Packing chromatic number under local changes in a graph

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that there exists a k-vertex coloring of G in which any two vertices receiving color i are at distance at least $i+1$. It is proved that in the class of subcubic graphs the packing chromatic number is bigger than 13, thus answering an open problem from [Gastineau, Togni, S-packing colorings of cubic graphs, Discrete Math. 339 (2016) 2461–2470]. In addition, the packing chromatic number is investigated with respect to several local operations. In particular, if $S_e(G)$ is the graph obtained from a graph G by subdividing its edge e, then $\lfloor \chi_\rho(G)/2 \rfloor + 1 \le \chi_\rho(S_e(G)) \le \chi_\rho(G) + 1$.

Key words: packing chromatic number; cubic graph; subdivision; contraction.

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1 Introduction

Many variations of the classical graph coloring have been introduced, several of which involve graph distance, which as a condition is usually imposed on the vertices that are given the same color. In this paper we study packing colorings defined as follows. The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k such that $V(G)$ can be partitioned into subsets $X_1, \ldots, X_k$, where X_i induces an i-packing; that is, vertices of X_i are pairwise at distance more than i. Equivalently, a k*-packing coloring* of G is a function $c : V(G) \to [k]$, where $[k] = \{1, \ldots, k\}$, such that if $c(u) = c(v) = i$, then $d_G(u, v) > i$, where $d_G(u, v)$ is the usual shortest-path distance between u and v in G. We mention that in distance-k colorings $V(G)$ is partitioned into k-packings.

The concept of the packing chromatic number was introduced in [10] and given the name in [3]. The problem intuitively appears more difficult than the standard coloring problem. Indeed, the packing chromatic number is intrinsically more difficult due to the fact that determining χ_ρ is NP-complete even when restricted to trees [7]. On the other hand, Argiroffo et al. discovered that the packing coloring problem is solvable in polynomial time for several nontrivial classes of graphs [2]. In addition, the packing chromatic number was studied on hypercubes [10, 16], Cartesian product graphs [11, 13], and distance graphs [6, 15].

In the seminal paper [10] the following problem was posed: does there exist an absolute constant M, such that $\chi_\rho(G) \leq M$ holds for any subcubic graph G. (Recall that a graph is *subcubic*, if its largest degree is bounded by 3.) This problem led to a lot of research but remains unsolved at the present. In particular, the packing chromatic number of the infinite hexagonal lattice is 7 (the upper bound being established in [8], the lower bound in [12]), hence the packing chromatic number of any subgraph of the hexagonal lattice is bounded by 7. The same bound also holds for subcubic trees as follows from a result of Sloper [14]. For the (subcubic) family of base-3 Sierpiński graphs the packing chromatic number was bounded by 9 in [4]. The exact value of the packing chromatic of some additional subcubic graphs was determined in [5]. Very recently, Gastineau and Togni [9] found a cubic graph with packing chromatic number equal to 13 and posed an open problem which intrigued us: does there exist a cubic graph with packing chromatic number larger than 13?

We proceed as follows. In the next section we prove that the answer to the above question is positive. More precisely, we construct a cubic graph on 78 vertices with packing chromatic number at least 14. A key technique in the related proof is edge subdivision. We hence give a closer look at this operation with respect to its effect on the packing chromatic number. In particular, the packing chromatic number does not increase by more than 1 when an edge of a graph is subdivided, but can decrease by at least 2. In addition, we prove that the lower bound for the packing chromatic number of an edge-subdivided graph is bigger than half of the packing chromatic number of the original graph. Then, in Section 3, we investigate the effect on the packing chromatic number of the following local operations: a vertex deletion, an edge deletion, and an edge contraction. In particular, we demonstrate that the difference $\chi_\rho(G) - \chi_\rho(G-e)$ can be arbitrarily large.

2 Edge subdivision

In this section we consider the packing chromatic number with respect to the edge-subdivision operation. If e is an edge of a graph G, then let $S_e(G)$ denote the graph obtained from G by subdividing the edge e. The graph obtained from G by subdividing all its edges is denoted $S(G)$.

