The Steiner k-eccentricity on trees *

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Abstract

We study the Steiner k-eccentricity on trees, which generalizes the previous one in the paper [X. Li, G. Yu, S. Klavžar, On the average Steiner 3-eccentricity of trees, arXiv:2005.10319, 2020]. To support the algorithm, we achieve much stronger properties for the Steiner k-ecc tree than that in the previous paper. Based on this, a linear time algorithm is devised to calculate the Steiner k-eccentricity of a vertex in a tree. On the other hand, the lower and upper bounds of the average Steiner k-eccentricity index of a tree on order n are established based on a novel technique which is quite different from that in the previous paper but much easier to follow.

Keywords: Steiner distance, Steiner tree, Steiner eccentricity, graph algorithms

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1 Introduction

In this paper we consider connected, simple, undirected graphs $G = (V(G), E(G))$. For basic graph notation and terminology we follow the book of West [28], while for algorithmic

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and computational terminology we use [4, 9].

The standard distance $d_G(u, v)$ between vertices u and v in graph G is the length of a shortest path between u and v in G. If $S \subseteq V(G)$, $|S| \geq 2$, then the *Steiner distance* $d_G(S)$ is the minimum size among all connected subgraphs of G containing S, that is,

$$d_G(S) = \min\{|E(T)| : T \text{ is a subtree of } G, S \subseteq V(T)\} .$$

Note that if $S = \{u, v\}$, then $d_G(S) = d_G(u, v)$. If $k \geq 1$, then the *Steiner k-eccentricity* of a vertex v in graph G is

$$\mathrm{ecc}_k(v, G) = \max\{d_G(S) : \ v \in S \subseteq V(G), |S| = k\} .$$

Note that, by definition, $\mathrm{ecc}_1(v, G) = 0$. $S \subseteq V(G)$ is a *Steiner k-ecc v-set* if $|S| = k$, $v \in S$, and $d_G(S) = \mathrm{ecc}_k(v, G)$. A corresponding minimum Steiner tree T is called a *Steiner k-ecc v-tree* (corresponding to the k-set S). We will also shorty say that T is a $MST(S, G)$. The average Steiner k-eccentricity of a graph G is the mean value of all vertices' Steiner k-eccentricities in G, that is,

$$\mathrm{aecc}_k(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \mathrm{ecc}_k(v, G) ,$$

which is an extention of the average eccentricity of a graph [7, 8].

The Steiner tree problem is NP-hard on general graphs [9, 16], but it can be solved in polynomial time on trees [2]. The Steiner distance on some special graph classes such as trees, joins, Corona products, threshold and product graphs, has been studied in [1, 3, 11, 23, 26]. The average Steiner k-distance is closely related to the k-th Steiner Wiener index. Both of them were studied on trees, complete graphs, paths, cycles and complete bipartite graphs [6, 12]. The average Steiner distance and the Steiner Wiener index were investigated in [5, 18, 20], while for some work on the Steiner diameter see [23, 26]. The Steiner k-diameter was compared with the Steiner k-radius in [15, 24]. Closely related invariants were also studied, for instance Steiner Gutman index [21], Steiner degree distance [13], Steiner hyper-Wiener index [25], multi-center Wiener indices [14], and Steiner (revised) Szeged index [10]. We especiall point to the substantial survey [22] on the Steiner distance and related results and to the recent investigation of isometric subgraphs for Steiner distance [27].

Very recently, the Steiner 3-eccentricity of trees was investigated in [19]. A linear-time algorithm was developed to calculate the Steiner 3-eccentricity of a vertex in a tree, and lower and upper bounds for the average Steiner 3-eccentricity index on trees were derived. In this paper we extend these results to arbitrary $k \geq 2$. In the next section we propose a linear algorithm to calculate the Steiner k-eccentricity of a vertex in a tree. In Section 3 we establish lower and upper bounds of the average Steiner k-eccentricity on trees. We conclude this paper by presenting several possibilities for future work.

