Applied Mathematics and Computation 488 (2025) 129131

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

Mutual-visibility and general position in double graphs and in Mycielskians

Dhanya Roy [a], Sandi Klavžar [b,c,d,*], Aparna Lakshmanan S [a]

[a] Department of Mathematics, Cochin University of Science and Technology, Cochin - 22, Kerala, India
[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[c] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

ARTICLE INFO

MSC:
05C12
05C69
05C76

Keywords:
General position
Mutual-visibility
Double graph
Mycielskian graph
Outer mutual-visibility
Total mutual-visibility

ABSTRACT

The general position problem in graphs is to find the largest possible set of vertices with the property that no three of them lie on a common shortest path. The mutual-visibility problem in graphs is to find the maximum number of vertices that can be selected such that every pair of vertices in the collection has a shortest path between them with no vertex from the collection as an internal vertex. Here, the general position problem and the mutual-visibility problem are investigated in double graphs and in Mycielskian graphs. Sharp general bounds are proved, in particular involving the total and the outer mutual-visibility number of base graphs. Several exact values are also determined, in particular the mutual-visibility number of the double graphs and of the Mycielskian of cycles.

1. Introduction

The graph general position problem reflects the Dudeney's no-three-in-line problem [10] as well as the general position subset selection problem from discrete geometry [11]. The problem was in a different context investigated on hypercubes [18], while it was introduced in its generality as follows [23]. A set S of vertices in a graph form a *general position set* if the graph contains no shortest path that contains at least three vertices of S. A largest general position set of a graph G is called a *gp-set* of G and its size is the *general position number* $\mathrm{gp}(G)$ of G. The same concept was in use two years earlier in [4] under the name geodetic irredundant sets, where it was defined in a different way.

In discrete geometry, a shortest path between two points is unique while in graphs there can be more than one shortest path between two vertices. This fact, as well as the computational concept of visibility between robots, brings the mutual-visibility problem in graphs into picture. This problem was introduced by Di Stefano [9] as follows. Given a set S of vertices in a graph G, two vertices u and v are *mutually-visible* or, more precisely, *S-visible*, if we can find a shortest u,v-path which contains no further vertices from S. The set S is *mutual-visibility* (*m-v set*, for short) if all pairs from S are S-visible. A largest m-v set is called a *μ-set* and its size is called the *mutual-visibility number* $\mu(G)$ of G (*m-v number*, for short).

* Corresponding author.
E-mail addresses: dhanyaroyku@gmail.com, dhanyaroyku@cusat.ac.in (D. Roy), sandi.klavzar@fmf.uni-lj.si (S. Klavžar), aparnaren@gmail.com, aparnals@cusat.ac.in (A. Lakshmanan S).

https://doi.org/10.1016/j.amc.2024.129131
Received 14 March 2024; Received in revised form 7 October 2024; Accepted 11 October 2024

Available online 17 October 2024
0096-3003/

In [5], a variety of m-v sets was introduced, we will use the following two variants. $S \subseteq V(G)$ is an *outer m-v set* if each $x, y \in S$ are S-visible and each $x \in S$ and $y \in V(G) \setminus S$ are also S-visible. A largest such set is a μ_o*-set*, its cardinality being the *outer m-v number* $\mu_o(G)$ of G. Further, $S \subseteq V(G)$ is a *total m-v set* provided that each $x, y \in V(G)$ are S-visible. A largest such set is a μ_t*-set*, its cardinality being the *total m-v number* $\mu_t(G)$ of G.

The general position problem and the mutual-visibility problem are well studied for different graph classes like diameter two graphs [1,8], cographs [1,9], Kneser graphs [12], line graphs of complete graphs [8,12], and maximal outerplane graphs [31]. Both problems were also investigated a lot on graph operations like the join of graphs [9,12], corona products [6,12,17], Cartesian products [6,15,16,19,29,30], and strong products [7,17]. In this paper we extend this line of research by investigating the problems on double graphs and on Mycielskian graphs which are respectively defined as follows.

Let G be a graph. The *double graph* $D(G)$ of G is constructed from the disjoint union of G and an isomorphic copy G' of G by adding edges uv' and $u'v$ for each edge $uv \in E(G)$, where w' is used to denote the copy of $w \in V(G)$. The *Mycielskian graph* $M(G)$ of G has $V(M(G)) = V(G) \cup V(G') \cup \{v^*\}$, where $V(G') = \{u' : u \in V(G)\}$, while $E(M(G)) = E(G) \cup \{uv' : uv \in E(G)\} \cup \{v'v^* : v' \in V(G')\}$. These two graph operators were respectively introduced in [24,25]. The Mycielskian has been studied in a couple of hundred papers and the trend is still continuing [2,3,13,14]. The double graphs have also received quite some attention, cf. [20,22].

In the following section, additional definitions required for this paper are listed, known results recalled, and some new observations stated. In Section 3 we prove that if G is not complete, then $\mu(D(G)) \geq n(G) + \mu_t(G)$. The bound is sharp as in particular follows from the proved formula $\mu(D(C_n)) = n$, $n \geq 7$. On the other hand, we construct graphs G for which the difference $\mu(D(G)) - (n(G) - \mu_t(G))$ is arbitrary large. In Section 4 we prove that $\mathrm{gp}(G) \leq \mathrm{gp}(D(G)) \leq 2\,\mathrm{gp}(G)$ and that the bounds are sharp. In the subsequent section, mutual-visibility in Mycielskian graphs is studied. In the main results we state that $\mu(M(P_n)) = n + \lfloor \frac{n+1}{4} \rfloor$ for $n \geq 5$, and that $\mu(M(C_n)) = n + \lfloor \frac{n}{4} \rfloor$ for $n \geq 8$. We also give bounds for $\mu(M(G))$, where $\mathrm{diam}(G) \leq 3$, in terms of $\mu_o(G)$ and $\mu(G)$.

2. Preliminaries

Let G be a connected graph. A partition $\mathcal{P} = \{T_1, \dots, T_k\}$ of $T \subseteq V(G)$ is said to be *distance-constant* provided that for each $i, j \in [k]$, $i \neq j$, and for every $x \in T_i$ and $y \in T_j$, the distance $d_G(x, y)$ does not depend on x and y. If so, we can declare that $d_G(T_i, T_j)$ is the distance between T_i and T_j. Having a distance-constant partition $\mathcal{P}$, we further say it is *in-transitive* provided that we have $d_G(T_i, T_t) \neq d_G(T_i, T_j) + d_G(T_j, T_t)$ for each $i, j, t \in [k]$. The characterization of general position sets that follows will be used either implicitly or explicitly in the rest of the paper. By $G[T]$ we denote the subgraph of G induced by $T \subseteq V(G)$. Now let's call up the following fundamental result from [1, Theorem 3.1].

Theorem 2.1. *$T \subseteq V(G)$ is a general position set of a connected graph G if and only if $G[T]$ is a union of disjoint complete subgraphs whose vertex sets form a distance-constant, in-transitive partition of T.*

Let G be a graph. Vertices x and y of G are *false twins* if $N_G(x) = N_G(y)$, where $N_G(x)$ stands for the open neighborhood of x in G. (Note that false twins are not adjacent.) Further, x and y are *true twins* if $N_G[x] = N_G[y]$, where $N_G[x]$ denotes the closed neighborhood of the vertex x in G. In [17], relations between true twins, the general position number, and the so-called strong resolving graphs were investigated. The following easy but useful general properties of twins hold.

