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Recent Developments on the Structure of Cartesian Products of Graphs

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Abstract. The Cartesian product of graphs was introduced more than 50 years ago and many fundamental results were obtained since then. Nevertheless, in the last years several basic problems on the Cartesian product were solved and interesting theorems proved on topics of contemporary interest in graph theory. Here we survey recent developments on the structure of the Cartesian product with emphasis on the connectivity, edge-connectivity and cancellation properties. Recognition algorithms, subgraph structure and the distinguishing number of Cartesian products are also mentioned.

Keywords. Cartesian product of graphs, connectivity, edge-connectivity, cancellation, recognition algorithm, distinguishing number.

1. Introduction

The Cartesian product of graphs is a straightforward operation, yet it is of utmost importance in graph theory and is definitely the most prominent graph product. Many important classes of graphs are Cartesian products (say hypercubes, Hamming graphs, grids, prisms), but it is even more important that Cartesian products are hosts for other graphs, say via isometric (that is, distance preserving) embeddings of graphs into products. For instance, graphs that allow such embeddings into hypercubes are known as partial cubes, one of the central classes of graphs in metric graph theory. Moreover, the Cartesian product is closely related to the most important other products of graphs and helps to understand them better. In particular, all known recognition algorithms for the strong and the direct product depend on that for the Cartesian one.

This graph product has been studied continuously since the 1950's and many structural theorems were proved along the way. From the 1960's we point out the unique prime factorization theorem [25,28]; in the (early) 1970's the structure of the automorphism group of the Cartesian product was clarified [10,23]; from the 1980's the canonical metric representation of a connected graph presents one of the highlights [9]; and from the 1990's we emphasize fixed box theorems [5,27]. We refer to the book [12] for more in depth information on the Cartesian product of graphs as well as to the forthcoming book [15] for a treatment of several recent results on Cartesian product graphs.

Despite all the research, many fundamental questions about Cartesian product of graphs remain unanswered or partially answered. For instance, how is the connectivity and the edge-connectivity of a Cartesian product graph dependent on invariants of its factors? Only in the last couple of years these two questions got their final answers. They are presented in Section 2..

From the algebraic point of view one of the natural questions about a graph product is whether the product obeys the cancellation law and if the r^{th} root is well-defined. The answers for connected Cartesian products follow directly from the unique prime factorization theorem, but what about disconnected graphs? This was answered in the last year and is explained in Section 3..

Many additional new, deep, and fundamental results on the structure of the Cartesian products were obtained recently. We briefly mention them in the last section. The first such result is a linear recognition algorithm for Cartesian products. Then subgraphs of Cartesian products are characterized. Finally, good understanding of the structure of the automorphism group of Cartesian products enabled to determine the distinguishing number for all powers of graphs with respect to the Cartesian product.

In the rest of this section we formally introduce the Cartesian product of graphs.

For given graphs G and H, the *Cartesian product* $G \,\square\, H$ has $V(G) \times V(H)$ for the vertex set; the edge set consists of pairs $(g,h)(g',h')$ where either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. The graphs G and H are called the *factors* of $G \,\square\, H$. The subgraph of $G \,\square\, H$ induced by $\{g\} \times V(H)$, where g is a vertex of G, is isomorphic to H and is called an *H-fiber*. Similarly one defines the *G-fibers*.

Since the Cartesian product is an associative operation, the product of several factors is well-defined. If the factors are the same we speak of powers with respect to the Cartesian product. More precisely, for a positive integer n, the n^{th} *Cartesian power* of a graph G is $G^n = \square_{i=1}^{n} G$.

2. Connectivity

Clearly, $G \,\square\, H$ is connected if and only if the factors G and H are connected. But for a given (product) graph we also wish to know its vertex and edge connectivity. Hence it is natural to wonder if these connectivities of $G \,\square\, H$ can be expressed in terms of the corresponding invariants of the factors. Although this is a natural and basic question, it was answered only recently. The reason is that despite the simple definition of the Cartesian product the connectivity problem is far away from being trivial.

We start with the vertex connectivity $\kappa(G \,\square\, H)$. The first result on it goes back to 1957 when Sabidussi [24] noticed that for arbitrary graphs G and H, $\kappa(G \,\square\, H) \geq \kappa(G) + \kappa(H)$. Then, in 1978, Liouville [22] announced that for any graphs G and H on at least two vertices,

$$\kappa(G \,\square\, H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\}, \tag{1}$$

where δ denotes the minimum degree of a given graph.

