{n+\ell}{d+2}$ (provided that $n \geq d$). In both cases, the trees that attain the equality are characterized. Using the fact that $\gamma_d(T) \leq \gamma_d^1(T)$, we also derive analogous bounds for $\gamma_d(T)$ and characterize trees attaining those bounds. In particular, if T is a tree with ℓ leaves and of order $n \geq d+\ell$, then

$$\gamma_d(T) \leq \gamma_d^1(T) \leq \begin{cases} \frac{n-\ell}{d}, & \text{if } n < (d+1)\ell, \\ \frac{n}{d+1}, & \text{if } n = (d+1)\ell, \\ \frac{n+\ell}{d+2}, & \text{if } n > (d+1)\ell, \end{cases}$$

and the upper bounds are best possible. We conclude the paper with a conjecture.

In the rest of the introduction additional definitions necessary for understanding the rest of the paper are given. For a positive integer n we will use the convention $[n] = \{1,\dots,n\}$. Let G be a graph. The degree of $u \in V(G)$ is denoted by $\deg_G(u)$ or $\deg(u)$ for short. Further, $\mathrm{diam}(G)$ is the diameter of G and $L(G)$ is the set of its leaves, that is, vertices of degree 1. We call a d-distance p-packing dominating set of G of size $\gamma_d^p(G)$ a *$\gamma_d^p(G)$-set*. When G is clear from the context, we may shorten it to *γ_d^p-set*. A *double star $D_{r,s}$* is a tree with exactly two vertices that are not leaves, with one adjacent to $r \geq 1$ leaves and the other to $s \geq 1$ leaves. When we say that a *path P is attached to a vertex v* of a graph G, we mean that P is disjoint from G and that we add an edge between v and an end vertex of P.

2. Bounding γ_d^1 for bipartite graphs

For the main result of this section, we first prove the following.

Theorem 2.1. *If $d \geq 1$ is an integer and G is a connected bipartite graph of order at least $d+1$, then $V(G)$ can be partitioned into $d+1$ d-distance independent dominating sets.*

Proof. Set $Z = \mathrm{diam}(G)$.

If $Z \leq d$, then each vertex is a d-distance dominating set of G. Since G is bipartite, a required partition of $V(G)$ can be constructed by considering a bipartition (X,Y) of G and partitioning X and Y into $d+1$ parts appropriately. Hence assume in the rest that $Z \geq d+1$.

Let P be a diametrical path of G, let x and y be its end-vertices, and root G at x. Let Γ_i, $0 \leq i \leq Z$, be the distance levels with respect to x, that is, $\Gamma_i = \{u \in V(G):\ d(x,u) = i\}$. Consider now the sets

$$S_i = \bigcup_{k \geq 0} \Gamma_{k(d+1)+i}, \quad i \in \{0,1,\dots,d\}.$$

We claim that $\{S_0, S_1, \dots, S_d\}$ is a partition of $V(G)$ as stated in the theorem.

Since distance levels of a bipartite graph form independent sets and as $d \geq 1$, each set S_i is independent. Hence it remains to prove that these sets are d-distance dominating sets.

Let u be an arbitrary vertex of G and assume that $u \in \Gamma_s$, where $0 \leq s \leq Z$. If $s \geq d$, then there exists a path of length d between u and a vertex from Γ_{s-d}. This already implies that u is d-distance dominated by each of the sets S_i, $i \in \{0,1,\dots,d\}$. Hence assume in the rest that $s < d$. Then by a parallel argument, u is d-distance dominated by each of the sets S_i, $i \in \{0,1,\dots,s\}$. It remains to verify that u is d-distance dominated by each of the sets S_i, $i \in \{s+1,\dots,d\}$. For this sake consider an arbitrary, fixed $t \in \{s+1,\dots,d\}$. Let Q be a shortest u,y-path and recall that by our assumption, $d(u,y) \leq Z$.

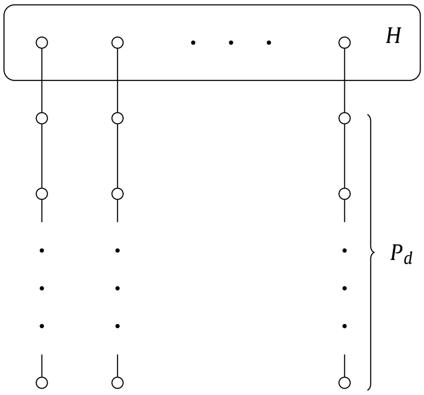

Fig. 1. The P_d-corona $H \circ P_d$ of a graph H.

Since every edge of G connects two vertices from consecutive distance levels Γ_i, the path Q necessarily contains a vertex $w \in \Gamma_t$. We claim that $d(u,w) \le d$. Suppose on the contrary that $d(u,w) > d$. Since Q is a shortest path, $d(w,y) \ge Z - t$. Using these facts together with $t \le d$, we get

$$Z \le d + (Z-t) < d(u,w) + d(w,y) = d(u,y) \le Z\,,$$

which is not possible. We can conclude that $d(u,w) \le d$. This means that u is d-distance dominated by S_t and we are done. □

In connection with Theorem 2.1 we add that Zelinka [14] proved that if $d \ge 1$ and G is a connected graph of order at least $d+1$, then $V(G)$ can be partitioned into $d+1$ disjoint d-distance dominating sets. In this general case, however, the partition need not be into independent sets.

The following is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let $d \ge 1$ be an integer. If G is a bipartite graph of order $n \ge d+1$, then $\gamma_d^1(G) \le \frac{n}{d+1}$.*

Corollary 2.2 generalizes [8, Theorem 71]. On the other hand, the upper bound in Corollary 2.2 may not hold if G is not bipartite. For example, for $n \ge d+2$ and $k \ge 2$, let $G_{n,k,d}$ be the complete graph K_n with k copies of P_d attached to each vertex. Clearly, $|V(G_{n,k,d})| = n(dk+1)$. While $\gamma_d(G_{n,k,d}) = n$, a d-distance independent domination needs much more vertices, and it is not hard to deduce that $\gamma_d^1(G_{n,k,d}) = 1 + (n-1)k$. As $n \ge d+2$ and $k \ge 2$, we infer that $\gamma_d^1(G_{n,k,d}) > \frac{|V(G_{n,k,d})|}{d+1}$.

3. Trees that attain equality in Corollary 2.2

Let $d \ge 1$ be an integer. The *P_d-corona $H \circ P_d$* of a graph H is the graph obtained from H and $|V(H)|$ disjoint copies of P_d, by attaching a copy of P_d to each vertex of H, see Fig. 1.

