On the average Steiner 3-eccentricity of trees[*]

Xingfu Li[a], Guihai Yu[a], Sandi Klavžar[b,c,d]

[a] College of Big Data Statistics, Guizhou University of Finance and Economics
Guiyang, Guizhou, 550025, China
xingfulisdu@qq.com; yuguihai@mail.gufe.edu.cn
[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[c] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
sandi.klavzar@fmf.uni-lj.si

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Abstract

The *Steiner k-eccentricity* of a vertex v of a graph G is the maximum Steiner distance over all k-subsets of $V(G)$ which contain v. In this paper Steiner 3-eccentricity is studied on trees. Some general properties of the Steiner 3-eccentricity of trees are given. A tree transformation which does not increase the average Steiner 3-eccentricity is given. As its application, several lower and upper bounds for the average Steiner 3-eccentricity of trees are derived.

Keywords: Steiner distance, Steiner tree, Steiner eccentricity, average Steiner eccentricity

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1 Introduction

Throughout this paper, all graphs considered are simple and connected. If $G = (V(G), E(G))$ is a graph, then its order and size will be denoted by $n(G)$ and $m(G)$, respectively. If $S \subseteq V(G)$, then the *Steiner distance* $d_G(S)$ of S is the minimum size among all connected subgraphs of G containing S, that is,

$$d_G(S) = \min\{m(T) : \ T \text{ subtree of } G \text{ with } S \subseteq V(T)\}.$$

If $k \geq 2$ is an integer and $v \in V(G)$, then the *Steiner k-eccentricity* $\mathrm{ecc}_k(v, G)$ of v in G is the maximum Steiner distance over all k-subsets of $V(G)$ which contain v, that is,

$$\mathrm{ecc}_k(v, G) = \max\{d_G(S) : \ v \in S \subseteq V(G), |S| = k\}.$$

Note that $\mathrm{ecc}_2(v, G)$ is the standard eccentricity of the vertex v, that is, the largest distance between v and the other vertices of G.

Li, Mao, and Gutman [18] proposed the *k-th Steiner Wiener index* $SW_k(G)$ of G as

$$SW_k(G) = \sum_{S \in \binom{V(G)}{k}} d_G(S)\,.$$

Note that $SW_2(G) = W(G)$, the celebrated Wiener index of G. Motivated by the k-th Steiner Wiener index, we introduce the *average Steiner k-eccentricity* $\mathrm{aecc}_k(G)$ of G as the mean value of all vertices' Steiner k-eccentricities in G, that is,

$$\mathrm{aecc}_k(G) = \frac{1}{n(G)} \sum_{v \in V(G)} \mathrm{ecc}_k(v, G)\,.$$

In this notation, $\mathrm{aecc}_2(G)$ is just the standard average eccentricity of G, cf. [4, 8, 9, 10, 26]).

The Steiner tree problem on general graphs is NP-hard to solve [11, 17], but it can be solved in polynomial time on trees [3]. The Steiner distance has been extensively studied on special graph classes such as trees, joins, standard graph products, corona products, and others, see [2, 5, 13, 24, 28]. The average Steiner k-distance and its close companion the k-th Steiner Wiener index have been studied on trees, complete graphs, paths, cycles, complete bipartite graphs, and others, see [7, 14]. The average Steiner distance and the Steiner Wiener index were also extensively studied, see [6, 18, 19, 21, 29, 30]. Some work on the Steiner diameter is present in [24, 28]. Other topological indices related to the Steiner distance have also been investigated: Steiner Gutman index in [25], Steiner degree distance in [15], Steiner hyper-Wiener index in [27], multi-center Wiener index in [16], Steiner Harary index in [23], and Steiner (revised) Szeged index in [12]. Y. Mao wrote an extensive survey paper on the Steiner distance in graphs [22].

In this paper we focus on the average Steiner 3-eccentricity of trees. In the rest of this section we list additional definitions needed in this paper. Then, in Section 2, we present several structural properties of the Steiner k-eccentricity of trees and discuss the complexity of computing the average Steiner 3-eccentricity of trees. In Section 3, the average Steiner 3-eccentricity of trees is investigated under a special transformation. Section 4 presents the existence and a extremal graph of the π-transformation. Relying on this behavior and the properties, in the subsequent section we establish several lower and upper bonds on the average Steiner 3-eccentricity of trees. We conclude by presenting several topics for future research.

A vertex of a graph of degree 1 is a *leaf* or a *pendent vertex*, and if it is of degree at least 2, then it is an *internal vertex*. With $\ell(G)$ we denote the number of leaves of a graph G. A vertex of a tree of degree at least 3 is a *branching vertex*. An edge is *pendent* if it is incident to a pendent vertex in a graph. A path P of a graph G is a *pendent path* if one endpoint of P has degree 1 and each internal vertex of P has degree 2.

If H_1 and H_2 are subgraphs of G, then the distance $d_G(H_1, H_2)$ between H_1 and H_2 is defined as $\min\{d_G(h_1, h_2) : h_1 \in V(H_1), h_2 \in V(H_2)\}$. In particular, if H_1 is the one vertex graph with u being its unique vertex, then we will write $d_G(u, H_2)$ for $d_G(H_1, H_2)$. The *eccentricity of a subgraph* H in G is $\mathrm{ecc}_G(H) = \max\{d_G(v, H) : v \in V(G)\}$.

If $S \subseteq V(G)$ and T is subtree of G with $S \subseteq V(T)$ and $m(T) = d_G(S)$, then we say that T is an *S-Steiner tree* and that a vertex of S is a *terminal* of T. If $k \geq 2$ and $v \in V(G)$, then a k-set $S \subseteq V(G)$ is a *Steiner k-ecc v-set* (or *k-ecc v-set* for short) if $v \in S$ and $d_G(S) = \mathrm{ecc}_k(v, G)$; a corresponding tree that realizes $\mathrm{ecc}_k(v, G)$ will be called a *Steiner k-ecc v-tree* (or *k-ecc v-tree* for short). A vertex v may have more than one k-ecc v-set, and each such set may have more than one Steiner k-ecc v-tree.

2 Preliminary results

The main topic of this paper is the average Steiner 3-eccentricity (of trees). We first give exact values of it for some classes of graphs, easy computations being omitted.

Proposition 2.1 *If $n \geq 3$, then $\mathrm{aecc}_3(K_n) = 2$, $\mathrm{aecc}_3(P_n) = n - 1$, $\mathrm{aecc}_3(K_{1,n-1}) = 3 - \frac{1}{n}$, and $\mathrm{aecc}_3(C_n) = \lceil \frac{3n}{4} \rceil$. Moreover, if $m, n \geq 3$, then $\mathrm{aecc}_3(K_{m,n}) = 3$.*

We now proceed with a series of lemmas.

Lemma 2.2 *If T is a tree and $S \subseteq V(T)$, then the S-Steiner tree is unique.*

Lemma 2.2 is implicitly used in the literature and also briefly mentioned in [22, p. 11]. It follows from the argument that two different S-Steiner trees would lead to a cycle in T. By Lemma 2.2, the formulation of the next lemma is justified.

Lemma 2.3 *Let T be a tree, $v \in V(T)$, and $v \in S \subseteq V(T)$, $|S| = k$. Let T_v be the unique S-Steiner tree and P a path in T, where $V(P) \neq \emptyset$, $V(P) \cap V(T_v) = \{x\}$, and x is an endpoint of the path P. If*

(1) $x \in S$ and $x \neq v$, or

(2) $x \notin S$ and T_v has an internal vertex which is in S and is different from v,

then there exists a k-set $S' \neq S$ with $v \in S'$, such that the size of the S'-Steiner tree is strictly larger than the size of T_v.

Proof. Suppose first that $x \in S$ and $x \neq v$. Let u be the pendent vertex of P not in T_v and set $S' = (S \cup \{u\}) - x$. Then the size of the S'-Steiner tree is $|E(T_v) \cup E(P)|$. Since $|V(P)| \geq 2$, we have $|E(T_v) \cup E(P)| \geq |E(T_v)| + 1 > |E(T_v)|$.

In the second case, let t be the internal vertex of T_v which is in S and different from v. Let again u be the pendent vertex of P not in T_v. In this case we set $S' = (S \cup \{u\}) - t$ and obtain another k-set which induces a larger size Steiner tree than the original k-set S. ∎

Recall that $\ell(T)$ denotes the number of leaves of a tree T.

Lemma 2.4 *Let T be a tree and $v \in V(T)$. If $k > \ell(T)$, then every k-ecc v-set contains all the leaves of T. The same conclusion holds if v is a leaf and $k = \ell(T)$.*

Proof. Trivially, $\mathrm{ecc}_k(v, T) \leq n(T) - 1$. Suppose that $k > \ell(T)$. Set $S = \{v\} \cup L \cup X$, where L is the set of leaves of T and X a set of arbitrary $k - \ell(T) - 1$ vertices from $V(T) \setminus (L \cup \{v\})$. Then $|S| = k$ and the S-Steiner tree is the whole tree T. Hence every k-ecc v-set is the whole tree T and thus contains all the leaves. If v is a leaf, then set $S = L \cup X$, where X a set of arbitrary $k - \ell(T)$ vertices from $V(T) \setminus L$ to reach the same conclusion. ∎

Lemma 2.5 *Let T be a tree, $v \in V(T)$, and $\ell(T) \geq k \geq 2$. If S is a k-ecc v-set, then every vertex from $S \setminus \{v\}$ is a leaf of T.*

Proof. If $k = \ell(T)$ and v is a leaf of T, then the conclusion follows by Lemma 2.4. In the rest we may hence assume that $k < \ell(T)$ or v is not a leaf of T.

