Packing chromatic number, $(1, 1, 2, 2)$-colorings, and characterizing the Petersen graph

Boštjan Brešar [a,b] Sandi Klavžar [a,b,c]
Douglas F. Rall [d] Kirsti Wash [e]

[a] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[c] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[d] Department of Mathematics, Furman University, Greenville, SC, USA
[e] Department of Mathematics, Trinity College, Hartford, CT, USA

Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set of G can be partitioned into sets $\Pi_1, \ldots, \Pi_k$, where Π_i, $i \in [k]$, is an i-packing. The following conjecture is posed and studied: if G is a subcubic graph, then $\chi_\rho(S(G)) \leq 5$, where $S(G)$ is the subdivision of G. The conjecture is proved for all generalized prisms of cycles. To get this result it is proved that if G is a generalized prism of a cycle, then G is $(1, 1, 2, 2)$-colorable if and only if G is not the Petersen graph. The validity of the conjecture is further proved for graphs that can be obtained from generalized prisms in such a way that one of the two n-cycles in the edge set of a generalized prism is replaced by a union of cycles among which at most one is a 5-cycle. The packing chromatic number of graphs obtained by subdividing each of its edges a fixed number of times is also considered.

Key words: packing chromatic number; cubic graph; subdivision; S-coloring; generalized prism; Petersen graph.

AMS Subj. Class: 05C70, 05C15, 05C12

1 Introduction

Given a graph G and a positive integer i, an *i-packing* in G is a subset W of the vertex set of G such that the distance between any two distinct vertices from W is greater than i. This generalizes the notion of an independent set, which is equivalent to a

1-packing. Now, the *packing chromatic number* of G is the smallest integer k such that the vertex set of G can be partitioned into sets $\Pi_1, \dots, \Pi_k$, where Π_i is an i-packing for each $i \in [k]$. This invariant is well defined on any graph G and is denoted $\chi_\rho(G)$. It was introduced in [13] under the name broadcast chromatic number, and subsequently studied under the current name, see [2–9, 17, 18, 20, 21].

One of the intriguing problems related to the packing chromatic number is whether it is bounded by a constant in the class of all cubic graphs. In particular, it was asked already in the seminal paper [13] what is the maximum packing chromatic number in the class of cubic graphs of a given order. Gastineau and Togni found a cubic graph G with $\chi_\rho(G) = 13$ and asked whether 13 is an upper bound for χ_ρ in the class of cubic graphs [12], which we answered recently in the negative [5]. More specifically, it was asked in [8] whether the invariant is bounded in the class of planar cubic graphs. A question of similar nature from [12] asks whether the subdivision $S(G)$ of any subcubic graph G (i.e., a graph with maximum degree 3) has packing chromatic number no more than 5. This question is the main motivation for the present paper. We suspect that the answer is positive, and pose it as the following conjecture.

Conjecture 1.1 *If G is a subcubic graph, then $\chi_\rho(S(G)) \leq 5$.*

The packing chromatic number of subdivided graphs has been studied in several papers. Using subdivided graphs the class of graphs with packing chromatic number equal to 3 was characterized in [13]. The effect on the invariant of the subdivision of an edge of a graph was analyzed in [5]. It was observed in [3] that $\chi_\rho(S(G)) \leq \chi_\rho(G) + 1$ for any graph G, and further proved that $\chi_\rho(S(K_n)) = n+1$. Consequently the packing chromatic number of subdivided graphs is generally not bounded, hence the restriction to subcubic graphs in Conjecture 1.1 is natural.

The paper is organized as follows. In the next section we introduce notation needed, list several facts related to Conjecture 1.1, and prove a connection between the packing chromatic number (of subdivided graphs) and the so-called $(1, 1, 2, 2)$-colorings. This connection is then used as our main tool while attacking the conjecture. Then, in Section 3, we prove that Conjecture 1.1 holds true for all generalized prisms of cycles. Along the way a characterization of the Petersen graph is obtained. (We refer to [14] for a recent characterization of the Petersen graph and to [22] for older characterizations.) Moreover, it is shown that any optimal packing coloring of the subdivided Petersen graph looks differently than one would expect. In Section 4 we then extend the main result of the previous section to the graphs obtained from generalized prisms in such a way that one of the two n-cycles in the edge set of a generalized prism is replaced by a union of cycles among which at most one is a 5-cycle. In the final section we consider the packing chromatic number of graphs obtained by subdividing each of its edges a fixed number of times.

2 Notation and preliminary results

All graphs considered in this paper are simple and connected, unless stated otherwise.

Let G be a graph and $S(G)$ its *subdivision*, that is, the graph obtained from G by replacing each edge with a disjoint path of length 2. In other words, $S(G)$ is obtained from G by subdividing each edge e of G with a new vertex to be denoted by v_e. The resulting vertex set $V(S(G))$ can thus be considered as $V(G) \cup \{v_e \mid e \in E(G)\}$. More generally, if $i \geq 1$, we define the graph $S_i(G)$ as the graph obtained from G by subdividing each of its edges precisely i times. In other words, $S_i(G)$ is obtained from G by replacing each edge with a disjoint path of length $i+1$. Note that $S_1(G) = S(G)$.

Observe that if H is a subgraph of G, then $\chi_\rho(H) \leq \chi_\rho(G)$. Indeed, this follows because $d_H(u,v) \geq d_G(u,v)$ holds for any vertices $u, v \in V(H)$. Consequently, a packing coloring of G restricted to H is a packing coloring of H. Since every subcubic graph is a subgraph of a cubic graph (easy exercise), it suffices to prove Conjecture 1.1 for cubic graphs. In addition, the following fact is a consequence of the characterization of the graphs of packing chromatic number 3 from [13].

Proposition 2.1 ([13]) *If G is a (connected) bipartite graph of order at least 3, then $\chi_\rho(S(G)) = 3$.*

Hence we can restrict our attention to cubic non-bipartite graphs. Since $\chi_\rho(S(K_4)) = 5$ (see [3]), Conjecture 1.1 reduces to 3-chromatic cubic graphs. Before we continue, we demonstrate that the conjecture does not hold for all 3-chromatic graphs.