If $d \ge 2$, then let $\mathcal{B}_d$ be the family of P_d-coronas of bipartite graphs, that is,

$$\mathcal{B}_d = \{H \circ P_d : H \text{ is a bipartite graph}\}.$$

Note that $P_{d+1} \in \mathcal{B}_d$. Observe also that each $G \in \mathcal{B}_d$, where $G = H \circ P_d$, is a bipartite graph with $|V(G)| = (d+1)|V(H)|$. The following proposition shows that the upper bound in Corollary 2.2 is best possible.

Proposition 3.1. *If $G \in \mathcal{B}_d$ is of order n, then $\gamma_d^1(G) = \frac{n}{d+1}$.*

Proof. Let $G = H \circ P_d$ for some bipartite graph H. By the definition of $H \circ P_d$, the set $L(G)$ is a d-distance independent dominating set of G. Thus, $\gamma_d^1(G) \le |L(G)| = |V(H)| = \frac{n}{d+1}$.

Conversely, for each $u \in V(H)$, let G_u be the subgraph of G induced by u and the vertices of the copy of P_d attached to u. Clearly, $G_u \cong P_{d+1}$. If D is a $\gamma_d^1(G)$-set, then $|D \cap V(G_u)| \ge 1$. Thus, $\gamma_d^1(G) = |D| \ge |V(H)| = \frac{n}{d+1}$. □

Note that if G is a connected bipartite graph of order $n = d+1$, then $\gamma_d^1(G) = 1 = \frac{n}{d+1}$, and $\gamma_d^1(C_{2d+2}) = 2 = \frac{2d+2}{d+1}$. Moreover, if $d = 1$, then $\gamma_1^1(K_{r,r}) = r = \frac{r+r}{2} = \frac{n}{2}$. In 2004, Ma and Chen gave an equivalent description of the bipartite graphs G of order n with $\gamma_1^1(G) = \frac{n}{2}$, see [9, Theorem 1]. They also proved an explicit characterization of such a family for the case of trees. To state the result, let ζ_1 be a family of trees defined by the following recursive construction.

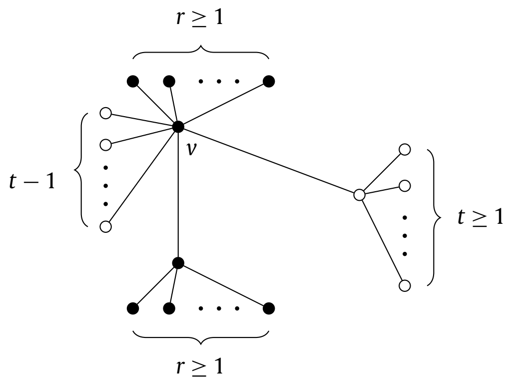

Fig. 2. A tree T from the family ζ_1, where T' is the double star induced by the black vertices.

(i) $K_2 \in \zeta_1$.
(ii) If $T' \in \zeta_1$, and T is obtained by joining the center of a new copy of $K_{1,t}$ $(t \geq 1)$ to a support vertex v of T' and adding $t-1$ leaves at v, then $T \in \zeta_1$.

See Fig. 2 for an illustration.[1]

The result of Ma and Chen for trees now reads as follows.

Theorem 3.2. ([9, Corollary 1]) *If T is a tree of order n, then $\gamma_1^1(T) = \frac{n}{2}$ if and only if $T \in \zeta_1$.*

We shall focus on the general case for $d \geq 2$, and give a complete characterization of the trees achieving equality in the upper bound of Corollary 2.2. Set

$$\mathcal{T}_d = \{T^* \circ P_d : T^* \text{ is a non-trivial tree}\}.$$

Note that $\mathcal{T}_d$ does not contain the path P_{d+1}. Since $\mathcal{T}_d \subset \mathcal{B}_d$, and by Proposition 3.1, $\gamma_d^1(T) = \frac{n}{d+1}$ for each tree $T \in \mathcal{T}_d$ of order n. Moreover, if T is a tree of order $n = d+1$, then we also have $\gamma_d^1(T) = 1 = \frac{n}{d+1}$.

In a tree T and for a vertex $v \in V(T)$, let $L(v)$ be the set of leaves of T that are neighbors of v in T. Root T at some vertex. Let T_v be the subtree induced in T by v and its descendants, and let $T - T_v = T - V(T_v)$. A vertex of T is called a *P_d-support vertex* if it is attached to a copy of P_d. For each $H \in \mathcal{T}_d$, every vertex of H^* is a P_d-support vertex of H, where $H = H^* \circ P_d$ for some non-trivial tree H^*. In particular, a P_1-support vertex of T is just a support vertex of T. A vertex of T is a *(P_i, P_j)-support vertex* if both a copy of P_i and a copy of P_j are attached to it. In particular, a (P_i, P_i)-support vertex has at least two copies of P_i attached. The *d-subdivision* of T is the tree obtained from T by subdividing each edge d-times. Then the 1-subdivision of T is just the subdivision of T.

Before proving the announced characterization of trees of order n with $\gamma_d^1(T) = \frac{n}{d+1}$, we state the following lemma which will also be used in the subsequent section.

Lemma 3.3. *Let $d \geq 2$ and let T be a tree with $s = \mathrm{diam}(T) \geq 2d+1$. Suppose that $P := v_1 v_2 \dots v_{s+1}$ is a diametrical path in T and the tree is rooted at v_{s+1}. If there is no P_{d+1}-support vertex and no (P_i, P_j)-support vertex in T with $i \in [d-1]$ and $j \in [d]$, then the following statements hold.*

(i) If $k \in \{2, \dots, d\} \cup \{s-d+2, \dots, s\}$, then $\deg(v_k) = 2$.
(ii) If $k \in \{d+1, s-d+1\}$, then $\deg(v_k) \geq 3$.
(iii) For every $v \in V(T)$, if v is the only vertex with $\deg_T(v) \geq 3$ in the subtree T_v, then T_v is isomorphic to the $(d-1)$-subdivision of a star $K_{1,t}$ with $t \geq 2$.
(iv) The subtree $T_{v_{d+1}}$ is isomorphic to the $(d-1)$-subdivision of a star $K_{1,t}$ with $t \geq 2$.
(v) If $s = 2d+1$, then T is obtained by taking the $(d-1)$-subdivisions of two stars K_{1,t_1} with $t_1 \geq 2$ and K_{1,t_2} with $t_2 \geq 2$, and adding an edge between the centers.

Proof. (i)–(ii) According to the conditions, there is no (P_1, P_1)-support vertex in T. That is, every vertex of T is adjacent to at most one leaf, and in particular, $\deg(v_2) = 2$. Further, if $d \geq 3$, then $\deg(v_3) = 2$, since otherwise v_3 would be a

[1] For the definition of ζ_1, we note that both vertices of a K_2 are support vertices. Then, (ii) can be applied to K_2, and this step results in the double star $D_{r,r}$ for every $r \geq 1$. It shows that the family ζ_1 is the same as $\{K_2\} \cup \zeta$ in [9].

