Some extremal results on the chromatic-stability index

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Abstract

The χ-stability index ${\rm es}_\chi(G)$ of a graph G is the minimum number of its edges whose removal results in a graph with the chromatic number smaller than that of G. In this paper three open problems from [European J. Combin. 84 (2020) 103042] are considered. Examples are constructed which demonstrate that a known characterization of k-regular ($k \le 5$) graphs G with ${\rm es}_\chi(G) = 1$ does not extend to $k \ge 6$. Graphs G with $\chi(G) = 3$ for which ${\rm es}_\chi(G) + {\rm es}_\chi(\overline{G}) = 2$ holds are characterized. Necessary conditions on graphs G which attain a known upper bound on ${\rm es}_\chi(G)$ in terms of the order and the chromatic number of G are derived. The conditions are proved to be sufficient when $n \equiv 2 \pmod 3$ and $\chi(G) = 3$.

Keywords: chromatic number; chromatic-stability index; regular graph

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1 Introduction

If $\mathcal{I}$ is a graph invariant and G a graph, then it is natural to consider the minimum number of vertices of G whose removal results in an induced subgraph G' with $\mathcal{I}(G') \neq \mathcal{I}(G)$ or with

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$E(G') = \emptyset$, see [2]. Let us call this number the *$\mathcal{I}$-stability number* of G and denote it by $vs_{\mathcal{I}}(G)$. Similarly one can be interested in the minimum number of edges that has to be removed in order to obtain a spanning subgraph G' with $\mathcal{I}(G') \neq \mathcal{I}(G)$ or with $E(G') = \emptyset$. In this case let us call the minimum number of edges the *$\mathcal{I}$-stability index* of G and denote it by $es_{\mathcal{I}}(G)$.

In this paper we are interested in the χ-stability index es_χ, spelled out as *chromatic-stability index*. The χ-stability index $\mathrm{es}_\chi(G)$ of a graph G with at least one edge is thus the minimum number of edges of G whose removal results in a graph with the chromatic number smaller than that of G. If $E(G) = \emptyset$, then $\mathrm{es}_\chi(G) = 0$. It should be noted that in some papers the term "chromatic edge-stability number" was used, but within the above proposed general framework, as well as since the investigation of the χ'-stability number has been initiated in [2], this earlier naming would lead to a confusing terminology.

The χ-stability index was first studied by Staton [10], who provided upper bounds es_χ for regular graphs in terms of the size of a given graph. The invariant was subsequently investigated in [3, 4, 8]. In this paper we continue this line of the research and are primarily interested in the following three open problems on the chromatic-stability index.

Problem 1.1 ([1, 3])**.** Characterize graphs G with $\mathrm{es}_\chi(G) = 1$.

Problem 1.2 ([1])**.** Characterize graphs G with $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$.

In [1] it was proved that if G is a graph of order n with $r = \chi(G)$, then

$$\mathrm{es}_\chi(G) \leq \begin{cases} \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor \,; & n \equiv r-1 \pmod{r}, \\ \lfloor \frac{n}{r} \rfloor^2 \,; & \text{otherwise}. \end{cases} \tag{1}$$

The third open problem of our interest now read as follows.

Problem 1.3 ([1])**.** Characterize graphs that attain the upper bound in (1).

In the rest of this section we recall definitions needed in this paper. In Section 2 we consider graphs G with $\mathrm{es}_\chi(G) = 1$ and construct examples which demonstrate that a known characterization of k-regular graphs G with $\mathrm{es}_\chi(G) = 1$ does not extend to $k \geq 6$. Then, in Section 3, we characterize graphs G with $\chi(G) = 3$ for which $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$ holds. In the concluding section we obtain necessary structural conditions on graphs G which attain the upper bound in (1). The conditions are proved to be sufficient when $n \equiv 2 \pmod 3$ and $\chi(G) = 3$.

The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G admits a proper coloring of its vertices using k colors. Unless stated otherwise, we will assume that the colors are from the set $[k] = \{1, \ldots, k\}$. A $\chi(G)$*-coloring*, or simply χ*-coloring* of G is a proper coloring using $\chi(G)$ colors. In a coloring of G, a set of vertices having the same color form a *color class*. If c is a k-coloring of G with color classes $C_1, \ldots, C_k$, then we will identify c with $(C_1, \ldots, C_k)$, that is, we will say that c is a coloring $(C_1, \ldots, C_k)$. When we will wish to emphasize that these color classes correspond to c, we will denote them by $(C_1^c, \ldots, C_k^c)$. If c is a coloring of G and $A \subseteq V(G)$, then let $c(A) = \bigcup_{a \in A} c(a)$. Let $c^*(G)$ denote the cardinality of a smallest color class among all χ-colorings of G. If $c^*(G) = 1$, then we say that G has a *singleton color class*. The *chromatic bondage number* $\rho(G)$ of G denotes the minimum number of edges between two color classes among all χ-colorings of a graph G. Note that $\mathrm{es}_\chi(G) \leq \rho(G)$ clearly holds.