Let T_v be a k-ecc v-tree and suppose that there exists a vertex $u \in S \setminus v$ which is an internal vertex of T. There there exists a leaf x in T which does not lie in T_v. Let P be the unique x, T_v-path in T. Then P is a pendent path with at least one edge not in T_v and hence we can use Lemma 2.3 to obtain a larger S-Steiner tree, a contradiction. ∎

Lemma 2.6 *Let T be a tree and $v \in V(T)$. Then every Steiner k-ecc v-tree contains a longest path starting at v.*

Proof. If $k = 2$, then $ecc_2(v, T)$ is the length of a longest path from v to all the other vertices in T, so there is nothing to be proved. In the sequel we may thus assume $k \geq 3$. Suppose on the contrary that T_v is a k-ecc v-tree which contains no longest path starting at v in T. Let S be the k-ecc v-set corresponding to T_v. Let P be a longest path starting at v in the tree T, and let v'' be the endpoint of P different from v. Let P_1 be the sub-path of P which is shared by T_v, and P_2 be the remaining sub-path of P. Then P_1 and P_2 share a unique vertex $v' \in V(P)$. The described situation is illustrated in Fig. 1.
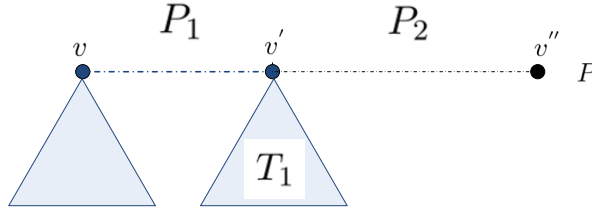

Figure 1: The situation from the proof of Lemma 2.6; the grey part is T_v

Note that v is an endpoint of P_1 and v'' is an endpoint of P_2. By the assumption, P_2 is not empty. Let F be a forest obtained by deleting all the edges in $E(P_1) \subseteq E(T_v)$ from the tree T_v. Let T_1 be the tree in F which contains the vertex v', cf. Fig. 1 again. We now distinguish two cases.

Suppose first that $n(T_1) = 1$. Then v' is a leaf of T_v. So v' must be in the set S. We claim that $v' \neq v$. Otherwise, the tree T_v would be a trivial tree and S contains the unique vertex v, which contradicts the fact that $k \geq 3$. Let $S' = (S \setminus \{v'\}) \cup \{v''\}$. Then S' is another k-set containing v and its S'-Steiner tree is $T_v \cup P_2$. Since S' is a larger tree than T_v, we have a contradiction to the fact that T_v is a k-ecc v-tree.

Suppose second that $n(T_1) \geq 2$. Then there must be a vertex $u \in V(T_1)$ such that u is a leaf of T_v. Then u lies in the k-set S. We construct a path P_3 as follows.

- If there is no branching vertex in T_v, then set P_3 to be the path from v' to u in T_v.

- Suppose that there is at least one branching vertex in T_v. Let $w \in V(T_v)$ be the branching vertex nearest to u in T_v. If w is on the path from v' to u, then let P_3 be the path from w to u in the tree T_v. Otherwise, let P_3 be the path from v' to u in the tree T_v.

Let $S' = (S \setminus \{u\}) \cup \{v''\}$. Then the tree $T_v' = (T_v \setminus P_3) \cup P_2$ is the S'-Steiner tree. Since P is a longest starting from v and T_v contains no such longest path from v, the length of P_2 is strictly larger than the length of P_3. So $m(T_v') > m(T_v)$, a final contradiction. ■

In the rest of the section we focus on the structure of 3-ecc v-trees. By Lemma 2.6, the endpoint x of some longest path starting at v must be in some 3-ecc v-set. Here is now a property of the third terminal in a 3-ecc v-set.

Lemma 2.7 *Let v be a vertex of a tree T and let $S = \{v, x, y\}$ be a 3-ecc v-set, where the v, x-path P is a longest path in T starting from v. Then $d_T(y, P) = ecc_T(P)$.*

Proof. Let T_v be the 3-ecc v-tree; so T_v contains P and the set $S = \{x, y, z\}$. The path P is thus fixed and hence the vertex y must be such that the y, P-path in T is as long as possible. But this in turn implies that for the third terminal y we must have $d_T(y, P) = \max\{d_T(s, P) : s \in V(T)\} = ecc_T(P)$. ■

Combining with Lemma 2.7, the next lemma asserts that in the case of 3-ecc v-sets, in Lemma 2.6 an arbitrary longest path starting from v can be used.

Lemma 2.8 *Let v be a vertex of a tree T, and let P_1 and P_2 be distinct longest paths having v as an endpoint. Then $\mathrm{ecc}_T(P_1) = \mathrm{ecc}_T(P_2)$.*

Proof. Let w be the last common vertex of P_1 and P_2. Clearly w exists, it is possible that $w = v$. Let t_1 and t_2 be the other endpoints of P_1 and P_2, respectively. As T is a tree and $P_1 \neq P_2$ we have $t_1 \neq t_2$. Let u and s be vertices of T such that $d_T(u, P_1) = \mathrm{ecc}_T(P_1)$ and $d_T(s, P_2) = \mathrm{ecc}_T(P_2)$. Let further P_u be the shortest u, P_1-path, P_s the shortest s, P_2-path, u_0 the endpoint of P_u different from u, and s_0 the endpoint of P_s different from s.

We claim that u_0 lies in the v, w-subpath of P_1 (or P_2 for that matter). Suppose on the contrary that u_0 is an internal vertex of the w, t_1-subpath of P_1. Since $d_T(u, P_1) = \mathrm{ecc}_T(P_1)$ it follows that the length of P_u is at least the length of the w, t_2-subpath of P_2. Since the latter path is of the same length as the w, t_1-subpath of P_1, we get that the concatenation of P_u with the v, u_0-subpath of P_1 is a path strictly longer than P_1, a contradiction.

We have thus proved that u_0 lies in the v, w-subpath of P_1. By a parallel argument we also get that s_0 lies in the v, w-subpath of P_2 (or in the v, w-subpath of P_1 for that matter). But his means that $d_G(u, P_1) = d_G(s, P_2)$ and hence $\mathrm{ecc}_T(P_1) = \mathrm{ecc}_T(P_2)$. ∎

To conclude the section, we briefly discuss the computation complexity of the average Steiner 3-eccentricity on trees. The problem is clearly polynomial. If T is a tree and v its vertex, then we can first compute $\mathrm{ecc}_3(v, T)$ by determining the S-Steiner tree for each 3-set S containing v, detecting in this way one of the largest size. This brute force strategy yields an $O(n^4)$ algorithm. Based on the results of this section, we have devised a faster, $O(n^2)$ algorithm. We do not present it here because in the follow up [1] of our paper, Aleksandar Ilić has made full use of results from this section and elaborately devised a linear-time algorithm to calculate the average Steiner 3-eccentricity of a tree. In another follow up paper [20], the results of this section were extended to the Steiner k-eccentricity, $k \geq 3$, and presented a linear-time algorithm to calculate the Steiner k-eccentricity of a vertex in a tree. It would be interesting to see however, whether there is a linear-time algorithm to calculate the average Steiner k-eccentricity of a tree similar to Ilić's algorithm from [1].

3 A transformation on trees

Let T be a tree with the structure as schematically depicted in Fig. 2. Here the w, v_0-path P is a pendant path for which we require that $0 \leq m(P) < \mathrm{ecc}_2(u, T_0)$ holds. (In case $m(P) = 0$, we have $v_0 = w$.) Then set $T' = T \setminus \{wx : x \in N_{T_1}(w)\} \cup \{uy : y \in N_{T_1}(w)\}$, see Fig. 2 again. We say that T' is obtained from T by a π-*transformation* and write $T' = \pi(T)$. The reverse transformation will be called a π^{-1}-*transformation*, that is, given T' as in Fig. 2, we set $T = T' \setminus \{ux : x \in N_{T_1}(u)\} \cup \{wy : y \in N_{T_1}(u)\}$ and write $T = \pi^{-1}(T')$.
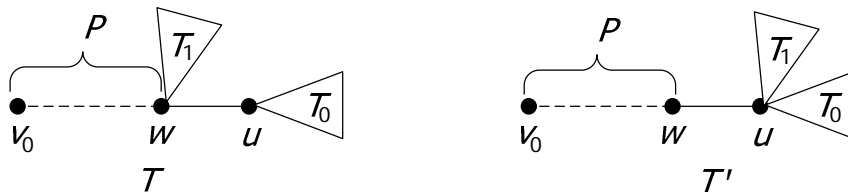

Figure 2: T and T'

Theorem 3.1 *Let T be a tree as in Fig. 2, and let $T' = \pi(T)$. Let P_0 be a longest path starting at u in T_0. If $\mathrm{ecc}_{T_0}(P_0) \le \mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0)$ and $\mathrm{ecc}_2(w, T_1) \le \mathrm{ecc}_2(w, P)$, then $\mathrm{aecc}_3(T') = \mathrm{aecc}_3(T)$. Otherwise, $\mathrm{aecc}_3(T') < \mathrm{aecc}_3(T)$.*

Proof. We are going to consider the behavior of the Steiner 3-eccentricity on the sets of vertices $V(P) \setminus \{w\}$, $V(T_1)$, and $V(T_0)$ on the following cases. (Recall that in the definition of the π-transformation we have required that $m(P) = \mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0)$ holds.)