Lemma 2.2. *Let G be a graph and $u, v \in V(G)$.*

(i) If u, v are false twins, and S is a general position (resp. m-v) set of G such that $S \cap \{u, v\} = \{u\}$, then $(S \setminus \{u\}) \cup \{v\}$ is also a general position (resp. m-v) set of G.

(ii) If u, v are true twins, and S is a general position set of G such that $u \in S$, then $S \cup \{v\}$ is also a general position set of G.

Proof. (i) Since $d_G(u, x) = d_G(v, x)$ for each $x \in V(G) \setminus \{u, v\}$, Theorem 2.1 yields that $(S \setminus \{u\}) \cup \{v\}$ is a general position set of G. Moreover, two vertices are S-visible if and only if they are $(S \setminus \{u\}) \cup \{v\}$-visible, hence $(S \setminus \{u\}) \cup \{v\}$ is a m-v set provided that S is a m-v set.

(ii) Using the fact that $d_G(u, x) = d_G(v, x)$ for each $x \in V(G) \setminus \{u, v\}$, Theorem 2.1 again can be used to realize that $S \cup \{v\}$ is a general position set. Assume $v \notin S$, for otherwise we are done. Let Q be the complete subgraph containing u from the partition of S corresponding to Theorem 2.1. Then $Q \cup \{v\}$ is also complete, therefore Theorem 2.1 yields that $S \cup \{v\}$ is as should be. □

Note that Lemma 2.2(ii) does not hold if general position sets are replaced by m-v sets. For example, consider the complete graph K_4 minus an edge with the vertex set $\{a, b, c, d\}$, where a and b are the non-adjacent pair. Then $S = \{a, b, c\}$ is a m-v set, c and d are true twins, but we cannot add d to S without affecting the mutual-visibility.

3. Mutual-visibility in double graphs

Here we consider mutual-visibility in double graphs. For this task recall that if G is a graph, then $V(D(G)) = V(G) \cup V(G')$ and that for each pair $u \in V(G)$ and $u' \in V(G')$ we have $N_{D(G)}(u) = N_{D(G)}(u')$.

If G is a graph, then, clearly, $\mu(G) = n(G)$ if and only if G is a complete graph, the same conclusion holds for the total mutual-visibility [21]. (Here and later, $n(G)$ denotes the order of G.) For each vertex $u \in V(D(G))$, clearly, $N_{D(G)}[u]$ is a m-v set of $D(G)$. Thus $\mu(D(G)) \geq 2\Delta(G) + 1$. Hence for the double graphs of graphs with a universal vertex, we have the following observation:

Observation 3.1. *If G is a graph with $n(G) \geq 2$ and with a universal vertex, then $\mu(D(G)) = 2n(G) - 1$.*

Theorem 3.2. *If G is not complete, then $\mu(D(G)) \geq n(G) + \mu_t(G)$ and the bound is sharp.*

Proof. Recall that $D(G)$ is the disjoint union of G and an isomorphic copy G' of G by adding edges uv' and $u'v$ for each edge $uv \in E(G)$, where w' is used to denote the copy of $w \in V(G)$.

Let S be a μ_t-set of G. We claim that the set $X = V(G') \cup S$ is a m-v set of $D(G)$. To prove it we make the following case analysis. If $u, v \in S$, then u and v are X-visible because S is a total m-v set of G.

Consider next vertices $u', v' \in V(G')$. If u' and v' are adjacent we are done. Assume next that u' and v' are not adjacent, and let $u = u_0, u_1, \ldots, u_k = v$ be a shortest u,v-path in G that makes the vertices u and v to be S-visible. Such a path exists because S is a total m-v set of G. Then the path $u', u_1, \ldots, u_{k-1}, v'$ is a shortest u', v'-path in $D(G)$ that makes the vertices u' and v' to be X-visible.

Consider finally a vertex $u \in S$ and a vertex $v' \in V(G')$. If u and v are adjacent, then also u and v' are adjacent and we are done. Otherwise, let $u = u_0, u_1, \ldots, u_k = v$ be a shortest u,v-path in G that makes S-visible u and v. Then, $u, u_1, \ldots, u_{k-1}, v'$ is a shortest u, v'-path in $D(G)$ that makes X-visible u and v'.

We have thus proved that $V(G') \cup S$ indeed forms a m-v set of $D(G)$, therefore $\mu(D(G)) \geq n + \mu_t(G)$. To demonstrate the sharpness, consider the path graph P_n, $n \geq 3$, with the vertices $u_1, \ldots, u_n$. Let S be an arbitrary m-v set of $D(P_n)$. If we would have indices $i < j < k$, such that $\{u_i, u_i', u_j, u_j', u_k, u_k'\} \in S$, then u_i and u_k would not be S-visible. Therefore, for at most two indices $i \in [n]$ we have $|S \cap \{u_i, u_i'\}| = 2$ which in turn implies that $\mu(D(P_n)) \leq n + 2$. Since $\mu_t(P_n) = 2$, the above proved bound yields $\mu(D(P_n)) \geq n + 2$, hence the bound is sharp. □

In the seminal paper on the mutual-visibility [9] it was proved that the mutual-visibility problem is NP-complete, while in [5] the same conclusion was obtained for each of the problems from the variety of mutual-visibility problems including the total mutual-visibility problem. Theorem 3.2 could indicate that the mutual-visibility problem is difficult also when restricted to double graphs.

The next result yields another family for which the bound of Theorem 4.2 is sharp.

Theorem 3.3. *If $n \geq 7$, then $\mu(D(C_n)) = n$.*

Proof. Let $V(D(C_n)) = V \cup V'$, where $V = V(C_n)$ and $V' = \{u' : u \in V\}$. Let S be a μ-set of $D(C_n)$ such that it contains as many vertices of V' as possible. For any $u \in V$, the vertices u and u' are false twins. Hence Lemma 2.2(i) implies that if $u \in S$ and $u' \notin S$, then $(S \cup \{u'\}) \setminus \{u\}$ is also a μ-set of $D(C_n)$. Since by the way S is selected, this is not possible, we thus infer that This $u \in V \cap S$, then u' also belongs to S.

If $V' \subseteq S$, then no vertex from V can be present in S, because if $v \in S$, then the two neighbors of v' in V' are not S-visible. Therefore, in this case $|S| = n$. By the same argument we also get that if $|S| > n$, then $|S \cap V| \geq 2$. We may hence assume in the rest that not all vertices from V' are in S. We now distinguish two cases.

Assume first that S contains at least three vertices from V, say $u, v, w \in V \cap S$. Then, by the maximality assumption, u', v' and w' are also in S. Now, if x' belongs to S, where $x \neq u, v, w$, then at least one of the shortest paths in V' from x' to u', v' or w' must contain at least one vertex among u', v' and w' as an internal vertex. We may assume without loss of generality that a shortest x', u'-path contains v' as an internal vertex. (It could be that also the other x', u'-path in $D(C_n)[V']$ is shortest. Then it contains w as an internal vertex, and the argument is parallel.) Since v is also in S, the vertices x' and u' are not S-visible. Therefore, $|S \cap V'| = 3$ so that $|S| = 6$, a contradiction with Theorem 3.2 which asserts that $|S| \geq n \geq 7$.