The following theorem is the key for the answer of the above mentioned question of Gastineau and Togni.

Theorem 2.1 *Suppose that there exists a constant M such that $\chi_\rho(H) \le M$ holds for any subcubic graph H. If G is a subcubic graph such that $\chi_\rho(G) = M$, then either $\chi_\rho(S_e(G)) \le M-2$ for any $e \in E(G)$, or* $\mathrm{diam}(G) \ge \lceil \frac{M}{2} \rceil - 2$.

Proof. Let G be a subcubic graph such that $\chi_\rho(G) = M$, where $\chi_\rho(H) \le M$ for every subcubic graph H. If $\chi_\rho(S_e(G)) \le M-2$ holds for any $e \in E(G)$, there is nothing to be proved. Hence assume that there exists an edge $e \in E(G)$ such that $\chi_\rho(S_e(G)) \ge M-1$. Let G' be the graph obtained from G by subdividing the edge e, and let x' be the new vertex. Let G'' be a copy of G', with x'' playing the role of x'. Let now $\widehat{G}$ be the graph obtained from the disjoint union of G' and G'' by connecting x' with x''.

Note first that $\widehat{G}$ is a subcubic graph, and hence by the theorem's assumption, $\chi_\rho(\widehat{G}) \le M$. Let c be an arbitrary optimal packing coloring of $\widehat{G}$. Because c restricted to G' (resp. G'') is a packing coloring of $G' = S_e(G)$ (resp. G''), c uses at least $M-1$ colors. We claim that $\mathrm{diam}(\widehat{G}) \ge M-1$. If c colors a vertex u' of G' and a vertex u'' of G'' by the color M, then $d_{\widehat{G}}(u', u'') > M$, and the claim follows. Otherwise, we assume that c restricted to G' does not use the color M. If also G'' does not use color M, then since $\chi_\rho(G') \ge M-1$ and $\chi_\rho(G'') \ge M-1$, there exist vertices v', v'' in G', resp. G'', with $c(v') = c(v'') = M-1$, and consequently, $\mathrm{diam}(\widehat{G}) > M-1$ as desired. So assume that color M is present on G'' (and not on G'). Color $M-1$ must be present on G', for otherwise $\chi_\rho(G') \le M-2$. If color $M-1$ is also used on G'', then it again follows that $\mathrm{diam}(\widehat{G}) > M-1$. Hence we are left with the situation that color M is present on G'' and not on G', while $M-1$ is used on G' and not on G''. We now claim that the color $M-2$ is present in both G' and G''. For if this is not the case, then in any of G' or G'' that is missing color $M-2$ relabeling all vertices colored with the highest color by the color $M-2$ would yield an $(M-2)$-packing coloring of G' or G'', which is again not possible. If w', w'' are the vertices in G', resp. G'', with $c(w') = c(w'') = M-2$, then $d_{\widehat{G}}(w', w'') \ge M-1$. This in turn implies $\mathrm{diam}(\widehat{G}) \ge M-1$, and so the claim is proved.

Consider again vertices w', w'' in G', resp. G'', with $c(w') = c(w'') \geq M - 2$. Since $\mathrm{diam}(G') \geq d_{\widehat{G}}(w', x')$ and $\mathrm{diam}(G'') \geq d_{\widehat{G}}(w'', x'')$, we infer that

$$\begin{aligned} 2\,\mathrm{diam}(G') + 1 &= \mathrm{diam}(G') + \mathrm{diam}(G'') + 1 \\ &\geq d_{\widehat{G}}(w', x') + d_{\widehat{G}}(w'', x'') + 1 \\ &= d_{\widehat{G}}(w', w'') \\ &\geq M - 1\,. \end{aligned}$$

Hence $\mathrm{diam}(G') \geq \lceil \frac{M}{2} \rceil - 1$. Since clearly $\mathrm{diam}(G') \leq \mathrm{diam}(G) + 1$ holds, we conclude that

$$\mathrm{diam}(G) \geq \mathrm{diam}(G') - 1 \geq \left\lceil \frac{M}{2} \right\rceil - 2\,.$$

□

Corollary 2.2 *There exists a cubic graph with packing chromatic number larger than 13.*

Proof. Let G_{38} be the cubic graph of order 38 with diameter 4 from [1] shown in Figure 1.