(P_i, P_2)-support vertex with $1 \le i \le 2 \le d-1$ contradicting the condition. Similarly, $\deg(v_k) = 2$ holds for all $k \in \{2, \dots, d\}$. By symmetry, the same is true for v_k if $k \in \{s-d+2, \dots, s\}$. This proves (i). The assumption that there is no pendant P_{d+1} in T directly implies (ii).

(iii) If $\deg_T(v) \ge 3$ and $\deg_{T_v}(u) \le 2$ for every further vertex u from $V(T_v)$, then at least two pendant paths are attached to v. Then, by the conditions in the lemma, every path attached is isomorphic to P_d.

(iv)–(v) As P is a diametrical path, a vertex $u \in V(T_{v_{d+1}})$ different from v_{d+1} cannot be a P_d-support vertex. Part (iii) then implies (iv). If we re-root T at the vertex v_1, the same property holds for the subtree induced by v_{s-d+1} and its descendants in the re-rooted tree. This directly implies (v) for the case of $s = 2d+1$. □

Theorem 3.4. *If $d \ge 2$ and T is a tree of order n, then $\gamma_d^1(T) = \frac{n}{d+1}$ holds if and only if $n = d+1$ or $T \in \mathcal{T}_d$.*

Proof. If T is of order $n = d+1$, then, clearly, $\gamma_d^1(T) = 1 = \frac{n}{d+1}$, and if $T \in \mathcal{T}_d$, then by Proposition 3.1, we have $\gamma_d^1(T) = \frac{n}{d+1}$. The proof of the necessity is by induction on n. If $\gamma_d^1(T) = \frac{n}{d+1}$, then $n = (d+1)q$ for some integer $q \ge 1$. If $q = 1$, then $n = d+1$. So, we may assume that $q \ge 2$ and $n \ge 2(d+1)$. If $\mathrm{diam}(T) \le 2d$, then $\gamma_d^1(T) = 1 < \frac{2(d+1)}{d+1} \le \frac{n}{d+1}$. In the continuation, we assume that $\mathrm{diam}(T) \ge 2d+1$ and $\gamma_d^1(T) = \frac{n}{d+1}$.

Claim A. *If $i \in [d-1]$ and $j \in [d]$, then there is no (P_i, P_j)-support vertex in T.*

Proof. Suppose, to the contrary, that v is a (P_i, P_j)-support vertex in T and $i \le j$. Let $P' := x_1x_2\dots x_i$ and $P'' := y_1y_2\dots y_j$ be two copies of P_i and P_j attached to v in T, where $x_iv, y_jv \in E(T)$. Note that $d(x_1, v) = i \le j = d(y_1, v)$. Consider $T' = T - V(P')$. Then $n' = |V(T')| = n - i \ge 2(d+1) - (d-1) = d+3$. Let D' be a $\gamma_d^1(T')$-set. If $v \in D'$, then D' is also a d-distance independent dominating set of T. If $v \notin D'$, then $|D'| = \gamma_d^1(T')$ implies $|D' \cap V(P'')| \le 1$. For the subcase $|D' \cap V(P'')| = 1$, we may assume that $y_j \in D'$. Then $d(x_k, y_j) \le d$ for each $k \in [i]$. For the subcase $|D' \cap V(P'')| = 0$, in order to d-distance dominate y_1 in T', there exists a vertex $u \in D'$ such that $d_{T'}(u, y_1) \le d$. Since $i \le j$, it holds that $d_T(x_k, u) \le d_T(u, y_1) = d_{T'}(u, y_1) \le d$ for each $k \in [i]$. Thus, D' is always a d-distance independent dominating set of T. Corollary 2.2 then implies

$$\gamma_d^1(T) \le |D'| = \gamma_d^1(T') \le \frac{n'}{d+1} < \frac{n}{d+1},$$

which contradicts the assumption $\gamma_d^1(T) = \frac{n}{d+1}$. This proves Claim A. (□)

Claim B. *If T has a P_{d+1}-support vertex v, then $T \in \mathcal{T}_d$.*

Proof. Let $P' := x_1x_2\dots x_{d+1}$ be a copy of P_{d+1} attached to v, where $x_{d+1}v \in E(T)$. Then $\deg(x_k) = 2$ for all $k \in [d+1] \setminus \{1\}$ and $\deg(x_1) = 1$. Consider $T' = T - V(P')$. Then $n' = |V(T')| = n - (d+1) \ge 2(d+1) - d - 1 = d+1$. Let D' be a $\gamma_d^1(T')$-set. Then $D' \cup \{x_1\}$ is a d-distance independent dominating set of T. By Corollary 2.2,

$$\gamma_d^1(T) \le |D'| + 1 = \gamma_d^1(T') + 1 \le \frac{n'}{d+1} + 1 = \frac{n-(d+1)}{d+1} + 1 = \frac{n}{d+1},$$

and the equality holds if and only if $\gamma_d^1(T) = \gamma_d^1(T') + 1$ and $\gamma_d^1(T') = \frac{n'}{d+1}$. The induction hypothesis therefore implies $n' = d+1$ or $T' \in \mathcal{T}_d$.

Suppose $n' = d+1$. Then $n = 2(d+1)$ and T is the tree obtained from a copy of P_{d+1} and a tree T' of order $d+1$ by joining x_{d+1} to a vertex v of T'. Note that $\mathrm{diam}(T') \le d$ with equality if and only if $T' \cong P_{d+1}$. Unless $T' \cong P_{d+1}$ and v is a leaf of T', $\{x_{d+1}\}$ is a d-distance independent dominating set of T, implying that $\gamma_d^1(T) = 1 < \frac{2(d+1)}{d+1} = \frac{n}{d+1}$, a contradiction. For the exception, we observe $T \cong P_{2(d+1)} \in \mathcal{T}_d$.

Suppose $T' \in \mathcal{T}_d$. Let $T' = T'_* \circ P_d$ for some non-trivial tree T'_*. If $v \in V(T'_*)$, then $T = T^* \circ P_d \in \mathcal{T}_d$, where T^* is the tree obtained from T'_* by adding a new vertex x_{d+1} and the edge $x_{d+1}v$ to it. If $v \notin V(T'_*)$, then let u_1 be the P_d-support vertex of T'_* such that the attached copy of P_d contains v. Since $|V(T'_*)| \ge 2$, there exists a neighbor $u_2 \in V(T'_*)$ of u_1. Let u'_1 and u'_2 be the leaves of T' corresponding to u_1 and u_2, respectively. Note that $v = u'_1$ is possible, and $D = (L(T') \setminus \{u'_1, u'_2\}) \cup \{x_{d+1}, u_2\}$ is a d-distance independent dominating set of T. Thus,

$$\gamma_d^1(T) \le |D| = |L(T')| = |V(T'_*)| = \frac{n'}{d+1} < \frac{n}{d+1}$$

that contradicts our assumption on T and finishes the proof of Claim B. (□)

Claim B shows that if $\gamma_d^1(T) = \frac{n}{d+1}$ and T contains a pendant path P_{d+1}, then $T \in \mathcal{T}_d$. The remaining part of the proof verifies that there is no tree T with $|V(T)| > d+1$ and $\gamma_d^1(T) = \frac{n}{d+1}$ that does not contain a pendant P_{d+1}. From now on, we suppose that there is no P_{d+1}-support vertex in T and that $\gamma_d^1(T) = \frac{n}{d+1}$.