2 Steiner k-eccentricity of vertices in trees

The techniques from [19] that enabled to calculate the Steiner 3-eccentricity in a tree are not suitable for calculating the Steiner k-eccentricity of a vertex in a tree for arbitrary $k \geq 2$. In this section we establish new, stronger structural properties for the Steiner k-ecc v-tree for a vertex v in a tree, and then apply them to devise a linear time algorithm to calculate the Steiner k-eccentricity of a vertex in a tree.

2.1 Two key structural properties

Before stating the two properties, let us introduce some notation and terminology on trees. A vertex of a tree of degree at least 3 is a *branching vertex*. Let $L(T)$ denote the set of pendent vertices (leaves) of a tree T. If u and v are vertices of a tree T, then we will denote the (unique) $u.v$-path in T by $P(u, v, T)$. Given a vertex $v \in V(T)$ and a leaf $u \in L(T)$, let w be the nearest branching vertex to u on $P(v, u, T)$. If there is no branching vertex on $P(v, u, T)$, we set $w = v$. Then we say that the sub-path $P(w, u, T)$ of $P(v, u, T)$ is a *quasi-pendent path (with respect to u and v)*.

In the rest we will use the following earlier lemma, also without explicitly mentioning it.

Lemma 2.1 [19, Lemmas 2.4, 2.5] *If T is a tree and $v \in V(T)$, then the following holds.*

(i) If $k > |L(T)|$, then every k-ecc v-set contains all the leaves of T. The same conclusion holds if v is a leaf and $k = |L(T)|$.

(ii) If $2 \leq k \leq |L(T)|$ and S is a k-ecc v-set, then every vertex from $S \setminus \{v\}$ is a leaf of T.

For our first structural result, we need one more lemma.

Lemma 2.2 *Let $k \geq 2$, let v be a vertex of a tree T, let T_v^k be a Steiner k-ecc v-tree, and let T_v^{k-1} be a Steiner $(k-1)$-ecc v-tree. Then there exists a leaf $u \in L(T_v^k) \setminus L(T_v^{k-1})$ such that the quasi-pendent path $P(w, u, T_v^k)$ has no common edge with T_v^{k-1}.*

Proof. If $k = 2$, then T_v^1 is a tree on a single vertex v, hence the conclusion is clear. Assume in the rest that $k \geq 3$ and suppose on the contrary that every leaf $u \in L(T_v^k) \setminus L(T_v^{k-1})$ satisfies that the quasi-pendant path $P(w, u, T_v^k)$ has common edges with T_v^{k-1}. Then to every leaf $u \in L(T_v^k)$ we can associate its private leaf of $L(T_v^{k-1})$. Hence the number of leaves in T_v^{k-1} is not less than that in T_v^k. This contradicts the fact (by Lemma 2.1) that the Steiner $(k-1)$-ecc v-set corresponding to T_v^{k-1} has one less element than the Steiner k-ecc v-set corresponding to T_v^k. ∎

Theorem 2.3 *Let $k \geq 2$, and let v be a vertex of a tree T. Then every Steiner k-ecc v-tree contains some Steiner $(k-1)$-ecc v-tree.*

Proof. The case $k = 2$ is trivial, hence assume in the rest that $k \geq 3$. Let T_v^k be a Steiner k-ecc v-tree and suppose on the contrary that it contains no Steiner $(k-1)$-ecc v-tree. If T_v^{k-1} is an arbitrary Steiner $(k-1)$-ecc v-tree, then, by Lemma 2.2, we may select a leaf u from T_v^k such that the quasi-pendant path $P(w, u, T_v^k)$ does not have common edges with T_v^{k-1}.