Assume second that $S \cap V = \{u, v\}$. Using the maximality assumption again, $u', v' \in S$. There is nothing to prove if $|S \cap V'| \leq n-2$, hence assume that $|S \cap V'| = n - 1$. Let $w' \in V'$ be the vertex not in S. If w is not adjacent to both u and v, then we may assume without loss of generality that the two neighbors of u' are in S, but then they are not S-visible. Similarly, if w is in C_n adjacent to both u and v, and z is the other neighbor of u, then z' and v' are not S-visible.

As none of the cases above is possible we can conclude that $\mu(D(C_n)) \leq n$. On the other hand, Theorem 4.2 yields $\mu(D(C_n)) \geq n$ and we are done. □

The proof of Theorem 3.3 asserts that for any $n \geq 4$, we have $\mu(D(C_n)) \leq \max\{6, \lfloor \frac{n}{2} \rfloor + 4, n\}$. Hence $\mu(D(C_4)) \leq 6$, $\mu(D(C_5)) \leq 6$ and $\mu(D(C_6)) \leq 7$. Also, $\{v_1', v_2', v_3', v_4', v_1, v_2\}$ is a m-v set of $D(C_4)$, $\{v_2', v_3', v_4', v_5', v_2, v_5\}$ is a m-v set of $D(C_5)$ and $\{v_2', v_3', v_4', v_5', v_6', v_2, v_6\}$ is a m-v set of $D(C_6)$. Therefore $\mu(D(C_4)) = \mu(D(C_5)) = 6$ and $\mu(D(C_6)) = 7$.

While the lower bound of Theorem 3.2 is sharp, it can, on the other hand, be arbitrarily bad, that is, the difference $\mu(D(G)) - (n(G) + \mu_t(G))$ can be arbitrarily large. For example, consider the balloon graph G_k, $k \geq 2$, constructed from the disjoint union of k copies of C_5 and a vertex which is adjacent to exactly one vertex of each of the k copies of C_5. Then we have:

Proposition 3.4. *If $k \geq 2$, then $\mu(D(G_k)) - (n(G_k) + \mu_t(G_k)) \geq k - 1$.*

Proof. Clearly, $n(G_k) = 5k + 1$. Using the characterization [28, Theorem 8] of graphs G with $\mu_t(G) = 0$ (or by verifying it directly), we can deduce that $\mu_t(G_k) = 0$. Further, we can use Fig. 1 to find out that $\mu(D(G_k)) \geq 6k$.

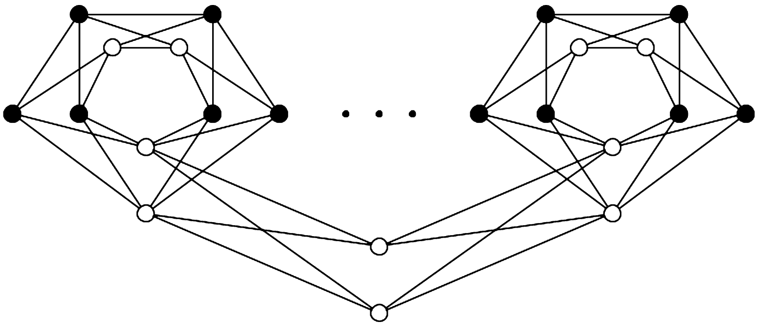

Fig. 1. A m-v set in the double graph of a balloon graph.

Hence we have

$$\mu(D(G_k)) - (n(G_k) + \mu_t(G_k)) \geq 6k - ((5k+1)+0) = k-1\,,$$

and we are done. □

4. General position in double graphs

Here we look at the general position number of double graphs. If $S \subseteq V(G) \subset V(D(G))$, we will set $S' = \{u' \in V(G') : \; u \in S\}$.

We first state a simple but useful lemma which easily follows from the fact that if $u \in V(G) \subset V(D(G))$ is not an isolated vertex, then $d_{D(G)}(u,u') = 2$.

Lemma 4.1. *Let G be a graph, $uv \in E(G)$, and let S be a general position set of $D(G)$. If $|S \cap \{u,v,u',v'\}| \geq 2$, then $|S \cap \{u,v,u',v'\}| = 2$.*

Note that Lemma 4.1 in particular implies that if S is a general position set of $D(G)$ such that $u,u' \in S$, then both u and u' are non-adjacent to all other vertices in S.

Theorem 4.2. *If G is a graph, then $\mathrm{gp}(G) \leq \mathrm{gp}(D(G)) \leq 2\,\mathrm{gp}(G)$ and the bounds are sharp. Moreover, $\mathrm{gp}(D(G)) = 2\,\mathrm{gp}(G)$ if and only if the gp-sets of $D(G)$ are of the form $X \cup X'$, where X is an independent gp-set of G.*

Proof. If S is a gp-set of G, then $S \subset V(D(G))$ is a general position set of $D(G)$. Hence $\mathrm{gp}(G) \leq \mathrm{gp}(D(G))$. Let now S be a gp-set of $D(G)$. Since G and G' are isometric subgraphs of $D(G)$, we infer that $V(G) \cap S$ is a general position set of G and $V(G') \cap S$ is such a set of G'. Hence

$$\mathrm{gp}(D(G)) = |S| = |S \cap V(G)| + |S \cap V(G')| \leq \mathrm{gp}(G) + \mathrm{gp}(G') = 2\,\mathrm{gp}(G)\,,$$

establishing the upper bound.

With the intention of seeing that the lower bound is sharp, note that $\mathrm{gp}(D(K_n)) = n$ holds for all $n \geq 2$ by Lemma 4.1.

Assume now that $\mathrm{gp}(D(G)) = 2\,\mathrm{gp}(G)$ and consider an arbitrary gp-set S of $D(G)$. As we already observed, $S \cap V(G)$ is a general position set of G and $S \cap V(G')$ is a general position set of G', therefore $S \cap V(G)$ is a gp-set of G and $S \cap V(G')$ a gp-set of G'. If $(S \cap V(G))' \neq S \cap V(G')$, then we may without loss of generality assume that there is a vertex $u \in S$ such that $u' \notin S$. But then an application of Lemma 2.2 yields a general position set in G' larger that $\mathrm{gp}(G') = \mathrm{gp}(G)$, a contradiction. Hence $S = (S \cap V(G)) \cup (S \cap V(G))'$. Moreover, by Lemma 4.1, $S \cap V(G)$ must be an independent set and we are done. □

There are many graphs admitting independent gp-sets which in turn explicitly demonstrate that the upper bound of Theorem 4.2 is sharp. This is in particular the case for paths P_n, $n \geq 3$, and for cycles C_n, $n \geq 6$. Hence by Theorem 4.2 we get $\mathrm{gp}(D(P_n)) = 4$, $n \geq 3$, and $\mathrm{gp}(D(C_n)) = 6$, $n \geq 6$. More on independent general position sets can be found in [26].