Let $s = \mathrm{diam}(T) \geq 2d+1$ and $P := v_1 v_2 \dots v_{s+1}$ be a diametrical path in T. Then $\deg(v_1) = \deg(v_{s+1}) = 1$. Root T at v_{s+1}. Our assumption on the non-existence of P_{d+1}-support vertices and Claim A imply that the properties stated in Lemma 3.3 (i)–(v) are valid for T.

If $s = \mathrm{diam}(T) = 2d+1$ then, by Lemma 3.3 (v), the tree T can be obtained from the $(d-1)$-subdivisions of two stars K_{1,t_1} and K_{1,t_2} with $t_1 \geq t_2 \geq 2$ by joining the centers with an edge. Then $N(v_{d+2})$ is a d-distance independent dominating set of T. Since $d \geq 2$, it gives the following contradiction:

$$\gamma_d^1(T) \leq |N(v_{d+2})| = t_2 + 1 = \frac{(t_2+1)d + t_2 + 1}{d+1} < \frac{2dt_2 + 2}{d+1} \leq \frac{d(t_1+t_2)+2}{d+1} = \frac{n}{d+1}.$$

So, we may assume that $\mathrm{diam}(T) \geq 2d+2$ and $n \geq 2d+3$. Regarding v_{d+2}, we divide the rest of the proof into two cases and prove that in both we get a contradiction.

Case 1. Each vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ is of degree at least 3.
By Lemma 3.3 (iii) and since P is a diametrical path, for each $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$, the subtree T_v is isomorphic to the $(d-1)$-subdivision of a star K_{1,t_v} with $t_v \geq 2$. Clearly, $T_{v_{d+1}}$ is contained in $T_{v_{d+2}}$, and therefore, $|V(T_{v_{d+2}})| \geq 2d+2$. Let $T' = T - T_{v_{d+2}}$. Since $\{v_{d+3}, \dots, v_{2d+3}\} \subseteq V(T')$, we obtain

$$d+1 \leq n' = |V(T')| \leq n - 2d - 2.$$

Let D' be a $\gamma_d^1(T')$-set. Then $D = D' \cup (N(v_{d+2}) \setminus \{v_{d+3}\})$ is a d-distance independent dominating set of T. Let $p = \deg(v_{d+2})$. Observe that $p \geq 2$ and $n' \leq n - (2d+1)(p-1) - 1$. By Corollary 2.2, we get the following contradiction:

$$\begin{aligned}\gamma_d^1(T) \leq |D| &= \gamma_d^1(T') + p - 1 \\ &\leq \frac{n'}{d+1} + p - 1 \\ &\leq \frac{n - (2d+1)(p-1) - 1}{d+1} + p - 1 \\ &= \frac{n - d(p-1) - 1}{d+1} \\ &< \frac{n}{d+1}.\end{aligned}$$

Case 2. There is a vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ with $\deg(v) \leq 2$.
If $\deg(v) = 2$ and T_v contains a vertex u with $\deg(u) \geq 3$, then Lemma 3.3 (iii) implies the existence of a leaf $w \in V(T_u)$ with $d(w,u) = d$. It follows then that $d(w, v_{d+2}) \geq d+2$ and $d(w, v_{s+1}) \geq s+1 = \mathrm{diam}(T)+1$, a contradiction. Therefore, $\deg(v) \leq 2$ implies that T_v is a path and v_{d+2} is a P_i-support vertex for some $i \geq 1$. By our assumption, $i \leq d$. Further, by Claim A, we have the following properties.

- If v_{d+2} is a P_i-support vertex of T for some $i \in [d-1]$, then there is only one pendant path attached to v_{d+2}, and it is clearly of order i.
- If v_{d+2} is a P_d-support vertex of T, then v_{d+2} is not a P_i-support vertex of T for any $i \in [d-1]$, and there is at least one copy of P_d attached to v_{d+2}.

Case 2.1. $L(v_{d+2}) \neq \emptyset$.
In this case v_{d+2} is a P_1-support vertex of T. Let $x \in L(v_{d+2})$ and $T' = T - x$. Now for each vertex $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$, the subtree T'_v is isomorphic to the P_d-subdivision of a star K_{1,t_v} for $t_v \geq 2$. Clearly, $n' = |V(T')| = n - 1 \geq 2d+2$.
Let D' be a $\gamma_d^1(T')$-set. If $v_{d+2} \in D'$, then D' is also a d-distance independent dominating set of T. If $v_{d+2} \notin D'$, then since $|D' \cap V(T'_{v_{d+1}})| \geq 1$, we may assume that $v_{d+1} \in D'$. The set D' is also a d-distance independent dominating set of T. For any subcase, $\gamma_d^1(T) \leq |D'| = \gamma_d^1(T') \leq \frac{n'}{d+1} = \frac{n-1}{d+1} < \frac{n}{d+1}$ by Corollary 2.2.

Case 2.2. $L(v_{d+2}) = \emptyset$.
In this case, v_{d+2} is a P_i-support vertex of T for some $i \in [d] \setminus \{1\}$ (where if $i = d$, then there could be multiple copies of P_d attached to v_{d+2}). Let $P' := x_1 x_2 \dots x_i$ be the (selected) copy of P_i attached to v_{d+2}, where $x_i v_{d+2} \in E(T)$. Then $\deg(x_k) = 2$ for all $k \in [i] \setminus \{1\}$ and $\deg(x_1) = 1$. Consider $T' = T - T_{v_d} - T_{x_i}$. Then $n' = |V(T')| = n - d - i \leq n - d - 2$ and $n' \geq d+3$ since $v_{d+1}, v_{d+2}, \dots, v_{2d+3} \in V(T')$.
Let D' be a $\gamma_d^1(T')$-set. Then $|D' \cap \{v_{d+1}, v_{d+2}\}| \leq 1$. If $v_{d+1} \in D'$ and $v_{d+2} \notin D'$, then let $D = D' \cup \{x_i\}$. If $v_{d+1} \notin D'$ and $v_{d+2} \in D'$, then let $D = D' \cup \{v_1\}$. If $v_{d+1}, v_{d+2} \notin D'$, then since v_{d+1} is attached to at least two copies of P_d, we have $D' \cap (V(T_{v_{d+1}}) \setminus V(T_{v_d})) \neq \emptyset$. Let $D = D' \cup \{v_{d+1}, x_i\} \setminus (V(T_{v_{d+1}}) \setminus V(T_{v_d}))$. For any subcase, D is a d-distance independent dominating set of T, and $\gamma_d^1(T) \leq |D'| + 1 = \gamma_d^1(T') + 1 \leq \frac{n'}{d+1} + 1 \leq \frac{n-d-2}{d+1} + 1 < \frac{n}{d+1}$ by Corollary 2.2.

