

Perfect graphs for domination games

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Abstract

Let $\gamma_g(G)$ and $\gamma_{tg}(G)$ be the game domination number and the total game domination number of a graph G , respectively. Then G is γ_g -perfect (resp. γ_{tg} -perfect), if every induced subgraph F of G satisfies $\gamma_g(F) = \gamma(F)$ (resp. $\gamma_{tg}(F) = \gamma_t(F)$). A recursive characterization of γ_g -perfect graphs is derived. The characterization yields a polynomial recognition algorithm for γ_g -perfect graphs. It is proved that every minimally γ_g -imperfect graph has domination number 2. All minimally γ_g -imperfect triangle-free graphs are determined. It is also proved that γ_{tg} -perfect graphs are precisely $\overline{2P_3}$ -free cographs.

Keywords: domination game; total domination game; perfect graph for domination game; triangle-free graph; cograph

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1 Introduction

The domination game on a graph G is played by Dominator and Staller. If Dominator (resp. Staller) starts the game, we speak of the D-game (resp. S-game). During

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the game, the players alternatively select vertices that are not dominated by the set of previously selected vertices. The game ends when no such vertex is available. Dominator's goal is to finish the game as soon as possible, while Staller wishes to play the game as long as possible. The unique number of moves played in the D-game (resp. S-game) when both players play optimally is the game domination number $\gamma_g(G)$ (resp. Staller-start game domination number $\gamma'_g(G)$) of G . The total domination game is defined analogously, the only difference being that when a new vertex is selected, it must totally dominate at least one vertex not yet totally dominated by the previously selected vertices; the corresponding game total domination numbers are denoted with $\gamma_{tg}(G)$ and $\gamma'_{tg}(G)$.

The seminal paper [4] on the domination game together with its follow-up [17] had a great impact, leading to several dozens of papers. Instead of listing (too) long list of references, we just point to [10, 16, 19, 23] and references therein. Similarly, the seminal papers on the total domination game [12, 13] led to its extensive investigation, cf. [5, 6, 14].

Perfect graphs lie in the very core of graph theory, papers [8, 20] being highlights of the theory. Now, just as $\chi(G) \geq \omega(G)$ holds trivially for every graph G , we infer from definitions that $\gamma_g(G) \geq \gamma(G)$. Hence we say that G is a γ_g -minimal graph if the equality $\gamma_g(G) = \gamma(G)$ holds. In this paper, we will call these graphs γ_g -graphs for short. The γ_g -minimal trees were characterized in [21], but a characterization of γ_g -minimal graphs is widely open. In this paper we study the hereditary version of this property via the following concept.

Definition 1.1 *A graph G is γ_g -perfect, if every induced subgraph F of G satisfies $\gamma_g(F) = \gamma(F)$.*

Note that G and/or F may be disconnected in the above definition.

Since the inequalities $\gamma'_g(G) \geq \gamma(G)$, $\gamma_{tg}(G) \geq \gamma_t(G)$, and $\gamma'_{tg}(G) \geq \gamma_t(G)$ also hold for every graph G (where, if the total domination is involved, G must of course be isolate-free), we introduce the analogous terminology for the Staller-start domination game and for the total domination games.

- G is a γ'_g -minimal graph if $\gamma(G) = \gamma'_g(G)$ holds, and is γ'_g -perfect if all of its induced subgraphs are γ'_g -minimal graphs.
- G is a γ_{tg} -minimal graph if $\gamma_t(G) = \gamma_{tg}(G)$ holds, and is γ_{tg} -perfect if all of its isolate-free induced subgraphs are γ_{tg} -minimal graphs.
- G is a γ'_{tg} -minimal graph if $\gamma_t(G) = \gamma'_{tg}(G)$ holds, and is γ'_{tg} -perfect if all of its isolate-free induced subgraphs are γ'_{tg} -minimal graphs.

In the literature, several similar problems were studied, that is, the equality between two covering and/or domination-type invariants is required to hold not only for a graph G but also for all induced subgraphs. See [22, 24] for earlier approaches and [1, 2, 3, 7, 11] for recently published results.

In this paper we characterize γ_g -perfect, γ'_g -perfect, γ_{tg} -perfect, and γ'_{tg} -perfect graphs. The main result is a characterization of the first class, it is proved in Section 3. The characterization describes a recursive structure of γ_g -perfect graphs that in particular yields a polynomial recognition algorithm for γ_g -perfect graphs. We also introduce minimally γ_g -imperfect graphs and prove that they have domination number 2. In Section 4 we discuss recognition complexity of γ_g -perfect graphs, and present some results on triangle-free graphs. Characterizations of γ'_g -perfect, γ_{tg} -perfect, and γ'_{tg} -perfect turned out to be simpler, we state them in Section 5. In particular, γ'_{tg} -perfect graphs are precisely cographs, and γ_{tg} -perfect graphs are precisely $\overline{2P_3}$ -free cographs.

2 Preliminaries

If v is a vertex of a graph $G = (V(G), E(G))$, then the *open neighborhood* $N_G(v)$ is the set of neighbors of v , while the *closed neighborhood* $N_G[v]$ is the open neighborhood supplemented with the vertex v itself. Two vertices, u and v , are (*true twins*) in G , if $N_G[u] = N_G[v]$, and they are *false twins* if $N_G(u) = N_G(v)$. The *degree* of v in G is $d_G(v) = |N_G(v)|$. The *closed neighborhood of a set* S of vertices is $N_G[S] = \bigcup_{v \in S} N_G[v]$. In this paper, the *open neighborhood of* S will be meant as $N'_G(S) = N_G[S] \setminus S$.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$ of G . A dominating set S is a *total dominating set* if every vertex from S has a neighbor in S . The smallest cardinality of a total dominating set is the *total domination number* $\gamma_t(G)$ of G , see the book [15].

The distance $d_G(u, v)$ between vertices u and v of a connected graph G is the minimum number of edges on a u, v -path. If H_1 and H_2 are subgraphs of a connected graph G , then the distance $d_G(H_1, H_2)$ between H_1 and H_2 is the minimum of the distances $d_G(v_1, v_2)$, where $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$.

We will say that a graph G is *minimally γ_g -imperfect*, if each of its proper induced subgraphs is γ_g -perfect but $\gamma(G) < \gamma_g(G)$. Since perfectness is a hereditary property, a graph is not γ_g -perfect if and only if it has an induced subgraph which is minimally γ_g -imperfect. This ensures that there exists a forbidden subgraph characterization for γ_g -perfect graphs. *Minimally γ'_g -imperfect*, *minimally γ_{tg} -imperfect* and

minimally γ'_{tg} -imperfect graphs are defined analogously to *minimally γ_g -imperfect* graphs.

We say that a graph G is *2- γ_g -perfect*, if every induced subgraph F of G with $\gamma(F) = 2$ is a γ_g -graph. By definition, every induced subgraph of a 2- γ_g -perfect graph is 2- γ_g -perfect, as well. This concept will be useful when proving our characterization theorem for γ_g -perfect graphs, but at the end we show that this property is equivalent to the γ_g -perfectness.

Cographs are, by definition, the graphs that contain no induced path P_4 . These graphs admit different characterizations, see [9]; the one to be applied here asserts that cographs are precisely the graphs that can be obtained from K_1 by means of the disjoint union and join of graphs.

3 Characterization of γ_g -perfect graphs

In this section our goal is to characterize γ_g -perfect graphs. The theorem, that is formulated and proved in Subsection 3.3, states an equivalence with a recursively defined graph class. The two operators used in the recursive definition are introduced in Subsection 3.1. As a consequence of the characterization theorem, we prove in Subsection 4.1 that γ_g -perfect graphs can be recognized in polynomial time.