A family of graphs for which the lower bound in Theorem 4.2 is sharp are the edge deleted complete graphs K_n^-, $n \geq 5$, that is, K_n^- is the graph obtained from K_n by deleting one of its edges. Clearly, $\mathrm{gp}(K_n^-) = n-1$. Considering $D(K_n^-)$, note first that since $D(K_n^-)$ contain a clique of order $n-1$ we have $\mathrm{gp}(D(K_n^-)) \geq n-1$. Let u and v be the non-adjacent pair of vertices in K_n^- and let S be a general position set of $D(K_n^-)$. If $w,w' \in S$, where $w \neq u,v$, then $S = \{w,w'\}$. Assume hence that for each $w \neq u,v$, the set S contains at most one vertex among w and w'. If $|S| \geq n$, then we must have that $|S \cap \{u,u',v,v'\}| \geq 2$. However, as soon as this is fulfilled we can infer that in each possible case we have $S \subseteq \{u,v,u',v'\}$. We conclude that $\mathrm{gp}(D(K_n^-)) = n-1$ for $n \geq 5$.

5. Mutual-visibility in Mycielskian graphs

The general position number of Mycielskian graphs was investigated in [27], in this section we complement this research by considering the m-v number of Mycielskian graphs. We find the exact value of μ of the Mycielskian graph of paths, cycles and graphs

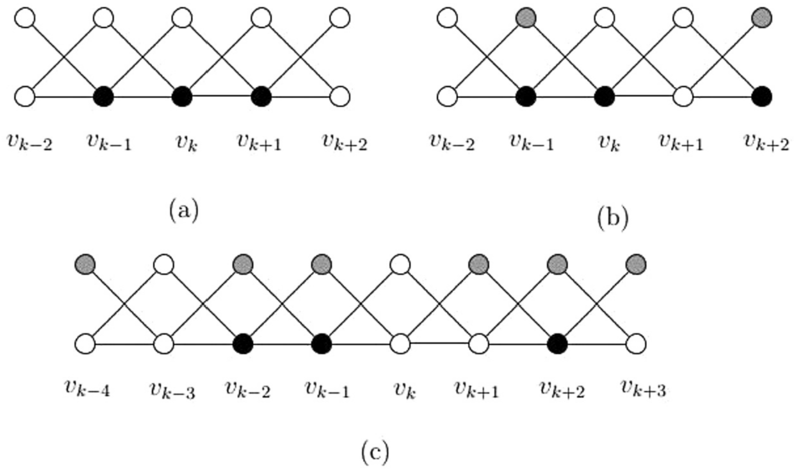

Fig. 2. Situations from the proof of Theorem 5.1. The black vertices denote the vertices in the m-v set, the grey vertices are those whose status is not known and the white vertices are those which cannot be present in the m-v set.

with universal vertices. Bounds of m-v number of Mycielskian graph of graphs having diameter at most three in terms of (outer) mutual-visibility of the graph are also presented.

Theorem 5.1. *If $n \geq 5$, then $\mu(M(P_n)) = n + \lfloor \frac{n+1}{4} \rfloor$.*

Proof. Let P_n have the vertices $v_1, \dots, v_n$ (connected with natural edges), so that $V(M(P_n)) = \{v_1, \dots, v_n\} \cup \{v'_1, \dots, v'_n\} \cup \{v^*\}$.

We will first prove that $\mu(M(P_n)) \leq n + \lfloor \frac{n+1}{4} \rfloor$. Consider an arbitrary m-v set S of $M(P_n)$ which contains the vertex v^*. Let $S = N \cup N'$, where $N \subseteq V(P_n)$ and $N' \subseteq V(P'_n) \cup \{v^*\}$. As we have assumed, the vertex v^* belongs to N'. Since v^* is in the unique shortest path connecting u' and v', where $d_{P_n}(u,v) \neq 2$, at most two vertices from $V(P'_n)$ are in N'. Also, since v^* lies on the unique shortest path connecting u and v, where $d_{P_n}(u,v) \geq 5$, at most four vertices from $V(P_n)$ are in N. Hence we have $|S| \leq 7$ which proves the assertion for $n \geq 7$. For $n = 5, 6$, using similar arguments we can prove that if $v^* \in N'$ then $|S| \leq 5$ and $|S| \leq 6$, respectively.

According to what has just been proven, in order to prove $\mu(M(P_n)) \leq n + \lfloor \frac{n+1}{4} \rfloor$, we may reduce our attention to those m-v sets which do not contain the vertex v^*. Moreover, if such a m-v set S contains only two vertices from $V(P_n)$ and they are adjacent, then we have $|S| \leq n$. So such m-v sets can also be excluded in the rest.

Claim. *To any m-v set S of $M(P_n)$, where $v^* \notin S$ and $|S \cap V(P_n)| \geq 3$, there exists a m-v set $T \cup T'$ of $M(P_n)$, where $T \subseteq V(P_n)$ and $T' \subseteq V(P'_n)$, such that $|T \cup T'| = |S|$ and $T \cap V(P_n)$ is an independent set.*

Let $S = N \cup N'$ be a m-v set of $M(P_n)$, where $N \subseteq V(P_n)$ and $N' \subseteq V(P'_n)$. If N is independent, there is nothing to prove. Otherwise, proceed as follows to replace the vertices in the μ-set so as to make a new mutual-visibility of the same cardinality and which is independent restricted to P_n. The construction is distinguished according to the following situations.

Assume first that three consecutive vertices of P_n lie in N. If $v_{k-1}, v_k, v_{k+1} \in N$, where $2 < k < n-2$, then none of the vertices v_{k-2}, v'_{k-2}, v'_{k-1}, v'_k, v'_{k+1}, v_{k+2}, v'_{k+2} lies in $N \cup N'$, see Fig. 2(a). Then we infer that $(N \cup N' \cup \{v'_k\}) \setminus \{v_k\}$ is a m-v set of $M(P_n)$ and hence a μ-set of $M(P_n)$. If $v_1, v_2, v_3 \in N$, then none of v'_1, v'_2, v'_3, v_4, v'_4 lies in $N \cup N'$. In this case we see that $(N \cup N' \cup \{v'_1\}) \setminus \{v_2\}$ is a m-v set of $M(P_n)$. Similarly, if $v_{n-2}, v_{n-1}, v_n \in N$, then $(N \cup N' \cup \{v'_n\}) \setminus \{v_{n-1}\}$ is a m-v set of $M(P_n)$. We have thus seen that $N \cup N'$ can be modified in such a way that no three consequent vertices from P_n are in N.

Assume next that $v_{k-1}, v_k, v_{k+2} \in N$, where $2 < k < n-2$. Then the vertices v_{k-2}, v'_{k-2}, v'_k, v_{k+1}, v'_{k+1} do no belong to $N \cup N'$, cf. Fig. 2(b), where v_{k-2} and v_{k+1} do not belong to N by the above modification. Then $(N \cup N' \cup \{v'_k\}) \setminus \{v_k\}$ is a m-v set of $M(P_n)$. If $v_1, v_2, v_4 \in N$, then $v'_1, v'_2, v_3, v'_3 \notin N \cup N'$. Then $(N \cup N' \cup \{v'_1\}) \setminus \{v_2\}$ is a m-v set of $M(P_n)$. Similarly, if $v_{n-3}, v_{n-1}, v_n \in N$, then $(N \cup N' \cup \{v'_n\}) \setminus \{v_{n-1}\}$ is a m-v set of $M(P_n)$. We can thus further modify $N \cup N'$ in such a way that no three vertices from P_n of the form v_{k-1}, v_k, v_{k+2} are in N.