Proposition 2.2 *If $K_{n,n,n}$ is the complete tripartite graph with all parts of order n, then $\chi_\rho(S(K_{n,n,n})) \xrightarrow[n\to\infty]{} \infty$.*

Proof. Let G_n denote $S(K_{n,n,n})$. Since $\mathrm{diam}(G_n) = 4$ for $n \geq 2$, we infer that in any packing coloring c of G_n every color bigger than 3 appears at most once. Let A, B and C be the tripartition of $V(K_{n,n,n})$. Suppose there is a vertex x from $A \cup B \cup C$ with $c(x) = 1$. Since $N[x]$ induces $K_{1,2n}$ with x as its center, in this case c uses at least $2n$ colors. Otherwise, we may assume without loss of generality that in $A \cup B$ there are vertices y and z with $c(y) = 2$, $c(z) = 3$. Clearly, then no vertex from C can receive colors 2 or 3, which in turn implies that c uses n different colors on C. □

Note that χ_ρ can be defined also in terms of a function on the vertex set of a graph G. Indeed, we say that a function $c : V(G) \to [k]$ is a *k-packing coloring* of G if for each i from the range of c, the set $c^{-1}(i)$ is an i-packing in G; we then also say that G is *k-packing colorable*. In this way, $\chi_\rho(G)$ is the smallest integer k such that there exists a k-packing coloring of G.

One approach to attack Conjecture 1.1 is by using the concept of an S-coloring, which generalizes that of a packing coloring. This concept was first briefly mentioned in [13] and later formally introduced in [15] as follows. Given a graph G and a non-decreasing sequence $S = (s_1, \dots, s_k)$ of positive integers, an *S-coloring* of G is a partition of the vertex set of G into k subsets $\Pi_1, \dots, \Pi_k$, where Π_i is an s_i-packing for each $i \in [k]$. We say that G is *S-colorable* if it has an S-coloring. Clearly, $\chi_\rho(G) \le k$ if and only if G is S-colorable for $S = (1, 2, \dots, k)$. For further results on the S-packing coloring see [10, 11, 16].

The following result shows in what way $(1,1,2,2)$-colorable graphs are related to Conjecture 1.1.

Proposition 2.3 *If G is $(1,1,2,2)$-colorable, then $\chi_\rho(S(G)) \le 5$.*

Proof. By [12, Proposition 1], every $(1,1,2,2)$-colorable graph G yields a $(1,3,3,5,5)$-colorable $S(G)$, which in turn implies that $S(G)$ is $(1,2,3,4,5)$-colorable, that is, $S(G)$ is 5-packing colorable. □

We next state a result that will be the main tool in our subsequent proofs. For its statement recall that the *square* G^2 of a graph G is the graph having the same vertex set as G and two vertices are adjacent in G^2 precisely when their distance in G is at most 2.

Lemma 2.4 *A graph G is $(1,1,2,2)$-colorable if and only if there is a partition $\{V_1, V_2, V_3\}$ of $V(G)$ such that V_2 and V_3 are independent sets and V_1 induces a bipartite graph in G^2.*

Proof. Suppose that $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ as stated above. Let A and B represent the partite sets of the graph $G^2[V_1]$. Note that A is a 2-packing in G for otherwise A would not be an independent set in G^2. Similarly, B is a 2-packing. Construct a $(1,1,2,2)$-coloring of G by assigning all the vertices of V_2 color 1, all the vertices of V_3 color 2, all the vertices of A color 3 and all the vertices of B color 4. Thus, (V_2, V_3, A, B) is a $(1,1,2,2)$-coloring of G.

Conversely, suppose that we have a $(1,1,2,2)$-coloring of G with color classes W_1, W_2, W_3, W_4. Since W_3 and W_4 are 2-packings in G, W_3 and W_4 are independent sets in G^2. It follows that $W_3 \cup W_4$ induces a bipartite graph in G^2. Let $V_1 = W_3 \cup W_4$, $V_2 = W_2$, and $V_3 = W_1$. By definition, $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ as claimed in the statement of the lemma. □

3 Generalized prisms and the Petersen graph

In this section we confirm Conjecture 1.1 for all generalized prisms of cycles, where a *generalized prism* is a cubic graph obtained from the disjoint union of two cycles of equal length by adding a perfect matching between the vertices of the two cycles. Along the way we prove that a generalized prism of a cycle is $(1,1,2,2)$-colorable unless it is the Petersen graph, thus characterizing the Petersen graph P in a new way. By separately verifying that $\chi_\rho(S(P)) = 5$, Conjecture 1.1 for generalized prisms then follows from Proposition 2.3. We begin with the following technical lemma.

Lemma 3.1 *If $C_n = v_1 \cdots v_n$ is a cycle on n vertices, then the following hold.*

(i) There exists a set $A \subset V(C_n)$ such that at most one pair of adjacent vertices in C_n is in the complement of A and $G^2[A]$ is an even cycle or a path.

(ii) If n is odd and $i \in \{3, \ldots, n-1\}$, there exists a set $A \subset V(G)$ such that $\{v_1, v_i, v_j\} \cap A = \emptyset$ for some $j \in \{i-1, i+1\}$, v_iv_j is the only adjacent pair of vertices in C_n, which is in the complement of A, and $G^2[A]$ is a path.

Proof. The result is trivial if $3 \le n \le 5$ so we may assume that $n \ge 6$. To prove statement (i), we first assume n is even, and let

$$A_1 = \begin{cases} \{v_i \mid i \text{ is odd}\} & \text{if } n \equiv 0 \pmod 4 \\ \{v_1, v_2, v_4, v_5\} \cup \{v_j \mid j \text{ is odd and } j \ge 7\} & \text{if } n \equiv 2 \pmod 4 \end{cases}.$$

Note that $G^2[A_1]$ is an even cycle.