Along the proof of the main theorem, we first consider 2- γ_g -perfect graphs and prove that they can be built from an isolated vertex by using the two specified operators. We also show that these operators applied to 2- γ_g -perfect graphs always result in γ_g -graphs. Then, using further statements on the structure of 2- γ_g -perfect graphs, we can prove the equivalence between γ_g -perfectness, 2- γ_g -perfectness, and the property of recursive constructability.

3.1 Preliminary observations

A *homogeneous clique* Q in a graph G is a clique in which every two vertices are true twins, that is, $N_G[u] = N_G[v]$ holds for every $u, v \in V(Q)$. In other words, a clique Q is homogeneous if there is a join between Q and $N'_G(V(Q))$. From now on, we will use the same notation Q when referring to the vertex set of the clique Q . A *maximal homogeneous clique* (shortly, MHC) is an inclusion-wise maximal homogeneous clique. By definition, two maximal homogeneous cliques are always vertex disjoint. A *perfect set of cliques* (shortly, PSC) in a graph G is a (possibly empty) set \mathcal{Q} of homogeneous cliques such that $d_G(Q, Q') = 3$ and there is a join between $N'_G(Q)$ and $N'_G(Q')$ for every $Q, Q' \in \mathcal{Q}$. Note that the empty set $\mathcal{Q} = \emptyset$ is a PSC in G , and if Q is a homogeneous clique in G , then the one-element set

$\mathcal{Q} = \{Q\}$ is also a PSC in G . If \mathcal{Q} is a nonempty PSC, we will usually use the notation $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ and $V(\mathcal{Q}) = \bigcup_{i=1}^k Q_i$. For every PSC \mathcal{Q} which consists of at least two cliques, every $v \in N'_G(Q_i)$ has a neighbor which is not adjacent to the vertices of Q_i and hence, the following statement holds by definitions.

Observation 3.1 *If \mathcal{Q} is a perfect set of cliques in a graph G and $|\mathcal{Q}| \geq 2$, then every $Q_i \in \mathcal{Q}$ is a maximal homogeneous clique in G .*

Given a graph G , its *MHC-contraction* is the graph \widehat{G} obtained from G by contracting every maximal homogeneous clique into one vertex. Equivalently, a graph isomorphic to \widehat{G} is obtained from G by sequentially deleting one of two true twins while such a pair exists. For a vertex $v \in V(G)$, the corresponding vertex in \widehat{G} will be denoted by \widehat{v} ; that is, for any two vertices $u, v \in V(G)$, the vertices \widehat{u} and \widehat{v} are identical in \widehat{G} , if and only if u and v are true twins in G . For any two non-twin vertices $u, v \in V(G)$, by definition, we have $d_G(u, v) = d_{\widehat{G}}(\widehat{u}, \widehat{v})$. Observe that if Q is a homogeneous clique in G and a vertex $v \in Q$ is dominated by a set $D \subseteq V(G)$, then every vertex $u \in Q$ is dominated by D . Similarly, if $v, u \in Q$ and $v \in D$, then $(D \cup \{u\}) \setminus \{v\}$ and D dominate the same set of vertices. It follows that $|D \cap Q| \leq 1$ holds for every minimum dominating set D and homogeneous clique Q . We also infer the following facts.

Observation 3.2 *Every graph G and its MHC-contraction \widehat{G} satisfy the following statements:*

- (i) $\gamma(G) = \gamma(\widehat{G})$;
- (ii) $\gamma_g(G) = \gamma_g(\widehat{G})$;
- (iii) $\gamma'_g(G) = \gamma'_g(\widehat{G})$;
- (iv) G is a γ_g -perfect graph if and only if \widehat{G} is γ_g -perfect;
- (v) G is a γ'_g -perfect graph if and only if \widehat{G} is γ'_g -perfect.

We will refer to the following graph operators:

- Given a graph G and a positive integer s , $G \uplus K_s$ denotes the vertex disjoint union of G and the complete graph K_s .
- If G is a graph, v a new vertex and \mathcal{Q} a perfect set of cliques in G , then the graph $\mathcal{O}(G, v, \mathcal{Q})$ is obtained from G by adding the vertex v and making it adjacent to all vertices in $V(G) \setminus V(\mathcal{Q})$.

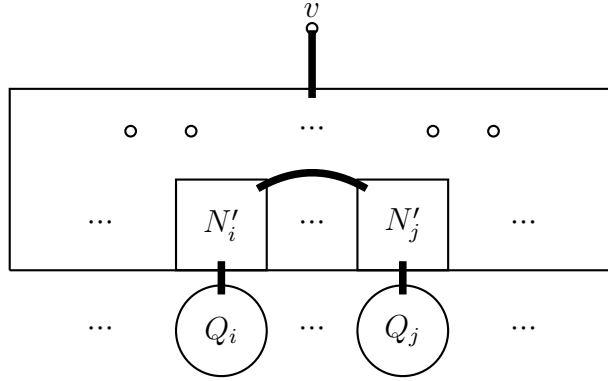


Figure 1: An illustration of the operator $\mathcal{O}(G, v, \mathcal{Q})$. The thick lines represent joins between the connected sets, $N'_i = N'_G(Q_i)$, and $N'_j = N'_G(Q_j)$.

When referring to the operators, we will always assume that $s \in \mathbb{N}$ and \mathcal{Q} is a PSC in G . Note that, under this assumption, $\gamma(G) \geq |\mathcal{Q}|$ must be true as $d_G(Q_i, Q_j) > 2$ for every homogeneous cliques Q_i and Q_j from \mathcal{Q} and therefore, the domination of the entire $V(\mathcal{Q})$ needs at least $|\mathcal{Q}|$ different vertices in G .

In Subsection 3.3 we will show that every γ_g -perfect graph can be built from an isolated vertex by using these two operators. In particular, we will prove that, if G is γ_g -perfect, then both $G \cup K_s$ and $\mathcal{O}(G, v, \mathcal{Q})$ are γ_g -perfect. As a preliminary result, we show that under the stronger condition $\gamma(G) > |\mathcal{Q}|$ the operator \mathcal{O} always gives a γ_g -graph even if G is not γ_g -perfect.

Proposition 3.3 *If \mathcal{Q} is a perfect set of cliques in G and $\gamma(G) > |\mathcal{Q}|$, then the graph $G' = \mathcal{O}(G, v, \mathcal{Q})$ is a γ_g -graph.*

Proof. If $\mathcal{Q} = \emptyset$, then v is a universal vertex and $\gamma(G') = \gamma_g(G') = 1$. Otherwise, let $\mathcal{Q} = \{Q_1, \dots, Q_k\}$. Choosing one vertex $x_i \in Q_i$ for every $i \in [k]$, we observe that v, x_1, \dots, x_k form a dominating set in G' and therefore, $\gamma(G') \leq k + 1$. On the other hand, any dominating set D which contains v , must contain a vertex d_i , which is different from v , to dominate x_i for every $i \in [k]$. If $j \neq \ell$, then $d_G(x_j, x_\ell) = 3$ and hence, $d_j \neq d_\ell$. This proves $|D| \geq k + 1$ if $v \in D$. If $v \notin D$, then D is a dominating set also in G and, by our condition, $|D| \geq \gamma(G) \geq k + 1$. Therefore, $\gamma(G') = k + 1$ holds.