For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and the open neighborhood of v in G, respectively. If $A \subseteq V(G)$, then let $N_G(A) = (\cup_{v \in A} N_G(v)) \setminus A$. For $A, B \subseteq V(G)$, let $E[A, B]$ be the set of edges which have one endpoint in A and the other in B, and let $e(A, B) = |E[A, B]|$. The subgraph of G induced by $A \subseteq V(G)$ will be denoted by $G[A]$. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G. The order of a largest complete subgraph in G is the *clique number* $\omega(G)$ of G. The *complement* of G is denoted by $\overline{G}$.

2 On Problem 1.1

Problem 1.1 which asks for a characterization of graphs G with $\mathrm{es}_\chi(G) = 1$ has been independently posed in [3, Problem 2.18] and in [1, Problem 5.3]. The two equivalent reformulations of the condition $\mathrm{es}_\chi(G) = 1$ from the next proposition are due to [8, Proposition 2.2] and [3, Remark 2.15], respectively. To be self-contained, we include a simple proof of the result.

Proposition 2.1. *If G is a graph with $\chi(G) \geq 2$, then the following claims are equivalent.*

(i) $\mathrm{es}_\chi(G) = 1$.

(ii) $\rho(G) = 1$.

(iii) *G admits a $\chi(G)$-coloring $(C_1, \ldots, C_{\chi(G)})$, where $|C_1| = 1$ and $e(C_1, C_2) = 1$.*

Proof. Let $\mathrm{es}_\chi(G) = 1$ and let $e = uv \in E(G)$ be an edge such that $\chi(G - e) = \chi(G) - 1$. If c is a $(\chi(G) - 1)$-coloring of $G - e$, then $c(u) = c(v)$, for otherwise c would be a proper coloring of G (using only $\chi(G) - 1$ colors). Recoloring u with a new color yields a coloring of G as required by (iii). Hence (i) implies (iii). The implication (iii) $\Rightarrow$ (ii) is obvious, and (ii) $\Rightarrow$ (i) follows from the already noted fact that $\mathrm{es}_\chi(G) \leq \rho(G)$ holds. $\square$

Although Proposition 2.1 formally gives two characterizations of graphs G with $\mathrm{es}_\chi(G) = 1$, it should be understood that Problem 1.1 asks for a *structural characterization* of such graphs. A partial solution of the problem is provided in the following result.

Theorem 2.2. ([1, Theorem 4.4]). *Let G be a connected, k-regular graph, $k \leq 5$. Then* $\mathrm{es}_\chi(G) = 1$ *if and only if G is K_2, G is an odd cycle, or $\chi(G) > 3$ and $c^\star(G) = 1$.*

The second part of [1, Problem 5.3] says: "In particular, for the regular case extend the classification of Theorem 2.2 to $k > 5$." We do not solve the problem, but demonstrate in the rest of the section that (i) the problem appears difficult and (ii) why $k = 5$ is the threshold for regular graphs. Let X be the graph as drawn in Fig. 1.

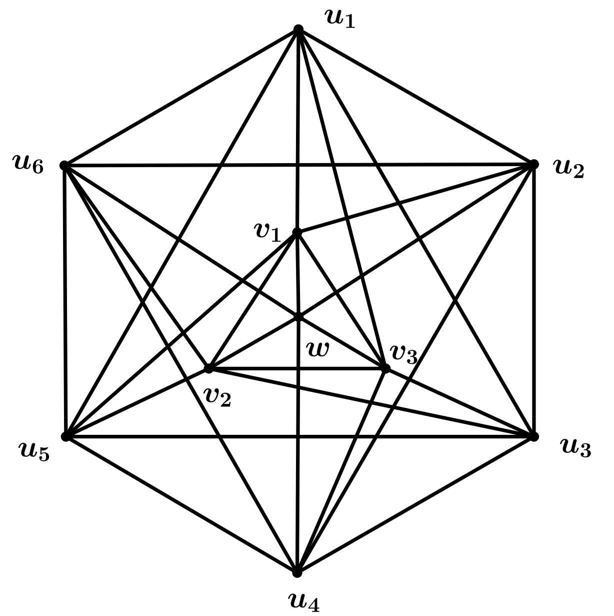

Figure 1: Graph X.