Suppose next that n is odd, $n \ge 7$. If we let

$$A_2 = \begin{cases} \{v_1, v_2, v_4, v_5, v_7, v_8\} \cup \{v_j \mid j \text{ is even and } j \ge 10\} & \text{if } n \equiv 1 \pmod 4, n \ge 9 \\ \{v_1, v_4\} \cup \{v_j \mid j \text{ is even and } j \ge 4\} & \text{if } n \equiv 3 \pmod 4 \end{cases},$$

then $G^2[A_2]$ is a path if $n \equiv 3 \pmod 4$ and $G^2[A_2]$ is an even cycle if $n \equiv 1 \pmod 4$. This concludes the proof of (i).

We next prove (ii) in which case n is odd. Let $i \in \{3, \ldots, n-1\}$. Suppose first that i is even. If $i \le n-3$, let

$$A_3 = \{v_2, v_{i+2}\} \cup \{v_j \mid 3 \le j \le i-1, j \text{ odd}\} \cup \{v_j \mid i+3 \le j \le n, j \text{ odd}\},$$

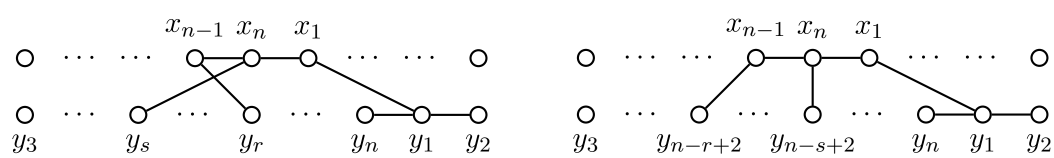
and if $i = n-1$, let

$$A_4 = \begin{cases} \{v_2, v_{n-3}, v_n\} \cup \{v_j \mid 3 \le j \le n-4, j \text{ odd}\} & \text{if } n \equiv 1 \pmod 4 \\ \{v_n\} \cup \{v_j \mid 2 \le j \le n-3, j \text{ even}\} & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

Finally, if i is odd, we let $A_5 = \{v_j \mid 2 \le j \le i-1, j \text{ even}\} \cup \{v_j \mid i+2 \le j \le n, j \text{ odd}\}$. In each case, $G^2[A_j]$ is a path for $j \in \{3, 4, 5\}$. □

Theorem 3.2 *If G is a generalized prism of a cycle, then G is $(1,1,2,2)$-colorable if and only if G is not the Petersen graph.*

Proof. Up to isomorphism there is only one generalized prism of the 3-cycle, and it is clearly $(1,1,2,2)$-colorable. So we may assume that C_n is a cycle on at least four vertices. By Lemma 2.4, it suffices to show that $V(G)$ can be partitioned into V_1, V_2, V_3, where V_2 and V_3 are independent sets and $G^2[V_1]$ is bipartite. In G, let $A = x_1 \cdots x_n$ and $B = y_1 \cdots y_n$ represent the two copies of C_n. By definition, there exists a perfect matching between A and B in G, and so we define $f : A \to B$ such that $f(x_i) = y_j$ if $x_i y_j \in E(G)$ for all $1 \leq i \leq n$. Without loss of generality we assume that $f(x_1) = y_1$. In addition, let $f(x_{n-1}) = y_r$ and $f(x_n) = y_s$ for some $\{r, s\} \subseteq \{2, \ldots, n\}$. We then draw A horizontally so that the indices increase from left to right and x_n is located in the middle of the cycle. Moreover, we can draw B horizontally and beneath A so that the indices increase from left to right and y_1 is drawn to the right of both y_r and y_s, as shown in Figure 1(a). If y_r is to the right of y_s, meaning $r > s$, then we can relabel the vertices of B so that $f(x_1)$ still has index 1, but the indices increase from right to left. Then we draw B so that the indices increase from left to right, as depicted in Figure 1(b), and y_r is to the left of both y_1 and y_s. So we may assume throughout the remainder of the proof that $r < s$.



(a) Original drawing of G (b) G after reindexing the vertices of B

Figure 1: Drawing of the generalized prism of a cycle

Suppose first that C_n is an even cycle. Let

- $X_2 = \{x_i \mid 1 \leq i \leq n,\ i \text{ is odd}\}$ and
- $X_3 = \{x_i \mid 1 \leq i \leq n,\ i \text{ is even}\}$.

Let Y_1 be the set A_1 from Lemma 3.1 and define

- $Y_2 = \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\}$ and
- $Y_3 = \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\}$.

One can easily verify that $G^2[Y_1]$ is an even cycle and $X_i \cup Y_i$ is independent for $i \in \{2, 3\}$. By Lemma 2.4 it follows that the generalized prism is $(1,1,2,2)$-colorable.

So we may assume that C_n is an odd cycle. Let $X_1 = \{x_n\}$,

- $X_2 = \{x_i \mid i \text{ is odd}, 1 \leq i \leq n-2\}$, and
- $X_3 = \{x_i \mid i \text{ is even}, 1 \leq i \leq n-1\}$.

In what follows, we partition the vertices of B into Y_1, Y_2, Y_3 depending on the parity of r and s. In each case, we let $V_i = X_i \cup Y_i$ for each $i \in [3]$ so that V_2 and V_3 are independent and $G^2[V_1]$ is bipartite.

Case 1. Suppose s is odd.

We shall assume first that r is odd as well. If $s \neq n$, let Y_1 be the set A_5 from Lemma 3.1 where $i = s$ so that $G^2[V_1]$ is a path that does not contain $\{y_1, y_s, y_{s+1}\}$. We then define

$$Y_2 = \begin{cases} \{y_{s+1}\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\} & \text{if } f^{-1}(y_{s+1}) \in X_3 \\ \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\} & \text{if } f^{-1}(y_{s+1}) \in X_2 \end{cases},$$

and

$$Y_3 = \begin{cases} \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\} & \text{if } f^{-1}(y_{s+1}) \in X_3 \\ \{y_{s+1}\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\} & \text{if } f^{-1}(y_{s+1}) \in X_2 \end{cases}.$$

If $s = n$ and r is odd or $r = n - 1$, we let $Y_1 = \{y_i \mid i \text{ is even}\}$,

- $Y_2 = \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\}$ and
- $Y_3 = \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\}$.