In order to cover all possibilities, the proof is based on the relationship among $\mathrm{ecc}_2(w, T_1)$, $\mathrm{ecc}_2(w, P)$ and $\mathrm{ecc}_2(u, T_0)$. By the definition, we have $\mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0)$. So on the whole, there are three kinds of relationship among them, i.e., (1) $\mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0) < \mathrm{ecc}_2(w, T_1)$, (2) $\mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(w, T_1) \le \mathrm{ecc}_2(u, T_0)$, and (3) $\mathrm{ecc}_2(w, T_1) \le \mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0)$. In the following, we elaborate that the theorem holds for each of these three cases.

Case 1: $\mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0) < \mathrm{ecc}_2(w, T_1)$.
To deal with this case, we need another variable $\mathrm{ecc}_{T_1}(P_1)$, where P_1 is a longest path starting at w in T_1. Since $\mathrm{ecc}_2(w, T_1)$ is the length of the longest path starting at w in T_1, we have $\mathrm{ecc}_{T_1}(P_1) \le \mathrm{ecc}_2(w, T_1)$. Considering the relationship among $\mathrm{ecc}_2(w, P)$, $\mathrm{ecc}_2(u, T_0)$, $\mathrm{ecc}_2(w, T_1)$, and $\mathrm{ecc}_{T_1}(P_1)$, there are the following three sub-cases to consider.

Case 1.1: $0 \le \mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0) < \mathrm{ecc}_{T_1}(P_1) \le \mathrm{ecc}_2(w, T_1)$.
In this case, the following three statements hold.

(i) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') = -1$ for every vertex $v \in V(P) \setminus \{w\}$.

By Lemmas 2.6 and 2.7, the other two terminals must be in T_1. This remains true after the π-transformation is performed. So for every $v \in V(P) \setminus \{w\}$, $\mathrm{ecc}_3(v, T)$ increases by 1 after the transformation.

(ii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \ge 0$ for every vertex $v \in V(T_1)$.

(iii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') = 1$ for every vertex $v \in V(T_0)$.

For every vertex $v \in V(T_0)$, by Lemma 2.6, the other endpoint of a longest path starting at v must be in T_1. This holds true after the π-transformation is performed. By Lemma 2.7, the third terminal could not be v_0. So after the π-transformation, $\mathrm{ecc}_3(v, T)$ decreases by 1.

Since we have assumed that $\mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0)$, we have $|V(P) \setminus \{w\}| < n(T_0)$. In summary, in this case, we have

$$
\begin{aligned}
\mathrm{aecc}_3(T) - \mathrm{aecc}_3(T') = \frac{1}{n} \Big\{ &\sum_{v \in V(P) \setminus \{w\}} [\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T')] + \\
&\sum_{v \in V(T_1)} [\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T')] + \sum_{v \in V(T_0)} [\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T')] \Big\} \\
\ge &\frac{1}{n} \Big[|V(T_0)| - |V(P) \setminus \{w\}| \Big] \\
> &\, 0.
\end{aligned}
$$

We conclude that $\mathrm{aecc}_3(T') < \mathrm{aecc}_3(T)$ holds in this case.

Case 1.2: $0 \le \mathrm{ecc}_2(w, P) < \mathrm{ecc}_{T_1}(P_1) \le \mathrm{ecc}_2(u, T_0) < \mathrm{ecc}_2(w, T_1)$.
Now we have:

(i) $\mathrm{ecc}_3(v, T) = \mathrm{ecc}_3(v, T')$ for every vertex $v \in V(P) \setminus \{w\}$.

(ii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \ge 0$ for every vertex $v \in V(T_1)$.

6

(iii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \geq 0$ for every vertex $v \in V(T_0) \setminus \{u\}$.

(iv) $\mathrm{ecc}_3(u, T) - \mathrm{ecc}_3(v, T') = 1$ for the vertex u.

Yet again $\mathrm{aecc}_3(T') < \mathrm{aecc}_3(T)$.

Case 1.3: $0 \leq \mathrm{ecc}_{T_1}(P_1) \leq \mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(u, T_0) < \mathrm{ecc}_2(w, T_1)$.
Now we have:

(i) $\mathrm{ecc}_3(v, T) = \mathrm{ecc}_3(v, T')$ for every vertex $v \in V(P) \setminus \{w\}$.

(ii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \geq 0$ for every vertex $v \in V(T_1)$.

(iii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \geq 0$ for every vertex $v \in V(T_0) \setminus \{u\}$.

(iv) $\mathrm{ecc}_3(u, T) - \mathrm{ecc}_3(v, T') = 1$.

Once more $\mathrm{aecc}_3(T') < \mathrm{aecc}_3(T)$.

So in Case 1 the theorem holds. In order to elaborate the other two cases, we use the variable $\mathrm{ecc}_{T_0}(P_0)$ instead of $\mathrm{ecc}_{T_1}(P_1)$ in the following proof. Since P_0 is a longest path starting at u in T_0, we have $\mathrm{ecc}_{T_0}(P_0) \leq \mathrm{ecc}_2(u, T_0)$.

Case 2: $\mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(w, T_1) \leq \mathrm{ecc}_2(u, T_0)$.
According to the relationship among $\mathrm{ecc}_2(w, P)$, $\mathrm{ecc}_2(w, T_1)$, $\mathrm{ecc}_2(u, T_0)$ and $\mathrm{ecc}_{T_0}(P_0)$, there are three sub-cases as following.

Case 2.1: $0 \leq \mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(w, T_1) < \mathrm{ecc}_{T_0}(P_0) \leq \mathrm{ecc}_2(u, T_0)$.
Now:

(i) $\mathrm{ecc}_3(v, T) = \mathrm{ecc}_3(v, T')$ for every vertex $v \in V(P) \setminus \{w\}$.

(ii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') = 1$ for every vertex $v \in V(T_1)$.

(iii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \geq 0$ for every vertex $v \in V(T_0)$.

Again $\mathrm{aecc}_3(T') < \mathrm{aecc}_3(T)$.

Case 2.2: $0 \leq \mathrm{ecc}_2(w, P) < \mathrm{ecc}_{T_0}(P_0) \leq \mathrm{ecc}_2(w, T_1) \leq \mathrm{ecc}_2(u, T_0)$.
Now:

(i) $\mathrm{ecc}_3(v, T) = \mathrm{ecc}_3(v, T')$ for every vertex $v \in V(P) \setminus \{w\}$.

(ii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') = 1$ for every vertex $v \in V(T_1)$.

(iii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \geq 0$ for every vertex $v \in V(T_0) \setminus \{u\}$.

(iv) $\mathrm{ecc}_3(u, T) - \mathrm{ecc}_3(v, T') = 1$ for the vertex u.

So also in this case $\mathrm{aecc}_3(T') < \mathrm{aecc}_3(T)$.

Case 2.3: $0 \leq \mathrm{ecc}_{T_0}(P_0) \leq \mathrm{ecc}_2(w, P) < \mathrm{ecc}_2(w, T_1) \leq \mathrm{ecc}_2(u, T_0)$.
Now:

(i) $\mathrm{ecc}_3(v, T) = \mathrm{ecc}_3(v, T')$ for every vertex $v \in V(P) \setminus \{w\}$.

(ii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \geq 0$ for every vertex $v \in V(T_1)$.

(iii) $\mathrm{ecc}_3(v, T) - \mathrm{ecc}_3(v, T') \geq 0$ for every vertex $v \in V(T_0) \setminus \{u\}$.

(iv) $\text{ecc}_3(u, T) - \text{ecc}_3(v, T') = 1$.

We conclude that $\text{aecc}_3(T') < \text{aecc}_3(T)$ in this case.

Case 3: $\text{ecc}_2(w, T_1) \leq \text{ecc}_2(w, P) < \text{ecc}_2(u, T_0)$.
This case is similar to Case 2, there are also there sub-cases which are elaborated in the following.

Case 3.1: $0 \leq \text{ecc}_2(w, T_1) \leq \text{ecc}_2(w, P) < \text{ecc}_{T_0}(P_0) \leq \text{ecc}_2(u, T_0)$.
Now:

(i) $\text{ecc}_3(v, T) = \text{ecc}_3(v, T')$ for every vertex $v \in V(P) \setminus \{w\}$.

(ii) $\text{ecc}_3(v, T) - \text{ecc}_3(v, T') = 1$ for every vertex $v \in V(T_1)$.