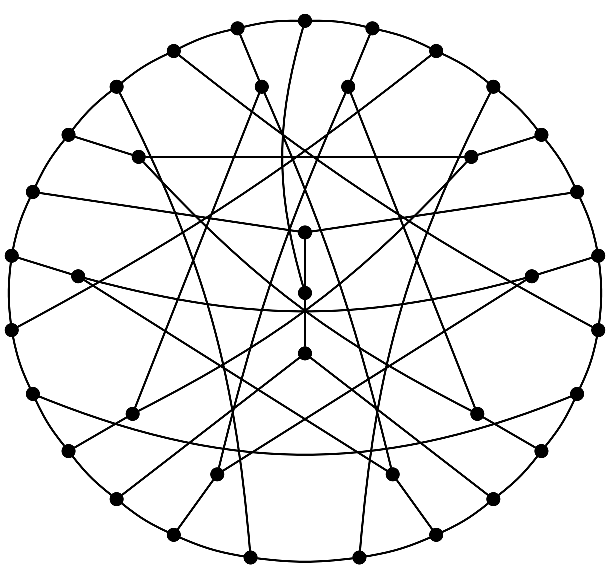

Figure 1: G_{38}

From [9, Proposition 6] we know that $\chi_\rho(G_{38}) = 13$. We have checked by computer that $\chi_\rho(S_e(G_{38})) = 12$ holds for any edge e of G_{38}. Assuming that $M = 13$ is the constant of Theorem 2.1, this theorem implies that $\mathrm{diam}(G_{38}) \geq \lceil \frac{13}{2} \rceil - 2 = 5$. However, since the diameter of G_{38} equals 4, we infer that M cannot be 13. $\square$

A closer look to the proof of Theorem 2.1 reveals that the graph constructed from two copies G'_{38} and G''_{38} of edge-subdivided G_{38} by connecting the vertices x' and x'' is a graph of order 78, say G_{78} schematically shown in Figure 2, such that $\chi_\rho(G_{78}) \geq 14$.

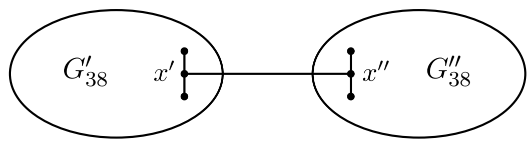

Figure 2: G_{78}

Motivated by the construction from the proof of Theorem 2.1, we next consider what happens with the packing chromatic number of an arbitrary graph when an edge is subdivided.

Theorem 2.3 *For any graph G with packing chromatic number j,*

$$\lfloor j/2 \rfloor + 1 \leq \chi_\rho(S_e(G)) \leq j + 1.$$

Moreover, for any $k \geq 2$ there exists a graph G with an edge e such that $k = \chi_\rho(G) = \chi_\rho(S_e(G)) - 1$.

Proof. Given a packing coloring c of G, a packing coloring of $S_e(G)$ can be obtained by using c on vertices of G and coloring the new vertex with an additional color. Hence we get the upper bound.

For the lower bound, let G be a graph with $\chi_\rho(G) = j$ and consider any edge $e = xy$ of G. Subdivide e to get the graph $H = S_e(G)$. That is, we remove the edge e from G and replace it with the path x, z, y. Let $W_1, \ldots, W_r$ be an optimal packing coloring of H.