In either case, $G^2[V_1]$ is a path. So we may assume that r is even and $r \neq n - 1$ (note that the case $r = n - 1$ is symmetric to the case when $s = n$, which was considered above).

If $s < n$, we let

- $Y_1 = \{y_i \mid 3 \leq i \leq s - 2, i \text{ is odd}\} \cup \{y_2, y_{s+1}\} \cup \{y_i \mid s + 2 \leq i \leq n, i \text{ is odd}\}$.

As above, $G^2[V_1]$ is a path. If $f^{-1}(y_{s-1}) \in X_3$, then let

- $Y_2 = \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\}$ and
- $Y_3 = \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\}$.

Otherwise, let

- $Y_2 = \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\}$ and
- $Y_3 = \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\}$.

Case 2. Suppose that s is even.

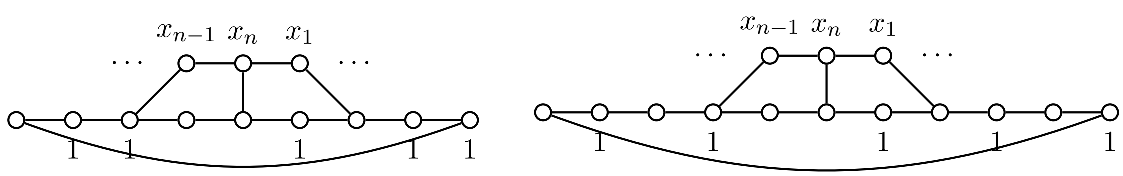
First, note that if r is odd, then we can define Y_1, Y_2, Y_3 similarly to those subcases given in Case 1 (which can be observed by reversing the roles of x_{n-1} and x_1). So we may assume that r is even. Suppose first that $n > 5$ and $r > 2$. Then one of the sets A_3 or A_4 given in Lemma 3.1 can be chosen for Y_1 so that $G^2[V_1]$ is a path and

Y_1 does not contain vertices $\{y_1, y_s, y_j\}$ where $j \in \{s-1, s+1\}$ (see Figure 2 for two corresponding examples). Thus, $G^2[V_1]$ is path. We then define

$$Y_2 = \begin{cases} \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\} & \text{if } s \le n-3, f^{-1}(y_{s+1}) \in X_3 \\ \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\} & \text{if } s \le n-3, f^{-1}(y_{s+1}) \in X_2 \\ \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\} & \text{if } s = n-1, f^{-1}(y_{s-1}) \in X_3 \\ \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_3\} & \text{if } s = n-1, f^{-1}(y_{s-1}) \in X_2 \end{cases},$$

and

$$Y_3 = \begin{cases} \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\} & \text{if } s \le n-3, f^{-1}(y_{s+1}) \in X_3 \\ \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\} & \text{if } s \le n-3, f^{-1}(y_{s+1}) \in X_2 \\ \{y_s\} \cup \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\} & \text{if } s = n-1, f^{-1}(y_{s-1}) \in X_3 \\ \{y_i \notin Y_1 \mid f^{-1}(y_i) \in X_2\} & \text{if } s = n-1, f^{-1}(y_{s-1}) \in X_2 \end{cases}.$$



(a) Y_1 when $n = 9$

(b) Y_1 when $n = 11$

Figure 2: The set Y_1 in the (1,1,2,2)-coloring of C_9 and C_{11}

Next, we suppose that $r = 2$ (while we may assume, by symmetry, that $s < n-1$), and let

$$Y_1 = \{y_{s+2}\} \cup \{y_i \mid s+3 \le i \le n, i \text{ odd}\} \cup \{y_i \mid 3 \le i \le s-1, i \text{ odd}\}.$$

Then choose Y_2 and Y_3 in the same way as above based on the index of s.

Finally, consider when C_n is a 5-cycle. If $f(x_2) = y_5$ and $f(x_3) = y_3$, then Figure 3 depicts a labeling of G where V_2 and V_3 are independent and $G^2[V_1]$ is bipartite. If $f(x_2) = y_3$ and $f(x_3) = y_5$, then G is the Petersen graph. The argument is complete by invoking the fact [12, Proposition 4] that the Petersen graph is not $(1,1,2,2)$-colorable. $\square$

By Theorem 3.2 and Proposition 2.3 we know that any subdivided generalized prism of a cycle but the subdivided Petersen graph P is 5-packing colorable. In addition, a 5-packing coloring of $S(P)$ is shown in Fig. 4. Hence we have the following result.

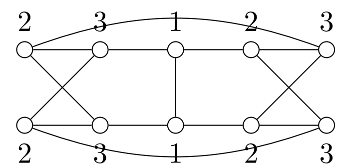


Figure 3: The labels depict a $(1,1,2,2)$-coloring

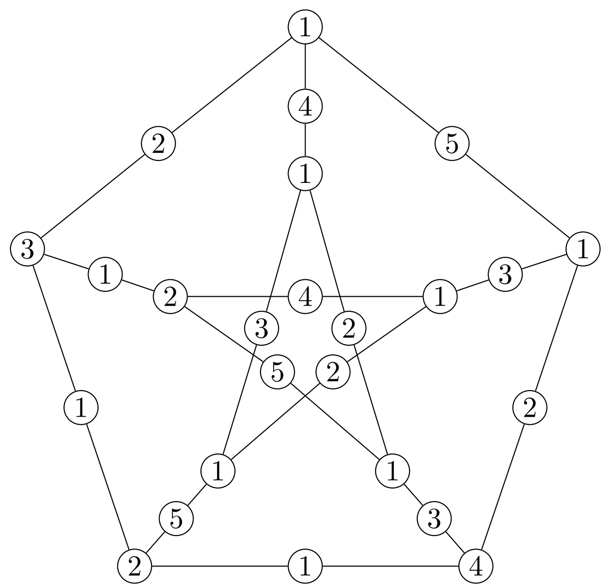


Figure 4: 5-packing coloring of $S(P)$

Corollary 3.3 *If G is a generalized prism of a cycle, then $\chi_\rho(S(G)) \leq 5$.*

Intuitively, it seems reasonable to expect that an optimal packing coloring of any subdivided graph colors all the subdivided vertices by 1. The example from Fig. 4 shows that an optimal coloring need not be like that. In fact, no optimal coloring of $S(P)$ colors all the subdivided vertices by 1. This is an immediate consequence of the following result.