In the domination game, let Dominator play v as his first move. In the later turns, no matter how both players play, the domination of the homogeneous cliques Q_1, \dots, Q_k needs exactly k further vertices. This strategy of Dominator shows that $\gamma_g(G') \leq k + 1 = \gamma(G')$. Since $\gamma_g(G') \geq \gamma(G')$ is also true, we conclude that $\gamma_g(G') = \gamma(G') = k + 1$. \square

Extending a graph G with a universal vertex, that is, constructing $\mathcal{O}(G, v, \emptyset)$, always results in a γ_g -graph. Already this simple fact shows that any graph can be embedded into a γ_g -graph and consequently, the class of γ_g -graphs does not admit a forbidden subgraph characterization.

In Subsection 3.3, we will often use the following lemma that gives characterizations of γ_g -graphs with small domination number. Each of these statements was either observed in [18] or can be obtained as a direct consequence of the earlier statements.

Proposition 3.4 *The following statements hold for every graph G .*

- (i) $\gamma(G) = \gamma_g(G) = 1$ holds if and only if $\gamma(G) = 1$.
- (ii) $\gamma(G) = \gamma_g(G) = 2$ holds if and only if G does not have a universal vertex but there exists a vertex $v \in V(G)$ such that $V(G) \setminus N[v]$ induces a homogeneous clique in G .
- (iii) If $\Delta(G) = |V(G)| - 2$, then $\gamma(G) = \gamma_g(G) = 2$.
- (iv) If G does not contain true twins, then $\gamma(G) = \gamma_g(G) = 2$ is true if and only if $\Delta(G) = |V(G)| - 2$.

3.2 Minimally γ_g -imperfect graphs

In this subsection, we identify a collection of minimally γ_g -imperfect graphs.

First, define the set \mathcal{F} of six bipartite graphs, see Fig. 2. The smallest one of them is $2P_3$, the largest is $K_{3,3}$, and all the remaining four members of \mathcal{F} are sandwiched between them.

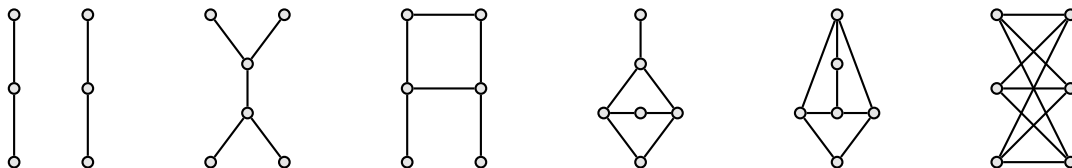


Figure 2: The six graphs contained in \mathcal{F} . From left to right, we denote them by F_1, \dots, F_6 .

Second, recall that the co-domino graph is the graph shown in Fig. 3 left, and that the complements of cycles $\overline{C_n}$, $n \geq 5$, are known as anti-holes; the anti-hole $\overline{C_6}$ is drawn in Fig. 3 right.

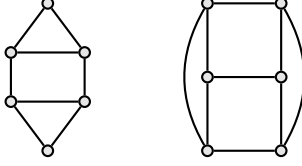


Figure 3: Co-domino (left) and the complement of C_6 (right).

Proposition 3.5 *The following graphs are minimally γ_g -imperfect:*

- (i) *the path P_5 ;*
- (ii) *the co-domino;*
- (iii) *the anti-hole $\overline{C_n}$ for every $n \geq 5$;*
- (iv) *each graph from \mathcal{F} .*

Proof. The graphs referred to in (i), (ii), and (iv) can be checked one-by-one by using Proposition 3.4. For an anti-hole $\overline{C_n}$ with $n \geq 5$, we first observe that $\gamma(\overline{C_n}) = 2$ as any two independent vertices form a dominating set. On the other hand, after playing an arbitrary vertex as a first move in the D-game, two non-twin vertices remain undominated and, as follows from Proposition 3.4 (ii), we have $\gamma_g(\overline{C_n}) = 3$. Taking an arbitrary proper induced subgraph H of $\overline{C_n}$, either $\gamma(H) = 1$ and Proposition 3.4 (i) implies $\gamma_g(H) = 1$, or there is a vertex with $d_H(v) = |V(H)| - 2$ and based on Proposition 3.4 (iii) we may infer $\gamma(H) = \gamma_g(H) = 2$. Note that the first case occurs when \overline{H} , which is a proper induced subgraph of the cycle C_n , contains an isolated vertex; the second case occurs if the minimum degree of \overline{H} equals 1. Therefore, $\overline{C_n}$ is not a γ_g -graph but its every proper induced subgraph is a γ_g -graph. This completes the proof for (iii). \square

In the next subsection we will prove that every minimally γ_g -imperfect graph has $\gamma(G) = 2$. We will often refer there to the following statement.

Proposition 3.6 *If there exist two different vertices u and v in a graph G such that both $N_G[u] \setminus N_G[v]$ and $N_G[v] \setminus N_G[u]$ contain two nonadjacent vertices, then G is not $2\text{-}\gamma_g$ -perfect.*

Proof. Suppose that x_1 and x_2 are two independent vertices from $N_G[u] \setminus N_G[v]$ and that y_1, y_2 are two independent vertices from $N_G[v] \setminus N_G[u]$. These assumptions directly imply that both $\{v, x_1, x_2\}$ and $\{u, y_1, y_2\}$ are independent vertex sets in G .

Hence, the six vertices induce a bipartite graph H with partite classes of size 3. Checking all the possibilities, we get that H is either isomorphic to a minimally γ_g -imperfect graph from \mathcal{F} , or contains P_5 . We may conclude that, under the given conditions, G cannot be $2\text{-}\gamma_g$ -perfect. \square

3.3 Characterization

Our goal here is to prove two main results, namely Theorems 3.7 and 3.8. Their proofs will be given at the end of the subsection.

Theorem 3.7 *The following statements are equivalent:*

- (i) G is γ_g -perfect.
- (ii) G is $2\text{-}\gamma_g$ -perfect.
- (iii) G can be obtained from an isolated vertex by repeatedly applying the following operators:
 - For a graph F , and for an $s \in \mathbb{N}$, take $F \cup K_s$;
 - For a graph F , and for a PSC \mathcal{Q} , take $\mathcal{O}(F, v, \mathcal{Q})$.

Theorem 3.8 *Every minimally γ_g -imperfect graph has domination number 2.*

Before we proceed with the proofs, let us consider an example. Figure 4 presents a construction of a γ_g -perfect graph on eight vertices with the consecutive application of operators described in Theorem 3.7 (iii). Note that by considering the same PSC and just iteratively applying operator \mathcal{O} , we get an infinite family of γ_g -perfect graphs.

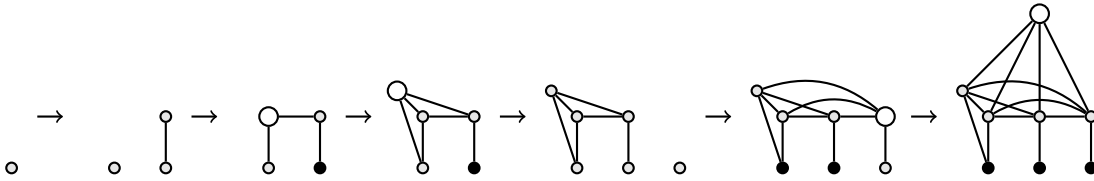


Figure 4: A construction of a γ_g -perfect graph. At the steps where operator \mathcal{O} is applied, the PSC is marked in black and the newly added vertex is slightly larger.

First, we prove that the operator $\mathcal{O}(G, v, \mathcal{Q})$ results in a γ_g -graph if G is 2- γ_g -perfect. We remark that the set of undominated homogeneous cliques referred to at the end of the theorem is not necessarily the same as \mathcal{Q} .

