# Characterizing Subgraphs of Hamming Graphs

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**Abstract:** Cartesian products of complete graphs are known as Hamming graphs. Using embeddings into Cartesian products of quotient graphs we characterize subgraphs, induced subgraphs, and isometric subgraphs of Hamming graphs. For instance, a graph $G$ is an induced subgraph of a Hamming graph if and only if there exists a labeling of $E(G)$ fulfilling the following two conditions: (i) edges of a triangle receive the same label; (ii) for any vertices $u$ and $v$ at distance at least two, there exist two labels which both appear on any induced $u, v$-path. © 2005 Wiley Periodicals, Inc. J Graph Theory 49: 302–312, 2005

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## 1. INTRODUCTION

Hamming graphs are, by definition, Cartesian products of complete graphs. For different characterizations of these graphs see [2,3,21,22,23]. The special case when all factor graphs are of the same order is treated in [9]. Hamming graphs can be recognized in linear time and space [15,16]. Isometric subgraphs of Hamming graphs, called *partial Hamming graphs*, have been intensively studied by now, *cf.* [1,4,6,25,26]. One of the most important subclasses of partial Hamming graphs is formed by weak retracts of Hamming graphs; these graphs are known as quasi-median graphs. Quasi-median structures have been independently discovered several times, *cf.* [3] and references therein.

Graham and Winkler [12] proved that every graph has a best representation as an isometric subgraph of a Cartesian product, *cf.* also [13,17]. The key construction is based on embeddings into Cartesian products of the so-called quotient graphs with respect to a certain relation defined on the edge set of a graph. Feder [10,11] followed with a similar approach in order to obtain such representations for stronger embeddability conditions: 2-isometric representation, weak retract representation, and Cartesian prime factorization.

In this paper, we take the opposite direction, treating isometry as the strongest property. We namely consider subgraphs (see also [24]), induced subgraphs, and isometric subgraphs of Hamming graphs. We show, roughly speaking, that embeddings into Cartesian product of quotient graphs can be applied also to subgraphs and induced subgraphs of Hamming graphs. Of course, as the embeddability conditions are rather weak in these two cases, we cannot expect to obtain some "best" (say unique) representation for subgraphs and induced subgraphs.

In the rest of this section, we fix the notation and introduce the scheme for embeddings into Cartesian products of quotient graphs. In Section 2, we first observe that a graph $G$ is a non-trivial subgraph of the Cartesian product of graphs if and only if $G$ is a non-trivial subgraph of the Cartesian product of two complete graphs. Then we characterize subgraphs of Hamming graphs via certain edge labelings of graphs. In Section 3, we give a similar characterization for induced subgraphs of Hamming graphs. As far as we know, no characterization of induced subgraphs of Hamming graphs was previously known. In the last section, we briefly discuss partial Hamming graphs, adding one more characterization of these graphs to the literature.

The *Cartesian product* $G \square H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ in which the vertex $(a, x)$ is adjacent to the vertex $(b, y)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. For a fixed vertex $a$ of $G$, the vertices $\{(a, x) \mid x \in V(H)\}$ induce a subgraph of $G \square H$ isomorphic to $H$, called an *H-layer* of $G \square H$. Analogously we define $G$-layers. A subgraph of $G \square H$ is called *non-trivial* if it intersects at least two $G$-layers and at least two $H$-layers. The map $p_G : G \square H \to G$ defined by $p_G(a, x) = a$, is called a *projection*. Clearly, the image of an edge under a projection is either an edge or a vertex.

As we already mentioned, Cartesian products of complete graphs are known as *Hamming graphs*. They can be alternatively described as follows. For $i = 1, 2, \ldots, n$ let $r_i \geq 2$ be given integers. Let $G$ be the graph whose vertices are the $n$-tuples $b_1 b_2 \cdots b_n$ with $b_i \in \{0, 1, \ldots, r_i - 1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one place—one coordinate. Then it is easy to see that $G$ is isomorphic to $K_{r_1} \Box K_{r_2} \Box \cdots \Box K_{r_n}$. For an edge $uv$ of $H = K_{r_1} \Box K_{r_2} \Box \cdots \Box K_{r_n}$ we define the *color map* $c : E(H) \to \{1, 2, \ldots, n\}$ with $c(uv) = i$, where $u$ and $v$ differ in coordinate $i$.

A subgraph $H$ of $G$ is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the length of a shortest $u, v$-path with respect to $G$. Note that an isometric subgraph is induced.