Proposition 3.4 *If G is not $(1,1,2,2)$-colorable, and $S(G)$ is $(1,2,3,4,5)$-colorable, then in every 5-packing coloring of $S(G)$ at least one of the subdivided vertices of $S(G)$ receives color bigger than 1.*

Proof. Suppose to the contrary that c is a 5-packing coloring of $S(G)$ with $c(u_e) = 1$ for every edge $e \in E(G)$. Then all vertices of $V(G)$ in $S(G)$ receive colors from $\{2,3,4,5\}$.

Consider the coloring c' of $V(G)$, obtained as the restriction of c to G. Note that vertices colored by the color 2, respectively 3, form an independent set in G, while the set of vertices colored by the color 4, respectively 5, is a 2-packing of G. This implies that c' is a $(1,1,2,2)$-coloring of G. □

4 A class larger than generalized prisms

In this section we confirm Conjecture 1.1 for a class of graphs larger than generalized prisms of cycles by proving that if G is a connected, cubic graph of order $2n$ with a 2-factor $\mathcal{F}$ and a perfect matching M, where $\mathcal{F}$ contains a cycle C of length n, no edge of M has both vertices in C, and $\mathcal{F}$ contains at most one 5-cycle, then G is $(1,1,2,2)$-colorable. In other words, our result extends Theorem 3.2 to the graphs obtained from generalized prisms in such a way that one of the two n-cycles in the edge set of a generalized prism is replaced by a union of cycles among which at most one is a 5-cycle.

Theorem 4.1 *Let G be a connected, cubic graph of order $2n$ with a 2-factor $\mathcal{F}$ and a perfect matching M. If $\mathcal{F}$ contains a cycle C of length n where no edge of M has both vertices in C, and $\mathcal{F}$ contains at most one 5-cycle, then G is $(1,1,2,2)$-colorable.*

Proof. Note that by Theorem 3.2, we may assume that $\mathcal{F}$ contains at least three cycles. Thus, $n \geq 6$. We let $C_n = x_1 \cdots x_n$ represent the cycle in $\mathcal{F}$ of order n, and let $Z_1, \ldots, Z_k$ be the remaining cycles of $\mathcal{F}$. Reindexing if necessary, we may assume that if $\mathcal{F}$ contains a 5-cycle that Z_1 is said 5-cycle. Otherwise, if Z_i is odd for some $i \in [k]$, we let Z_1 represent the smallest odd cycle among all Z_i, $i \in [k]$. In any case, we let $Z_1 = y_1 \cdots y_p$ for some $3 \leq p < n$.

Assume first that Z_1 is a 5-cycle so that $p = 5$. Note that there exists $x_i \in C_n$ such that $f(x_i) \in Z_1$ and $f(x_j) \notin Z_1$ for some $j \in \{i-1, i+1\}$. Reindexing $x_1, \ldots, x_n$ if necessary, we may assume $f(x_n) \in Z_1$ and, redrawing G if necessary, $f(x_{n-1}) \notin Z_1$.

As in Theorem 3.2, we let $X_1 = \{x_n\}$, $X_2 = \{x_i \mid 1 \leq i < n, i \text{ is odd}\}$ and $X_3 = \{x_i \mid 2 \leq i < n, i \text{ is even}\}$. In what follows, we partition the vertices of $\cup_{i=1}^k Z_i$ into Y_1, Y_2, Y_3, and let $V_i = X_i \cup Y_i$ for each $i \in [3]$. In each case, $G^2[V_1]$ will be bipartite and V_2, V_3 will be independent sets.

Case 1. Suppose that $f(x_1) \in Z_1$. Without loss of generality, we may assume $f(x_1) = y_1$, and reindexing Z_1 if necessary, $f(x_n) = y_s$ where $s \in \{4, 5\}$. For each $i \in \{2, \ldots, k\}$ let T_i be one of the sets A_1 or A_2 from Lemma 3.1 depending on the congruence class of n modulo 4. Note that for each $i \in \{2, \ldots, k\}$, $G^2[T_i]$ is bipartite.

Next, we assume for the time being that $s = 4$ and let $T_1 = \{y_2, y_5\}$. We let

- $Y_1 = \cup_{i=1}^k T_i$,
- $Y_2 = W_2 \cup \bigcup_{i=2}^k \{v_j \in Z_i - T_i \mid f^{-1}(v_j) \in X_3\}$ and

- $Y_3 = W_3 \cup \bigcup_{i=2}^{k} \{v_j \in Z_i - T_i \mid f^{-1}(v_j) \in X_2\}$,

where

- $W_2 = \{y_3\}$ and $W_3 = \{y_1, y_4\}$, if $f^{-1}(y_3) \in X_3$;
- $W_2 = \{y_4\}$ and $W_3 = \{y_1, y_3\}$, if $f^{-1}(y_3) \in X_2$.

In $G^2[V_1]$, all edges incident to x_n are bridges to either the K_2 induced by $T_1 - \{x_n\}$ or to a bipartite component induced by T_i for some $i \in \{2, \dots, k\}$. Thus, $G^2[V_1]$ is bipartite. Furthermore, V_i where $i \in \{2, 3\}$ is independent.