Theorem 3.9 *If G is a 2- γ_g -perfect graph and \mathcal{Q} is a perfect set of cliques in G , then $\mathcal{O}(G, v, \mathcal{Q})$ is a γ_g -graph. Further, there is an optimal start vertex for Dominator in the D-game on $\mathcal{O}(G, v, \mathcal{Q})$ such that after this first move only a set of homogeneous cliques remains undominated.*

Proof. Consider graphs G and $G' = \mathcal{O}(G, v, \mathcal{Q})$ that satisfy the conditions of the theorem. First suppose that $\mathcal{Q} = \emptyset$. Then, by Proposition 3.4 (i), we have $\gamma(G') = \gamma_g(G')$ and the optimal start vertex for Dominator is v . If $\mathcal{Q} = \{Q_1\}$, then $\gamma(G')$ equals either 1 or 2 and, by Proposition 3.4 (i) or (ii), the equality $\gamma(G') = \gamma_g(G')$ is true. If $\gamma(G') = 1$, there is a universal vertex which is an optimal start vertex; if $\gamma(G') = 2$, v is an optimal start vertex and $N_{G'}[v]$ omits only Q_1 .

From now on, we assume that $\mathcal{Q} = \{Q_1, \dots, Q_k\}$, where $k \geq 2$. For every $i \in [k]$ we introduce the notation $N'_i = N'_G(Q_i)$ and $N' = \bigcup_{i=1}^k N'_i$. A part of the second neighborhood of a homogeneous clique Q_i is specified as $N''_i = N'_G(N'_i) \setminus (N' \cup V(\mathcal{Q}))$; we also define $N'' = \bigcup_{i=1}^k N''_i$. By definition, and since any two different cliques Q_i and Q_j are at distance 3, the sets N'_1, \dots, N'_k are pairwise disjoint and the same is true for the sets $Q_1 \cup N'_1, \dots, Q_k \cup N'_k$ but it does not hold necessarily for N''_1, \dots, N''_k . Observe further that, by definition of PSC and by the condition $k \geq 2$, the set N''_i is not empty for any $i \in [k]$.

By Proposition 3.3, $\gamma(G') = \gamma_g(G')$ is true if $\gamma(G) > k$. Note that in this case v is an optimal start vertex in the D-game and $V(G') \setminus N_{G'}[v] = V(\mathcal{Q})$.

From now on, we may assume that there is a dominating set D of cardinality k in G . We will also suppose that, under this condition, D is chosen such that $|D \cap N'|$ is maximum. To dominate all the cliques Q_1, \dots, Q_k , the set D must contain at least one vertex from each $Q_i \cup N'_i$. Consequently, $D \subseteq V(\mathcal{Q}) \cup N'$ and D contains exactly one vertex, say d_i , from each $Q_i \cup N'_i$.

Under the condition $\gamma(G) = k \geq 2$, we continue the proof with a series of claims.

Claim 1 *There exists a vertex y in $N' \cap D$ such that $N'' \subseteq N[y]$.*

Proof. Suppose that d_i and d_j are different vertices with $d_i \in N'_i \cap D$, and $d_j \in N'_j \cap D$ and let x_i be a vertex from Q_i . If d_i has a neighbor $z_i \in N''$ which is not a neighbor of d_j , then x_i and z_i are two independent vertices from $N[d_i] \setminus N[d_j]$. Similarly, if d_j has a neighbor $z_j \in N''$ which is not a neighbor of d_i , then $N[d_j] \setminus N[d_i]$ contains two independent vertices. These two facts together, by Proposition 3.6,

would contradict the $2\text{-}\gamma_g$ -perfectness of G . Hence, at least one of $(N[d_i] \setminus N[d_j]) \cap N''$ and $(N[d_j] \setminus N[d_i]) \cap N''$ is empty, hence $N[d_i] \cap N''$ is a subset of $N[d_j] \cap N''$ or vice versa. This defines a linear ordering for the sets $N[d_i] \cap N''$, $i \in [k]$, even if $|N' \cap D| = 1$. As D dominates all vertices from N'' , there exists a vertex in $N' \cap D$ that dominates the entire N'' . \square

Claim 2 *There is at most one $i \in [k]$ such that the set N'_i does not induce a complete subgraph in G .*

Proof. Suppose for a contradiction that neither $G[N'_i]$ nor $G[N'_j]$ is complete. Then, for every $x_i \in Q_i$ and $x_j \in Q_j$, both sets $N[x_i] \setminus N[x_j]$ and $N[x_j] \setminus N[x_i]$ contain nonadjacent vertices. By Proposition 3.6, this contradicts the $2\text{-}\gamma_g$ -perfectness of G . \square

Claim 3 *If $N'' = \emptyset$, then $\gamma(G') = \gamma_g(G')$.*

Proof. By Claim 2 and since $k \geq 2$, there is a set N'_j that induces a complete subgraph in G . As \mathcal{Q} is a perfect set of cliques in G , any vertex $y_j \in N'_j$ dominates the entire $N' \cup Q_j \cup \{v\}$ in G' . If Dominator first plays such a vertex y_j , only the $k-1$ homogeneous cliques different from Q_j remain undominated because $\gamma(G) = k$. In the continuation of the game, under any strategy of Staller and Dominator, exactly one homogeneous clique will be dominated with each move. Therefore, this (optimal) first move of Dominator ensures that the game finishes within k moves. Consequently, we have $\gamma_g(G') \leq k = \gamma(G)$ and may conclude $\gamma(G') = \gamma_g(G')$. \square

Claim 4 *If $N'' \neq \emptyset$, then $\gamma(G') = \gamma_g(G')$.*

Proof. First, suppose that there is no incomplete N'_i and, consequently, N' induces a complete subgraph in G' . Then, by Claim 1, Dominator may choose a vertex $y \in N'$ which dominates the entire $N'' \cup N' \cup \{v\}$ and also dominates one homogeneous clique Q_i . Note that, by the condition $\gamma(G) = |\mathcal{Q}|$, all the vertices of G' are contained in $V(\mathcal{Q}) \cup N' \cup N'' \cup \{v\}$. Thus, after the first move y of Dominator, only $k-1$ cliques from \mathcal{Q} remain undominated and the game will be finished with k moves. This proves $\gamma(G') = \gamma_g(G')$.

In the other case, when N' is not complete, we might also have a vertex from N' that dominates the entire $N'' \cup N' \cup \{v\}$. This implies $\gamma(G') = \gamma_g(G')$, again.

What remains is to consider the case when we do not have a vertex in N' which dominates the entire $N' \cup N''$. By Claim 1, we have a vertex $y \in N'$ that dominates N'' . We may assume, without loss of generality, that $y \in N'_1$. Then there is a

vertex $y' \in N'_1$ which is not dominated by y . Since by Claim 2, N'_2 induces a clique, any vertex z from N'_2 dominates the entire N' but does not dominate N'' . Fixing any such vertex z , there exists a vertex w from N'' which is nonadjacent to z but adjacent to y . Further, let $x_1 \in Q_1$ and $x_2 \in Q_2$ be two arbitrary vertices from the homogeneous cliques. Observe that the two independent vertices x_1 and w belong to $N_G[y] \setminus N_G[z]$ and also that the independent vertices x_2 and y' are contained in $N_G[z] \setminus N_G[y]$. By Proposition 3.6, this case is not possible as it contradicts the $2\text{-}\gamma_g$ -perfectness of G . \square

Our previous discussions on the cases $|\mathcal{Q}| = 0$ and $|\mathcal{Q}| = 1$, Proposition 3.3, Claims 3 and 4 together imply $\gamma(G') = \gamma_g(G')$ for any $G' = \mathcal{O}(G, v, \mathcal{Q})$, where G is $2\text{-}\gamma_g$ -perfect. An optimal start vertex with the required property was identified for all cases. This finishes the proof of Theorem 3.9. \square

After proving a lemma, we will show that every $2\text{-}\gamma_g$ -perfect graph G can be obtained from another $2\text{-}\gamma_g$ -perfect graph F by using the operator disjoint union with a complete graph or the operator \mathcal{O} . We say that F is a $\gamma\text{-}2\text{-maximal subgraph}$ of G if it is an induced subgraph of G with $\gamma(F) \leq 2$ and inclusion-wise maximal with this property. That is, for any induced subgraph F' of G with $V(F) \not\subseteq V(F')$ we have $\gamma(F') \geq 3$.