(iii) $\text{ecc}_3(v, T) = \text{ecc}_3(v, T')$ for every vertex $v \in V(T_0)$.

Therefore, $\text{aecc}_3(T') < \text{aecc}_3(T)$.

Case 3.2: $0 \leq \text{ecc}_2(w, T_1) < \text{ecc}_{T_0}(P_0) \leq \text{ecc}_2(w, P) < \text{ecc}_2(u, T_0)$.
In this case we obtain the same conclusions as the Case 1. Hence we conclude that $\text{aecc}_3(T') = \text{aecc}_3(T)$ holds also in this case.

Case 3.3: $0 \leq \text{ecc}_{T_0}(P_0) \leq \text{ecc}_2(w, T_1) \leq \text{ecc}_2(w, P) < \text{ecc}_2(u, T_0)$.
In this case it is evident that the following three statements hold.

(i) $\text{ecc}_3(v, T) = \text{ecc}_3(v, T')$ for every vertex $v \in V(P) \setminus \{w\}$.

(ii) $\text{ecc}_3(v, T) = \text{ecc}_3(v, T')$ for every vertex $v \in V(T_1)$.

(iii) $\text{ecc}_3(v, T) = \text{ecc}_3(v, T')$ for every vertex $v \in V(T_0)$.

By the definition of the average Steiner 3-eccentricity it thus follows that $\text{aecc}_3(T') = \text{aecc}_3(T)$ holds in this case.

Since all the possibilities have been verified, the theorem holds. ∎

Corollary 3.2 *Let T' be a tree as in Fig. 2, and let $T = \pi^{-1}(T')$. Let P_0 be a longest path starting at u in T_0. If $\text{ecc}_{T_0}(P_0) \leq \text{ecc}_2(w, P) < \text{ecc}_2(u, T_0)$ and $\text{ecc}_2(w, T_1) \leq \text{ecc}_2(w, P)$, then $\text{aecc}_3(T') = \text{aecc}_3(T)$. Otherwise, $\text{aecc}_3(T) > \text{aecc}_3(T')$.*

4 More on the π-transformation

In the section, we present several properties of the transformation introduced in the previous section. In Section 4.1 we list conditions under which the transformation exists, while in Section 4.2 we study an extremal graph at which we would arrive after a sequence of π-transformations.

4.1 The existence of π-transformation

If a vertex of a tree has degree at least 3, then the vertex is said to be a *branching vertex*. If two distinct pendent paths are attached to the same branching vertex, then the branching vertex is said to be a *pendent branching vertex*. Referring to the definition of the π-transformation in Section 3, if the subtree T_1 is a pendent path of T, then the π-transformation will be refereed to as a *pendent π-transformation* on T.

Lemma 4.1 *If there are two distinct branching vertices u and v in a tree, then there are two distinct pendent branching vertices u_1 and u_2 such that the distance between u_1 and u_2 is not less than the distance between u and v.*

Proof. Let P be the path between u and v. Since u and v are distinct, there must be two distinct leaves u' and v' such that the path between u and u' shares no edge with P, and the path between v and v' shares no edge with P. Moreover, there is a pendent branching vertex on the path between u and u', while there is a pendent branching vertex on the path between v and v', because u and v are both branching vertices. Therefore, there are two distinct pendent branching vertices u_1 and u_2 such that the distance between u_1 and u_2 is not less than the distance between u and v. ■

Lemma 4.2 *If there are two distinct branching vertices of a tree T such that the distance between them is at least 2, then there exists a pendent π-transformation on T.*

Proof. By Lemma 4.1, there are two distinct pendent branching vertices $u_1, u_2 \in V(T)$ such that the distance between u_1 and u_2 is at least 2, cf. Fig. 3. The paths between v_3 and u_1 and between w_1 and u_1 are two pendent paths attached to the pendent branching vertex u_1, while the paths between v_4 and u_2 and between w_2 and u_2 are two pendent paths attached to the pendent branching vertex u_2. Let P and Q be the pendent paths between w_1 and u_1 and between w_2 and u_2, respectively. The neighbors of u_1 and u_2 on the path between them are respectively marked as v_1 and v_2. Since the distance between u_1 and u_2 is more than 1, we have $v_1 \neq u_2$ and $v_2 \neq u_1$.
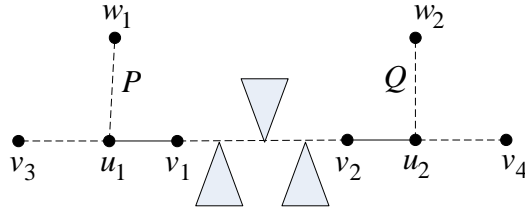

Figure 3: The configuration of two pendent paths P and Q

If the length of the path between v_3 and u_1 is strictly less than the length of the path between v_1 and v_4, then a new tree T' can be achieved by a pendent π-transformation, i.e., $T' = T \setminus \{(u_1, x) : x \in N_P(u_1)\} \cup \{(v_1, y) : y \in N_P(u_1)\}$.

Otherwise, since $v_1 \neq u_2$ and $v_2 \neq u_1$, the length of the path between v_3 and v_2 is strictly larger than the length of the path between u_2 and v_4. Then a new tree T'' can be achieved by a pendent π-transformation, i.e., $T'' = T \setminus \{(u_2, x) : x \in N_Q(u_2)\} \cup \{(v_2, y) : y \in N_Q(u_2)\}$. ■

As trees of course contain no triangles, Lemma 4.2 immediately gives the following consequence.

Lemma 4.3 *If a tree T has more than two branching vertices, then there exists a pendent π-transformation on T.*

Lemma 4.4 *Let T be a tree with exact two branching vertices, where the two branching vertices are adjacent. If there are two pendent paths such that they have different length and attach to distinct branching vertices, then there exists a π-transformation on T. Otherwise, there is no π-transformation on T.*

Proof. Let u_1 and u_2 be two unique (adjacent) branching vertices of T, see Fig. 4.

Suppose first that there are two distinct leaves v_1 and v_2 such that the length of the path between v_1 and u_1 is not equal to length between v_2 and u_2. We may assume without loss of generality that

9

the first of these paths is longer than the second. . Then the length of the path between v_2 and u_2 is strictly less than the length between u_2 and v_1. Let T_1 be the subtree rooted at u_2 obtained by deleting all the edges on the path between v_1 and v_2, see Fig. 4 again. A new tree T' can be obtained by a π-transformation on T as $T' = T \setminus \{(u_2, x) : x \in N_{T_1}(u_2)\} \cup \{(u_1, y) : y \in N_{T_1}(u_2)\}$.
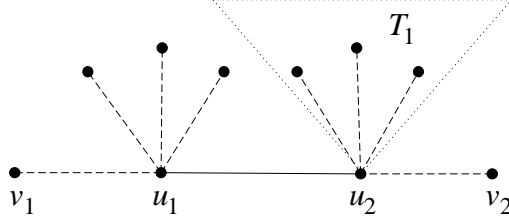

Figure 4: The configuration of a tree with exact two branching verticces

In the second case all pendent paths attaching to both u_1 and u_2 have the same length. Then the precondition of the π-transformation does not hold and hence there is no π-transformation on T. ■

Lemma 4.5 *Let T be a tree with exactly one branching vertex. If there are two distinct pendent paths such that their lengths differ by at least 2, then here exists a π-transformation on T.*

Proof. Let u be the unique branching vertex of T. Let v_1 and v_2 be two leaves such that the length of the path between v_1 and u has at least two more edges than the path between v_2 and u, see Fig. 5.
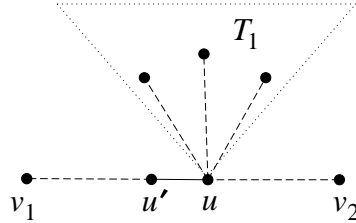

Figure 5: The configuration of a tree with exact one branching verticces

Let T_1 be the subtree rooted at u obtained by deleting all the edges on the path between v_1 and v_2. Let u' be the unique neighbor of u on the path between u and v_1. Then the length of the path between v_1 and u' is strictly larger than that between u and v_2. So a new tree T' can be obtained by a π-transformation on T as $T' = T \setminus \{(u, x) : x \in N_{T_1}(u)\} \cup \{(u', y) : y \in N_{T_1}(u)\}$. ■

4.2 Properties of a bi-stars

We say that a tree T is a *bi-star* if T contains exactly two branching vertices, the two branching vertices are adjacent, and all its pendent paths are of the same length. (Note that in the literature bi-stars are often restricted to the case when all the pendent paths are of length 1.) By Lemma 4.4, bi-stars admit no π-transformation. In this subsection we study the Steiner 3-eccentricity in a bi-stars.