Then we have:

Proposition 2.3. *The graph X is a 6-regular graph with $\chi(X) = 4$, $c^\star(X) = 1$, and* $\mathrm{es}_\chi(X) = 2$.

Proof. Since $\omega(X) = 4$, $\chi(X) \geq 4$. We give a 4-coloring c of X as follows: $c(w) = 4$, $c(v_1) = c(u_3) = c(u_6) = 1$, $c(v_3) = c(u_2) = c(u_5) = 2$, $c(v_2) = c(u_1) = c(u_4) = 3$. Since color 4 is used exactly once, $\chi(X) = 4$ and $c^*(X) = 1$. It remains to prove that $\mathrm{es}_\chi(X) = 2$.

Let X' be the graph obtained from X by deleting the edges wv_1, wu_6. Then we can get a 3-coloring c' of X' as follows: $c'(w) = c'(v_1) = c'(u_3) = c'(u_6) = 1$, $c'(v_3) = c'(u_2) = c'(u_5) = 2$, and $c'(v_2) = c'(u_1) = c'(u_4) = 3$. Hence $\mathrm{es}_\chi(X) \leq 2$.

Suppose now on the contrary that $\mathrm{es}_\chi(X) = 1$. Then by Proposition 2.1(iii), there exists a coloring $c = (C_1, C_2, C_3, C_4)$, such that $|C_1| = 1$ and $e(C_1, C_2) = 1$. Since $X[\{v_1, v_2, v_3, w\}] \cong$

K_4, we have $c(w) = 1$ or $c(v_i) = 1$ for some $i \in [3]$. If $c(w) = 1$, then $\chi(X[N(w)]) = 3$ and color 2 appears only once in $N(w)$. But this is impossible because $X[v_1, v_2, v_3] \cong K_3$ and $X[u_2, u_4, u_6] \cong K_3$. If $c(w) \neq 1$, then by symmetry we may without loss of generality assume that $c(v_1) = 1$. Then we consider the coloring of $N(v_1)$. If $c(u_5) \neq c(v_3)$, say $c(u_5) = a \in \{2, 3, 4\}$ and $c(v_3) = b \in \{2, 3, 4\} \backslash \{a\}$, then $c(v_2) = c(u_1) = c = \{2, 3, 4\} \backslash \{a, b\}$, $c(w) = a$, and $c(u_2) = b$, contradicting the fact that $e(C_1, C_2) = 1$. If $c(u_5) = c(v_3)$, say $c(u_5) = c(v_3) = a \in \{2, 3, 4\}$, then $c(v_2) = b \in \{2, 3, 4\} \backslash \{a\}$, and $c(w) = c = \{2, 3, 4\} \backslash \{a, b\}$. Since $\{w, v_2, u_5\} \subseteq N(u_6)$, we have $c(u_6) = 1$, a contradiction with the fact that $|C_1| = 1$. So $\mathrm{es}_\chi(G) \geq 2$ and we are done. □

Proposition 2.3 shows that Theorem 2.2 does not extend to 6-regular graphs. On the other hand, consider the following example to see that there exist 4-chromatic, 6-regular (and of higher regularity) graphs with $\mathrm{es}_\chi(G) = 1$. A graph $G = C(n; a_0, a_1, \ldots, a_k)$ is called a *circulant* if $V(G) = [n]$ and $E(G) = \{(i, j) : |i - j| \in \{a_0, a_1, \ldots, a_k\} \pmod{n}\}$, where $1 \leq a_0 < a_1 < \cdots < a_k \leq n/2$. If $a_k < n/2$, then G is a $(2k + 2)$-regular graph; otherwise, G is $(2k + 1)$-regular. In [5, Theorem 2.1], Dobrynin, Melnikov, and Pyatkin constructed 4-critical r-regular circulants for $r \in \{6, 8, 10\}$. (Recall that a graph G with $\chi(G) = k$ is called *edge-critical* (or simply *k-critical*) if its chromatic number is strictly less than k after removing any edge.) Hence these regular graphs satisfy $\mathrm{es}_\chi(G) = 1$.

3 On Problem 1.2

Let G be a graph with $\mathrm{es}_\chi(G) = 1$ and $\chi(G) = r$. We say that a χ-coloring of G is *a good coloring* if it satisfies the conditions of Proposition 2.1(iii). Let $\mathcal{C}(G)$ be the set of good colorings of G. If $c = (C_1^c, \ldots, C_r^c) \in \mathcal{C}(G)$, then we may always without loss of generality assume that $|C_1^c| = 1$ and $e(C_1^c, C_2^c) = 1$.

Clearly, $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$ holds if and only if $\mathrm{es}_\chi(G) = \mathrm{es}_\chi(\overline{G}) = 1$. We first characterize disconnected graphs G for which $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$ holds.