Let S_v^k be the Steiner k-ecc v-set corresponding to T_v^k and set $S_1 = S_v^k \setminus \{u\}$. Then S_1 is a $(k-1)$-set containing the vertex v. Moreover, the tree $T_1 = T_v^k \setminus (P(w, u, T_v^k) \setminus \{w\})$ is a $MST(S_1, T)$. By the assumption, the size of T_1 is strictly less than that of T_v^{k-1}, that is,

$$|E(T_1)| < |E(T_v^{k-1})| \,. \tag{1}$$

Let $S_2 = S_v^{k-1} \cup \{u\}$, where S_v^{k-1} is the Steiner $(k-1)$-ecc v-set corresponding to the tree T_v^{k-1}. Then S_2 is a k-set which contains the vertex v. Let T_2 be a $MST(S_2, T)$. In the following we are going to show that the size of T_2 is larger than that of T_v^k.

Since the quasi-pendant path $P(w, u, T_v^k)$ does not share any edge with T_v^{k-1} and must be a sub-path of the quasi-pendant path $P(w', u, T_2)$, the size of T_2 satisfies

$$|E(T_2)| \geq |E(T_v^{k-1})| + |E(P(w, u, T_v^k))| \,. \tag{2}$$

Combining (1) and (2) we obtain that

$$\begin{aligned} |E(T_2)| &\geq |E(T_v^{k-1})| + |E(P(w, u, T_v^k))| \\ &> |E(T_1)| + |E(P(w, u, T_v^k)) \\ &= |E(T_v^k)| \,. \end{aligned} \tag{3}$$

Hence $|E(T_2)| > |E(T_v^k)|$. Since T_2 is a minimum Steiner tree on a k-set containing v, (3) contradicts the fact that T_v^k is a Steiner k-ecc v-tree. ■

Theorem 2.3 thus asserts that a Steiner k-ecc v-tree contains some Steiner $(k-1)$-ecc v-tree. The question now is, how to determine such a Steiner $(k-1)$-ecc v-tree. The message of the next result is that for our purposes, any Steiner $(k-1)$-ecc v-tree will do it. Before stating the theorem, we need some more notation. If H is a subgraph of a graph G, and $v \in V(G)$, then the distance from v to H is $d_G(v, H) = \min\{d_G(v, u) : u \in V(H)\}$. The eccentricity of H in G is $\mathrm{ecc}_G(H) = \max\{d_G(v, H) : v \in V(G)\}$.

Theorem 2.4 *Let $k \geq 1$, and let v be a vertex of a tree T. If T_1 and T_2 are Steiner k-ecc v-trees of T, then $\mathrm{ecc}_T(T_1) = \mathrm{ecc}_T(T_2)$.*

Proof. There is nothing to be proved if $T_1 = T_2$. Hence assume in the rest that T_1 and T_2 are different Steiner k-ecc v-trees of T. If $k = 1$, then a (unique) Steiner 1-ecc v-tree is induced by the vertex v itself. Since all longest paths starting from v have the same length, the assertion of the theorem is clear for $k = 1$. Hence we may also assume in the rest of the proof that $k \geq 2$.

Let P_1 and P_2 be longest paths from vertices of $V(T)$ to trees T_1 and T_2, respectively. Let u_1 and u_2 be the two endpoints of P_1 with $u_1 \in V(T_1)$, and let w_1 and w_2 be the two endpoints of P_2 with $w_1 \in V(T_2)$. Set $T_0 = T_1 \cap T_2$. To prove the theorem it suffices to prove that $u_1 \in V(T_0)$ and $w_1 \in V(T_0)$. By symmetry, it suffices to prove the first assertion, that is, $u_1 \in V(T_0)$.

Suppose on the contrary that $u_1 \in V(T_1) \setminus V(T_0)$. Let s be a leaf of T_1 such that u_1 is on the path $P(v, s, T_1)$. Then there must be a vertex $w_0 \in V(T_0)$ and a leaf t of T_2 such that $E(P(w_0, s, T_1)) \cap E(P(w_0, t, T_2)) = \emptyset$, see Fig. 1. Note that w_0 may be the vertex v.