Let u_1 and u_2 be the two branching vertices in the bi-star T. Let S_1 and S_2 be the subtrees respectively rooted at u_1 and u_2 obtained by deleting the edge $u_1 u_2$. Then S_1 and S_2 are both star-like trees. By definition, all pendent paths in S_1 and S_2 have the same length. Let v be a leaf in $V(S_1) \cap V(T)$ and w be a leaf in $V(S_2) \cap V(T)$. Let u_1' be the neighbor of u_1 on the path between

u_1 and v in T. Let T_1 be the sub-tree rooted at u_1 which is obtained by deleting the edges $u_1 u'_1$ and $u_1 u_2$, see Fig. 6.
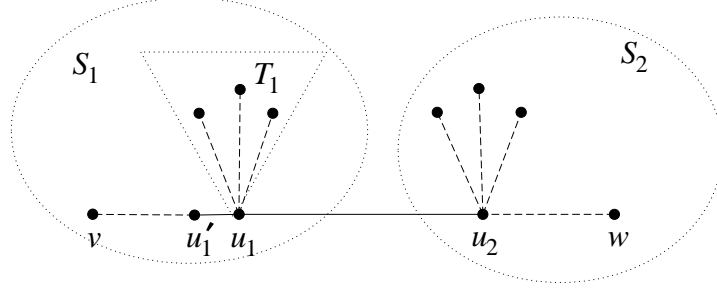

Figure 6: The configuration of a bi-star

Lemma 4.6 *Let T be a bi-star, and S_1, S_2 its corresponding star-like trees. If $v \in V(S_i)$, $i \in \{1, 2\}$, then a Steiner 3-ecc v-set has a vertex in $V(S_{3-i})$.*

Proof. By symmetry we may assume without loss of generality that $v \in V(S_1)$. Let S_v be a Steiner 3-ecc v-set. By Lemma 2.3 and the definition of bi-stars, the other two vertices in S_v must be leaves of T. Suppose on the contrary that no vertex in $S_v \setminus \{v\}$ is in S_2. Then there must be a vertex $u \in S_v \setminus \{v\}$ such that u is a leaf in S_1. Let $S'_v = S_v \setminus \{u\} \cup \{w\}$. Then S'_v is a 3-set containing v and the Steiner tree on S'_v is strictly larger than that on S_v, since there is an edge between u_1 and u_2, a contradiction. ∎

We can define a τ-transformation on a bi-star T, which is to transform T into a new tree T', with $T' = \tau(T) = T \setminus \{(u_1, x) : x \in N_{T_1}(u_1)\} \cup \{(u_2, y) : y \in N_{T_1}(u_1)\}$, see Fig. 7. Obviously, T' is a star-like tree with exactly one branching vertex.
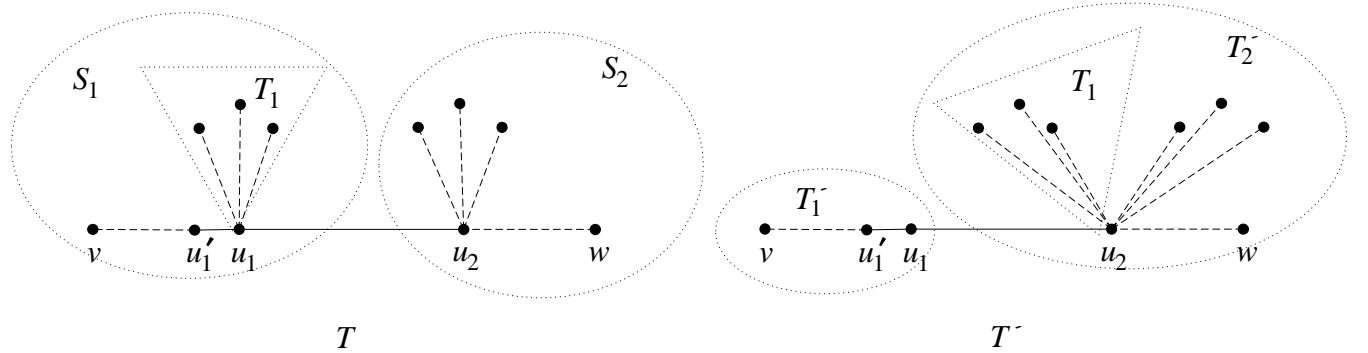

Figure 7: The τ-transformation: $T' = \tau(T)$ where T is bi-star

Deleting the edge $u_1 u_2$ from T' results in two subtrees. Let T'_1 be the subtree rooted at u_1, and T'_2 the subtree rooted at u_2, see Fig. 7. The proof of the following lemma is similar to that of Lemma 4.6, hence we omit it.

Lemma 4.7 *Let T be a bi-star, let $T' = \tau(T)$, and let T'_1, T'_2, be the the subtrees defined in the above paragraph. If $v \in V(T'_i)$, $i \in \{1, 2\}$, then a Steiner 3-ecc v-set has a vertex in $V(T'_{3-i})$.*

Theorem 4.8 *Let T be a bi-star and let $T' = \tau(T)$. Then for every vertex $v \in V(T) = V(T')$ the Steiner 3-eccentricity of v is the same in both T and T'.*

11

Proof. In this proof, we use the notations from Fig. 7. Let P_1 be the path between u_1 and v, and let P_2 be the path between u_2 and w. The vertex set $V(T)$ is partitioned into the following four subsets: $V_1 = V(P_1)$, $V_2 = V(P_2)$, $V_3 = V(S_1) \setminus V_1$ and $V_4 = V(S_2) \setminus V_2$. We are going to prove that for each set V_i, $i \in \{1, 2, 3, 4\}$, and for every vertex $v \in V_i$, we have $\mathrm{ecc}_3(v, T) = \mathrm{ecc}_3(v, T')$.

Case 1: $v \in V_1 = V(P_1)$.
The position of v does not change after the τ-transformation. Let $S_T(v)$ and $S_{T'}(v)$ be Steiner 3-ecc v-sets in T and T', respectively. By Lemmas 4.6, 4.7 and 2.3, without loss of generality, let w be in both $S_T(v)$ and $S_{T'}(v)$. Now we consider the third vertex in $S_T(v)$ as well as $S_{T'}(v)$. Obviously, in $S_{T'}(v)$ the third vertex must be a leaf in $V(T'_2) \setminus \{w\}$. So no matter whether the third vertex of $S_T(v)$ is in V_3 or in V_4, the size of Steiner tree on $S_T(v)$ and that on $S_{T'}(v)$ are the same.

Case 2: $v \in V_2 = V(P_2)$.
The proof in this case is similar to Case 1.

Case 3: $v \in V_2 = V(S_1) \setminus V_1$.
After the τ-transformation, the position of v is changed. Let $S_T(v)$ and $S_{T'}(v)$ be Steiner 3-ecc v-sets in T and T', respectively. By Lemmas 4.6, 4.7 and 2.3, without loss of generality, let w be in $S_T(v)$ and v be in $S_{T'}(v)$. Now consider the third vertex in $S_T(v)$ as well as in $S_{T'}(v)$. It is obvious that the third vertex in $S_T(v)$ is a leaf of $S_1 \cup S_2 \setminus \{w\}$, while the third vertex in $S_{T'}(v)$ is a leaf of $S_1 \cup S_2 \setminus \{v\}$. Therefore, the sizes of Steiner trees on $S_T(v)$ and on $S_{T'}(v)$ are the same.

Case 4: $v \in V_2 = V(S_2) \setminus V_2$.
The proof in this case is also similar to Case 1. ∎

5 Some applications of the π-transformation

As an application of the π-transformation, we establish in this section several lower and upper bounds on the average Steiner 3-eccentricity of trees in terms of the order, the maximum degree, the number of pendent vertices, the matching number, the independent number, the diameter, and the radius.

5.1 Lower and upper bounds on general trees

Theorem 5.1 *If T is a tree on n vertices, then*

$$3 - \frac{1}{n} \leq \mathrm{aecc}_3(T) \leq n - 1\,.$$

The lower bound is achieved when $T \cong K_{1,n-1}$, while the upper bound is achieved when $T \cong P_n$.

Proof. Let T be an arbitrary tree of order n. Then repeatedly apply the π-transformation on T until no further π-transformation is possible.