Now, one can easily see that Y_1, Y_2, Y_3 can be defined in a similar fashion if instead $s = 5$. (□)

Case 2. Suppose that $f(x_1) \notin Z_1$. Without loss of generality, we may assume $f(x_n) = y_1$. In this case, we define T_i as in Case 1 for each $i \in \{2, \dots, k\}$ and we let $T_1 = \{y_2, y_4\}$. We let

- $Y_1 = \bigcup_{i=1}^{k} T_i$,
- $Y_2 = W_2 \cup \bigcup_{i=2}^{k} \{v_j \in Z_i - T_i \mid f^{-1}(v_j) \in X_3\}$ and
- $Y_3 = W_3 \cup \bigcup_{i=2}^{k} \{v_j \in Z_i - T_i \mid f^{-1}(v_j) \in X_2\}$,

where $y_3 \in W_2$ if and only if $f^{-1}(y_3) \in X_3$, and otherwise $y_3 \in W_3$; and y_1 and y_5 are in different sets W_2, W_3, depending on $f^{-1}(y_5)$.

As in Case 1, $G^2[V_1]$ is bipartite and V_i is independent for $i \in \{2, 3\}$. (□)

Now consider the case, when at least one of the cycles Z_i is odd and none of them is a 5-cycle. Recall that Z_1 is a shortest odd cycle from $\mathcal{F}$. We shall assume that $f(x_n) = y_s$ for some $s \in [p]$. Whether or not $f(x_1) \in Z_1$, we may choose T_1 to be the set A_3, A_4, or A_5 from Lemma 3.1 so that $\{y_s, y_{s+1}\} \cap T_1 = \emptyset$ if $f(x_1) \notin Z_1$, $\{f(x_1), y_s, y_{s+1}\} \cap T_1 = \emptyset$ if $f(x_1) \in Z_1$, and $G^2[T_1 \cup X_1]$ is bipartite. Then for each $i \in \{2, \dots, k\}$, we let T_i be one of the sets A_1 or A_2 from Lemma 3.1 depending on the congruence class of n modulo 4. Defining Y_1, Y_2, and Y_3 similarly as in Case 1, one can verify that $G^2[V_1]$ is indeed bipartite.

Finally, consider the case that Z_1 is even, in which case all the cycles Z_i are even, and so n is also even. In this and only in this case, we let $X_1 = \emptyset$, and $X_2 = \{x_i \mid 1 \leq i \leq n-1, i \text{ is odd}\}$ and $X_3 = \{x_i \mid 2 \leq i \leq n, i \text{ is even}\}$. Next, for each $i \in [k]$, we let T_i be the set A_1 from Lemma 3.1 and we define $Y_1 = \bigcup_{i=1}^{k} T_i$ (note that $V_1 = Y_1$). Letting

- $Y_2 = \bigcup_{i=1}^{k} \{v_j \in Z_i - T_i \mid f^{-1}(v_j) \in X_3\}$ and
- $Y_3 = \bigcup_{i=1}^{k} \{v_j \in Z_i - T_i \mid f^{-1}(v_j) \in X_2\}$

we obtain a $(1,1,2,2)$-coloring of G. □

Note that the graphs from Theorem 4.1 are 2-connected. We suspect that a similar approach might work to prove that an arbitrary 2-connected cubic graph (except the Petersen graph) has a $(1,1,2,2)$-packing coloring. (Recall that by Petersen's theorem [19] the edge set of any such graph can be partitioned into a 2-factor and a perfect matching.)

One class of cubic graphs covered by the result in Theorem 4.1 are some subclasses of generalized Petersen graphs. Let k and n be positive integers such that $k < n/2$. The *generalized Petersen graph* $P(n,k)$ has vertex set $\{u_1, v_1, \ldots, u_n, v_n\}$. The edge set of $P(n,k)$ is the set

$$\{u_i u_{i+1} \mid i \in [n]\} \cup \{u_i v_i \mid i \in [n]\} \cup \{v_i v_{i+k} \mid i \in [n]\}\,,$$

where addition on the subscripts is computed modulo n. The set $\{u_i \mid i \in [n]\}$ induces a cycle of order n, while the set $\{v_i \mid i \in [n]\}$ induces a disjoint union of cycles. The order and the number of this latter collection of cycles depends on the relationship between n and k. It is easy to see that if n and k are relatively prime, then $\{v_i \mid i \in [n]\}$ induces a single cycle of order n. In this case $P(n,k)$ is a generalized prism of C_n and satisfies the hypotheses of Theorem 3.2 unless $n = 5$, in which case $P(5,k)$ is either the ordinary prism of C_5 (that is, the Cartesian product of C_5 and K_2) or the famous Petersen graph. If n and k are not relatively prime, then the subgraph of $P(n,k)$ induced by $\{v_i \mid i \in [n]\}$ consists of the disjoint union of n/r cycles each of order r, where r is the smallest positive integer such that rk is divisible by n. Hence, these will be 5-cycles if and only if n is a multiple of 5.

Corollary 4.2 *If n and k are positive integers such that $k < n/2$ and n is not a multiple of 5, then $P(n,k)$ has a $(1,1,2,2)$-coloring and hence $\chi_\rho(S(P(n,k))) \le 5$.*

5 Multiple subdivisions

We have already remarked that $\chi_\rho(S(K_n)) = n+1$. We next consider $\chi_\rho(S_i(K_n))$ for $i \ge 2$.

Proposition 5.1 *If $n \ge 3$ and $i \ge 3$, then*

$$\chi_\rho(S_i(K_n)) = \begin{cases} 3; & \text{if } i \equiv 3 \pmod 4, \\ 4; & \text{otherwise.} \end{cases}$$

Moreover, $\chi_\rho(S_2(K_n)) \xrightarrow[n\to\infty]{} \infty$.

Proof. Clearly, $\chi_\rho(S_i(K_n)) \ge 3$ for $n \ge 3$ and $i \ge 3$. Note that $S_i(K_n)$ contains a cycle of length $3i+3 = 3(i+1)$. Since $\chi_\rho(C_n) = 3$ if $n \equiv 0 \pmod 4$, and $\chi_\rho(C_n) = 4$

otherwise (see [13]), we get that $\chi_\rho(S_i(K_n)) \geq 3$ if $i \equiv 3 \pmod 4$, and $\chi_\rho(S_i(K_n)) \geq 4$ otherwise.