We will construct a packing coloring of G. Note that $\{x, y\} \not\subseteq W_n$ for any $n \geq 2$. Fix i such that $2 \leq i \leq r$ and suppose there are vertices $u, v \in W_i$ such that $d_G(u, v) = i$. Since W_i is an i-packing in H, we know that every shortest (u, v)-path in H contains the vertex z. From among all pairs of vertices in W_i that are at distance i in G select

a_i, b_i such that $d_G(a_i, x) = t$ is the minimum and $d_G(b_i, y) = s$, and so $t \le s$. Thus $d_H(a_i, b_i) = t+s+2 = i+1$. Let $c \in W_i - \{a_i, b_i\}$. It now follows that $d_G(c, x) > t$, for otherwise $d_H(c, a_i) \le 2t < i+1$, a contradiction. Similarly, $d_G(c, y) \ge s$, or otherwise it follows that $d_H(a_i, c) = d_G(a_i, x)+2+d_G(y, c) < t+2+s = i+1$, again a contradiction. For each such value of i we remove the vertex a_i from W_i and place it into a set X of vertices that will eventually be "recolored." For all pairs u, v remaining in W_i it follows that $d_G(u, v) \ge i + 1$. If a_2 as defined above exists, then $a_2 = x$. It follows that W_1 is independent in G and $|X| = m \le r - 1$. Otherwise if x and y belong to W_1 place vertex x in the set X. In this case W_2 is a 2-packing in G and $|X| = m \le r - 1$. Hence we can recolor the vertices in X using colors $r + 1, \dots, r + m$ and this gives a packing coloring of G using at most $2r - 1$ colors. That is, $\chi_\rho(G) \le 2r - 1$.

To prove the last assertion of the theorem, consider the following examples. For $k = 2$, we have $2 = \chi_\rho(P_3) = \chi_\rho(P_4) - 1$, and $P_4 = S_e(P_3)$. Let now $k \ge 3$. Recall [3, Lemma 6] asserting that $\chi_\rho(S(K_k)) = k + 1$. Consider now the process of obtaining $S(K_k)$ from K_k by subdividing each of the edges of K_k one by one, and observe that in the beginning of this process $\chi_\rho(K_k) = k$, and at the end we have $\chi_\rho(S(K_k)) = k + 1$. Since in each step the packing chromatic number can increase by at most one, at some stage of the process we have graphs G_i and G_{i+1}, such that $G_{i+1} = S_e(G_i)$ for some edge e of G_i, and $\chi_\rho(G_i) = k$, $\chi_\rho(G_{i+1}) = k + 1$. □

We do not know if the lower bound of Theorem 2.3 is sharp. On the other hand, it is possible that the subdivision of an edge decreases the packing chromatic number by 2. Consider the following examples. Let $n \ge 5$, and let X_n be the graph obtained from the disjoint union of two copies of K_n, denoted by U and V, by first joining a vertex u of U with a vertex v of V, and then subdividing the edge uv twice. Figure 3 depicts the graph X_5. Let $e = xy$ be the edge, where x is adjacent to u, and y is adjacent to v.

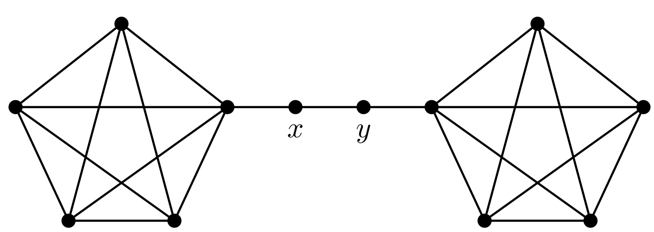

Figure 3: X_5

We claim that $\chi_\rho(X_n) = 2n - 3$. Let c be an optimal packing coloring of X_n.

Suppose first that $c(x) = 1$ and $c(y) = 2$. Clearly, c restricted to U uses all the colors from $[n]$. On the other hand, c uses colors $1, 3, 4$ on V, while the other $n - 3$ vertices must receive new colors. Hence in this case, c uses $2n - 3$ colors. Suppose next that one of the vertices x and y is colored with a color a, where $a > 2$. Then c uses n colors on U, different from a, and $n-4$ new colors on V. Hence also in this case c uses $2n-3$ colors, which proves the claim.