Proposition 3.1. *Let G be a graph with components $G_1, \ldots, G_s$, $s \geq 2$, and let $\mathcal{G} = \{G_i : \chi(G_i) = \chi(G), i \in [s]\}$. Then $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$ if and only if*

(i) *$|\mathcal{G}| = 1$ and $\mathrm{es}_\chi(G_i) = 1$ for $G_i \in \mathcal{G}$, and*

(ii) *there exists a G_j such that $\mathrm{es}_\chi(\overline{G_j}) = 1$, or there exist components G_j and G_k, $j \neq k$, such that $c^*(\overline{G_j}) = 1$ and $c^*(\overline{G_k}) = 1$.*

Proof. The following fact is essential for the rest of the argument: if c is a proper coloring of $\overline{G}$, then $c(V(G_i)) \cap c(V(G_j)) = \emptyset$ for every $i, j \in [s]$, $i \neq j$. If G satisfies (i) and (ii), then (i) yields $\mathrm{es}_\chi(G) = 1$, while (ii) gives $\mathrm{es}_\chi(\overline{G}) = 1$. Conversely, suppose that $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$. Then $\mathrm{es}_\chi(G) = 1$ and $\mathrm{es}_\chi(\overline{G}) = 1$. If $|\mathcal{G}| \geq 2$ or $\mathrm{es}_\chi(G_i) \geq 2$ for any $G_i \in \mathcal{G}$, then $\chi(G-e) = \chi(G)$ for any $e \in E(G)$, a contradiction. This means that (i) holds. Since $\mathrm{es}_\chi(\overline{G}) = 1$, there exists an a edge $\overline{e} \in E(\overline{G})$ such that $\chi(\overline{G} - \overline{e}) < \chi(\overline{G})$. We consider two cases for the edge $\overline{e}$. If $\overline{e} \in E(\overline{G_j})$ for some $j \in [s]$, then $\mathrm{es}_\chi(\overline{G_j}) = 1$. In the other case the two endpoints of $\overline{e}$ lie in different components, say in G_j and in G_k, $j \neq k$. But then $c^*(\overline{G_j}) = 1$ and $c^*(\overline{G_k}) = 1$. Thus (ii) holds as well. □

In the main result of this section we now characterize connected graphs G with $\chi(G) = 3$ for which $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$ holds.

Theorem 3.2. *Let G be a connected graph of order n, with $\chi(G) = 3$. Then $\mathrm{es}_\chi(G) + \mathrm{es}_\chi(\overline{G}) = 2$ if and only if*

(i) *all odd cycles in G share one edge,*

(ii) $c^*(\overline{G}) = 1$,

(iii) $\chi(\overline{G}) \geq \lceil \frac{n}{2} \rceil$,

(iv) *if n is even, $\chi(\overline{G}) = \frac{n}{2}$, and $||C_2^c| - |C_3^c|| = 1$ for each $c = (C_1^c, C_2^c, C_3^c) \in \mathcal{C}(G)$, then $g(G) = 3$, and for any proper coloring of $\overline{G}$, if $\{x_1, x_2, x_3\}$ is a color class, then $d_G(v) \geq 2$ for each $v \in N_G(\{x_1, x_2, x_3\})$.*

Proof. Necessity: Since $\mathrm{es}_\chi(G) = 1$ and $\chi(G) = 3$, there is an edge $e \in E(G)$ such that $G - e$ has no odd cycles. So (i) holds. It was observed in [1, Lemma 4.3] that $\mathrm{es}_\chi(G) = 1$ implies $c^*(G) = 1$, hence (ii) holds. Let $c = (C_1^c, C_2^c, C_3^c) \in \mathcal{C}(G)$. We have $\omega(\overline{G}) \geq \lfloor \frac{n}{2} \rfloor$ since $|C_1^c| = 1$. So, $\chi(\overline{G}) \geq \omega(\overline{G}) \geq \lceil \frac{n}{2} \rceil$ when n is even. In the case of n is odd and $\omega(\overline{G}) = \frac{n-1}{2}$, we have $|C_2^c| = |C_3^c| = \frac{n-1}{2}$ and $\overline{G}[C_2^c] \cong K_{C_2^c}$, $\overline{G}[C_3^c] \cong K_{C_3^c}$. Note that for any proper coloring of $\overline{G}$, there is at most one color class with 3 vertices, and the number of vertices in other color classes must be smaller than 3. By Proposition 2.1(iii), there exists a χ-coloring of $\overline{G}$ such that some color class has exactly one vertex. Then $\chi(\overline{G}) \geq \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$.