To prove that these lower bounds are tight, we color $S_i(K_n)$ as follows. If $i \equiv 3 \pmod 4$, then color the vertices $v \in V(K_n)$ with color 3; otherwise color all these vertices with 4. Colorings of the subdivided vertices are done based on the parity of $i \pmod 4$ as follows. If $i \equiv 3 \pmod 4$, then for each original edge of K_n color the subdivided vertices consecutively by $1, 2, 1$, and add the block of colors $3, 1, 2, 1$ as many times as required. If $i = 4$, use colors $1, 2, 3, 1$. For any even $i \geq 6$, alternatively attach to the four colors $1, 2, 3, 1$ the pairs $2, 1$ and $3, 1$ as many times as required. Finally, let $i \equiv 1 \pmod 4$. If $i = 5$, then use the pattern $1, 3, 1, 2, 1$, and if $i \geq 9$, then add the block $3, 1, 2, 1$ as many times as required. In all of the cases it is straightforward to verify that the constructed colorings are packing colorings.

It remains to consider the case $i = 2$. If $e = uv \in E(K_n)$, then let u_e and v_e be the vertices of $S_2(K_n)$ obtained by subdividing the edge e, where u_e is adjacent to u and v_e to v. Let c be an arbitrary packing coloring of $S_2(K_n)$. Then for any edge $e = uv \in E(K_n)$ we must have $c(u_e) \neq 1$ or $c(v_e) \neq 1$. Define now the orientation of K_n as follows. If for the edge $e = uv$ we have $c(u_e) \neq 1$, then in K_n orient the edge uv from u to v. Otherwise we must have $c(v_e) \neq 1$ in which case we orient the edge uv from v to u. (In the case that both $c(u_e) \neq 1$ and $c(v_e) \neq 1$ hold, we orient the edge uv arbitrarily.) By the degree sum formula for digraphs and the pigeon-hole principle there exists a vertex u with out-degree at least $\lceil (n-1)/2 \rceil$. This means that in $S_2(K_n)$ u has at least that many neighbors colored with different colors bigger than 1. Hence $\chi_\rho(S_2(K_n)) > \lceil (n-1)/2 \rceil$. □

By using our earlier observation that the packing chromatic number of a subgraph is bounded above by the packing chromatic number of the original graph, we get the following immediate corollary of Proposition 5.1.

Corollary 5.2 *If G is a connected graph of order at least 3 and $i \geq 3$, then*

$$3 \leq \chi_\rho(S_i(G)) \leq 4\,.$$

In the case of trees we can further strengthen the result of Corollary 5.2 by including the parameter $i = 2$ (and $i = 1$), and by showing that for any odd i, the packing chromatic number is always 3. More precisely, we have the following result.

Theorem 5.3 *If $i \geq 1$, then*

$$\max\{\chi_\rho(S_i(T)) \mid T \text{ tree}\} = \begin{cases} 3; & i \text{ odd}, \\ 4; & i \text{ even}. \end{cases}$$

Proof. Let T be a tree on at least three vertices. Then, as already mentioned, $\chi_\rho(S_1(T)) = 3$, hence the assertion holds for $i = 1$.

Let $i = 2$ and let T be an arbitrary tree. To see that $\chi_\rho(S_2(T)) \le 4$ let v be an arbitrary vertex of T and consider the BFS-tree of $S_2(T)$ rooted in v. Then set

$$c(x) = \begin{cases} 1; & d_T(v,x) \equiv 1 \pmod{3}, \\ 2; & d_T(v,x) \equiv 2 \pmod{3}, \\ 3; & d_T(v,x) \equiv 0 \pmod{6}, \\ 4; & d_T(v,x) \equiv 3 \pmod{6}. \end{cases}$$

It is straightforward to verify that c is a packing coloring of $S_2(T)$. Let now T be a tree with a vertex u of degree at least 3, let v be a neighbor of u and let w be a neighbor of v different from u. Recall that if $xy \in E(T)$, then we denote with e_{xy} and e_{yx} the vertices of $S_2(T)$ obtained by subdividing xy, where e_{xy} is the vertex adjacent to x. Let c be a packing coloring of $S_2(T)$. If $c(u) = 1$, then considering the neighbors of u (in $S_2(T)$) we see that $\chi_\rho(S_2(T)) \ge 4$. The same conclusion also follows if $\{c(u), c(v)\} = \{2, 3\}$. Suppose next that $c(u) = c(v) = 2$. Then we may without loss of generality assume that $c(e_{vu}) = 3$ which in turn implies that $c(e_{vw}) = 1$. But then $c(e_{wv}) \ge 4$. Finally, let $c(u) = c(v) = 3$. Assuming without loss of generality that $c(e_{vu}) = 1$, we get $c(e_{vw}) = 2$, $c(e_{wv}) = 1$, but then $c(w) \ge 4$. This settles the case $i = 2$.

Suppose now that $i \ge 3$. Then by Corollary 5.2, $\chi_\rho(S_i(T)) \le 4$. We first deal with i odd in which case we need to prove that $\chi_\rho(S_i(T)) \le 3$ for any tree T. In the first subcase assume that $i \equiv 3 \pmod{4}$. By Corollary 5.2 we know that $3 \le \chi_\rho(S_i(T))$. Since $\chi_\rho(S_i(T)) \le \chi_\rho(S_i(K_n))$, where T has order n, and $\chi_\rho(S_i(K_n)) = 3$ by Proposition 5.1, we conclude that $\chi_\rho(S_i(T)) = 3$ when $i \equiv 3 \pmod{4}$. The second subcase to consider is when $i \equiv 1 \pmod{4}$, $i \ge 5$. Again root $S_i(T)$ in a vertex of T, say u, and consider the corresponding BFS tree. Consider the following sequence S of $i+1$ colors: first repeat the block $2, 1, 3, 1$ as many times as necessary and finish it with colors $2, 1, 3$. Note that $|S| \equiv 3 \pmod{4}$. Let now $e = xy$ be an edge of T, where $d_T(x,u) < d_T(y,u)$. If $d_T(x,u)$ is even, then color the vertices in $S_i(T)$ between x and y (including x and y) with the sequence of colors S, otherwise (if $d_T(x,u)$ is odd), color the vertices in $S_i(T)$ between x and y (including x and y) with the sequence of colors obtained by reversing S. Note that this gives a well-defined coloring of $V(S_i(T))$, that is, each vertex of T receives a unique color and that c is a packing coloring.