We claim that in the last step we must necessarily arrive at $K_{1,n-1}$. See Fig. 9 for an example of such a procedure. Suppose on the contrary that in the last step we do not arrive at $K_{1,n-1}$. Let T' be the tree achieved in the last step. In this case, there are at least two internal vertices. Without loss of generality, let u_1 and u_2 be two distinct internal vertices in T'. Let u_3 be the unique neighbor of u_1 in the unique path between u_1 and u_2 in T'. Let T'_1 and T'_2 be the two subtrees respectively rooted at u_1 and u_3 in the forest which is obtained by deleting the edge $u_1 u_2$ from T' (see Fig. 8(a)). Now one can do a π-transformation to obtain a new tree T'' defined as $T'' = T \setminus \{(u_1, x) : x \in N_{T'_1}(u_1)\} \cup \{(u_3, y) : y \in N_{T'_1}(u_1)\}$. This contradicts the fact that T' is achieved in the last step. Hence, in the last step we must arrive at $K_{1,n-1}$.
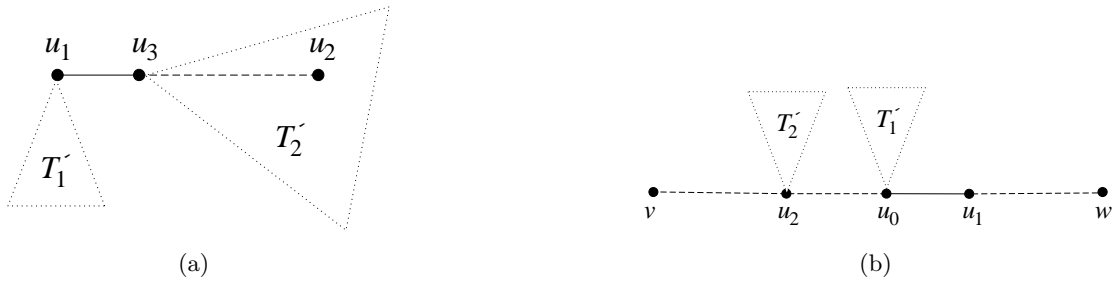
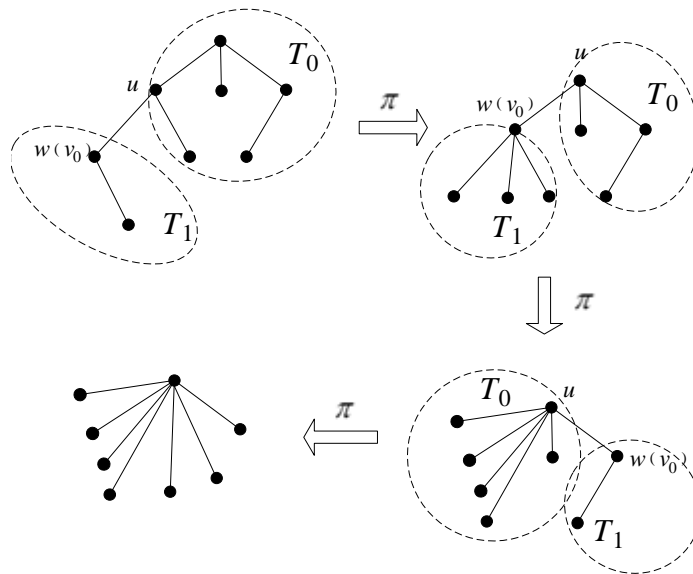Figure 8: To prove the bounds on general graphs

Figure 9: Transforming a tree with a sequences of π-transformations to a star

By Theorem 3.1, during the process, each π-transformation does not increase the average Steiner 3-eccentricity. Hence $\mathrm{aecc}_3(K_{1,n-1}) \leq \mathrm{aecc}_3(T)$.

On the other hand, repeatedly apply the π^{-1}-transformation on T as long as it is possible. We claim that in the last step we must necessarily arrive at the path P_n, see Fig. 10 for an example. Suppose on the contrary that in the last step we do not arrive at P_n. Let T' be the tree obtained in the last step. Then T' contains at least one vertex of degree at least 3, let u_0 be such a vertex. Let u_1 be the unique neighbor of u_0 on a longest path starting at u_0 in T', and let w be the other endpoint of the longest path, see Fig. 8(b). Let v be the leave of T' such that the path between v and u_0 has the smallest length among all paths starting at u_0 to all leaves. If the path between v and u_0 is a pendent path of T', then let u_2 be the neighbor of u_0 on this path. Otherwise, let u_2 be a branching vertex of T' such that u_2 is on the path between v and u_0 and the path between v and u_2 is a pendent path of T', see Fig. 8(b). Hence the length of the pendent path between v and u_2 is strictly less than that between u_0 and w. This implies that one more π^{-1}-transformation can be done on T'. This contradicts the fact that T' is the tree obtained in the last step. So we finally must arrive at the path P_n.

By Corollary 3.2, at each step of this process the average Steiner 3-eccentricity does not decrease, hence $\mathrm{aecc}_3(P_n) \geq \mathrm{aecc}_3(T)$. Using the values from Proposition 2.1 we thus have $3 - \frac{1}{n} =$
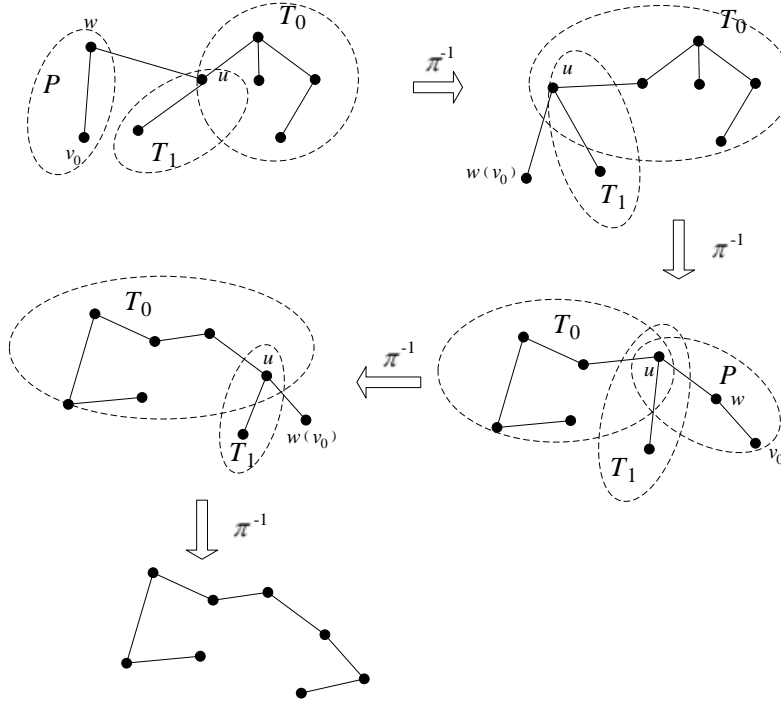
Figure 10: Transforming a tree with a sequences of π^{-1}-transformations to a path

$\mathrm{aecc}(S_{1,n-1}) \leq \mathrm{aecc}_3(T) \leq \mathrm{aecc}_3(P_n) = n - 1$ and we are done. ∎

5.2 An upper bound on trees with maximum degree

A *broom* $B(n, \Delta)$ is a tree obtained from $K_{1,\Delta}$ by attaching a path of length $n - \Delta - 1$ to an arbitrary pendent vertex of the star. See Fig. 11 for an example.
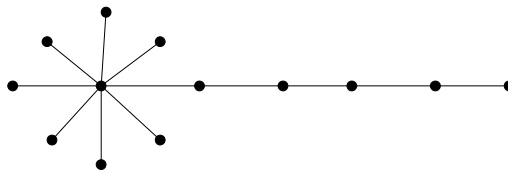

Figure 11: The broom $B(13, 8)$

Theorem 5.2 *If T is a tree of order $n = n(T)$ and maximum degree $\Delta = \Delta(T)$, then*

$$\mathrm{aecc}_3(T) \leq \mathrm{aecc}_3(B(n, \Delta)) = n - \Delta + 1 + \frac{\Delta}{n}.$$

Proof. Let T be a tree with $n = n(T)$ and $\Delta = \Delta(T)$, and let r be the vertex of T with degree Δ. Consider T as a tree rooted in r. Let $T_1, \ldots, T_\Delta$ be the maximal subtrees of T each of which do not contain r, but exactly one of the neighbor is r. We may also consider these Δ trees to be rooted at r. Repeatedly apply the π^{-1}-transformation on each subtree T_i, until no more such transformations could be done. By the proof of Theorem 5.1, in the last step, every T_i becomes a path. Let T' be the final tree in this procedure.

When all subtrees turn into paths, we can further proceed the π^{-1}-transformation on T' until we arrive at the broom $B(n, \Delta)$, see Fig. 12 for an example. Now we will elaborate the process.
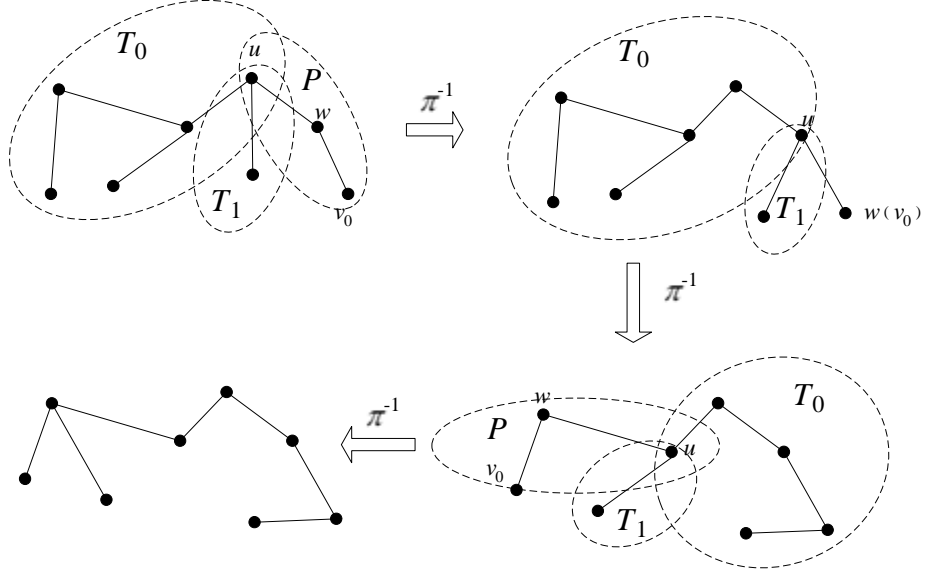