Suppose now that n is even, $\chi(\overline{G}) = \frac{n}{2}$, and $||C_2^c| - |C_3^c|| = 1$ for any $c = (C_1^c, C_2^c, C_3^c) \in \mathcal{C}(G)$. Let $C_1^c = \{x_1\}$, and let x_2 be the vertex of C_2^c such that $x_1x_2 \in E(G)$. Let $\bar{c} \in \mathcal{C}(\overline{G})$ and let the color set used by $\bar{c}$ be $[\frac{n}{2}]$. We claim that $\bar{c}(x_1) = \bar{c}(x_2)$ and $\bar{c}(x_1) \in \bar{c}(C_3^c)$. Notice that x_1 is in $\overline{G}$ adjacent to all vertices of C_2^c except x_2. If $|C_2^c| - |C_3^c| = 1$, then $|C_2^c| = \frac{n}{2}$. Then the claim holds because $\chi(\overline{G}) = \frac{n}{2} = |\bar{c}(C_2^c)|$. Suppose second that $|C_3^c| - |C_2^c| = 1$. Then $|C_3^c| = \frac{n}{2}$ and $|C_2^c| = \frac{n-2}{2}$. We have $|\bar{c}(C_3^c)| = \frac{n}{2}$. If $\bar{c}(x_1) \neq \bar{c}(x_2)$, then $\bar{c}(x_1 \cup C_2^c) = [\frac{n}{2}]$, contradicting the fact that $\bar{c} \in \mathcal{C}(\overline{G})$ because there is no singleton color class. Hence $\bar{c}(x_1) = \bar{c}(x_2)$ and $\bar{c}(x_1) \in \bar{c}(C_3^c)$ since $\bar{c}(C_3^c) = [\frac{n}{2}]$. Thus $g(G) = 3$. We might as well set $x_3 \in C_3^c$ and $\bar{c}(x_1) = \bar{c}(x_2) = \bar{c}(x_3) = 1$ in the following. Suppose there is a vertex $v \in N_G(\{x_1, x_2, x_3\})$ such that $d_G(v) = 1$. If $|C_2^c| - |C_3^c| = 1$, then we have $C_2^c \subseteq N_{\overline{G}}(v)$ when $v \in N_G(x_1)$, $(C_2^c \backslash \{x_2\}) \cup x_3 \subseteq N_{\overline{G}}(v)$ when $v \in N_G(x_2)$ and $V(G)\backslash\{x_3\} = N_{\overline{G}}(v)$ when $v \in N_G(x_3)$. Thus $\chi(\overline{G}) > \frac{n}{2}$ when $v \in N_G(x_1) \cup N_G(x_2)$, a contradiction. When $v \in N_G(x_3)$, we may without loss of generality assume that $\bar{c}(C_3^c) = [\frac{n-2}{2}]$. Then $\bar{c}(v) = \frac{n}{2}$ since $x_2 \in N_{\overline{G}}(v)$. But every color in $[\frac{n-2}{2}]$ appears exactly twice in $N_{\overline{G}}(v)$, contradicting the fact that $\bar{c} \in \mathcal{C}(\overline{G})$. If $|C_3^c| - |C_2^c| = 1$, then we have $\chi(\overline{G}) > \frac{n}{2}$ when $v \in N_G(x_3)$ and $\bar{c} \notin \mathcal{C}(\overline{G})$ when $v \in N_G(x_1) \cup N_G(x_2)$ by the same analysis above, a contradiction.

Sufficiency: Suppose an edge e is shared by all odd cycles of G. Then $\chi(G - e) \leq 2$. Hence $\mathrm{es}_\chi(G) = 1$ holds by definition. Suppose $\chi(\overline{G}) \geq \lceil \frac{n}{2} \rceil$. In [1, Lemma 4.2] it was proved that if $\chi(\overline{G}) \geq \frac{n+2}{2}$, then $\mathrm{es}_\chi(\overline{G}) = 1$. So we may assume $\chi(\overline{G}) = \lceil \frac{n}{2} \rceil$ in the following.