This completes the proof of Theorem 3.4. □

4. Upper bounds on γ_d and γ_d^1 of trees in terms of the order and the number of leaves

For any tree T of order n and with ℓ leaves, the set of non-leaves is a dominating set of T. Hence, $\gamma_1(T) \le n-\ell$. Note that the equality holds if and only if each vertex of T is either a leaf or a support vertex. If there exists a vertex $u \in V(T)$ that is neither a leaf nor a support vertex, then $V(T) \setminus (\{u\} \cup L(T))$ is a dominating set of T, implying that $\gamma_1(T) < n-\ell$. On the other hand, the upper bound $\gamma_1^1(T) \le n-\ell$ is not true for every tree T. For example, let $T' = T^* \circ P_1 \in \mathcal{T}_1$ for some tree T^*, and let T be the tree obtained from T' by adding $r \ge 2$ leaves to each vertex of T'. It can be checked that

$$\gamma_1^1(T) = |V(T^*)| + r|V(T^*)| > 2|V(T^*)| = 2(r+1)|V(T^*)| - 2r|V(T^*)| = n-\ell.$$

Set now

$$\mathcal{F}_2 = \{T : \ T - L(T) \in \zeta_1\}\ ,$$

and if $d \ge 3$, then set

$$\mathcal{F}_d = \left\{T : \ T - L(T) \text{ is a tree of order } d \text{ or belongs to } \mathcal{T}_{d-1}\right\}\ .$$

Note that each graph from $\mathcal{F}_d$, $d \ge 2$, is a tree, and the following property is equivalent to the definition of $\mathcal{F}_d$.

($\star$) If $d \ge 3$, a tree T belongs to $\mathcal{F}_d$ if and only if it can be obtained from some tree T', which satisfies $|V(T')| = d$ or $T' \in \mathcal{T}_{d-1}$, by adding at least one pendant vertex to each leaf of T', and some number (possibly zero) to other vertices of T'. For $d=2$, a tree T belongs to $\mathcal{F}_2$ if and only if it can be obtained similarly from a tree $T' \in \zeta_1$.

For $d \ge 2$, we prove the following result.

Theorem 4.1. *Let $d \ge 2$ be an integer and T be a tree of order n and with ℓ leaves. If $n-\ell \ge d$, then $\gamma_d^1(T) \le \frac{n-\ell}{d}$ with equality if and only if $T \in \mathcal{F}_d$.*

Proof. Consider the tree $T' = T - L(T)$. Let $n' = |V(T')| = n-\ell \ge d$. Let D' be a $\gamma_{d-1}^1(T')$-set. By Corollary 2.2, $|D'| = \gamma_{d-1}^1(T') \le \frac{n'}{d}$. Moreover, D' is also a d-distance independent dominating set of T, implying that

$$\gamma_d^1(T) \le |D'| = \gamma_{d-1}^1(T') \le \frac{n'}{d} = \frac{n-\ell}{d}. \tag{1}$$

Assume that $\gamma_d^1(T) = \frac{n-\ell}{d}$ holds for a tree T. Inequalities in (1) therefore imply $\gamma_d^1(T) = \gamma_{d-1}^1(T') = \frac{n'}{d}$. By Theorems 3.2 and 3.4, we know that $T' \in \zeta_1$ when $d=2$, and T' is a tree of order d or $T' \in \mathcal{T}_{d-1}$ when $d \ge 3$. Since $T' = T - L(T)$, we conclude $T \in \mathcal{F}_d$.

It remains to prove that $\gamma_d^1(T) \ge \frac{n-\ell}{d}$ holds for every $T \in \mathcal{F}_d$. Consider first a tree T from $\mathcal{F}_2$ and let $T' = T - L(T)$. Hence $T' \in \zeta_1$. We will prove the inequality by induction on T' according to the recursive definition of ζ_1. If $T' \cong K_2$, then T is a double star and $\gamma_2^1(T) = 1 = \frac{n-\ell}{2}$. If $T' \cong D_{r,r}$, for $r \ge 1$, then any $\gamma_1^1(T')$-set is a smallest 2-distance independent dominating set of T, implying that

$$\gamma_2^1(T) = \gamma_1^1(T') = r+1 = \frac{2r+2}{2} = \frac{n'}{2} = \frac{n-\ell}{2}.$$

Assume next that $T' = T - L(T)$ is a tree from ζ_1 which is neither K_2 nor a double star. Let $T_2' = T'$ and let T_1' be the tree from ζ_1 such that T_2' is obtained from T_1' by the recursive construction of ζ_1, that is, T_2' can be obtained by joining the center u of a new copy of $K_{1,t}$ $(t \ge 1)$ to a support vertex v of T_1', and adding $t-1$ leaves at v. For $i \in [2]$, let T_i be a tree from $\mathcal{F}_2$, which is obtained from T_i' according to ($\star$). Moreover, let $n_i' = |V(T_i')|$, $n_i = |V(T_i)|$, and $\ell_i = |L(T_i)|$, $i \in [2]$.