Consider now the graph $S_e(X_n)$, and the following coloring c of this graph. Let $c(x) = c(y) = 1$, and $c(v_{xy}) = 2$, where v_{xy} is the vertex obtained by subdividing the edge xy. The mapping c restricted to U uses colors from $[n]$ and restricted to V uses colors from $\{1, 2, 3, 4, 5\}$ together with $n - 5$ new colors. Hence $\chi_\rho(S_e(X_n)) \leq 2n - 5$. The opposite inequality follows by the observation that however $S_e(X_n)$ is colored, n different colors must be used on U, and out of these at most five colors can be used also on V. This shows that $\chi_\rho(S_e(X_n)) = \chi_\rho(X_n) - 2$.

Let us call an edge e of graph G *weak* if $\chi_\rho(S_e(G)) < \chi_\rho(G) - 1$. We have not been able to find a graph G such that all its edges are weak. We are inclined to believe that there are no such graphs. From this point of view the following consequence of Theorem 2.1 is relevant.

Corollary 2.4 *Suppose that there exists a constant M such that $\chi_\rho(H) \leq M$ holds for any subcubic graph H, and let G be a subcubic graph such that $\chi_\rho(G) = M$. If there are no subcubic graphs in which all edges are weak then $M \leq 2\,\mathrm{diam}(G) + 4$.*

3 Vertex deletion, edge deletion and contraction

Since the distances in a graph when an edge is removed can only increase, it is clear that for any graph G, any vertex v of G, and any edge e of G, we have

$$\chi_\rho(G - v) \leq \chi_\rho(G) \quad \text{and} \quad \chi_\rho(G - e) \leq \chi_\rho(G).$$

On the other hand, there are no lower bounds for $\chi_\rho(G - v)$ and $\chi_\rho(G - e)$. For the former operation, let G_n, $n \geq 4$, be the graph obtained from the path P_n by adding a vertex x and making it adjacent to all vertices of the path. Note that $\chi_\rho(G_n) \geq \lceil \frac{n}{2} \rceil + 1$, and since $G_n - x$ is isomorphic to P_n, we have $\chi_\rho(G_n - x) = 3$. To deal with edge removal we state

Proposition 3.1 *For every positive integer r there exists a graph G with an edge e such that $\chi_\rho(G) - \chi_\rho(G - e) \geq r$.*

Proof. Consider the following construction. Let $k \geq 4$, and $n \geq 2k-2$. Let A and B be two copies of the graph K_n, and $a, a' \in V(A)$, $b, b' \in V(B)$. The graph $G_{n,k}$ is obtained from the disjoint union of A and B by connecting with an edge vertices a and b and also connecting vertices a' and b', and then replacing the edge $a'b'$ with a path of length $2k-1$. Figure 4 depicts the graph $G_{6,4}$.

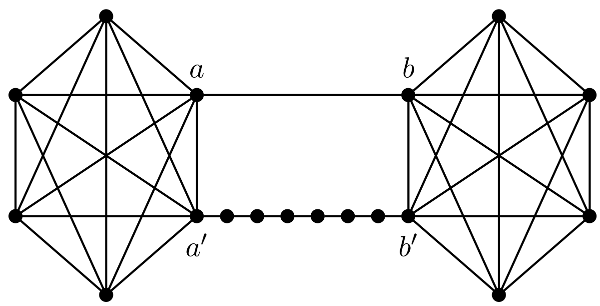

Figure 4: $G_{6,4}$

We first claim that $\chi_\rho(G_{n,k}) \geq 2n-2$. Note that n colors are used in any packing coloring on A. Since the distance between a vertex of A and a vertex of B is at most 3, we derive that only colors 1 and 2 can be repeated in B, hence the claim.