Suppose first that n is odd. Let $\bar{c}$ be a proper coloring of $\overline{G}$. Since $\chi(\overline{G}) = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, the complement $\overline{G}$ has a singleton color class under $\bar{c}$. If $\overline{G}$ has two singleton color classes under $\bar{c}$, then $\rho(\overline{G}) = 1$. Otherwise, other color classes have exactly two vertices. At this time, since G is connected, $\Delta(\overline{G}) < n-1$, thus $\rho(\overline{G}) = 1$. Suppose second that n is even. We have $||C_2^c| - |C_3^c|| = 1$ for any $c_i \in \mathcal{C}(G)$ since $\chi(\overline{G}) = \frac{n}{2}$. Since $c^*(\overline{G}) = 1$, there is a proper coloring such that some color class contains three vertices. Let $\bar{c}$ be the proper coloring and $\bar{c}(x_1) = \bar{c}(x_2) = \bar{c}(x_3)$, where $x_s \in C_s^c$ for $s \in [3]$. Let $\{\alpha, \beta\} = \{2, 3\}$. If $|C_\alpha^c| - |C_\beta^c| = 1$, then $|C_\alpha^c| = \frac{n}{2}$ and $|C_\beta^c| = \frac{n-2}{2}$. Since $\chi(\overline{G}) = \frac{n}{2}$, we may assume $\bar{c}(C_\alpha^c) = [\frac{n}{2}]$ and $\frac{n}{2} \notin C_\beta^c$, say $\bar{c}(u) = \frac{n}{2}$. Since G is connected and $d_G(v) \geq 2$ for any $v \in N_G(\{x_1, x_2, x_3\})$, we have $N_{C_\beta^c}(u)\backslash\{x_\beta\} \neq \emptyset$ or $\{x_1, x_\beta\} \subseteq N_G(u)$. Thus $\rho(\overline{G}) = 1$, and by Proposition 2.1 we conclude that $\mathrm{es}_\chi(\overline{G}) = 1$. □

4 On Problem 1.3

Obviously, when $r = 2$, the upper bound in (1) is attained if and only if the graph in question is a complete bipartite graph in which the orders of its bipartition sets differ by at most one. For an arbitrary r we have:

Theorem 4.1. *Let G be a graph of order n and with $r = \chi(G)$.*

(i) *Suppose that $n \equiv r - 1 \pmod r$ and $\mathrm{es}_\chi(G) = \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor$. Then for any r-coloring $(C_1, \ldots, C_r)$ of G, where $|C_1| \leq \cdots \leq |C_r|$, we have*

- **(1)** $|C_1| = \lfloor \frac{n}{r} \rfloor$, *and* $|C_2| = \cdots = |C_r| = \lfloor \frac{n}{r} + 1 \rfloor$.
- **(2)** *If $2 \leq i \leq r$, then $G[C_1 \cup C_i]$ is a complete bipartite graph with bipartition (C_1, C_i).*
- **(3)** *If $v \in C_i$ and $j \in [r]\backslash\{i\}$, then $e(v, C_j) \geq \lfloor \frac{n}{r} \rfloor$.*

(ii) *Suppose that* $n \not\equiv r-1 \pmod r$ *and* $\mathrm{es}_\chi(G) = \lfloor \frac{n}{r} \rfloor^2$. *Then for any r-coloring* $(C_1, \ldots, C_r)$ *of G, where* $|C_1| \le \cdots \le |C_r|$, *we have*

(1) $|C_1| = |C_2| = \lfloor \frac{n}{r} \rfloor$.

(2) *If* $|C_i| = \lfloor \frac{n}{r} \rfloor$, *and* $v \in C_i$ *and* $j \in [r] \backslash \{i\}$, *then* $e(v, C_j) \ge \lfloor \frac{n}{r} \rfloor$. *If* $|C_i| > \lfloor \frac{n}{r} \rfloor$, *then* $\sum_{v_s \in C_i} \ell_s \ge \lfloor \frac{n}{r} \rfloor^2$, *where* $\ell_s = \min\{e(v_s, C_j) : \ v_s \in C_i,\ j \in [r] \backslash \{i\}\}$.

Proof. (i) Consider an r-coloring $(C_1, \ldots, C_r)$ of G, where $|C_1| \le \cdots \le |C_r|$.

(1) Since $n \equiv r-1 \pmod r$, we have $n = r\lfloor n/r \rfloor + r - 1$. From here it was deduced in the proof of [1, Theorem 2.1] that there exists at least one pair of color class C_i and C_j, $i < j$, such that $|C_i| + |C_j| \le \lfloor \frac{n}{r} \rfloor + \lfloor \frac{n}{r} + 1 \rfloor$. Since $\mathrm{es}_\chi(G) = \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor$, we have $|C_i| = \lfloor \frac{n}{r} \rfloor$ and $|C_j| = \lfloor \frac{n}{r} + 1 \rfloor$. Moveover, we have $i = 1$ and $|C_k| \ge \lfloor \frac{n}{r} + 1 \rfloor$ for $2 \le k \le r$, since otherwise $\mathrm{es}_\chi(G) \le |C_1||C_2| \le \lfloor \frac{n}{r} \rfloor^2$, a contradiction. Thus $|C_2| = \cdots = |C_r| = \lfloor \frac{n}{r} + 1 \rfloor$ because $n = r\lfloor n/r \rfloor + r - 1$.