Assume that $\gamma_2^1(T_1) = \frac{n_1-\ell_1}{2}$. We are going to prove that $\gamma_2^1(T_2) \ge \frac{n_2-\ell_2}{2}$. Note that $n_i' = n_i - \ell_i$ and $n_2' = n_1' + 2t$. Let D_2 be a $\gamma_2^1(T_2)$-set that contains as few leaves from T_2 as possible and let $D_1 = D_2 \cap V(T_1)$. If $v \in D_2$, then $u \notin D_2$ and, by the minimality of $|D_2 \cap L(T_2)|$, we have $L_{T_2'}(u) \subset D_2$. Now $D_1 = D_2 \setminus L_{T_2'}(u)$ is a 2-distance independent dominating set of T_1, implying that $\gamma_2^1(T_1) \le |D_1| = |D_2| - t$. If $v \notin D_2$, then we may assume that $u \in D_2$. Also, $L_{T_2'}(v) \setminus L_{T_1'}(v) \subset D_2$ holds by the minimality of $|D_2 \cap L(T_2)|$. Further, $\emptyset \ne L_{T_1'}(v) \subset D_1$, and v and the leaves added to $L_{T_1'}(v)$ in T_1 will be independently dominated by $L_{T_1'}(v)$. Hence, $D_1 = D_2 \setminus \{u\} \setminus (L_{T_2'}(v) \setminus L_{T_1'}(v))$ is a 2-distance independent dominating set of T_1, implying that $\gamma_2^1(T_1) \le |D_1| = |D_2| - t$. Hence no matter whether v belongs to D_2 or not, we have

$$\gamma_2^1(T_2) \ge \gamma_2^1(T_1) + t = \frac{n_1-\ell_1}{2} + \frac{n_2'-n_1'}{2} = \frac{n_2'}{2} = \frac{n_2-\ell_2}{2}.$$

Assume now that $T \in \mathcal{F}_d$ and $d \ge 3$. For $T' = T - L(T)$, let $n' = |V(T')| = n-\ell \ge d$. If $n' = d$, then $\gamma_d^1(T) \ge 1 = \frac{n-\ell}{d}$. If $T' \in \mathcal{T}_{d-1}$, then let $T' = T^* \circ P_{d-1}$ for some non-trivial tree T^*. For each $u \in V(T^*)$, let T_u' be the subtree of T' induced by u and the vertices of the copy of P_{d-1} attached to u, and let T_u be the subtree of T induced by $V(T'_u)$ and the leaves added to $V(T'_u)$ in T. If D is a $\gamma_d^1(T)$-set, then $|D \cap V(T_u)| \geq 1$ for every $u \in V(T^*)$. Thus, we have

$$\gamma_d^1(T) = |D| \geq |V(T^*)| = \frac{n'}{d} = \frac{n-\ell}{d}.$$

This completes the proof of Theorem 4.1. □

We note that the condition of $n \geq d + \ell$ is necessary in Theorem 4.1. Let T' be a tree of order at most $d-1$. Consider the tree T obtained from T' by adding at least one pendant vertex to each leaf of T' and some number to other vertices of T'. Then $n' = |V(T')| = n - \ell \leq d - 1$ and we may infer $\gamma_d^1(T) \geq 1 > \frac{d-1}{d} \geq \frac{n-\ell}{d}$.

Favaron [6] proved that if T is a tree of order $n \geq 2$ and with ℓ leaves, then $\gamma_1^1(T) \leq \frac{n+\ell}{3}$, and gave the full list of extremal trees for this bound. Our next theorem extends Favaron's result to all $d \geq 2$.

Theorem 4.2. *Let $d \geq 2$ be an integer and T a tree of order n and with ℓ leaves. If $n \geq d$, then $\gamma_d^1(T) \leq \frac{n+\ell}{d+2}$ with equality if and only if $T \in \{P_d\} \cup \mathcal{T}_d$.*

Proof. The proof is similar to that of Theorem 3.4, the only difference lies in the fact that here, while proving the upper bound, we simultaneously detect the extreme trees. First, we prove the sufficiency of the equality. If $T \cong P_d$, then $\gamma_d^1(T) = 1 = \frac{d+2}{d+2} = \frac{n+\ell}{d+2}$. If $T \in \mathcal{T}_d$, then by Proposition 3.1, $\gamma_d^1(T) = \frac{n}{d+1} = \frac{n+\frac{n}{d+1}}{d+2} = \frac{n+\ell}{d+2}$.

To prove the upper bound and that the equality implies $T \in \{P_d\} \cup \mathcal{T}_d$, we will clarify the structure of T in two claims. On the other hand, we will consider the diameter of T, and pay special attention to the two terminals of a diametrical path (by then we will be able to use the two proved claims). We proceed with the proof by induction on n. If $\mathrm{diam}(T) \leq 2d$, then $\gamma_d^1(T) = 1 = \frac{d+2}{d+2} \leq \frac{n+\ell}{d+2}$. The equality holds if and only if $n = d$ and $\ell = 2$, implying that $T \cong P_d$. So, we may assume that $\mathrm{diam}(T) \geq 2d+1$ and $n \geq 2d+2$. Note that if $\ell > \frac{n}{d+1}$, then by Corollary 2.2, $\gamma_d^1(T) \leq \frac{n}{d+1} < \frac{n+\ell}{d+2}$.

Claim C. *Let $i \in [d-1]$ and $j \in [d]$ with $i \leq j$. If T has a vertex v that is a (P_i, P_j)-support vertex, then $\gamma_d^1(T) < \frac{n+\ell}{d+2}$.*

Proof. Let $P' := x_1x_2\ldots x_i$ and $P'' := y_1y_2\ldots y_j$ be a copy of P_i and P_j, respectively, attached to v in T, where $x_iv, y_jv \in E(T)$. Since $n \geq 2d+2$ and $|V(P')| \leq |V(P'')| \leq d$, we have $\deg(v) \geq 3$. Consider $T' = T - V(P')$. Then $\ell' = |L(T')| = \ell - 1$ and $n' = |V(T')| = n - i \geq d+3$. As in the proof of Claim A it can be proved that there exists a $\gamma_d^1(T')$-set D' that is a d-distance independent dominating set of T. Using the induction hypothesis, we have

$$\gamma_d^1(T) \leq |D'| = \gamma_d^1(T') \leq \frac{n'+\ell'}{d+2} = \frac{n-i+\ell-1}{d+2} < \frac{n+\ell}{d+2}.$$

This proves Claim C. (□)

Claim D. *If T has a P_{d+1}-support vertex v, then $\gamma_d^1(T) \leq \frac{n+\ell}{d+2}$ and if equality holds, then $T \in \mathcal{T}_d$.*

Proof. Let $P' := x_1x_2\ldots x_{d+1}$ be a copy of P_{d+1} attached to v, where $x_{d+1}v \in E(T)$. Then $\deg_T(x_k) = 2$ for all $k \in [d+1] \setminus \{1\}$ and $\deg_T(x_1) = 1$. Consider $T' = T - V(P')$. Since $n \geq 2d+2$, we have $\deg_T(v) \geq 2$. Then $\ell' = |L(T')| = \ell$ if $\deg_T(v) = 2$ and $\ell' = \ell - 1$ if $\deg_T(v) \geq 3$. We observe that $n' = |V(T')| = n - (d+1) \geq d+1$ and consider two cases according to the degree of v.

Case D1. $\deg_T(v) \geq 3$.