In the third case assume that $v_{k-2}, v_{k-1}, v_{k+2} \in N$, where $k \neq 3, n-2$. Then $v'_{k-3}, v'_k \notin N \cup N'$. Moreover, we also infer that by the above modifications, $v_{k-3}, v_k, v_{k+1} \notin N$, see Fig. 2(c). Then $(N \cup N' \cup \{v_k, v'_k\}) \setminus \{v_{k-1}, v'_{k+1}, v_{k+3}\}$ is a m-v set of $M(P_n)$. (Note that only one among v'_{k+1} and v_{k+3} will be present initially in $N \cup N'$ which implies that the cardinality does not change.) If $v_1, v_2, v_5 \in N$, then $v'_1, v'_2, v_3, v'_3, v_4 \notin N \cup N'$. Then $(N \cup N' \cup \{v_3, v'_3\}) \setminus \{v_{4'}, v_6\}$ is a m-v set of $M(P_n)$. (Note that only one among v'_4 and v_6 will be present initially in $N \cup N'$.) Similarly, if $v_{n-4}, v_{n-1}, v_n \in N$, then $(N \cup N' \cup \{v_{n-2}, v'_{n-2}\}) \setminus \{v_{n-1}, v'_{n-3}, v_{n-5}\}$ is a m-v set of $M(P_n)$. (Note that only one among v'_{n-3} and v_{n-5} will be present initially in $N \cup N'$.)

In the last case to be considered assume that $v_{k-2}, v_{k-1}, v_{k+l} \in N$ for some $l \geq 3$. Then $v_k, v'_k, v_{k+1}, v_{k+2} \notin N \cup N'$. In addition, $v_{k-5}, v_{k-4}, v_{k-3}, v'_{k-3} \notin N \cup N'$, if those vertices are present in the graph. Then $(N \cup \{v_k\}) \setminus \{v_{k-1}\}$ is a m-v set of $M(P_n)$. Thus modify $N \cup N'$ in such a way that no three vertices from P_n of this form are in N. This proves the claim.

In view of the proved Claim, it remains to consider a m-v set $T \cup T'$ of $M(P_n)$, where $T \subseteq V(P_n)$ and $T' \subseteq V(P'_n)$, such that T is an independent set. Now, if there are two vertices u and v in T such that $d_G(u,v) \geq 5$, then each of u and v must have a neighbor in $V(P'_n)$ which is not in T' in order that u are v are visible. Also, if there are two vertices u and v' in T such that $d_G(u,v) = 4$, then u must have a neighbor in $V(P'_n)$ which is not in T' in order that u are v' are visible. Such a vertex from $V(P'_n) \setminus T'$ can be used or shared by at most two vertices from T. Hence, the cardinality of $T \cup T'$ will be maximum when the vertices $v_1, v_3, v_5, \dots$ are included in T and the vertices $v'_2, v'_6, v'_{10}, \dots$ are excluded from T'. We can conclude that

$$\mu(M(P_n)) \leq \left\lceil \frac{n}{2} \right\rceil + n - \left\lceil \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil \right\rceil = n + \left\lfloor \frac{n+1}{4} \right\rfloor .$$

The above consideration also gives rise to the following construction of a largest m-v set of $M(P_n)$. Let $R = \{v_1, v_3, \dots, v_k\}$, where $k = n-1$ when n is even, and $k = n$ when n is odd. Let further $R' = \{v'_{4\ell+2} : \ell \in \mathbb{N}_0, 2 \leq 4\ell + 2 \leq n\}$. In the case when $n \bmod 4 = 3$, we further add the vertex v'_{n-1} to R'. Set now

$$S = R \cup (V(P'_n) \setminus R').$$

By considering all the cases (up to symmetry) and having the discussion from the previous paragraph in mind, we infer that S is a m-v set of $M(P_n)$. Since $|R| = \lceil \frac{n}{2} \rceil$ and $|R'| = \lceil \frac{1}{2} \lceil \frac{n}{2} \rceil \rceil$, we have

$$\mu(M(P_n)) \geq |S| = \left\lceil \frac{n}{2} \right\rceil + n - \left\lceil \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil \right\rceil = n + \left\lfloor \frac{n+1}{4} \right\rfloor ,$$

and we are done. □

Theorem 5.2. *If $n \geq 8$, then $\mu(M(C_n)) = n + \lfloor \frac{n}{4} \rfloor$.*

Proof. Let C_n have the vertices $v_1, \dots, v_n$ (connected with natural edges), so that $V(M(C_n)) = \{v_1, \dots, v_n\} \cup \{v'_1, \dots, v'_n\} \cup \{v^*\}$. We will assume that indices are computed modulo n.

We first prove $\mu(M(C_n)) \leq n + \lfloor \frac{n}{4} \rfloor$ and consider for this sake all m-v sets of $M(C_n)$. In the first case, let S be a m-v set of $M(C_n)$ which contains the vertex v^*. Let $S = N \cup N'$, where $N \subseteq V(C_n)$ and $N' \subseteq V(C'_n) \cup \{v^*\}$. By our assumption, the vertex v^* belongs to N'. Hence at most two vertices from $V(C'_n)$, say v'_k and v'_{k+2}, are in N'. If $n \in \{8,9\}$, it can easily be verified that $|S| \leq 8$. For $n \geq 10$, at most five vertices from $V(C_n)$ are in N, since no two vertices $u, v \in V(C_n)$ such that $d_{C_n}(u,v) \geq 5$, are S-visible. So in any case the claimed inequality holds. Because of that, in the rest we may restrict our attention to m-v sets of $M(C_n)$ which do not contain v^*.

Claim. *To any m-v set S of $M(C_n)$, where $v^* \notin S$, there exists a m-v set $T \cup T'$ of $M(C_n)$, where $T \subseteq V(C_n)$ and $T' \subseteq V(C'_n)$, such that $|T \cup T'| = |S|$ and the following holds. As soon as $v_k, v_{k+1} \in T$, the vertices v_{k-2}, v_{k-1}, v_{k+2}, and v_{k+3} do not belong to T.*

Let $S = N \cup N'$ be a m-v set of $M(C_n)$, where $N \subseteq V(C_n)$ and $N' \subseteq V(C'_n)$. Then we are going to modify $N \cup N'$ such that the modified m-v set of $M(C_n)$ will satisfy the condition of the claim. To this end, we distinguish a few cases.