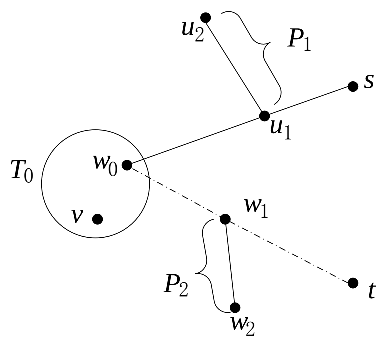

Figure 1: The configuration of the vertices w_0, u_1, u_2, w_1, w_2, s and t.

We claim that $V(P_1) \cap V(T_2) = \emptyset$. Otherwise, let $x \in V(T_2) \cap V(P_1)$. Then the path $E(P(x, v, T_1)) \setminus E(P(x, v, T_2)) \neq \emptyset$, since $E(P(w_0, u_1, T_1)) \neq \emptyset$. So the two paths $P(x, v, T_1)$ and $P(x, v, T_2)$ form a cycle in the original graph T. This contradicts to the fact that T is a tree. In the same way, we obtain that $V(P_2) \cap V(T_2) = \emptyset$.

Since $|E(P_1)| = d_T(u_2, T_1) = \mathrm{ecc}_T(T_1)$ and $|E(P(w_0, t, T_2))| = d_T(t, T_1)$, we have

$$|E(P_1)| \geq |E(P(w_0, t, T_2))| . \tag{4}$$

Moreover, since we have assumed that $u_1 \in V(T_1) \setminus V(T_0)$, we infer that $|E(P(u_1, w_0, T))| > 0$. Together with (4) this yields

$$\begin{aligned} |E(P(u_2, w_0, T))| &= |E(P_1)| + |E(P(u_1, w_0, T))| \\ &\geq |E(P(w_0, t, T_2))| + |E(P(u_1, w_0, T))| \\ &> |E(P(w_0, t, T_2))| . \end{aligned} \tag{5}$$

Now we pay attention to the tree T_2. Let S be the Steiner k-ecc v-set corresponding to the tree T_2. Let $S' = S \setminus \{t\} \cup \{u_2\}$. Then S' is a k-set containing the vertex v. In the following, we will establish a contradition that the tree $T_2' = MST(S', T)$ has more edges than the tree T_2. Recall that T_2 is a Steiner k-ecc v-tree.

Let $P(w, t, T_2)$ be the quasi-pendant path with respect to v in T_2 and distinguish the following cases.

Case 1: $w \in V(P(w_0, t, T_2)) \setminus \{w_0\}$.
In this case the tree $T_2' = MST(S', T)$ can be represented as $T_2' = T_2 \setminus P(w, t, T_2) \cup P(w_0, u_2, T)$. Since the path $P(w, t, T_2)$ is a sub-path of $P(w_0, t, T_2)$, $|E(P(w_0, t, T_2))| \geq |E(P(w, t, T_2))|$ holds. Combining this fact with (5) we have:

$$\begin{aligned}
|E(T_2')| &= |E(T_2)| - |E(P(w, t, T_2))| + |E(P(w_0, u_2, T))| \\
&\geq |E(T_2)| - |E(P(w_0, t, T_2))| + |E(P(w_0, u_2, T))| \\
&> |E(T_2)| .
\end{aligned}$$

Case 2: $w \in V(T_0)$.
Now the tree $T_2' = MST(S', T)$ can be represented as $T_2' = T_2 \setminus P(w, t, T_2) \cup P(w, u_2, T)$. Recall that the path $P(w, t, T_2)$ is composed of two sub-paths which are $P(w, w_0, T_2)$ and $P(w_0, t, T_2)$ respectively. And $P(w, u_2, T)$ is also composed of two sub-paths which are $P(w, w_0, T_2)$ and $P(u_2, w_0, T)$. By (5) we can estimate as follows:

$$\begin{aligned}
|E(T_2')| &= |E(T_2)| - |E(P(w, t, T_2))| + |E(P(w, u_2, T))| \\
&= |E(T_2)| - (|E(P(w, w_0, T_2))| + |E(P(w_0, t, T_2))|) \\
&\quad + (|E(P(w, w_0, T_2))| + |E(P(u_2, w_0, T))|) \\
&= |E(T_2)| - |E(P(w_0, t, T_2))| + |E(P(u_2, w_0, T))| \\
&> |E(T_2)| .
\end{aligned}$$