Let D' be a $\gamma_d^1(T')$-set. The set $D' \cup \{x_1\}$ is a d-distance independent dominating set of T. By the induction hypothesis,

$$\gamma_d^1(T) \leq |D' \cup \{x_1\}| = \gamma_d^1(T') + 1 \leq \frac{n'+\ell'}{d+2} + 1 = \frac{n-(d+1)+\ell-1}{d+2} + 1 = \frac{n+\ell}{d+2},$$

and the equality holds if and only if $\gamma_d^1(T) = \gamma_d^1(T') + 1$ and $\gamma_d^1(T') = \frac{n'+\ell'}{d+2}$. Note that $n' \geq d+1$, so $T' \ncong P_d$ and $T' \in \mathcal{T}_d$.

Let $T' = T'_* \circ P_d$ for some non-trivial tree T'_*. Then $\ell' = \frac{n'}{d+1}$. Since $\deg(v) \geq 3$, we infer that $v \notin L(T')$. If $v \in V(T'_*)$, then $T = T_* \circ P_d \in \mathcal{T}_d$, where T_* is the tree obtained from T'_* by adding a new vertex x_{d+1} to it such that $x_{d+1}v \in E(T_*)$. If $v \notin V(T'_*) \cup L(T')$, then let u be the P_d-support vertex of T'_* attached to the copy of P_d containing v, and u' be the leaf of T' corresponding to u. Note that $v \neq u'$ and $D = (L(T') \setminus \{u'\}) \cup \{x_{d+1}\}$ is a d-distance independent dominating set of T. Thus,

$$\gamma_d^1(T) \leq |D| = |L(T')| = \frac{n'}{d+1} = \frac{n'+\ell'}{d+2} = \frac{n-(d+1)+\ell-1}{d+2} < \frac{n+\ell}{d+2}.$$

Case D2. $\deg_T(v) = 2$.

Let $P'' := x_1x_2 \dots x_{d+1}v$ be a copy of P_{d+2} attached to v', where $vv' \in E(T)$. Consider $T'' = T - V(P'') = T' - v$. Then $n'' = |V(T'')| = n - (d+2) \geq d$, and $\ell'' = |L(T'')| \leq \ell$ with equality if and only if $\deg_T(v') = 2$. Let D'' be a $\gamma_d^1(T'')$-set. The set $D'' \cup \{x_2\}$ is a d-distance independent dominating set of T. By the induction hypothesis,

$$\gamma_d^1(T) \leq |D'' \cup \{x_2\}| = |D''| + 1 = \gamma_d^1(T'') + 1 \leq \frac{n'' + \ell''}{d+2} + 1 \leq \frac{n - (d+2) + \ell}{d+2} + 1 = \frac{n+\ell}{d+2},$$

and the equality holds if and only if $\gamma_d^1(T) = \gamma_d^1(T'') + 1$, $\ell'' = \ell$, and $\gamma_d^1(T'') = \frac{n''+\ell''}{d+1}$, i.e., $T'' \in \{P_d\} \cup \mathcal{T}_d$.

Note that $\deg_T(v') = 2$ and $\deg_{T''}(v') = 1$. If $T'' \cong P_d$, then $T \cong P_{2d+2} \in \mathcal{T}_d$. Suppose that $T'' \in \mathcal{T}_d$. Let $T'' = T''_* \circ P_d$ for some non-trivial tree T''_*. Then $\ell'' = \frac{n''}{d+1}$. Clearly, $v' \in L(T'')$. Let $u'_1 \in V(T''_*)$ be the P_d-support vertex in T'', which is attached to the copy of P_d containing v'. Since $|V(T''_*)| \geq 2$, there exists a neighbor $u'_2 \in V(T''_*)$ of u'_1. It is clear that v' is the leaf of T' corresponding to u'_1. Let u''_2 be the leaf of T' corresponding to u'_2. Since $d \geq 2$, the set $D = (L(T'') \setminus \{v', u''_2\}) \cup \{u'_2, x_{d+1}\}$ is a d-distance independent dominating set of T. Thus, we have

$$\gamma_d^1(T) \leq |D| = |L(T'')| = \frac{n''}{d+1} = \frac{n'' + \ell''}{d+2} = \frac{n - (d+2) + \ell}{d+2} < \frac{n+\ell}{d+2}.$$

This completes the proof of Claim D. (□)

In the continuation, we may suppose that there is no P_{d+1}-support vertex in T and also that if v is a (P_i, P_j)-support vertex, then $i = j = d$. Let $s = \mathrm{diam}(T) \geq 2d+1$ and let $P := v_1v_2 \dots v_{s+1}$ be a diametrical path in T. Root T at v_{s+1}. Hence, by Lemma 3.3, $\deg(v_k) \leq 2$ for each $k \in [d] \cup ([s+1] \setminus [s-d+1])$, and $\deg(v_k) \geq 3$ for each $k \in \{d+1, s-d+1\}$. It also follows that the subtree $T_{v_{d+1}}$ is isomorphic to the $(d-1)$-subdivision of a star $K_{1,t}$ for some $t \geq 2$.

If $s = \mathrm{diam}(T) = 2d+1$, then by Lemma 3.3 (v), T is obtained from the $(d-1)$-subdivision of a star K_{1,t_1} and the $(d-1)$-subdivision of a star K_{1,t_2} by joining the centers v_{d+1} and v_{d+2}. We may assume that $t_1 \geq t_2 \geq 2$. Then $N(v_{d+2})$ is a d-distance independent dominating set of T. Since $d \geq 2$, we have

$$\begin{aligned}\gamma_d^1(T) &\leq |N(v_{d+2})| = \deg(v_{d+2}) = t_2 + 1 = \frac{(d+1)t_2 + d + t_2 + 2}{d+2} \\ &< \frac{(d+1)t_2 + dt_2 + t_2 + 2}{d+2} = \frac{2(d+1)t_2 + 2}{d+2} \\ &\leq \frac{d(t_1+t_2) + 2 + (t_1+t_2)}{d+2} = \frac{n+\ell}{d+2}.\end{aligned}$$

So, we may assume that $\mathrm{diam}(T) \geq 2d+2$ and $n \geq 2d+3$. Regarding v_{d+2}, we divide the rest of the proof into two cases and prove that the strict inequality $\gamma_d^1(T) < \frac{n+\ell}{d+2}$ holds in each case.

Case 1. Every vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ is of degree at least 3.

For each vertex $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$ we have $\deg(v) \geq 3$, and the subtree T_v is isomorphic to the $(d-1)$-subdivision of a star K_{1,t_v} for $t_v \geq 2$. Let $T' = T - T_{v_{d+2}}$ and $p = \deg(v_{d+2})$. It holds that

$$d+1 \leq n' = |V(T')| \leq n - 1 - (2d+1)(p-1).$$

Moreover, we have

$$\ell' = |L(T')| \leq \ell - 2(p-1) + 1 = \ell - 2p - 1,$$

with equality if and only if $\deg(v_{d+3}) = 2$, and for each $v \in N(v_{d+2}) \setminus \{v_{d+3}\}$, $t_v = 2$.