We now introduce the central concept of this paper. Let $G$ be a connected graph and let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a partition of $E(G)$. The *quotient graph* $G/F_i$ has connected components of $G \backslash F_i$ as vertices, two components $C$ and $C'$ being adjacent whenever there exists an edge of $F_i$ connecting a vertex of $C$ with a vertex of $C'$. For each $i$, define a map $f_i : V(G) \to V(G/F_i)$ by $f_i(v) = C$, where $C$ is the component of $G \backslash F_i$ containing $v$. Then let

$$f : G \to G/F_1 \Box G/F_2 \Box \cdots \Box G/F_k$$

be the natural coordinate-wise mapping, that is,

$$f(v) = (f_1(v), f_2(v), \ldots, f_k(v)).$$

We call $f$ the *quotient map* of $G$ with respect to $\mathcal{F}$. Note that $f$ need not be one-to-one in general and that it is possible that some quotient graphs are the one vertex graph. However, all the partitions $\mathcal{F}$ introduced later will lead to one-to-one mappings with non-trivial quotient graphs.

A partition $\{F_1, F_2, \ldots, F_k\}$ of $E(G)$ naturally leads to an edge-labeling $\ell : E(G) \to \{1, 2, \ldots, k\}$ by setting $\ell(e) = i$, where $e \in F_i$. Unless stated otherwise, a *labeling* (or more precisely a *k-labeling*) of $G$ will mean an edge-labeling (with $k$ labels).

Finally, two edges $e = xy$ and $f = uv$ of a graph $G$ are in the Djoković and Winkler [8,26] relation $\Theta$ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Relation $\Theta$ is reflexive and symmetric, $\Theta^*$ stands for the transitive closure of $\Theta$. Let $\mathcal{F}$ be the partition of $E(G)$ induced by $\Theta^*$. Graham and Winkler [12] proved that the corresponding quotient map is an isometry and called it the *canonical isometric embedding* of $G$.

## 2. SUBGRAPHS OF HAMMING GRAPHS

Subgraphs of Cartesian product graphs were first investigated by Lamprey and Barnes [19,20] and later characterized in [18] using certain vertex-labelings. Additional results are given in [5]. In this section, we give another characterization of them, this time using edge-labelings. Before giving the result, we note that the general problem can be reduced to that of characterizing subgraphs of Hamming graphs, as the next lemma asserts. Recall that a subgraph $G$ of $G_1 \square G_2$ is non-trivial if the projections $p_{G_1}(G)$ and $p_{G_2}(G)$ both contain at least two vertices. More generally, $G$ is a non-trivial subgraph of $G_1 \square G_2 \square \cdots \square G_k$, if $p_{G_i}(G)$ contains at least two vertices for $i = 1, 2, \ldots, k$.

**Lemma 2.1.** *A graph $G$ is a non-trivial subgraph of the Cartesian product of graphs if and only if $G$ is a non-trivial subgraph of a Hamming graph with two factors.*

***Proof.*** Clearly, we only need to prove that a non-trivial subgraph of the Cartesian product of graphs can be embedded as a non-trivial subgraph into the Cartesian product of two complete graphs. So let $G$ be a non-trivial subgraph of $G_1 \square G_2 \square \cdots \square G_k = G_1 \square (G_2 \square \cdots \square G_k)$. Setting $H_1 = G_2 \square \cdots \square G_k$, we get that $G$ is a non-trivial subgraph of $G_1 \square H_1$. But now we can connect any non-adjacent vertices of $G_1$ and of $H_1$ without violating that $G$ is a non-trivial subgraph. We conclude that $G$ is a non-trivial subgraph of $K_{|V(G_1)|} \square K_{|V(H_1)|}$. ■

By Lemma 2.1 we only need to consider 2-labelings of graphs with respect to their embeddability into Cartesian products. Thus, for a given 2-labeling of $G$ we pose the following condition.

**Condition A.** *Let $G$ be a 2-labeled graph. Let $C$ be an induced cycle of $G$ that possesses both labels. Then the labels change at least three times while passing the cycle.*

Note that if $G$ has a 2-labeling obeying Condition A, then the edges of a triangle have the same label, and the opposite edges of an induced quadrangle have the same label.

**Theorem 2.2.** *Let $G$ be a connected graph. Then $G$ is a non-trivial subgraph of the Cartesian product of graphs if and only if there exists a 2-labeling of $G$ that fulfills Condition A.*

***Proof.*** Let $G$ be a non-trivial subgraph of the Cartesian product of graphs. By Lemma 2.1 we can restrict to products of two complete graphs.