(2, 3) Observe that $G[C_1 \cup C_i]$ is a complete bipartite graph with bipartition (C_1, C_i) for any $2 \le i \le r$, since otherwise, $\mathrm{es}_\chi(G) \le |C_1||C_i| \le \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor - 1$, a contradiction. Therefore, we have $e(v, C_j) \ge \lfloor \frac{n}{r} \rfloor$ when $v \in C_1$ or $j = 1$. If $e(v, C_j) < \lfloor \frac{n}{r} \rfloor$ for some $v \in C_i$ and $j \in [r] \backslash \{i\}$ $(i, j > 1)$, then by deleting the edge set $E(v, C_j) \cup E(C_i \backslash \{v\}, C_1)$, we get an $(r-1)$-coloring with the color class set $\{C_1 \cup (C_i \backslash \{v\}), C_2, \ldots, C_j \cup \{v\}, \ldots, C_r\} \backslash \{C_i\}$. Notice that $|E(v, C_j) \cup E(C_i \backslash \{v\}, C_1)| < \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor$. Thus $\mathrm{es}_\chi(G) < \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor$, a contradiction.

(ii) Suppose $n \not\equiv r - 1 \pmod r$ and $\mathrm{es}_\chi(G) = \lfloor \frac{n}{r} \rfloor^2$. Consider an r-coloring $(C_1, \ldots, C_r)$ of G, where $|C_1| \le |C_2| \le \cdots \le |C_r|$.

(1) By the proof of [1, Theorem 2.1], there exists at least one pair of color class C_i and C_j $(i \le j)$ in which $|C_i| + |C_j| \le 2\lfloor \frac{n}{r} \rfloor$. Since $\mathrm{es}_\chi(G) = \lfloor \frac{n}{r} \rfloor^2$, we have $|C_i| = |C_j| = \lfloor \frac{n}{r} \rfloor$. Moveover, we have $|C_k| \ge \lfloor \frac{n}{r} \rfloor$ for $1 \le k \le r$, since otherwise $\mathrm{es}_\chi(G) \le |C_1||C_2| < \lfloor \frac{n}{r} \rfloor^2$, a contradiction. Thus $|C_1| = |C_2| = \lfloor \frac{n}{r} \rfloor$.

(2) Suppose $|C_i| = \lfloor \frac{n}{r} \rfloor$ and there exists some $v \in C_i$ and $j \in [r] \backslash \{i\}$ such that $e(v, C_j) < \lfloor \frac{n}{r} \rfloor$. We take a color class C_k with $\lfloor \frac{n}{r} \rfloor$ vertices, which is different from C_i. This is possible because $|C_1| = |C_2| = \lfloor \frac{n}{r} \rfloor$. Note that k and j are not necessarily distinct. Then we delete the edge set $E(v, C_j) \cup E(C_i \backslash \{v\}, C_k)$ and get an $(r-1)$-coloring with color class set $\{C_1, \ldots, C_k \cup (C_i \backslash \{v\}), \ldots, C_j \cup \{v\}, \ldots, C_r\} \backslash \{C_i\}$. Notice that $|E(v, C_j) \cup E(C_i \backslash \{v\}, C_k)| < \lfloor \frac{n}{r} \rfloor^2$. Thus $\mathrm{es}_\chi(G) < \lfloor \frac{n}{r} \rfloor^2$, a contradiction.

Suppose $|C_i| > \lfloor \frac{n}{r} \rfloor$ and $\sum_{v_s \in C_i} \ell_s < \lfloor \frac{n}{r} \rfloor^2$, where $\ell_s = \min\{e(v_s, C_j) : \ v_s \in C_i,\ j \in [r] \backslash \{i\}\}$. Let C^s be one of the corresponding color classes when ℓ_s is taken for v_s. Then for any $v_s \in C_i$, we delete the edge set $E(v_s, C^s)$ and get an $(r-1)$-coloring by putting v_s in C^s. Thus $\mathrm{es}_\chi(G) < \lfloor \frac{n}{r} \rfloor^2$, a contradiction. $\square$

Recall that a graph coloring $(C_1, \ldots C_k)$ is *equitable* [6] if $||C_i| - |C_j|| \leq 1$ holds for all $i \neq j$. Hence all the colorings from Theorem 4.1(i) are equitable and consequently, the corresponding extremal graphs have the same chromatic number and the equitable chromatic number. (See [7, 9] for a couple of recent investigations of the equitable chromatic number.)

Theorem 4.2. *Let G be a graph of order n, where $n \equiv 2 \pmod 3$, and with $\chi(G) = 3$. If any 3-coloring of G satisfies (1)-(3) of Theorem 4.1(i), then* $\mathrm{es}_\chi(G) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{3} + 1 \rfloor$.