Let D' be a $\gamma_d^1(T')$-set. Then $D = D' \cup (N(v_{d+2}) \setminus \{v_{d+3}\})$ is a d-distance independent dominating set of T. Since $d \geq 2$ and $p \geq 2$, by the induction hypothesis we get

$$\begin{aligned}\gamma_d^1(T) \leq |D| = |D'| + p - 1 = \gamma_d^1(T') + p - 1 &\leq \frac{n' + \ell'}{d+2} + p - 1 \\ &\leq \frac{n - 1 - (2d+1)(p-1) + \ell - 2p - 1}{d+2} + p - 1 \\ &= \frac{n + \ell - d(p-1) - p - 3}{d+2} < \frac{n+\ell}{d+2}.\end{aligned}$$

Case 2. There is a vertex v in $N(v_{d+2}) \setminus \{v_{d+1}, v_{d+3}\}$ with $\deg(v) \leq 2$.

Since v_{d+2} is not a P_{d+1}-support vertex and P is a diametrical path, Lemma 3.3 (iii) implies that T_v is a pendant path P_i for some $i \in [d]$. Moreover, we have the following.

- If v_{d+2} is a P_i-support vertex of T for some $i \in [d-1]$, then there is no other pendant path attached to v_{d+2}.
- If v_{d+2} is a P_d-support vertex of T, then v_{d+2} is not a P_i-support vertex of T for any $i \in [d-1]$, and there is at least one copy of P_d attached to v_{d+2}.

Case 2.1. $L(v_{d+2}) \neq \emptyset$.
Let $x \in L(v_{d+2})$ and $T' = T - x$. Clearly, $\deg_T(v_{d+2}) \geq 3$ and $\deg_{T'}(v_{d+2}) \geq 2$. Then $\ell' = |L(T')| = \ell - 1$ and $n' = |V(T')| = n - 1 \geq 2d + 2$. Let D' be a $\gamma_d^1(T')$-set. By considering whether v_{d+2} is in D' or not, we observe that D' can be chosen such that it is also a d-distance (independent) dominating set of T. By the induction hypothesis, we have $\gamma_d^1(T) \leq |D'| = \gamma_d^1(T') \leq \frac{n'+\ell'}{d+2} = \frac{n-1+\ell-1}{d+2} < \frac{n+\ell}{d+2}$.

Case 2.2. $L(v_{d+2}) = \emptyset$.
Let $P' := x_1 x_2 \ldots x_i$ be a copy of P_i attached to v_{d+2}, where $x_i v_{d+2} \in E(T)$. Then $i \in [d] \setminus \{1\}$, and $\deg(x_k) = 2$ for all $k \in [i] \setminus \{1\}$, while $\deg(x_1) = 1$. Consider $T' = T - T_{v_d} - T_{x_i}$. By Lemma 3.3 (ii), $\deg(v_{d+1}) \geq 3$ and, by our condition, $\deg(v_{d+2}) \geq 3$. Therefore, $\ell' = |L(T')| = \ell - 2$. We also know that $n' = |V(T')| = n - d - i \leq n - d - 2$ and $n' \geq d + 3$.

Let D' be a $\gamma_d^1(T')$-set. Then $|D' \cap \{v_{d+1}, v_{d+2}\}| \leq 1$. As in Case 2.2 of Theorem 3.4, let

$$D = \begin{cases} D' \cup \{x_i\}, & \text{if } v_{d+1} \in D' \text{ and } v_{d+2} \notin D', \\ D' \cup \{v_1\}, & \text{if } v_{d+1} \notin D' \text{ and } v_{d+2} \in D', \\ D' \cup \{v_{d+1}, x_i\} \setminus (V(T_{v_{d+1}}) \setminus V(T_{v_d})), & \text{if } v_{d+1}, v_{d+2} \notin D'. \end{cases}$$

For any subcase, D is a d-distance independent dominating set of T. By the induction hypothesis, we have $\gamma_d^1(T) \leq |D| \leq |D'| + 1 = \gamma_d^1(T') + 1 \leq \frac{n'+\ell'}{d+2} + 1 \leq \frac{n-d-2+\ell-2}{d+2} + 1 < \frac{n+\ell}{d+2}$.

This completes the proof of Theorem 4.2. □

Now we set

$$\mathcal{F}_2' = \{T : \; T - L(T) \in \{K_2\} \cup \mathcal{T}_1\} \,,$$

and if $d \geq 3$, then set

$$\mathcal{F}_d' = \mathcal{F}_d.$$

By Theorems 4.1 and 4.2, we have the following two corollaries, respectively.

Corollary 4.3. *Let $d \geq 2$ be an integer and T be a tree of order n and with ℓ leaves. If $n - \ell \geq d$, then $\gamma_d(T) \leq \frac{n-\ell}{d}$ with equality if and only if $T \in \mathcal{F}_d'$.*

Corollary 4.4. *Let $d \geq 2$ be an integer and T be a tree of order n and with ℓ leaves. If $n \geq d$, then $\gamma_d(T) \leq \frac{n+\ell}{d+2}$ with equality if and only if $T \in \{P_d\} \cup \mathcal{T}_d$.*

Combining the above results with Corollary 2.2, we obtain

Corollary 4.5. *If $d \geq 2$, and T is a tree with ℓ leaves and of order $n \geq d + \ell$, then*

$$\gamma_d(T) \leq \gamma_d^1(T) \leq \begin{cases} \frac{n-\ell}{d}, & \text{if } n < (d+1)\ell, \\ \frac{n}{d+1}, & \text{if } n = (d+1)\ell, \\ \frac{n+\ell}{d+2}, & \text{if } n > (d+1)\ell. \end{cases}$$

Moreover, these bounds are best possible.

5. A conjecture

Recall that Ma and Chen [9] described equivalently bipartite graphs G of order n with $\gamma_1^1(G) = \frac{n}{2}$. For $d \geq 2$ we pose:

Conjecture 5.1. *If $d \geq 2$ and G is a connected bipartite graph of order n, then $\gamma_d^1(G) = \frac{n}{d+1}$ if and only if $G \in \{C_{2d+2}\} \cup \mathcal{B}_d$ or $n = d+1$.*

Since $\gamma_1^1(K_{r,r}) = r = \frac{r+r}{2} = \frac{n}{2}$, the condition of $d \geq 2$ of the conjecture above is necessary. If Conjecture 5.1 holds true, then it generalizes Theorem 3.4. Moreover, the result [13, Theorem 3] due to Topp and Volkmann, restricted to bipartite graphs, gives exactly the same characterization for graphs G with $\gamma_d(G) = \frac{n}{d+1}$ as we pose in Conjecture 5.1 for the d-distance independent domination.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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