Figure 12: Transforming a tree with a sequences of π^{-1}-transformations to a broom

One can imagine that T' is a tree in which there are Δ paths attached to a star $K_{1,\Delta}$. The star has the central vertex r. If there is no more than one neighbor of r each of which has degree 2, then T' is just a broom. In this case, the theorem holds trivially. Now we consider that there are at least two neighbors in $N_{T'}(r)$ each of which has degree 2. Without loss of generality, let $r_1, r_2 \in N_{T'}(r)$ be two distinct neighbors of r, see Fig. 13. In the figure, u and v are both leaves of T'. If the size of the path between r_1 and u is less than that between r and v, then apply a π^{-1}-transformation on T' to obtain a new tree T'' with $T'' = T' \setminus \{(r, x) : x \in N_{T_1'}(r)\} \cup \{(r_1, y) : y \in N_{T_1'}(r)\}$. Otherwise, also apply a π^{-1}-transformation on T' to obtain another new tree T''' with $T''' = T' \setminus \{(r, x) : x \in N_{T_1'}(r)\} \cup \{(r_2, y) : y \in N_{T_1'}(r)\}$. Whenever there are two such different neighbors of r, whose degrees are both 2, we employ the above procedure on the tree.
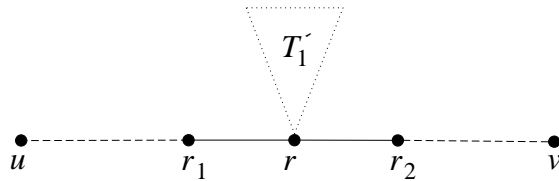

Figure 13: To achieve a broom

We claim that in the last step we will achieve a broom with maximum degree Δ. Obviously, the degree of the center vertex in the star $K_{1,\Delta}$ remains to be Δ all the way in the whole procedure. So the maximum degree of the tree obtained in the last step is Δ. Now we show that in the final step we must arrive at a broom. Suppose on the contrary that we do not achieve a broom in the last step. Let T_1 be the tree in the last step and r be the central vertex of the star $K_{1,\Delta}$. In this case, there must be two neighbors of r each of which has degree 2. This means that one more π^{-1}-transformation can be done on T_1, just like the procedure in the above paragraph. This contradicts the fact that the tree is achieved in the last step. So finally we must achieve a broom.

By Corollary 3.2, during the whole process the Steiner 3-eccentricity does not decrease. This implies that $\mathrm{aecc}_3(T) \leq \mathrm{aecc}_3(B(n, \Delta))$. Finally, the broom $B = B(n, \Delta)$ has Δ leaves, and $\mathrm{ecc}_3(v, B) = n - \Delta + 2$ holds for each of its leaves v. For each of the other $n - \Delta$ vertices w of B we have $\mathrm{ecc}_3(w, B) = n - \Delta + 1$. Hence $\mathrm{aecc}_3(B) = (\Delta(n - \Delta + 2) + (n - \Delta)(n - \Delta + 1))/n = n + 1 - \Delta + \frac{\Delta}{n}$, and we are done. ∎

5.3 A lower bound on trees with constant number of leaves

A *starlike tree* is a tree with exactly one vertex of degree at least three. In other words, a starlike tree is a tree obtained by attaching to an isolated vertex $t \geq 3$ pendant paths. If the lengths of these pendant paths pairwise differ by at most one, then the starlike tree is called *balanced*. Note that if T is a balanced starlike tree of order n and with p leaves, then it is uniquely determined (up to isomorphism); we will denote it by $BS_{n,p}$.

Theorem 5.3 *Let T be a tree of order $n \geq 2$ and with p pendent vertices. Then*

$$\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(BS_{n,p}).$$

Proof. If T has more than two branching vertices, by Lemma 4.3, one can do a pendent π-transformation on T. Recall that the π-transformation does not increase the average Steiner 3-eccentricity.

Now suppose T has exact two branching vertices. If the distance of these two branching vertices is at least two, then by Lemma 4.2, one can do a pendent π-transformation on T. If the distance of these two branching vertices is exact one, and there are two pendent paths such that they have different length and attach to distinct branching vertices, then by Lemma 4.4, one can also do a pendent π-transformation on T. All the processes does not increase the average Steiner 3-eccentricity. If the distance of these two branching vertices is exact one, but every pendent path has the same length, then by Theorem 4.8, one can do a τ-transformation to obtain a new tree with only one branching vertices.

Now we consider the case that there is an unique branching vertex in the tree T. If there are two distinct pendent paths such that their lengths differ by at most one, then T is trivially a balanced star-like tree. Otherwise, by Lemma 4.5, one can also do a π-transformation on T.

To sum all, all the transformations do not increase the average Steiner 3-eccentricity. Finally we arrive at the starlike tree $BS_{n,p}$, see Fig. 14 for an example.

Since in all the transformations made to reach $BS_{n,p}$ the average Steiner 3-eccentricity has not increased, we conclude that $\mathrm{aecc}_3(BS_{n,p}) \leq \mathrm{aecc}_3(T)$. ∎

5.4 Lower bounds on trees with matching and independence number

If $m \geq 3$ and $m + 2 \leq n \leq 2m + 1$, then let $T_{n,m}$ be a tree obtained from $K_{1,m}$ by respectively adding a pendent edge to its $n - m - 1$ pendent vertices. Hence we add a leaf to at least one and not more than m leafs of $K_{1,m}$. Note also that $n(T_{n,m}) = n$. Observe further that $\alpha(T_{n,m}) = m$ and $\beta(T_{n,m}) = n - m$, where $\alpha(G)$ and $\beta(G)$ are the independence and the matching number of G, respectively.

Theorem 5.4 *If T is a tree with $n = n(T)$ and $\beta = \beta(T) \geq 2$, then*

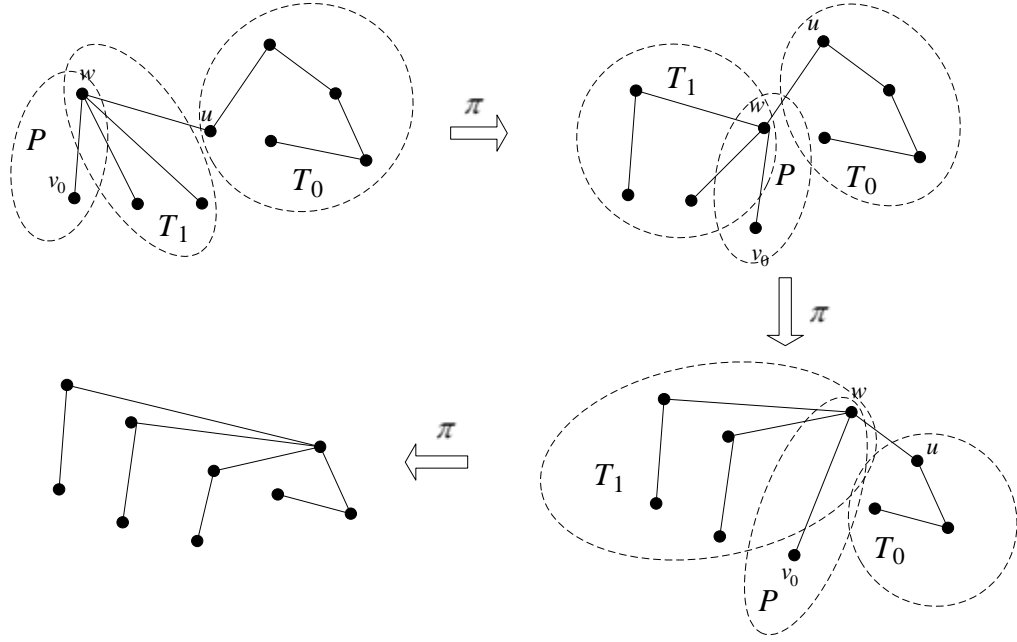$$\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(T_{n,n-\beta}).$$

Figure 14: Transforming a tree with a sequences of π-transformations to a balanced starlike tree

Proof. Let M be a maximum matching of T, so that $|M| = \beta$. Set further $\ell = \ell(T)$. If $e \in M$, then at most one of the endpoints of e is a leaf, hence $\ell \leq \beta + (n - 2\beta) = n - \beta$. By Theorem 5.3, we have $\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(BS_{n,\ell})$. Applying Theorem 3.1 again we can then estimate that

$$\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(BS_{n,\ell}) \geq \mathrm{aecc}_3(BS_{n,n-\beta}) = \mathrm{aecc}_3(T_{n,n-\beta}),$$

and we are done. ∎

For trees with perfect matchings, Theorem 5.4 together with a straightforward computation of $\mathrm{aecc}_3(T_{n,n/2})$ yields the following consequence.

Corollary 5.5 *If T be a tree of order n with a perfect matching, then*

$$\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(T_{n,n/2}) = \begin{cases} 3; & n = 4, \\ \frac{9}{2}; & n = 6, \\ \frac{11}{2} - \frac{2}{n}; & n \geq 8. \end{cases}$$

We next give a bound with the independence number of a tree.