Proof. Let c be a 3-coloring of G satisfying (1)-(3) of Theorem 4.1(i). Let $\{i, j\} = \{2, 3\}$. For $v \in C_i$ we may let $e(v, C_j) = \lfloor \frac{n}{3} \rfloor$ (as adding edges to a graph cannot decrease its χ-stability index). Since for any $e \in E(G)$, e lies in exactly $\lfloor \frac{n}{3} \rfloor$ subgraphs K_3, the graph $G - e$ has at most $\lfloor \frac{n}{3} \rfloor$ fewer subgraphs isomorphic to K_3 than G. Let $F \subseteq E(G)$ with $|F| = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{3} + 1 \rfloor - 1$. Then the graph $G \backslash F$ has at most $\lfloor \frac{n}{3} \rfloor (\lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{3} + 1 \rfloor - 1)$ fewer subgraphs K_3 than G. Since G has $\lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{3} + 1 \rfloor$ subgraphs K_3, we thus infer that $G \backslash F$ has at least one subgraph K_3 and consequently $\chi(G \backslash F) = 3$. Hence, $\mathrm{es}_\chi(G) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{3} + 1 \rfloor$. □

Let G be a graph with n vertices and $r = \chi(G)$. Note that when $r = 5$ and $n \equiv 4 \pmod 5$, the conditions (1)-(3) in Theorem 4.1(i) are not sufficient. Let G_{12} be the graph from Fig. 2, and let G_{14} be obtained from G_{12} by adding two new vertices u_0 and v_0, and connecting u_0 and v_0 to all vertices of G_{12}. Then we have the following result.

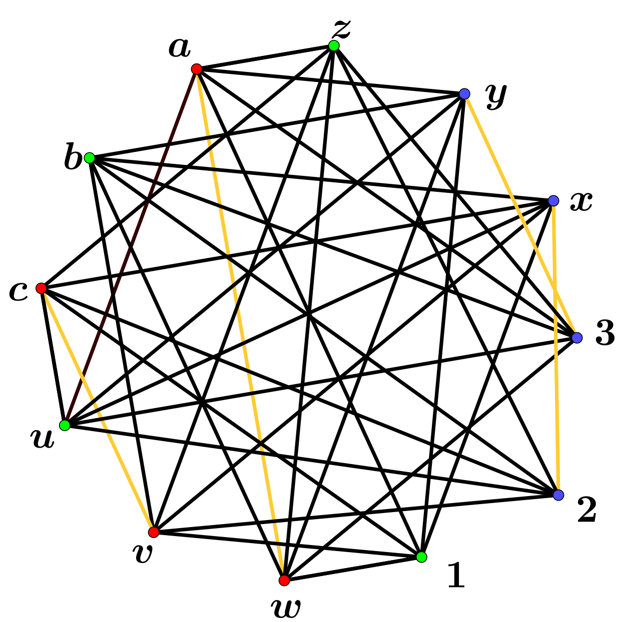

Figure 2: Graph G_{12}.

Proposition 4.3. *The graph G_{14} satisfies conditions (1)-(3) of Theorem 4.1(i), but* $\mathrm{es}_\chi(G_{14}) < \lfloor \frac{n}{r} \rfloor \lfloor \frac{n}{r} + 1 \rfloor = 6$.

Proof. We first show that $\chi(G_{14}) = 5$. Let $A = \{a, b, c\}$, $B = \{u, v, w\}$, $C = \{1, 2, 3\}$, and $D = \{x, y, z\}$. We claim that $\chi(G_{14}) = 5$ and that G_{14} has a unique 5-coloring. With a computer search (using SageMath), we found all independent sets of G_{14} with at least three vertices: A, B, C, D, $\{b, u, 1, z\}$, and each $X \subseteq \{b, u, 1, z\}$ with $|X| = 3$. So, if any three vertices of $\{b, u, 1, z\}$ have the same color under some proper coloring $c : V(G_{14}) \to [k]$ of G_{14}, then $k \geq 6$. Thus $\chi(G_{14}) = 5$ and the unique 5-coloring has color classes $\{u_0, v_0\}$, A, B, C, D. Therefore, the graph G_{14} satisfies conditions (1)-(3) of Theorem 4.1(i).

On the other hand, by deleting the edges cv, aw, $3y$, and $2x$ (colored orange in the figure), we can get a 4-coloring with color classes $\{u_0, v_0\}$, $\{a, c, v, w\}$, $\{b, u, 1, z\}$, $\{2, 3, x, y\}$. Therefore, $\mathrm{es}_\chi(G_{14}) \leq 4$. □

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