Lemma 3.10 *Let G be a $2\text{-}\gamma_g$ -perfect graph and let v be a vertex in G such that $d_{\widehat{G}}(\widehat{v}) = \Delta(\widehat{G})$. If F is a $\gamma\text{-}2\text{-maximal subgraph}$ of G which contains the entire $N[v]$, then \widehat{v} is a vertex of maximum degree in \widehat{F} and v is an optimal start vertex in the D -game on F .*

Proof. Under the given conditions, $\gamma(F) = \gamma_g(F) = 2$. By Observation 3.2 and Proposition 3.4, this implies $\Delta(\widehat{F}) = |V(\widehat{F})| - 2$ and hence, there is a vertex \widehat{u} which is adjacent to all but one vertex of \widehat{F} . Now, assume for a contradiction that $d_{\widehat{F}}(\widehat{v}) < d_{\widehat{F}}(\widehat{u})$. We consider two cases under this assumption.

- First suppose that $N_G[v] \subseteq N_G[u]$. This implies $N_{\widehat{G}}[\widehat{v}] \subseteq N_{\widehat{G}}[\widehat{u}]$ and, since \widehat{v} is of maximum degree in \widehat{G} , we have $d_{\widehat{G}}(\widehat{v}) = d_{\widehat{G}}(\widehat{u})$. Consequently, $N_{\widehat{G}}[\widehat{v}] = N_{\widehat{G}}[\widehat{u}]$ and $N_G[v] = N_G[u]$ hold. The latter equality implies $N_F[v] = N_F[u]$ that contradicts the assumption $d_{\widehat{F}}(\widehat{v}) < d_{\widehat{F}}(\widehat{u})$.
- In the other case, we suppose that u is not adjacent to all vertices of $N_G[v]$. Since \widehat{u} has degree $|V(\widehat{F})| - 2$ in \widehat{F} , there is exactly one maximal homogeneous clique Q in F such that $Q \subseteq N_G[v] \setminus N_G[u]$. Note that, since $\gamma(F) = 2$, there exist some vertices in F (and in G) which are adjacent to u but not adjacent to v . Then, $d_{\widehat{F}}(\widehat{v}) < d_{\widehat{F}}(\widehat{u})$ and $d_{\widehat{G}}(\widehat{v}) \geq d_{\widehat{G}}(\widehat{u})$ imply that the clique

Q is homogeneous in F and non-homogeneous in G . That is, there are two vertices, say x_1 and x_2 in Q such that x_1 has a neighbor z with $zx_2 \notin E(G)$ and $z \notin V(F)$. Then, $N_G[u] \cup N_G[x_1]$ contains $V(F)$ as a proper subset. The subgraph $G[N_G[u] \cup N_G[x_1]]$ is dominated by u and x_1 and, therefore, F cannot be a γ -2-maximal subgraph of G .

As both possible cases were concluded with contradictions, we infer that \widehat{v} is a vertex of maximum degree in \widehat{F} and, since \widehat{F} is a γ_g -graph, $d_{\widehat{F}}(\widehat{v}) = |V(\widehat{F})| - 2$. Thus, in the D-game on F , the vertex v is an optimal start vertex for Dominator. \square

Theorem 3.11 *Every 2- γ_g -perfect graph G can be constructed in the following way:*

- (i) *If G is disconnected, then it can be obtained as $G' \cup K_s$, where G' is also 2- γ_g -perfect.*
- (ii) *If G is connected, then for every vertex $v \in V(G)$ with $d_{\widehat{G}}(\widehat{v}) = \Delta(\widehat{G})$, there exists a perfect set of cliques \mathcal{Q} in $G - v$ so that $G = \mathcal{O}(G - v, v, \mathcal{Q})$. Further, $G - v$ is a 2- γ_g -perfect graph.*

Proof. (i) Since every induced subgraph F of G of domination number 2 is a γ_g -graph, G is $2P_3$ -free and hence, it cannot contain more than one non-complete component. This yields that a disconnected G can always be obtained as a disjoint union $G' \cup K_s$. Since G' is an induced subgraph of G , it is 2- γ_g -perfect as well.

(ii) Consider a connected graph G satisfying the conditions in the theorem and consider a vertex v with $d_{\widehat{G}}(\widehat{v}) = \Delta(\widehat{G})$. We prove that the vertices outside $N_G[v]$ form a PSC in G . It is clearly true, if $\gamma(G) = 1$ and v is a universal vertex.

First, suppose that there are two vertices x and x' in $V(G) \setminus N_G[v]$ such that x and x' are not twins in G but they are adjacent. Then, we have a vertex z in G which is adjacent to exactly one of x and x' ; we may suppose that $xz \in E(G)$ and $x'z \notin E(G)$. Since $\{v, x\}$ is a 2-element dominating set in the subgraph induced by $N_G[v] \cup \{z, x, x'\}$, we may consider a γ -2-maximal subgraph F of G which contains all the vertices from $N_G[v] \cup \{z, x, x'\}$. By Lemma 3.10, the vertex \widehat{v} must be of maximum degree in \widehat{F} and in particular, $d_{\widehat{F}}(\widehat{v}) = |V(\widehat{F})| - 2$ must hold. This contradicts the fact that v is not adjacent in F to at least two non-twin vertices, namely to x and x' . This contradiction proves that $V(G) \setminus N_G[v]$ consists of components Q_1, \dots, Q_k which are homogeneous cliques in G .

Secondly, we prove that no two of the homogeneous cliques Q_1, \dots, Q_k are at distance 2. Suppose, to the contrary, that two vertices, x_i from Q_i and x_j from Q_j , where $i \neq j$, have a common neighbor y in G . All vertices in $N_G[v] \cup \{x_i, x_j\}$ are dominated by v and y . Therefore, we may consider again a γ -2-maximal subgraph

F of G which contains all the vertices from $N_G[v] \cup \{x_i, x_j\}$. By Lemma 3.10, $d_{\widehat{F}}(\widehat{v}) = |V(\widehat{F})| - 2$ must hold. On the other hand, as the non-twin vertices x_i and x_j are outside $N_G[v]$, we have $d_{\widehat{F}}(\widehat{v}) \leq |V(\widehat{F})| - 3$. This contradiction proves that $d(Q_i, Q_j) \geq 3$ for any two different indices i and j . This is true when either G or $G - v$ is considered.