Theorem 5.6 *If T be a tree of order n and $\alpha = \alpha(T)$, then*

$$\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(T_{n,\alpha}).$$

Proof. Set again $\ell = \ell(T)$. Clearly, $\alpha \geq \ell(T)$. By Theorem 5.3, we have $\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(BS_{n,\ell})$. By the aid of Theorem 3.1 we conclude that $\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(BS_{n,\ell}) \geq \mathrm{aecc}_3(BS_{n,\alpha})$. ∎

17

5.5 Lower bounds on trees with constant diameter or radius

Recall that the *diameter* $\mathrm{diam}(G)$ and the *radius* $\mathrm{rad}(G)$ of a graph G are the maximum and the minimum, respectively, eccentricity of the vertices of G. The *center* of G is the set of its vertices with minimum eccentricity. Recall also that the center of a tree consists either of a single vertex or of two adjacent vertices.

Let $T_{n,d}(p_1, \ldots, p_{d-1})$ be a tree of order n obtained from a path $P_{d+1} = v_0 v_1 \ldots v_d$ by attaching $p_i \geq 0$ pendent vertices to v_i for every $i \in [d-1]$. Clearly, as the order of $T'_{n,d}(p_1, \ldots, p_{d-1})$ is n, we must have $\Sigma_{i=1}^{d-1} p_i = n - d - 1$. In the special case when d is even and all the $n - d - 1$ vertices are attached to the vertex $v_{d/2}$, we briefly denote the tree with $T'_{n,d}$. Similarly, if d is odd, then let $T'_{n,d}$ denote the graph in which $\lfloor (n-d-1)/2 \rfloor$ vertices are attached to $v_{\lceil d/2 \rceil}$ and $\lceil (n-d-1)/2 \rceil$ vertices are attached to $v_{\lceil d/2 \rceil}$.

Theorem 5.7 *If T is a tree of order n and $\mathrm{diam}(T) = d$, then*

$$\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(T'_{n,d}).$$

Proof. Let T be a tree as stated and let $P = v_0 v_1 \ldots v_d$ be a longest path in T. Since P is a longest path in T, both v_0 and v_d are leaves of T. For $i \in [d-1]$ let T_i be a maximal subtree of T that contains v_i but no other vertex of P. Consider T_i as a rooted tree with the root v_i. Then the depth of the rooted tree T_i is at most the minimum of the lengths of the v_0, v_i-subpath of P and the v_i, v_d-subpath of P, that is, at most $\min\{i, d-i\}$. Therefore, for each $i \in [d-1]$, we can repeatedly apply the π-transformation on the subtree T_i respected to T so than T_i turns into a star rooted at v_i. The average Steiner 3-eccentricity has not increased along the way. After this procedure is over, $T_{n,d}(p_1, \ldots, p_{d-1})$ is constructed. Afterwards we repeatedly apply the π-transformation on each pendent vertex attached to v_i for each $i \in [d-1]$, to arrive at $T'_{n,d}$. ∎

Note that if d is odd, then we could define $T'_{n,d}$ also by arbitrary distributing the $n - d - 1$ vertices that are attached to $v_{\lceil d/2 \rceil}$ and to $v_{\lceil d/2 \rceil}$. That is, any such tree can serve for the lower bound of Theorem 5.7.

If the center of a tree T contains only one vertex, then $\mathrm{diam}(T) = 2\,\mathrm{rad}(T)$, and if the center of T consists of two vertices, $\mathrm{diam}(T) = 2\,\mathrm{rad}(T) - 1$. Hence Theorem 5.7 yields the following consequence.

Corollary 5.8 *If T is a tree of order n and $r = \mathrm{rad}(G)$, then $\mathrm{aecc}_3(T) \geq \mathrm{aecc}_3(T'_{n,2r-1})$.*

6 Concluding remarks

We have derived several lower and upper bounds for the average Steiner 3-eccentricity on a tree with different constrained parameters. It would be interesting to see if and how these bounds extend to $k \geq 4$.

Just a little research has been done by now on the (average) Steiner k-eccentricity for $k \geq 3$. Hence a lot of work still has to be done.

References

[1] A. Ilić, Optimal algorithm for computing Steiner 3-eccentricities of trees, or arXiv:2008.09299v2 [cs.DS] (21 Feb 2021).

[2] B. S. Anand, M. Changat, S. Klavžar, I. Peterin, Convex sets in lexicographic product of graphs, Graphs Combin. 28 (2012) 77–84.

[3] L. W. Beineke, O. R. Oellermann, R. E. Pippert, On the Steiner median of a tree, Discrete Appl. Math. 68 (1996) 249–258.

[4] R. M. Casablanca, P. Dankelmann, Distance and eccentric sequences to bound the Wiener index, Hosoya polynomial and the average eccentricity in the strong products of graphs, Discrete Appl. Math. 263 (2019) 105–117.

[5] G. Chartrand, O. R. Oellermann, S. Tian, H. B. Zou, Steiner distance in graphs, Časopis Pěst. Mat. 114 (1989) 399–410.

[6] P. Dankelmann, H. C. Swart, O. R. Oellermann, The average Steiner distance of graphs with prescribed properties, Discrete Appl. Math. 79 (1997) 91–103.

[7] P. Dankelmann, O. R. Oellermann, H. C. Swart, The average Steiner distance of a graph, J. Graph Theory 22 (1996) 15–22.

[8] P. Dankelmann, W. Goddard, H. C. Swart, The average eccentricity of a graph and its subgraphs, Util. Math. 65 (2004) 41–51.

[9] P. Dankelmann, S. Mukwembi, Upper bounds on the average eccentricity, Discrete Appl. Math. 167 (2014) 72–79.

[10] P. Dankelmann, F. J. Osaye, S. Mukwembi, B. G. Rodrigues, Upper bounds on the average eccentricity of K_3-free and C_4-free graphs, Discrete Appl. Math. 270 (2019) 106–114.

[11] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman & Company, New York, 1979.

[12] M. Ghorbani, X. Li, H. R. Maimani, Y. Mao, S. Rahmani, M R. Parsa, Steiner (revised) Szeged index of graphs, MATCH Commun. Math. Comput. Chem. 82 (2019) 733–742.

[13] T. Gologranc, Steiner convex sets and Cartesian product, Bull. Malays. Math. Sci. Soc. 41 (2018) 627–636.

[14] I. Gutman, X. Li, Y. Mao, Inverse problem on the Steiner Wiener index, Discuss. Math. Graph Theory 38 (2017) 83–95.

[15] I. Gutman, On Steiner degree distance of trees, Appl. Math. Comput. 283 (2016) 163–167.

[16] I. Gutman, B. Furtula, X. Li, Multicenter Wiener indices and their applications, J. Serb. Chem. Soc. 80 (2015) 1009–1017.

[17] F. K. Hwang, D. S. Richards, P. Winter, The Steiner Tree Problem, North-Holland, Amsterdam, 1992.

[18] X. Li, Y. Mao, I. Gutman, The Steiner Wiener index of a graph, Discuss. Math. Graph Theory 36 (2016) 455–465.

[19] X. Li, Y. Mao, I. Gutman, Inverse problem on the Steiner Wiener index, Discuss. Math. Graph Theory 38 (2018) 83–95.

[20] X. Li, G. Yu, S. Klavžar, J. Hu and B. Li, The Steiner k-eccentricity on trees, arXiv:2008.07763 [math.CO] (18 Aug 2020).

[21] L. Lu, Q. Huang, J. Hou, X. Chen, A sharp lower bound on the Steiner Wiener index for trees with given diameter, Discrete Math. 341 (2018) 723–731.

[22] Y. Mao, Steiner distance in graphs—a survey, arXiv:1708.05779 [math.CO] (18 Aug 2017).

[23] Y. Mao, Steiner Harary index, Kragujevac J. Math. 42 (2018) 29–39.

[24] Y. Mao, E. Cheng, Z. Wang, Steiner distance in product networks, Discrete Math. Theor. Comput. Sci. 20 (2018) paper 8, 25 pp.

[25] Y. Mao, K. C. Das, Steiner Gutman index, MATCH Commun. Math. Comput. Chem. 79 (2018) 779–794.

[26] L.-D. Pei, X.-F. Pan, J. Tian G.-Q. Peng, On AutoGraphiX conjecture regarding domination number and average eccentricity, Filomat 33 (2019) 699–710.

[27] N. Tratnik, On the Steiner hyper-Wiener index of a graph, Appl. Math. Comput. 337 (2019) 360–371.

[28] Z. Wang, Y. Mao, C. Melekian, E. Cheng, Steiner Distance in join, corona, cluster, and threshold graphs, J. Inform. Sci. Eng. 35 (2019) 721–735.

[29] C. Wang, C. Ye, S. Zhang, Z. Wang, H. Li, The Steiner Wiener index of trees with given diameter, Util. Math. 108 (2018) 45–52.

[30] J. Zhang, G.-J. Zhang, H. Wang, X. D. Zhang, Extremal trees with respect to the Steiner Wiener index, Discrete Math. Algorithms Appl. 11 (2019) paper 1950067, 16 pp.