It remains to consider the case when $i \ge 4$ is even. To complete the argument we need to show that $\chi_\rho(S_i(T)) \ge 4$ for some tree T. Suppose on the contrary that $\chi_\rho(S_i(T)) = 3$ holds for any tree T on at least three vertices. If u is a vertex of degree at least 3, then as above we infer that if c is a 3-packing coloring of $S_i(T)$, then $c(u) > 1$. In the rest we will also use the fact that if $c(x_1) = 3$ for some vertex of $S_i(T)$, and $x_1, x_2, x_3, x_4, \ldots$ is a path in $S_i(T)$, then $c(x_2) = 1$. Indeed, for otherwise $c(x_2) = 2$, but then $c(x_3) = 1$ and we would have $c(x_4) \ge 4$. Consider an arbitrary edge uv of T and consider the following subcases.

Let $c(u) = c(v) = 3$. Then the subdivided vertices between u and v must receive the sequence of colors $1, 2, 1, 3, 1, 2, 1, \ldots$. But then the number of subdivided vertices between u and v is odd, a contradiction.

Let $c(u) = c(v) = 2$ and let w be the vertex adjacent to u on the u,v-path. Assume first that $c(w) = 1$. Then the vertices between u and v receive colors $1, 3, 1, 2, 1, 3, 1, \ldots$ which again mean that there are an odd number of these subdivided vertices. Assume next that $c(w) = 3$. We may assume that $y \neq v$ is another neighbor of u in T. Then the neighbor of u on the u,y-path in $S_i(T)$ receives color 1. But then we need color at least 4 for the next vertex on the u,y-path.

Suppose finally that $c(u) = 2$ and $c(v) = 3$. If $c(w) = 1$, then the sequence of colors on the u,v-path is $1, 3, 1, 2, 1, 3, \ldots$ and we would have an odd number of subdivided vertices. While if $c(w) = 3$ we get the same contradiction as in the above paragraph. □

Acknowledgements

This work was supported in part by the Ministry of Science of Slovenia under the grant ARRS-BI-US/16-17-013. B.B. and S.K. are also supported in part by the Ministry of Science of Slovenia under the grant P1-0297. D.F.R. is supported by a grant from the Simons Foundation (Grant Number #209654 to Douglas F. Rall). The authors wish to express their appreciation to Jernej Azarija for the computations of an optimal packing coloring of the subdivided Petersen graph.

References

[1] I. Alegre, M-A. Fiol, J.L.A. Yebra, Some large graphs with given degree and diameter, J. Graph Theory 10 (1986) 219–224.

[2] G. Argiroffo, G. Nasini, P. Torres, The packing coloring problem for lobsters and partner limited graphs, Discrete Appl. Math. 164 (2014) 373–382.

[3] B. Brešar, S. Klavžar, D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, Discrete Appl. Math. 155 (2007) 2303–2311.

[4] B. Brešar, S. Klavžar, D.F. Rall, Packing chromatic number of base-3 Sierpiński graphs, Graphs Combin. 32 (2016) 1313–1327.

[5] B. Brešar, S. Klavžar, D.F. Rall, K. Wash, Packing chromatic number under local changes in a graph, submitted.

[6] J. Ekstein, P. Holub, O. Togni, The packing coloring of distance graphs $D(k, t)$, Discrete Appl. Math. 167 (2014) 100–106.

[7] J. Fiala, P.A. Golovach, Complexity of the packing coloring problem for trees, Discrete Appl. Math. 158 (2010) 771–778.

[8] J. Fiala, S. Klavžar, B. Lidický, The packing chromatic number of infinite product graphs, European J. Combin. 30 (2009) 1101–1113.

[9] A. Finbow, D.F. Rall, On the packing chromatic number of some lattices, Discrete Appl. Math. 158 (2010) 1224–1228.

[10] N. Gastineau, Dichotomies properties on computational complexity of S-packing coloring problems, Discrete Math. 338 (2015) 1029–1041.

[11] N. Gastineau, H. Kheddouci, O. Togni, Subdivision into i-packing and S-packing chromatic number of some lattices, Ars Math. Contemp. 9 (2015) 331–354.

[12] N. Gastineau, O. Togni, S-packing colorings of cubic graphs, Discrete Math. 339 (2016) 2461–2470.

[13] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, Broadcast chromatic numbers of graphs, Ars Combin. 86 (2008) 33–49.

[14] W. Goddard, M.A. Henning, A characterization of cubic graphs with paired-domination number three-fifths their order, Graphs Combin. 25 (2009) 675–692.

[15] W. Goddard, H. Xu, The S-packing chromatic number of a graph, Discuss. Math. Graph Theory 32 (2012) 795–806.

[16] W. Goddard, H. Xu, A note on S-packing colorings of lattices, Discrete Appl. Math. 166 (2014) 255–262.

[17] Y. Jacobs, E. Jonck, E.J. Joubert, A lower bound for the packing chromatic number of the Cartesian product of cycles, Cent. Eur. J. Math. 11 (2013) 1344–1357.

[18] D. Korže, A. Vesel, On the packing chromatic number of square and hexagonal lattice, Ars Math. Contemp. 7 (2014) 13–22.

[19] J. Petersen, Die Theorie der regulären Graphen, Acta Math. 15 (1891) 193–220.

[20] O. Togni, On packing colorings of distance graphs, Discrete Appl. Math. 167 (2014) 280–289.

[21] P. Torres, M. Valencia-Pabon, The packing chromatic number of hypercubes, Discrete Appl. Math. 190–191 (2015) 127–140.

[22] D. Torri, N. Zagaglia Salvi, New characterizations of the Petersen graph, Bull. Inst. Combin. Appl. 19 (1997) 79–82.