Finally, observe that there is an edge between any two vertices y_i and y_j whenever $y_i \in N'_i$ and $y_j \in N'_j$, $i \neq j$. Indeed, in case of $y_i y_j \notin E(G)$ we would have an induced P_5 , namely $x_i y_i v y_j x_j$ in G . Since $\gamma(P_5) = 2 < \gamma_g(P_5)$, this contradicts the condition in the theorem. We conclude that, under the conditions of part (ii), $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ is a PSC in $G - v$ and G can be obtained as $\mathcal{O}(G - v, v, \mathcal{Q})$. \square

Proposition 3.12 *If G is a $2\text{-}\gamma_g$ -perfect graph, then $G \uplus K_s$ is a γ_g -graph.*

Proof. If G consists of c complete components, then $\gamma(G \uplus K_s) = \gamma_g(G \uplus K_s) = c + 1$. Otherwise, G contains c complete components ($c \geq 0$) and exactly one component, say G' , which is not a complete graph. Since G' is $2\text{-}\gamma_g$ -perfect and connected, by Theorem 3.11 (ii) and Theorem 3.9, there is an optimal first move v for Dominator such that $V(G') \setminus N[v]$ consists of k homogeneous cliques. If Dominator plays this vertex v as his first move on the entire $G \uplus K_s$, then $k + c + 1$ homogeneous cliques remain undominated and we have $\gamma(G \uplus K_s) = \gamma_g(G \uplus K_s) = k + c + 2$. This proves that $G \uplus K_s$ is a γ_g -graph. \square

Now we are ready to prove the characterization theorem.

Proof of Theorem 3.7. First, we show that (ii) implies (i). Note that the statement is clearly true for graphs of small order and then, we may proceed by induction on the number of vertices in G . Suppose that G is $2\text{-}\gamma_g$ -perfect. By definition, it is also true for every induced subgraph of G . Hence, for every proper induced subgraph F of G , the induction hypothesis implies that F is γ_g -perfect. As for G itself, if it is disconnected then, by Theorem 3.11 (i), G can be obtained as $G' \uplus K_s$ from a $2\text{-}\gamma_g$ -perfect G' . Proposition 3.12 then implies $\gamma(G) = \gamma_g(G)$. In the other case, G is connected and, by Theorem 3.11 (ii) and Theorem 3.9, we have $\gamma(G) = \gamma_g(G)$ again. This proves that every $2\text{-}\gamma_g$ -perfect graph is γ_g -perfect, that is, (ii) \Rightarrow (i). The other direction immediately follows from the definitions. Therefore, G is γ_g -perfect if and only if it is $2\text{-}\gamma_g$ -perfect.

Now, we prove that (ii) is equivalent with (iii). By Theorem 3.11, each $2\text{-}\gamma_g$ -perfect graph G can be obtained from an appropriate smaller $2\text{-}\gamma_g$ -perfect graph G' as $G = G' \uplus K_s$ or as $G = \mathcal{O}(G', v, \mathcal{Q})$. Applying the theorem for the $2\text{-}\gamma_g$ -perfect G' and then, repeatedly, for the smaller graphs, the process ends with K_1 . This proves the implication (ii) \Rightarrow (iii).

To prove the other direction, we proceed by induction on the order of the graph. Suppose that a graph H can be built from K_1 by using the two operators specified in (iii). As K_1 is a γ_g -graph and Theorem 3.9 and Proposition 3.12 say that the operators preserve this property, H is a γ_g -graph. We consider an arbitrary induced proper subgraph F of H which satisfies $\gamma(F) = 2$.

- If $H = H' \cup K_s$ and H' can be built from K_1 by using the two operators, then, by the induction hypothesis, H' is $2\text{-}\gamma_g$ -perfect. If F is an induced subgraph of H' , then F is also $2\text{-}\gamma_g$ -perfect. In the other case, since $\gamma(F) = 2$, F contains some vertices from K_s and meets H' in a subgraph of domination number 1. In either case, F is a γ_g -graph.
- Suppose that $H = \mathcal{O}(H', v, \mathcal{Q})$, where H' is built from K_1 by using the two operators. We have two cases again. First, assume that F is a subgraph of H' . Since H' is $2\text{-}\gamma_g$ -perfect by the induction hypothesis, the same is true for F . In the second case, F contains v and also contains some (outer) vertices which are outside $N_H[v]$, because otherwise it would hold $\gamma(F) = 1$. Since $\gamma(F) = 2$, these outer vertices belong to either one or two homogeneous cliques from \mathcal{Q} . If F meets only one homogeneous clique from \mathcal{Q} , then, no matter whether F is connected or not, v is an optimal start vertex in the D-game on F and we have $\gamma_g(F) = \gamma(F) = 2$. Finally, suppose that F meets two cliques from \mathcal{Q} , say Q_i and Q_j . If F contains vertices from both $N'_i = N'_H(Q_i)$ and $N'_j = N'_H(Q_j)$, then F also contains the join between $N'_i \cap F$ and $N'_j \cap F$. Thus, $V(F) \cap V(H')$ induces a subgraph H'' of H' such that $\mathcal{Q}' = \{V(F) \cap Q_i, V(F) \cap Q_j\}$ is a PSC in H'' and $F = \mathcal{O}(H'', v, \mathcal{Q}')$. Since H'' is an induced subgraph of H' , it is $2\text{-}\gamma_g$ -perfect, and Theorem 3.9 implies that F is a γ_g -graph. If F does not meet N'_i but, as it was assumed, F meets Q_i , then F is disconnected. In this case, $V(F) \cap Q_i$ induces a clique component in F , while the another component must have a universal vertex. This universal vertex is an optimal start vertex in the D-game and we have $\gamma_g(F) = \gamma(F) = 2$.

Hence, in H , every induced subgraph F of domination number 2 is a γ_g -graph. In other words, H is a $2\text{-}\gamma_g$ -perfect graph. We may conclude that (iii) implies (ii) and, therefore, (i), (ii), and (iii) are equivalent. \square

Proof of Theorem 3.8. Assume for a contradiction that there exists a graph G with $\gamma(G) \geq 3$ which is minimally γ_g -imperfect. That is, every proper induced subgraph of G , including all induced subgraphs of domination number 2, are γ_g -perfect. But then, Theorem 3.7 implies that G is also γ_g -perfect. This contradiction completes the proof. \square

4 Some applications

In this section we present some applications of the characterization from Section 3.

4.1 Recognition complexity

Here we demonstrate that the characterizations of γ_g -perfect graphs are constructive, more precisely we have:

Theorem 4.1 *Graphs that are γ_g -perfect can be recognized in polynomial time.*

Proof. Let G be an arbitrary graph. Then we first determine its connected components and in view of Theorem 3.7 discard the components that induce complete graphs. Clearly, this can be done in polynomial time. Then G is γ_g -perfect if and only if the remaining non-complete component H is γ_g -perfect. Let $v \in V(H)$ be an arbitrary vertex with the property that $d_{\widehat{H}}(\widehat{v}) = \Delta(\widehat{H})$. Clearly, such a vertex can be found in polynomial time. By Theorem 3.11, H is $2\text{-}\gamma_g$ -perfect (and hence γ_g -perfect by Theorem 3.7) if and only if $H - N[v]$ consists of a perfect set of cliques in $H - v$ and $H - v$ is $2\text{-}\gamma_g$ -perfect. Since checking whether $H - N[v]$ consists of a perfect set of cliques can be done in polynomial time, we have reduced in polynomial time the problem of whether H is γ_g -perfect to verifying whether $H - v$ is γ_g -perfect. Repeating the procedure on $H - v$ yields a polynomial recognition algorithm. \square

4.2 Triangle-free graphs

In this section, we study triangle-free γ_g -perfect and minimally γ_g -imperfect graphs. In particular, we also discuss trees. For this, we need the following notation. A graph $KC_{m,n}$, $m \geq 1, n \geq 0$, has vertices $\{c, d, u_1, \dots, u_m, v_1, \dots, v_n\}$ and edges $c \sim u_j \sim d$, $j \in [m]$, and $c \sim v_i$, $i \in [n]$. Additionally, set $KC_{0,n} = K_{1,n}$. Note that the graph $KC_{1,n}$ is the star $K_{1,n}$ with a pendant P_2 attached to the central vertex, and graphs $KC_{m,n}$, $m \geq 2$, contain 4-cycles.