In both cases we have thus proved that $|E(T_2')| > |E(T_2)|$, a contradiction to the fact that T_2 is a Steiner k-ecc v-tree. ■

2.2 A linear time algorithm

By Theorems 2.3 and 2.4, the problem to calculate the Steiner k-eccentricity of a given vertex of a tree can be reduced to recursively finding a longest path starting at a given vertex. This is formally done in Algorithm 1.

Algorithm 1: k-ECC(v, T, k)

```
   Input: A vertex v, a tree T, and an integer k ≥ 2
   Output: The Steiner k-eccentricity of v in T
1  if the number of leaves is less than k then
2  |   return |V(T)| − 1;
3  end
4  else
5  |   ecc = 0;
6  |   for i = 1 to k − 1 do
7  |   |   Longest_Path(v, T, path);
8  |   |   ecc = ecc + |E(P)|;
9  |   |   Path_Shrinking(v, T, path);
10 |   end
11 |   return ecc
12 end
```

To explain Steps 1-3 of Algorithm 1, we state the following lemma.

Lemma 2.5 *Let $k \geq 3$ and let v be a vertex of a tree T. If $|L(T)| < k$, then the Steiner k-ecc v-tree is the entire tree T.*

Proof. The cardinality of the set $S = \{v\} \cup L(T)$ is at most k, since $|L(T)| < k$. Moreover, the $MST(S, T)$ is the entire tree T. Hence the Steiner k-ecc v-tree is the entire T. ∎

Steps 4-12 form the recursive reduction which consists of finding $k-1$ times a longest path starting at a vertex. In Step 7 we use the depth-first search (DFS) algorithm [4] to find a longest path starting at a given vertex, tte details are present in Algorithm 2. Step 9 shrinks the path obtained in Step 7 into a single vertex for the purpose of the next loop, the details are presented in Algorithm 3. Algorithms 2 and 3 are borrowed from [19] where one can find additional details on them. For the statement of these algorithms we recall that if v is a vertex of a graph G, then the set of its neighbours is denoted by $N_G(v)$.

Algorithm 2: Longest_Path(v, T, path)

Input: A vertex v, a tree T rooted at v, and an array named *path* to store a longest path starting at v

Output: the length of a longest path starting at v

```
max=0; temp=max;
for each vertex u ∈ N_T(v) which has not been visited till now do
    temp=Longest_Path(u, T, path);
    if temp>max then
        path[v]=u;
        max=temp;
    end
end
return max+1;
```

Algorithm 3: Path_Shrinking(v, T, path)

Input: A tree T, a vertex v, and an array named *path* to store a longest path starting at v

Output: A new tree obtained by shrinking the longest path into the single vertex v

```
w=v;
while path[w]≠ ∅ do
    for each vertex x ∈ N_T(w) do
        remove the edge (w, x) from T;
        add a new edge between x and v in T;
    end
    w=path[w];
end
```

Theorem 2.6 *Algorithm 1 computes the Steiner k-eccentricity of a vertex in a tree and can be implemented to run in $O(k(n+m))$ time, where n and m are the order and the size of the tree, respectively.*

Proof. The correctness of Algorithm 1 is ensured by Theorems 2.3 and 2.4.

By Lemma 2.5, the Steiner k-eccentricity of a vertex in a tree is equal to the size of the tree if its number of leaves is less than k. There is a linear-time algorithm to find all leaves of a tree by the depth-first search (DFS) algorithm [4]. Hence Steps 1-3 can be implemented in $O(n+m)$ time. Similarly, each loop in Steps 6-9 can be implemented in $O(n+m)$ time, thus all loops require $O(k(n+m))$ time. ■

To conlcude the section we again point out that the structural properties to support the algorithm(s) from [19] only ensure calculation of the Steiner 3-eccentricity. Hence we need to develop a new approach that works for general k.