Let $G$ be a non-trivial subgraph of $K_n \square K_m$, $n, m \geq 2$ and let $e \in E(G)$. Then we set

$$\ell(e) = \begin{cases} 1; & p_{K_n}(e) \text{ is an edge,} \\ 2; & p_{K_n}(e) \text{ is a vertex}. \end{cases}$$

Let $C = v_1 v_2 \cdots v_k$ be an induced cycle of $G$ that possesses both labels. Suppose that labels change only twice on $C$, that is, the labels along $C$ are $1, \ldots, 1, 2, \ldots, 2$, where $v_1 v_2$ is the first edge with label 1. By the definition of the Cartesian product and $\ell$, vertex $v_1$ is of degree at least 4 on $C$. As this is not possible, the labeling $\ell$ fulfills Condition A.

Conversely, let $\ell$ be a 2-labeling of a connected graph $G$ that satisfies Condition A. Let $\mathcal{F} = \{F_1, F_2\}$ be the partition of $E(G)$ induced by $\ell$ and let $f$ be the quotient map of $G$ with respect to $\mathcal{F}$. We are going to show that $G$ is a non-trivial subgraph of $G/F_1 \,\square\, G/F_2$.

We claim that $f$ is one-to-one. Let $u \neq v$ be vertices of $G$. Suppose first that $e = uv \in E(G)$ and assume without loss of generality that $\ell(uv) = 1$, that is, $uv \in F_1$. We claim that $u$ and $v$ are in different components of $G \backslash F_1$. Let $P$ be an arbitrary path connecting the endvertices of $e$ not containing $e$. Suppose that every edge of $P$ has label 2 and select $P$ to be shortest possible among all such paths. The edge $e$ together with the path $P$ forms a cycle $C$. By Condition A, $C$ is not induced. Let $w$ and $w'$ be two non-consecutive but adjacent vertices of $C$. Select $w$ and $w'$ such that the distance between $w$ and $w'$ along the cycle is as short as possible. Then $\ell(ww') = 1$ by the minimality of $P$. Let $C'$ be the cycle containing the edge $ww'$ and the $w, w'$-subpath of $C$ containing only labels 2. Then $C'$ is an induced cycle on which the labels change only twice, a contradiction with Condition A. Hence, $f(u) \neq f(v)$ if $uv$ is an edge.

Suppose next $d_G(u, v) \geq 2$. If every $u, v$-path contains at least one edge with label 1, or if every $u, v$-path contains at least one edge with label 2, we are done. Indeed, then $u$ and $v$ are mapped into different vertices in at least one of $G/F_1$ and $G/F_2$. Thus assume that there are $u, v$-paths $P$ and $Q$ such that all edges of $P$ receive label 1 and all edges of $Q$ label 2. Let $P$ and $Q$ be shortest among all such paths. Let $w$ be the first common vertex of $P$ and $Q$ after $u$ traversing these two paths from $u$ to $v$. (Note that we may have $w = v$.) Then the $u, w$-subpath $P'$ of $P$ together with the $u, w$-subpath $Q'$ of $Q$ form a cycle $C$. If $C$ is induced we violate Condition A. Hence, assume $C$ is not induced and consider an edge that connects non-adjacent vertices of $C$ of minimal possible distance. Then we have three possibilities. The first is that there is an edge between non-consecutive vertices of $P'$. By the minimality of $P$, the label of this edge is 2. The second case is that there is an edge between non-consecutive vertices of $Q'$. By the minimality of $Q$, the label of this edge must be 1. The last possibility is that there is an edge between a vertex of $P'$ and a vertex of $Q'$. Such an edge can be labeled 1 or 2. In any of the three cases, we find an induced cycle that violates Condition A and conclude that $f(u) \neq f(v)$ holds also in this case, which proves the claim.

To conclude the proof we just need to observe that an edge of $G$ is mapped by the quotient map to an edge of $G/F_1 \,\square\, G/F_2$. Hence $G$ is a subgraph of $G/F_1 \,\square\, G/F_2$. Moreover, since each of $G/F_1$ and $G/F_2$ contains at least two vertices, it also follows that $G$ is a non-trivial subgraph. ∎

**Corollary 2.3.** *A graph G is a non-trivial subgraph of a Hamming graph if and only if there exists a 2-labeling of G that fulfills Condition A.*