Proposition 4.2 *A connected triangle-free graph is γ_g -perfect if and only if it is isomorphic to $KC_{m,n}$ for some $m, n \geq 0$.*

Proof. It can easily be checked, that graphs $KC_{m,n}$, for some $m, n \geq 0$, are all connected, triangle-free, and γ_g -perfect.

Now suppose G is a connected triangle-free γ_g -perfect graph. By Theorem 3.7, $G = \mathcal{O}(F, v, \mathcal{Q})$ for some triangle-free γ_g -perfect graph F . As G is triangle-free,

$|\mathcal{Q}| \in \{0, 1\}$ and each clique in \mathcal{Q} can have only one vertex. In addition, the vertices $V(F) \setminus V(\mathcal{Q})$ must induce an independent set, otherwise G would again contain a triangle.

If $|\mathcal{Q}| = 0$, then F is just a graph with no edges, thus $\mathcal{O}(F, v, \mathcal{Q})$ is a star, i.e. a graph $KC_{0,n}$. If $|\mathcal{Q}| = 1$, then F can only be a star with some additional independent vertices, where the only clique in \mathcal{Q} consists of the center of the star. In this case $\mathcal{O}(F, v, \mathcal{Q})$ is isomorphic to $KC_{m,n}$, $m \geq 1$. \square

Corollary 4.3 *A tree T is γ_g -perfect if and only if it is isomorphic to $KC_{0,n}$ or $KC_{1,n}$.*

Note that the last corollary can also be obtained from the characterization of γ_g -minimal trees [21].

In the rest of the subsection we investigate minimally γ_g -imperfect triangle-free graphs. We first prove a more general result.

Theorem 4.4 *The only disconnected minimally γ_g -imperfect graph is $2P_3$.*

Proof. Let G be a disconnected minimally γ_g -imperfect graph. From Theorem 3.8 it follows that $\gamma(G) = 2$. Thus G has exactly two connected components. At least one of these two components is non-complete, otherwise $\gamma_g(G) = 2$ and G is not imperfect. If both components are non-complete, then either $G = 2P_3$, or G contains $2P_3$ as an induced subgraph and is thus not a minimally γ_g -imperfect. The only remaining case is that one of the components is complete, and the other component, denote it with G' , is not. But then if Dominator starts the domination game by playing the universal vertex of G' , Staller has no other option but to finish the game. Thus $\gamma_g(G) = 2$ and G is not imperfect. \square

We now list all minimally γ_g -imperfect trees and then use this result to determine all triangle-free minimally γ_g -imperfect graphs.

Proposition 4.5 *The only minimally γ_g -imperfect trees are P_5 and the tree F_2 from the family \mathcal{F} .*

Proof. Let v be a leaf of a minimally γ_g -imperfect tree T . Then $T - v$ is a perfect tree, hence it is by Corollary 4.3 either $KC_{0,n}$ or $KC_{1,n}$. If $T - v = KC_{0,n}$, then attaching v either to c or to v_i , we get a perfect tree. If $T - v = KC_{1,n}$, then attaching v to c yields a perfect tree. On the other hand, attaching v to d or v_i results in a graph, which contains P_5 as an induced subgraph. Thus T is minimally γ_g -imperfect only if it is equal to P_5 . The remaining case is that v is attached to

u_1 . In this case, T contains the tree F_2 from the family \mathcal{F} as an induced subgraph, hence it is minimally γ_g -imperfect only if it is isomorphic to this tree. \square

Theorem 4.6 *The only minimally γ_g -imperfect triangle-free graphs are P_5 , C_5 and the graphs from \mathcal{F} .*

Proof. Let G be a minimally γ_g -imperfect triangle-free graph. If G is disconnected, then the only possibility is $F_1 = 2P_3$ by Theorem 4.4. If all non-pendant vertices of a connected graph G are cut vertices, then G is a tree and Proposition 4.5 settles this case. The only remaining case is that G is connected and has a vertex v which is not a cut vertex. This means that $G - v$ is a connected triangle-free γ_g -perfect graph, thus by Proposition 4.2, $G - v = KC_{m,n}$ for some $m, n \geq 0$. We consider different possibilities for $G - v$ and analyze how v can be added back to obtain the graph G . Note that as G is triangle-free, the neighborhood of v is an independent set. We use the notation introduced at the beginning of Subsection 4.2 and additionally set $\mathcal{V} = \{v_1, \dots, v_n\}$ and $\mathcal{U} = \{u_1, \dots, u_m\}$. Recall also the graphs from the family $\mathcal{F} = \{F_1, \dots, F_6\}$, see Fig. 2 again.

Note that whenever we determine a minimally γ_g -imperfect subgraph H of G , the only possibility for G to be minimally γ_g -imperfect is that $H = G$. Hence in the remaining part of the proof, we only determine minimally γ_g -imperfect subgraphs.

If $G - v = KC_{0,n}$, then, if v is connected only to c or if v is connected to some vertices in \mathcal{V} , we see that G is perfect. If $G - v = KC_{1,n}$, then we only need to consider cases where v belongs to a cycle in G (otherwise G is a tree and Proposition 4.5 settles this case). If v is adjacent to c and d , then G is perfect. If $N(v) \subseteq \mathcal{V}$, then G contains an induced P_5 . If v is adjacent to d and some vertices in \mathcal{V} , then G contains an induced C_5 . The remaining options yield a graph with triangles.

We now study the case $G - v = KC_{m,n}$, $m \geq 2$. We distinguish the following cases.

Case 1: v has only one neighbor in G .

- $n = 0$:

If v is adjacent to c or d , then G is perfect. The same holds if v is adjacent to u_1 or u_2 , and $m = 2$. But if $m \geq 3$ and v is adjacent to one of the vertices in \mathcal{U} , then G contains an induced F_4 .

- $n \geq 1$:

If v is adjacent to c , then G is perfect. If v is adjacent to d or a vertex in \mathcal{V} , then G contains an induced P_5 . If v is adjacent to a vertex in \mathcal{U} , then G contains an induced F_3 .

Case 2: v has at least two neighbors in G .

- $n = 0$:

If $m = 2$, then all cases can be easily checked, and we see that non of the obtained graphs is minimally γ_g -imperfect. Hence we suppose $m \geq 3$.

If v is adjacent to c and d , then G is perfect. If v is adjacent to exactly two vertices in \mathcal{U} , then G contains an induced F_5 . If v is adjacent to three or more vertices in \mathcal{U} , then G contains an induced F_6 .

- $n \geq 1$:

If v is adjacent to c and d , then G is perfect. If $N(v) \subseteq \mathcal{V} \cup \mathcal{U}$ and v is not adjacent to all vertices in \mathcal{U} , then G contains one of P_5 , F_3 , or F_5 as an induced subgraph. If $\mathcal{U} \subseteq N(v) \subseteq \mathcal{V} \cup \mathcal{U}$, then there are a few cases. If $m = 2$, then either G contains an induced F_4 (if v has no neighbors in \mathcal{V}) or G contains an induced F_5 (if v has at least one neighbor in \mathcal{V}). If $m \geq 3$, then G contains an induced F_6 . And finally, if v is adjacent to d and some vertices in \mathcal{V} , then G contains an induced C_5 .