3 Upper and lower bounds

In this section we establish an upper and a lower bound on the average Steiner k-eccentricity index of a tree for $k \geq 3$. These bounds were earlier proved in [19] in the special case $k = 3$. It is appealing that to obtain the bound for the general case, the proof idea is quite different and significantly simpler that the one in [19]. For the new approach, the following construction is essential.

π-transformation: Let T be a tree and let $P = P(u, v, T)$ be a path with at least one edge, such that every internal vertex of P is of degree 2 in T. Let X be the maximal subtree containing u in the tree $T \setminus E(P)$, and Y be the maximal subtree containing v in the graph $T \setminus E(P)$. We may without loss of generality assume that $\mathrm{ecc}_T(u, X) \leq \mathrm{ecc}_T(v, Y)$. Then the *$\pi$-transformation* $\pi(T)$ of T is defined as $T' = \pi(T) = T \setminus \{(u, w) : w \in N_X(u)\} \cup \{(v, w) : w \in N_X(u)\}$. The inverse transformation is is $T = \pi^{-1}(T') = T' \setminus \{(v, w) : w \in N_X(v)\} \cup \{(u, w) : w \in N_X(v)\}$. See Fig. 2.

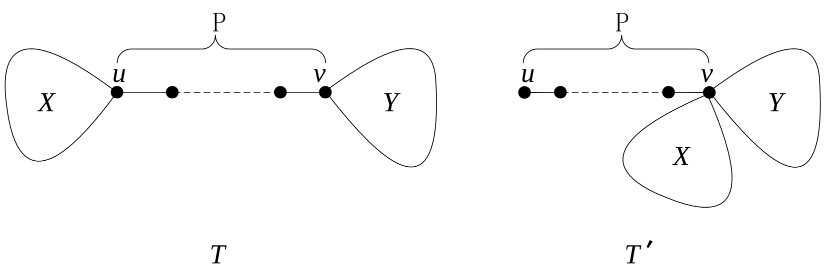

Figure 2: $T' = \pi(T)$ and $T = \pi^{-1}(T')$

Lemma 3.1 *Let T, P, u, v, X, Y, and T' be as in the definition of the π-transformation. If $w \in V(P) \cup V(X)$, then in T' there exists a Steiner k-ecc w-set S such that $S \cap (V(Y) \setminus \{v\}) \neq \emptyset$.*

Proof. Let S' be a Steiner k-ecc w-set in T' such that $S \cap (V(Y) \setminus \{v\}) = \emptyset$, and set $Q = S' \setminus \{w\}$. Since $k \geq 3$, the cardinality of Q is at least two. Let $v' \in V(Y)$ such that the distance between v and v' is $\mathrm{ecc}_{T'}(v, Y)$. Consider the following two cases.

Case 1: $Q \cap V(X) = \emptyset$.
In this case the vertices of Q are all in P. Let $w' \in Q$ be the nearest vertex to v. Construct a new vertex set $S'' = (S' \setminus \{w'\}) \cup \{v'\}$.

Case 2: $Q \cap V(X) \neq \emptyset$.
Let $w' \in Q \cap V(X)$. Construct a new vertex set $S'' = (S' \setminus \{w'\}) \cup \{v'\}$.

In each of the two cases, the size of $MST(S'', T')$ is not less than the size of $MST(S', T')$, hence the assertion. ∎

Lemma 3.2 *Under the notation of Lemma 3.1,* $\text{aecc}_k(T) \geq \text{aecc}_k(T')$.

Proof. If w is a vertex in $V(Y) \setminus \{v\}$, then for any Steiner k-ecc w-set S' in T', the size of a minimum Steiner tree on S' in graph T is not less than that in T'. So the Steiner k-eccentricity of every vertex $w \in V(Y)$ in T is not less than that in T'.