Letting $G'_{n,k} = G_{n,k} - ab$ we next claim that $\chi_\rho(G'_{n,k}) \leq 2(n-k)+4$. Consider the following packing coloring of $G'_{n,k}$. First color the path of length $2k-1$ between a' and b' with colors from $\{1,2,3\}$. Because in $G'_{n,k}$ every vertex in $A \setminus \{a'\}$ is at distance $2k+1$ from any vertex in $B \setminus \{b'\}$, we can use colors $4, \ldots, 2k$ in both A and B. Note that this is possible because we have assumed that $n \geq 2k-2$, and hence the number of vertices in $A \setminus \{a'\}$ and in $B \setminus \{b'\}$ is at least $2k-3$, respectively. This in turn implies that the colors $4, \ldots, 2k$ can indeed be used twice. The remaining vertices are then colored by unique colors. Consequently,

$$\chi_\rho(G'_{n,k}) \leq 3 + (2k-3) + [2(n-1) - 2(2k-3)] = 2(n-k)+4\,.$$

It follows that $\chi_\rho(G_{n,k}) - \chi_\rho(G'_{n,k}) \geq (2n-2) - [2(n-k)+4] = 2k-6$. The assertion now follows. $\square$

We next turn our attention to edge contractions. We denote the graph obtained from G by contracting its edge e by $G|e$.

Theorem 3.2 *If G is a graph and e an edge in G, then*

$$\chi_\rho(G) - 1 \leq \chi_\rho(G|e) \leq 2\chi_\rho(G)\,.$$

Proof. Let $e = xy$ be the edge that is contracted in a graph G, and v_{xy} the resulting vertex. For the lower bound, let c be an optimal packing coloring of $G|e$. We define the coloring c' of G by letting $c'(x) = c(v_{xy})$, $c'(y) = \chi_\rho(G|e) + 1$, and $c'(z) = c(z)$ for any other vertex in G. Since the distances in G are at least as large as the distances in $G|e$ between the corresponding vertices, c' is packing coloring of G. It follows that $\chi_\rho(G) \leq \chi_\rho(G|e) + 1$.

For the upper bound let c be an optimal packing coloring of G. We define the coloring c' of $G|e$ in two steps. First, let $c'(v_{xy}) = c(y)$, and $c'(z) = c(z)$ for any other vertex z of $G|e$. Let $i \in [\chi_\rho(G)]$, and let x_i be a vertex of $G|e$ that minimizes $d_{G|e}(z, v_{xy})$ over all $z \in V(G|e)$ with $c(z) = i$. (Note that x_i coincides with v_{xy} for exactly one $i \in [\chi_\rho(G)]$.) Then, in the second step, set $c'(x_i) = \chi_\rho(G) + i$. We claim that c' is a packing coloring of $G|e$.

Note that for any two vertices a and b of $G|e$ we have that $d_{G|e}(a, b)$ is either $d_G(a, b)$ or $d_G(a, b) - 1$. Moreover, in the latter case there exists a shortest (a, b)-path in G that contains the edge xy. Suppose that there exist vertices u and v, both different from x_i, with $c'(u) = c'(v) = i$ such that $d_{G|e}(u, v) = i$. Clearly, then in G the edge xy must lie on some shortest (u, v)-path P of length $i + 1$. Hence we may assume that P is of the form $u - P' - x - y - P'' - v$. We may also assume without loss of generality that $d_G(x_i, x) \leq d_G(x_i, y)$. Since x_i is a closest vertex to v_{xy} among all vertices colored by i, we derive that $d_G(x_i, x) < d_G(v, x)$, hence $d_G(u, x_i) \leq d_G(u, x) + d_G(x, x_i) < d_G(u, x) + d_G(x, v) = i + 1$. This is a contradiction with c being a packing coloring of G, in which u and x_i are both colored by color i. This shows that c' is a packing coloring of $G|e$ with $2\chi_\rho(G)$ colors, hence the proof of the upper bound is also complete. $\square$

Note that Theorem 3.2 is in some sense dual to Theorem 2.3. To see that the lower bound of Theorem 3.2 is sharp, just consider complete graphs. For the upper bound, similarly as in Theorem 2.3, we are not aware of any example of a graph such that after the contraction of its edge the packing chromatic number would increase by more than 2. On the other hand, the graphs $S_e(X_n)$, as presented in Section 2 show that the contraction of the edge e, yielding the graph X_n, increases their packing chromatic number by 2.