Assume first that $v_{k-1}, v_k, v_{k+1} \in N$. In this case, v_{k-2}, v'_{k-2}, v'_{k-1}, v'_k, v'_{k+1}, v_{k+2}, and v'_{k+2} do not lie in $N \cup N'$, just as shown in Fig. 2(a). Now we consider the set $(N \cup N' \cup \{v'_k\}) \setminus \{v_k\}$ and show that it is a m-v set of $M(C_n)$. If $v_{k+l} \in N$, for $l = 3$ or for any $l \geq 5$, then since v_k and v_{k+l} are visible, we get v'_k and v_{k+l} are visible. Similarly, if $v_{k-l} \in N$, for $l = 3$ or for any $l \geq 5$, we get that v'_k and v_{k-l} are visible. Now, if $n \geq 10$, then v'_k and v_{k+4} are visible since v_{k-1} and v_{k+4} are visible. Similarly, v'_k and v_{k-4} are visible because v_{k-1} and v_{k-4} are visible. It can be verified directly that for $n = 8$, the above possibility along with $v_{k-4} \in N$ or $v_{k+4} \in N$ implies $|N \cup N'| \leq 7$, which is a contradiction. Similarly, for $n = 9$, the above possibility along with $v_{k-4} \in N$ or $v_{k+4} \in N$ implies $|N \cup N'| \leq 10$, which is a contradiction.

Assume second that $v_{k-3}, v_{k-1}, v_k, v_{k+2} \in N$. In this case, v_{k-2}, v'_{k-2}, v'_{k-1}, v'_k, v_{k+1}, and v'_{k+1} do not belong to $N \cup N'$, see Fig. 3(a). We now consider $(N \cup N' \cup \{v'_k\}) \setminus \{v_k\}$ and assert that it is a m-v set of $M(C_n)$. The vertices v'_k and v_{k+2} are visible since $v_{k+1} \notin N$. If $v_{k+l} \in N$, for $l = 3$ or for any $l \geq 5$ then, since v_k and v_{k+l} are visible, we get that v'_k and v_{k+l} are visible. Similarly, if $v_{k-l} \in N$, for $l = 3$ or for any $l \geq 5$, we get that v'_k and v_{k-l} are visible. If $v_{k-4} \in N$, then either v'_{k-5} or v'_{k-3} not in N'. Hence, v'_k and v_{k-4} are visible. If $n \geq 10$, then v'_k and v_{k+4} are visible since v_{k-1} and v_{k+4} are visible. It can be directly verified that for $n = 9$, the above possibility along with $v_{k+4} \in N$ implies $|N \cup N'| \leq 10$.

Assume next that $v_{k-4}, v_{k-1}, v_k, v_{k+2} \in N$. In this case, v_{k-3}, v_{k-2}, v'_{k-2}, v'_k, v_{k+1}, v'_{k+1} do not lie in $N \cup N'$, cf. Fig. 3(b)). We now consider two subcases.

In the first subcase assume that $v'_{k-3} \notin N'$ or $v'_{k-5} \notin N'$. Then we assert that $(N \cup N' \cup \{v'_k\}) \setminus \{v_k\}$ is a m-v set of $M(C_n)$. The vertices v'_k and v_{k+2} are visible since $v_{k+1} \notin N$. If $v_{k+l} \in N$, for $l = 3$ or for any $l \geq 5$, then, since v_k and v_{k+l} are visible, we get v'_k and v_{k+l} are visible. Similarly, if $v_{k-l} \in N$, for $l = 3$ or for any $l \geq 5$, we get v'_k and v_{k-l} are visible. Also, v'_k and v_{k-4} are visible since $v'_{k-3} \notin N'$ or $v'_{k-5} \notin N'$. Now, for $n \geq 10$, the vertices v'_k and v_{k+4} are visible since v_{k-1} and v_{k+4} are visible. It can be easily verified that for $n = 9$, the above possibility along with $v_{k+4} \in N$ implies $|N \cup N'| \leq 10$

In the second subcase assume that $v'_{k-5}, v'_{k-3} \in N'$. Then $v'_{k-1} \notin N'$, since v_k and v_{k-4} are visible. We now assert that $(N \cup N' \cup \{v'_{k-1}\}) \setminus \{v_{k-1}\}$ is a m-v set of $M(C_n)$. If $v_{k+l} \in N$, for $l = 2$ or for any $l \geq 4$, then, since v_{k-1} and v_{k+l} are visible, we get that v'_{k-1} and v_{k+l} are visible. Similarly, if $v_{k-l} \in N$, for $l = 4$ or for any $l \geq 6$, we find that v'_{k-1} and v_{k-l} are visible. Also, v'_k and v_{k+3} are visible since $v'_{k+2} \notin N'$ or $v'_{k+4} \notin N'$. Similarly, v'_k and v_{k-5} are visible since $v'_{k-4} \notin N'$ or $v'_{k-6} \notin N'$. The claim is proved.

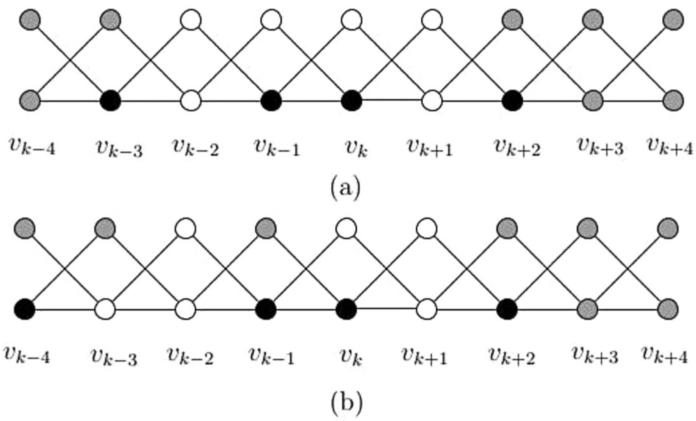

Fig. 3. Situations from the proof of Theorem 5.2. The black vertices again denote the vertices in the m-v set, the grey vertices are those whose status is not known and the white vertices are those which cannot be present in the m-v set.

We have thus proved that there exists a m-v set $T \cup T'$ of $M(C_n)$, where $T \subseteq V(C_n)$ and $T' \subseteq V(C_n')$, such that if $v_k, v_{k+1} \in T$, then $v_{k-2}, v_{k-1}, v_{k+2}, v_{k+3} \notin T$. We are now going to show that for this set we have $|T \cup T'| \le \lfloor \frac{n}{2} \rfloor + n - \lceil \frac{1}{2} \lfloor \frac{n}{2} \rfloor \rceil$. If $v_k, v_{k+1} \in T$, then v_{k-2}, v_{k-1}, v_{k+2}, v_{k+3} do not belong to T. Also, $v'_{k-1}, v'_{k+2} \notin T'$. Now, if $T' = V(C_n')$, then $T = \emptyset$, for $n \ge 8$. If $T' = V(C_n') \setminus \{v_2'\}$, then $T = \{v_1, v_3\}$. If $T' = V(C_n') \setminus \{v_2', v_6'\}$, then $T = \{v_1, v_3, v_5, v_7\}$. This process can be continued until the $\lceil \frac{1}{2} \lceil \frac{n}{2} \rceil \rceil$ vertices $v_2', v_6', v_{10}', \dots$ are excluded from $V(C_n')$ so that the $\lfloor \frac{n}{2} \rfloor$ vertices $v_1, v_3, v_5, \dots$ can be included in T. Hence $\mu(M(C_n)) \le \lfloor \frac{n}{2} \rfloor + n - \lceil \frac{1}{2} \lfloor \frac{n}{2} \rfloor \rceil = n + \lfloor \frac{n}{4} \rfloor$.