If w is a vertex in $V(P) \cup V(X)$, then by Lemma 3.1, there exists a Steiner k-ecc w-set S' in T', such that $S' \cap (V(Y) \setminus \{v\}) \neq \emptyset$. The size of a minimum Steiner tree on S' in T is not less than that in T'. Therefore the Steiner k-eccentricity of every vertex $w \in V(P) \cup V(X)$ in T is not less than that in T'.

In any case, the Steiner k-eccentricity of every vertex $v \in V(T')$ is not larger than that in T. As the average Steiner k-eccentricity index is the mean value of all vertices' Steiner k-eccentricities, the average Steiner k-eccentricity of T' is not large than that of T. ∎

If the order of a tree T is not larger than k, then a Steiner k-ecc v-set contains all vertices of T for every $v \in V(T)$. Then every Steiner k-ecc v-tree is the entire tree T for every vertex v. So for a given $k \geq 3$, we just consider the trees where the order of each is more than k.

Theorem 3.3 *If $k \geq 3$ is an integer, and T a tree on order $n > k$, then*

$$k - \frac{1}{n} \leq \text{aecc}_k(T) \leq n - 1.$$

Moreover, the star S_n attains the lower bound, and the path P_n attains the upper bound.

Proof. Repeatedly applying the π-transformation on T until it is possible, we obtain the star S_n. On the other hand, repeatedly applying the π^{-1} transformation on T until it is possible, we obtain the path P_n. By Lemma 3.2, the π-transformation does not increase the average Steiner k-eccentricity of T. Hence the star S_n attains the minimum Steiner k-eccentricity, and the path P_n attains the maximum Steiner k-eccentricity. Finally, we obtain $\text{aecc}_k(S_n) = k - \frac{1}{n}$ and $\text{aecc}_k(P_n) = n - 1$ by straightforward computation. ∎

In Fig. 3 an example is given in which the process of constructing extremal graphs, that is, a start and a path, by means of the π-transformation and the π^{-1}-transformation.

In [17] the average Steiner 2-eccentricity of trees was investigated. For the sake of our final result, we recall the following result.

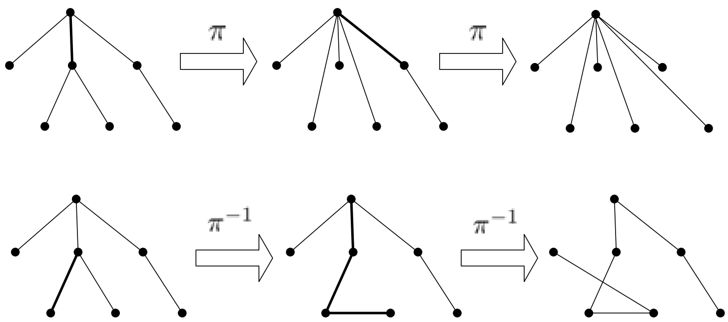

Figure 3: Constructing extremal graphs using the π transformation and the π^{-1} transformation. Bold edges denote the paths defined in the transformations.

Lemma 3.4 ([17]) *Let T be a tree of order n. Then* $\text{aecc}_2(S_n) \leq \text{aecc}_2(T) \leq \text{aecc}_2(P_n)$. *The left equality holds if and only if $T \cong S_n$, while the right equality holds if and only if $T \cong P_n$.*

Combining Theorem 3.3 with Lemma 3.4, we have the following result.

Corollary 3.5 *If $k \geq 2$ is an integer, then S_n (resp. P_n) attains the minimum (resp. the maximum) average Steiner k-eccentricity in the class of trees.*

4 Conclusion

In this paper we have derived a linear-time algorithm to calculate the Steiner k-eccentricity of a vertex in a tree, and established lower and upper bounds for the average Steiner k-eccentricity of a tree. These results extend those from [19] for the case $k = 3$. It remains open to determine the extremal graphs for the average Steiner k-eccentricity index on trees for $k \geq 2$. Moreover, the general problem to compute the Steiner k-eccentricity of a general graph is widely open, in particular, it is not known whether it is NP-hard.

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