Note that in (1) the "$\leq$" easily follows from the fiber structure of the Cartesian product and the fact that the connectivity cannot be larger that the minimum degree.

(Of course, $\delta(G \,\square\, H) = \delta(G) + \delta(H)$.) But years have passed by and the announced proof of (1) never appeared. Finally, Špacapan gave a proof of it, so now the assertion (1) can be formulated as a theorem.

Theorem 2.1 ([26]). *For any graphs G and H on at least two vertices,*

$$\kappa(G \,\square\, H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\}\,.$$

It is natural to expect that in "most" cases the minimum in Theorem 2.1 will be attained by the minimum degree. In fact, this holds for all powers of an arbitrary nontrivial graph.

Theorem 2.2 ([19]). *Let G be a connected graph on at least two vertices. Then for any $n \geq 2$, $\kappa(G^n) = n\,\delta(G)$.*

The edge-connectivity of the Cartesian product was determined in 2006 by Xu and Yang:

Theorem 2.3 ([30]). *For any graphs G and H on at least two vertices,*

$$\kappa'(G \,\square\, H) = \min\{\kappa'(G)|V(H)|, \kappa'(H)|V(G)|, \delta(G) + \delta(H)\}\,.$$

Just as with Theorem 2.1, the "$\leq$" part of the proof is straightforward. To prove the other inequality Xu and Yang applied the edge version of Menger's theorem and by a careful analysis found enough edge-disjoint paths for any pair of vertices. Later, in [19] a short proof (independent of Menger's theorem) of Theorem 2.3 was given. Moreover, the structure of minimum edge cuts was clarified.

Theorem 2.4 ([19]). *Let S be a minimum edge cut of $G \,\square\, H$. Then either S is induced by a minimum edge cut of a factor, or S is the set of edges incident to a vertex of $G \,\square\, H$.*

Just as for the vertex connectivity, for the edge connectivity of powers of graphs the minimum degree always does the job.

Theorem 2.5 ([19]). *Let G be a connected graph on at least two vertices. Then for any $n \geq 2$, $\kappa'(G^n) = n\,\delta(G)$.*

3. Prime factorization and cancellation

We have already mentioned the following fundamental result on the Cartesian product.

Theorem 3.1 ([25,28]). *Every connected graph has a unique prime factor decomposition with respect to the Cartesian product.*

Theorem 3.1 cannot be extended to disconnected graphs which is demonstrated with the next example

$$(K_1 + K_2 + K_2^2) \,\square\, (K_1 + K_2^3) = (K_1 + K_2^2 + K_2^4) \,\square\, (K_1 + K_2)\,,$$

where the operation $+$ stands for the disjoint union of graphs.

Theorem 3.1 easily implies that for any connected graphs G and H and any positive integer r,

$$G^r = H^r \Rightarrow G = H\,, \tag{2}$$

and that for any connected graphs G, H and K,

$$G \,\Box\, K = H \,\Box\, K \Rightarrow G = H\,. \tag{3}$$

It is now natural to ask what can be said about the roof property (2) and the cancellation (3) property for arbitrary graphs. Contrary to the non-unique factorization of disconnected graphs, Fernández, Leighton and López-Presa in a recent paper [7] proved:

Theorem 3.2 ([7]). *Let G and H be arbitrary graphs and $r \geq 1$. Then $G^r = H^r$ implies $G = H$.*

The key idea of the proof of Theorem 3.2 is that finite graphs form a commutative semiring with respect to Cartesian multiplication and disjoint union. In this way finite graphs can be embedded into a polynomial ring with integer coefficients that is compatible with Cartesian multiplication and disjoint union. Then a given graph is uniquely embedded into the polynomial ring which in turn possesses the unique factorization property that is then lifted back to graphs.

Using the same algebraic approach one can also prove:

Theorem 3.3 ([14]). *Let G, H, K be arbitrary graphs such that $G \,\Box\, K = H \,\Box\, K$. Then $G = H$.*

4. Other recent results

4.1 *Recognizing Cartesian product graphs*

Another fundamental question on the Cartesian product is the following: given a connected graph G, is it a Cartesian product graph, and if so, find its unique prime factorization.