To complete the argument we construct a m-v set of $M(C_n)$ of cardinality $n + \lfloor \frac{n}{4} \rfloor$ as follows. Let $R \subseteq V(C_n)$ be an independent set of maximum cardinality of C_n and let R' be a smallest set of vertices from $V(C_n')$ which dominate all the vertices from R. Then $R \cup (V(C_n') \setminus R')$ is a m-v set of $M(C_n)$. Since $|R| = \lfloor \frac{n}{2} \rfloor$ and $|R'| = \lceil \frac{1}{2} \lfloor \frac{n}{2} \rfloor \rceil$ we have

$$|R \cup (V(C_n') \setminus R')| = \left\lfloor \frac{n}{2} \right\rfloor + n - \left\lceil \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rceil = n + \left\lfloor \frac{n}{4} \right\rfloor$$

which in turn implies that $\mu(M(C_n)) \ge n + \lfloor \frac{n}{4} \rfloor$. □

Proposition 5.3. *If G is a graph with $n(G) \ge 2$ and with a universal vertex, then $\mu(M(G)) = 2n(G) - 1$.*

Proof. Let v be a universal vertex of G. Then it is straightforward to verify that $(V(G) \setminus \{v\}) \cup V(G')$ is a m-v set of $M(G)$ which implies that $\mu(M(G)) \ge 2n(G) - 1$.

Let S be an arbitrary m-v set of $M(G)$. If $V(G) \subseteq S$, then $S \cap V(G') = \emptyset$, hence in this case $|S| \le n(G) + 1 \le 2n(G) - 1$. The same conclusion holds when $S \cap V(G) = \emptyset$. Assume in the rest that $1 \le |S \cap V(G)| \le n(G) - 1$. Then if $v^* \notin S$, we immediately get $|S| \le 2n(G) - 1$ and if $v^* \in S$, then $|S \cap V(G')| \le n(G) - 1$, and we obtain the same conclusion. In any case $\mu(M(G)) \ge 2n(G) - 1$. □

Theorem 5.4. *If G is not a complete graph and* $\mathrm{diam}(G) \le 3$*, then*

$$n(G) + \mu_o(G) \le \mu(M(G)) \le n(G) + \mu(G) + 1\,.$$

Moreover, if $\mu(M(G)) = n(G) + \mu(G) + 1$, then every μ-set of $M(G)$ contains v^*.

Proof. Let M be a μ_o-set of G. Then, having in mind that $\mathrm{diam}(G) \le 3$, it is straightforward to verify that $M \cup V(G')$ is a m-v set of $M(G)$ and therefore $\mu(M(G)) \ge n + \mu_o(G)$.

Let S be a μ-set of $M(G)$. Assume first that $S \cap V(G)$ is a m-v set of G. Then $|S \cap V(G)| \le \mu(G)$ and hence, $|S| \le n + \mu(G) + 1$. Assume second that $S \cap V(G)$ is not a m-v set of G. Then there exists $u, v \in S \cap V(G)$ such that u and v are not mutually visible in G but are mutually visible in $M(G)$. Denoting by $I_G[u, v]$ the set of all vertices that lie on shortest u, v-paths in G, we then have $x \in I_G[u, v]$ such that $x \in S$ but $x' \notin S$. If $(S \setminus \{x\}) \cap V(G)$ is a m-v set of G then $|S \cap V(G)| \le \mu(G) + 1$. Hence, $|S| \le n + \mu(G) + 1$. If $(S \setminus \{x\}) \cap V(G)$ is not a m-v set of G then proceed as above, that is, at each step we detect a vertex $y' \in V(G') \setminus S$ corresponding to a vertex $y \in S$. Therefore, $\mu(M(G)) \le n + \mu(G) + 1$.

Concerning the second part of the statement, assume now that $\mu(M(G)) = n(G) + \mu(G) + 1$ and suppose by way of contradiction that S is a μ-set of $M(G)$ such that $v^* \notin S$. Then $|S \cap V(G)| = \mu(G) + k$, for some $k \ge 1$. It follows that $|S \cap V(G')| = n(G) - k + 1$. Using a parallel argument as above, we are now going to show that $|S \cap V(G')| \le n(G) - k$, which becomes a contradiction. Since $|S \cap V(G)|$ is not a m-v set of G, there exists $u, v \in S \cap V(G)$ such that u and v are not mutually visible in G but are mutually visible in $M(G)$. Then there exists $x \in I_G[u, v]$ such that $x \in S$ but $x' \notin S$. If $k \ne 1$ then $(S \cap V(G)) \setminus \{x\}$ is still not a m-v set of

G and hence the above process can be repeated. Thus there exist distinct vertices $x'_1, \ldots, x'_k$ that are not in $S \cap V(G')$ and hence $|S \cap V(G')| \le n(G) - k$. □

We now give some examples how Theorem 5.4 can be applied. First, from the theorem we read that $\mu(M(P_4)) \in \{6, 7\}$ and that, moreover, if $\mu(M(P_4)) = 7$, then every μ-set of $M(P_4)$ contains v^*. But if a m-v set S of $M(P_4)$ contains v^*, then we infer that $|S| \le 5$. Indeed, note first that S contains at most two vertices from $V(P'_4)$, and if so, these two vertices must be either v'_1, v'_3 or v'_2, v'_4. From here it readily follows that $|S| \le 5$. We can conclude that $\mu(M(P_4)) = 6$.

Let $r_1 \ge r_2 \ge 3$ and set $n = r_1 + r_2$. Using Theorem 5.4 we get that $\mu(M(K_{r_1,r_2})) \in \{2n-2, 2n-1\}$ and that if $\mu(M(K_{r_1,r_2})) = 2n-1$, then every μ-set of $M(K_{r_1,r_2})$ contains v^*. Let S be an arbitrary m-v set of $M(K_{r_1,r_2})$ with $v^* \in S$, and let $uv \in E(K_{r_1,r_2})$. Then at most one among u' and v' can be in S, hence $|S| \le n + r_1 - 1$. We conclude that $\mu(M(K_{r_1,r_2})) = 2n - 2$.

In Theorem 5.2 we have determined $\mu(M(C_n))$ for $n \ge 8$. We now do the same for shorter cycles. By Theorem 5.4, $\mu(M(C_n)) \ge n+2$, for $4 \le n \le 7$. We claim that here equality always holds. Let S be a μ-set of $M(C_n)$. If $v^* \in S$ then at most two vertices from $V(C'_n)$, say v'_k and v'_{k+2}, are in S. If $v'_k, v'_{k+2} \in S$ then at least one vertex, say v_{k+1}, is not in S. Thus in this case, $|S| \le n+2$. Now, suppose $v^* \notin S$. Then as in the proof of Theorem 5.2, there exists a μ-set S of $M(C_n)$ such that if $v_k, v_{k+1} \in S$, then $v_{k-2}, v_{k-1}, v_{k+2}, v_{k+3} \notin S$. If $V(C'_n) \nsubseteq S$, then $|S| \le n + \lfloor \frac{n}{4} \rfloor = n$. If $V(C'_n) \subseteq S$, then $S \cap V(C_n)$ forms an outer m-v set of C_n. Hence $|S| \le n+2$ and consequently $\mu(M(C_n)) = n+2$, for $4 \le n \le 7$.