It follows from the above case analysis, that G is minimally γ_g -imperfect only if it is equal to one of the graphs from $\{P_5, C_5\} \cup \mathcal{F}$. \square

5 Further types of perfectness

Proposition 5.1 *A graph G is γ'_g -perfect, if and only if it is the disjoint union of cliques.*

Proof. Since $\gamma(P_3) = 1 < \gamma'_g(P_3) = 2$, P_3 is a minimally γ'_g -imperfect graph. Then, G is P_3 -free; it is a disjoint union of some, say k , clique components. It is easy to see that $\gamma(G) = \gamma'_g(G) = k$. \square

Proposition 5.2 *An isolate-free graph G is γ_{tg} -perfect, if and only if it is $(P_4, \overline{2P_3})$ -free.*

Proof. Since $\gamma_t(P_4) = 2 < 3 = \gamma_{tg}(P_4)$ and $\gamma_t(\overline{2P_3}) = 2 < 3 = \gamma_{tg}(\overline{2P_3})$, it follows that an isolate-free, γ_{tg} -perfect graphs is $(P_4, \overline{2P_3})$ -free.

Suppose now that G is an isolate- and $(P_4, \overline{2P_3})$ -free graph. To prove that G is γ_{tg} -perfect we proceed by induction on $n(G)$, where the cases with $n(G) \leq 3$, that is, P_2 , P_3 , and K_3 , are clear.

Let $n(G) \geq 4$ and assume first that G is connected. Then, since G is P_4 -free, that is, a cograph, G is the join of two smaller cographs G_1 and G_2 . In the first

subcase assume that one of G_1 and G_2 is connected, say G_1 . Then G_1 is a smaller, connected cograph that satisfies the induction assumptions, hence G_1 is γ_{tg} -perfect with $\gamma_t(G_1) = 2$. Let Dominator start the total domination game played on G with an optimal start vertex from the game played on G_1 . If Staller replies with a move in G_2 , the game is over. And if Staller replies with a move in G_1 , the game is also over because $\gamma_{tg}(G_1) = 2$. In any case, $\gamma_{tg}(G) = \gamma_t(G) = 2$. In the second subcase none of G_1 and G_2 is connected. Since G contains no induced $\overline{2P_3}$, we infer that at least one of G_1 and G_2 , say G_1 , is edge-less. Then the first move of Dominator on a vertex from G_1 forces Staller to play her first move on G_2 , so that $\gamma_{tg}(G) = \gamma_t(G) = 2$ holds again.

If G is not connected, then it consists of $k \geq 2$ components, each being an isolate- and $(P_4, \overline{2P_3})$ -free graph. Then by the above, for each component G' we have $\gamma_{tg}(G') = \gamma_t(G') = 2$. Taking into account that each connected component is a co-graph and hence a join, it is now straightforward that $\gamma_{tg}(G) = \gamma_t(G) = 2k$.

We have thus proved that if G is an isolate- and $(P_4, \overline{2P_3})$ -free graph, then $\gamma_t(G) = \gamma_{tg}(G)$. Since each induced subgraph of G is also a $(P_4, \overline{2P_3})$ -free graph, G is γ_{tg} -perfect. \square

With similar but simpler arguments as used in the proof of Proposition 5.2 we also get the following.

Proposition 5.3 *An isolate-free graph G is γ'_{tg} -perfect, if and only if it is a cograph.*

6 Concluding remarks

By a computer search we have obtained all γ_g -perfect and all minimally γ_g -imperfect graphs on up to 9 vertices. The results are presented in Table 1.

In Proposition 3.5 we have proved that the anti-holes $\overline{C_n}$, $n \geq 5$, are minimally γ_g -imperfect graphs and also identified eight additional examples of minimally γ_g -imperfect graphs. Using computer we have verified that there are no additional such graphs on up to nine vertices. Based on these facts, as well as on the results of Theorems 4.4 and 4.6, we pose:

Conjecture 6.1 *There are no other minimally γ_g -imperfect graphs but those listed in Proposition 3.5.*

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n	perfect all	perfect connected	min. imperfect
3	4	2	0
4	11	6	0
5	32	19	2
6	122	81	8
7	536	386	1
8	2754	2102	1
9	15752	12476	1

Table 1: The number of all, and the number of connected γ_g -perfect graphs on n vertices. The last column represents the number of all minimally γ_g -imperfect graphs on n vertices.

References

- [1] J. D. Alvarado, S. Dantas, D. Rautenbach, Perfectly relating the domination, total domination, and paired domination numbers of a graph, *Discrete Math.* 338 (2015) 1424–1431.
- [2] S. Arumugam, Bibin K. Jose, Cs. Bujtás, Zs. Tuza, Equality of domination and transversal numbers in hypergraphs, *Discrete Appl. Math.* 161 (2013) 1859–1867.
- [3] G. Boruzanlı Ekinci, Cs. Bujtás, On the equality of domination number and 2-domination number, arXiv:1907.07866.
- [4] B. Brešar, S. Klavžar, D.F. Rall, Domination game and an imagination strategy, *SIAM J. Discrete Math.* 24 (2010) 979–991.
- [5] Cs. Bujtás, On the game total domination number, *Graphs Combin.* 34 (2018) 415–425.
- [6] Cs. Bujtás, M.A. Henning, Zs. Tuza, Transversal game on hypergraphs and the $\frac{3}{4}$ -conjecture on the total domination game, *SIAM J. Discrete Math.* 30 (2016) 1830–1847.
- [7] E. Camby, F. Plein, A note on an induced subgraph characterization of domination perfect graphs, *Discrete Appl. Math.* 217 (2017) 711–717.

- [8] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, *Ann. of Math.* 164 (2006) 51–229.
- [9] D. G. Corneil, H. Lerchs, L. Stewart Burlingham, Complement reducible graphs, *Discrete Appl. Math.* 3 (1981) 163–174.
- [10] P. Dorbec, G. Košmrlj, G. Renault, The domination game played on unions of graphs, *Discrete Math.* 338 (2015) 71–79.
- [11] M.A. Henning, S. Jäger, D. Rautenbach, Hereditary equality of domination and exponential domination, *Discuss. Math. Graph Theory* 38 (2018) 275–285.
- [12] M.A. Henning, S. Klavžar, D.F. Rall, Total version of the domination game, *Graphs Combin.* 31 (2015) 1453–1462.
- [13] M.A. Henning, S. Klavžar, D.F. Rall, The $4/5$ upper bound on the game total domination number, *Combinatorica* 37 (2017) 223–251.
- [14] M. A. Henning, S. Klavžar, D. F. Rall, Game total domination critical graphs, *Discrete Appl. Math.* 250 (2018) 28–37.
- [15] M. A. Henning, A. Yeo, *Total Domination in Graphs*, Springer, New York, 2013.
- [16] T. James, S. Klavžar, A. Vijayakumar, The domination game on split graphs, *Bull. Aust. Math. Soc.* 99 (2019) 327–337.
- [17] W.B. Kinnersley, D.B. West, R. Zamani, Extremal problems for game domination number, *SIAM J. Discrete Math.* 27 (2013) 2090–2107.
- [18] S. Klavžar, G. Košmrlj, S. Schmidt, On graphs with small game domination number, *Appl. Anal. Discrete Math.* 10 (2016) 30–45.
- [19] S. Klavžar, D.F. Rall, Domination game and minimal edge cuts, *Discrete Math.* 342 (2019) 951–958.
- [20] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* 2 (1972) 253–267.
- [21] A.J. Nadjafi-Arani, M. Siggers, H. Soltani, Characterisation of forests with trivial game domination numbers, *J. Comb. Optim.* 32 (2016) 800–811.
- [22] D. P. Sumner, J. I. Moore, Domination perfect graphs, *Notices Amer. Math. Soc.* 26 (1979) A-569.

- [23] K. Xu, X. Li, On domination game stable graphs and domination game edge-critical graphs, *Discrete Appl. Math.* 250 (2018) 47–56.
- [24] I. E. Zverovich, V. E. Zverovich, An induced subgraph characterization of domination perfect graphs, *J. Graph Theory* 20 (1995) 375–395.