4 Concluding remarks

In this paper we answered a question of Gastineau and Togni [9] by showing that there is a graph whose packing chromatic number is greater than 13. However, the problem from [10] concerning the existence of a constant upper bound for the packing chromatic number on the class of cubic graphs remains an interesting, unresolved problem. It is possible that using Theorem 2.1 leads to subcubic graphs with increasing packing chromatic number. However, to prove this would require new methods since our approach in part uses a computer.

Several open problems arise from considering local operations on graphs and how these affect the packing chromatic number. For instance, the graph G_{38} from Section 2 has the property that the subdivision of an arbitrary edge produces a graph whose packing chromatic number is exactly one less than that of G_{38}. Cycles of the form C_{4k+3} also share this property. It would be interesting to know more about this class of subdivision critical graphs. The examples X_n from Section 2 show that there exist graphs that have an edge whose subdivision decreases the packing chromatic number by 2. As mentioned in Section 2 we suspect that there does not exist a graph for which the subdivision of any of its edges decreases the packing chromatic number by more then 1. In other words, we suspect that there are no graphs with only weak edges.

Following the definition of graphs that are critical with respect to ordinary chromatic number (i.e., the chromatic number of any subgraph is less than that of the original graph) it is natural to study graphs that are critical with respect to the packing chromatic number. For graphs with no isolated vertices this is equivalent to requiring that the packing chromatic number decreases upon the removal of any edge. Examples of these are cycles whose order is not congruent to 0 modulo 4, complete graphs, and the Petersen graph.

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References

[1] I. Alegre, M-A. Fiol, J.L.A. Yebra, Some large graphs with given degree and diameter, J. Graph Theory 10 (1986) 219–224.

[2] G. Argiroffo, G. Nasini, P. Torres, The packing coloring problem for lobsters and partner limited graphs, Discrete Appl. Math. 164 (2014) 373–382.

[3] B. Brešar, S. Klavžar, D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, Discrete Appl. Math. 155 (2007) 2303–2311.

[4] B. Brešar, S. Klavžar, D.F. Rall, Packing chromatic number of base-3 Sierpiński graphs, Graphs Combin., in press; DOI: 10.1007/s00373-015-1647-x.

[5] L. Daouya, I. Bouchemakh, E. Sopena, Packing coloring of some undirected and oriented coronae graphs, arXiv:1506.07248.

[6] J. Ekstein, P. Holub, O. Togni, The packing coloring of distance graphs $D(k,t)$, Discrete Appl. Math. 167 (2014) 100–106.

[7] J. Fiala, P.A. Golovach, Complexity of the packing coloring problem for trees, Discrete Appl. Math. 158 (2010) 771–7789.

[8] J. Fiala, S. Klavžar, B. Lidický, The packing chromatic number of infinite product graphs, European J. Combin. 30 (2009) 1101–1113.

[9] N. Gastineau, O. Togni, S-packing colorings of cubic graphs, Discrete Math. 339 (2016) 2461–2470.

[10] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, Broadcast chromatic numbers of graphs, Ars Combin. 86 (2008) 33–49.

[11] Y. Jacobs, E. Jonck, E.J. Joubert, A lower bound for the packing chromatic number of the Cartesian product of cycles, Cent. Eur. J. Math. 11 (2013) 1344–1357.

[12] D. Korže, A. Vesel, On the packing chromatic number of square and hexagonal lattice, Ars Math. Contemp. 7 (2014) 13–22.

[13] Z. Shao, A. Vesel, Modeling the packing coloring problem of graphs, Appl. Math. Model. 39 (2015) 3588–3595.

[14] C. Sloper, An eccentric coloring of trees, Austral. J. Combin. 29 (2004) 309–321.

[15] O. Togni, On packing colorings of distance graphs, Discrete Appl. Math. 167 (2014) 280–289.

[16] P. Torres, M. Valencia-Pabon, The packing chromatic number of hypercubes, Discrete Appl. Math. 190–191 (2015) 127–140.