6. Concluding remarks

In Theorem 3.2 we have bounded the mutual-visibility number of an arbitrary, non-complete graph G with $\mu(D(G)) \ge n(G) + \mu_t(G)$ and demonstrated the bound to be sharp. Since $\mu(D(G))$ can be arbitrary larger than $\ge n(G) + \mu_t(G)$, we pose the problem to find some other lower bound (in term of some other invariants), a function of which can also be used as an upper bound.

In Theorem 3.3, we have determined $\mu(D(C_n))$. To determine $\mu(D(G))$ for an arbitrary graph G seems out of reach, hence computing the mutual-visibility number of the double graph of some other standard graphs would be of interest.

In Theorem 4.2, we have bounded $\mathrm{gp}(D(G))$ from below and from above, and characterized the graphs that achieve the upper bound. It remains open to characterize the graphs that achieve the lower bound of the theorem.

In Theorem 5.4, the mutual-visibility number of the Mycielskian of graphs with diameter at most 3 is bounded. To derive such bounds for general graphs remains a challenging problem. Moreover, determining $\mu(M(G))$ for other graphs G than paths and cycles is also something to be investigated in the future.

Acknowledgements

Dhanya Roy thanks Cochin University of Science and Technology for providing financial support under University JRF Scheme. Sandi Klavžar was supported by the Slovenian Research Agency ARIS (research core funding P1-0297 and projects N1-0218, N1-0285).

Data availability

No data was used for the research described in the article.

References

[1] B.S. Anand, U. Chandran S.V., M. Changat, S. Klavžar, E.J. Thomas, Characterization of general position sets and its applications to cographs and bipartite graphs, Appl. Math. Comput. 359 (2019) 84–89.
[2] E.-Z. Bidine, T. Gadi, K. Taoufiq, M. Kchikech, The exponential growth of the packing chromatic number of iterated Mycielskians, Discrete Appl. Math. 341 (2023) 232–241.
[3] D. Boutin, S. Cockburn, L. Keough, S. Loeb, K.E. Perry, P. Rombach, Determining number and cost of generalized Mycielskian graphs, Discuss. Math., Graph Theory 44 (2024) 127–149.
[4] U. Chandran S.V., G.J. Parthasarathy, The geodesic irredundant sets in graphs, Int. J. Math. Combin. 4 (2016) 135–143.
[5] S. Cicerone, G. Di Stefano, L. Drožđek, J. Hedžet, S. Klavžar, I.G. Yero, Variety of mutual-visibility problems in graphs, Theor. Comput. Sci. 974 (2023) 114096.
[6] S. Cicerone, G. Di Stefano, S. Klavžar, On the mutual-visibility in Cartesian products and in triangle-free graphs, Appl. Math. Comput. 438 (2023) 127619.
[7] S. Cicerone, G. Di Stefano, S. Klavžar, I.G. Yero, Mutual-visibility in strong products of graphs via total mutual-visibility, Discrete Appl. Math. 358 (2024) 136–146.
[8] S. Cicerone, G. Di Stefano, S. Klavžar, I.G. Yero, Mutual-visibility problems on graphs of diameter two, Eur. J. Comb. 120 (2024) 103995.
[9] G. Di Stefano, Mutual visibility in graphs, Appl. Math. Comput. 419 (2022) 126850.
[10] H.E. Dudeney, Amusements in Mathematics, Nelson, Edinburgh, 1917.
[11] V. Froese, I. Kanj, A. Nichterlein, R. Niedermeier, Finding points in general position, Int. J. Comput. Geom. Appl. 27 (2017) 277–296.
[12] M. Ghorbani, H.R. Maimani, M. Momeni, F.R. Mahid, S. Klavžar, G. Rus, The general position problem on Kneser graphs and on some graph operations, Discuss. Math., Graph Theory 41 (2021) 1199–1213.
[13] T. James, A. Vijayakumar, Mycielskian of graphs with small game domination number, Art Discrete Appl. Math. 6 (2023) 2.07.
[14] D.A. Kalarkop, R. Rangarajan, A counterexample on the conjecture and bounds on χ_{gd}-number of Mycielskian of a graph, Commun. Comb. Optim. 9 (2024) 197–204.
[15] S. Klavžar, B. Patkós, G. Rus, I.G. Yero, On general position sets in Cartesian products, Results Math. 76 (2021) 123.
[16] S. Klavžar, G. Rus, The general position number of integer lattices, Appl. Math. Comput. 390 (2021) 125664.
[17] S. Klavžar, I.G. Yero, The general position problem and strong resolving graphs, Open Math. 17 (2019) 1126–1135.
[18] J. Körner, On the extremal combinatorics of the Hamming space, J. Comb. Theory, Ser. A 71 (1995) 112–126.

[19] D. Korže, A. Vesel, Mutual-visibility sets in Cartesian products of paths and cycles, Results Math. 79 (2024) 116.
[20] S. Kozerenko, A. Serdiuk, New results on imbalance graphic graphs, Opusc. Math. 43 (2023) 81–100.
[21] D. Kuziak, J.A. Rodríguez-Velázquez, Total mutual-visibility in graphs with emphasis on lexicographic and Cartesian products, Bull. Malays. Math. Soc. 46 (2023) 197.
[22] Z. Lou, N. Obata, Q. Huang, Quadratic embedding constants of graph joins, Graphs Comb. 38 (2022) 161.
[23] P. Manuel, S. Klavžar, A general position problem in graph theory, Bull. Aust. Math. Soc. 98 (2018) 177–187.
[24] E. Munarini, C.P. Cippo, A. Scagliola, N. Zagaglia Salvi, Double graphs, Discrete Math. 308 (2008) 242–254.
[25] J. Mycielski, Sur le coloriage des graphs, Colloq. Math. 3 (1955) 161–162.
[26] E.J. Thomas, U. Chandran S.V., On independent position sets in graphs, Proyecciones 40 (2021) 385–398.
[27] E.J. Thomas, U. Chandran S.V., J. Tuite, G. Di Stefano, On the general position number of Mycielskian graphs, Discrete Appl. Math. 353 (2024) 29–43.
[28] J. Tian, S. Klavžar, Graphs with total mutual-visibility number zero and total mutual-visibility in Cartesian products, Discuss. Math., Graph Theory 44 (2024) 1277–1291.
[29] J. Tian, K. Xu, The general position number of Cartesian products involving a factor with small diameter, Appl. Math. Comput. 403 (2021) 126206.
[30] J. Tian, K. Xu, S. Klavžar, The general position number of the Cartesian product of two trees, Bull. Aust. Math. Soc. 104 (2021) 1–10.
[31] J. Tian, K. Xu, D. Chao, On the general position numbers of maximal outerplane graphs, Bull. Malays. Math. Sci. Soc. 46 (2023) 198.