This question has a long history, the original question was whether the problem is polynomial or not. The problem was solved independently by Feigenbaum, Hershberger and Schäffer [6] and Winkler [29] by providing recognition algorithms of complexity $(n^{4.5})$ and (n^4), respectively. Here n stands for the number of vertices of a given graph, while m will denote the number of edges.

In 1992 Feder [4] followed with a new idea (to consider certain relations Θ and τ) and reduced the recognition complexity to $O(mn)$. At about the same time, Aurenhammer, Hagauer and Imrich [3] carefully considered the structure of Cartesian product graphs and further reduced the complexity to $O(m \log n)$. Finally, the story was rounded in 2007 when Imrich and Peterin [16] succeeded to recognize Cartesian products in linear time:

Theorem 4.1 ([16]). *Let G be a connected graph with m edges. Then its unique prime factor decomposition can be determined in $O(m)$ time.*

4.2 *Subgraphs of Cartesian products*

Given a graph G, we can ask the following question: Is G a subgraph of some Cartesian product graph? Of course, G is a subgraph of $G \,\square\, H$, so the question only makes sense if we require that G is a nontrivial subgraph. More precisely, we say that a subgraph of $G \square H$ is *nontrivial* if it intersects at least two G-fibers and at least two H-fibers.

A plotting of a graph is its drawing in the plane such that the endvertices of any edge have the same abscissa or the same ordinate. Now, calling a plotting to be trivial if all the vertices either have the same abscissa or the same ordinate, it is straightforward to observe that a graph is a nontrivial subgraph of a Cartesian product if and only if it has a nontrivial plotting [21].

In fact, a graph is a nontrivial subgraph of the Cartesian product of graphs if and only if it is a nontrivial subgraph of the Cartesian product of two complete graphs. But more can be said. Calling a labeled path, in which any two consecutive vertices receive different labels, a *properly colored path*, the following theorem was proved.

Theorem 4.2 ([17,18]). *Let X be a connected graph on at least three vertices. Then the following statements are equivalent.*

(i) *X is a nontrivial Cartesian subgraph.*
(ii) *$E(X)$ can be labeled with two labels such that on any induced cycle C of X that possesses both labels, the labels change at least three times while traversing the edges of C.*
(iii) *$V(X)$ can be labeled with k labels, $2 \le k \le |X| - 1$, such that the endvertices of any properly colored path receive different labels.*

We add that in [18] induced and isometric subgraphs of products of complete graphs are also characterized.

4.3 *Distinguishing Cartesian products*

The distinguishing number $D(G)$ of a graph G is a relatively recently introduced, yet already well-studied, graph invariant [2]. It is defined as the least integer d such that G has a labeling with d labels that is preserved only by a trivial automorphism.

Albertson [1] proved that for a connected prime graph G, $D(G^r) = 2$ for all $r \ge 4$, and, if $|V(G)| \ge 5$, then $D(G^r) = 2$ for all $r \ge 3$. In [20] the result was extended by showing that eventually $D(G^r) = 2$ for any connected graph $G \ne K_2$ and any $r \ge 3$. The story was completed by the following result.

Theorem 4.3 ([13]). $D(G^k) = 2$ *for* $k \ge 2$ *for all nontrivial, connected graphs* $G \ne K_2, K_3$. *Furthermore,* $D(K_n^k) = 2$ *for* $n = 2, 3$, *if* $n + k \ge 6$ *and* $D(K_2^2) = D(K_2^3) = D(K_3^2) = 3$.

The distinguishing number of the Cartesian product of two complete graphs was also determined independently by two groups of authors.

Theorem 4.4 ([8,11]). *Let k, n, d be integers so that $d \geq 2$ and $(d-1)^k < n \leq d^k$. Then*

$$D(K_k \,\square\, K_n) = \begin{cases} d, & \text{if } n \leq d^k - \lceil \log_d k \rceil - 1; \\ d+1, & \text{if } n \geq d^k - \lceil \log_d k \rceil + 1. \end{cases}$$

If $n = d^k - \lceil \log_d k \rceil$ then $D(K_k \,\square\, K_n)$ is either d or $d+1$ and can be computed recursively in $O(\log^*(n))$ time.

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