***Proof.*** Combine Theorem 2.2 with Lemma 2.1. ∎

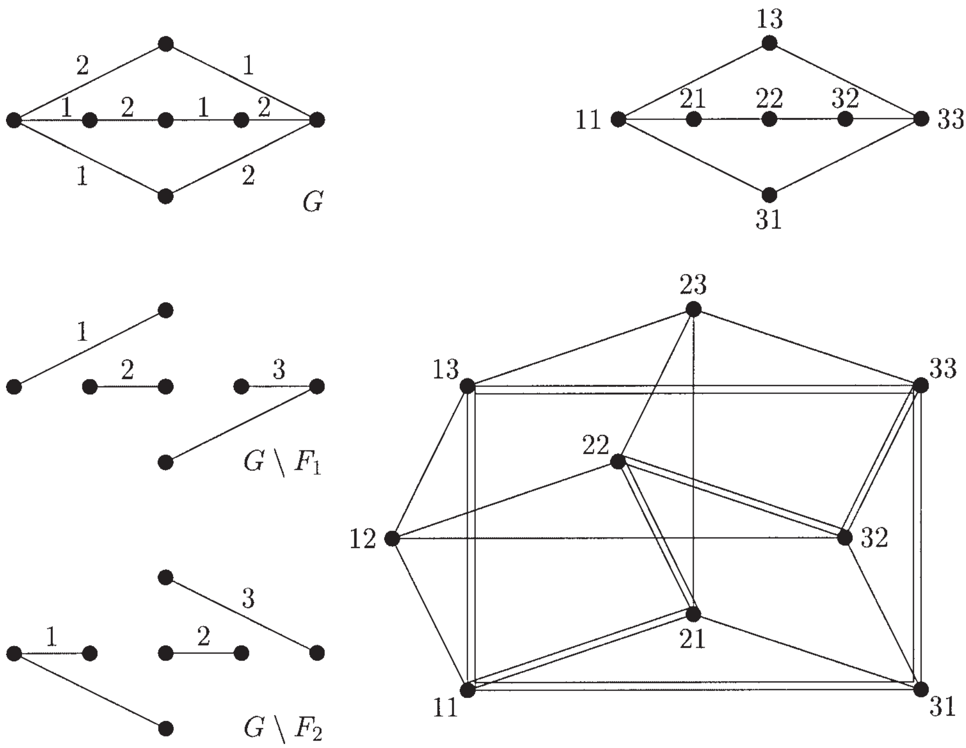


FIGURE 1. Subgraph of a Hamming graph and its quotient embedding.

Theorem 2.2 and its proof are illustrated in Figure 1. A graph $G$ is given together with a 2-labeling fulfilling Condition A. On the figures of $G\backslash F_1$ and $G\backslash F_2$, the connected components are assigned numbers 1, 2, and 3, that represent the vertices of $G/F_1 = K_3$ and $G/F_2 = K_3$. The images of vertices under the quotient map are also given and finally the embedding of $G$ into $K_3 \,\square\, K_3$ is shown.

The characterization of subgraphs of Cartesian product graphs $G$ from [18] involves vertex-labelings, where labels can use integers between 2 and $|V(G)| - 1$. Hence, the present approach seems to be more convenient which we demonstrate on the following example ([18, Corollary 3]).

Let $G$ be a bipartite graph with radius 2 and suppose that $G$ contains no subgraph isomorphic to $K_{2,3}$. We claim that $G$ is a non-trivial subgraph of the Cartesian product of two complete graphs. Let $u$ be a vertex of $G$ such that all vertices of $G$ are at distance at most two from it and let $v$ be a neighbor of $u$. Assign label 1 to $uv$ and to the edges between vertices in $N(u)\backslash v$ and $N(v)\backslash u$. Assign label 2 to all the remaining edges. Note that all induced cycles of $G$ are 4-cycles. Moreover, because $G$ is $K_{2,3}$-free, it follows immediately that every induced 4-cycle is labeled 1,2,1,2. Hence, $G$ is a non-trivial subgraph of a Hamming graph by Corollary 2.3.

## 3. INDUCED SUBGRAPHS OF HAMMING GRAPHS

In this section, we characterize induced subgraphs of Hamming graphs. We first state two labeling conditions needed for the result.

**Condition B.** *Let G be a labeled graph. Then edges of a triangle have the same label.*

**Condition C.** *Let G be a labeled graph and let u and v be arbitrary vertices of G with $d_G(u, v) \geq 2$. Then there exist different labels i and j which both appear on any induced $u, v$-path.*

Let $G$ be a labeled graph fulfilling Condition C. Let $C_k$, $k \geq 4$, be an induced cycle of $G$ and let $u$ and $v$ be vertices of $C_k$ with $d_{C_k}(u, v) = 2$. Then the labels of a $u, v$-path of length 2 on $C_k$ are different. Hence, by Condition C, the other $u, v$-path along $C_k$ contains these two labels. Therefore we infer:

**Lemma 3.1.** *Let G be a labeled graph fulfilling Condition C and let $C_k$, $k \geq 4$, be an induced cycle of G. Then every label of $C_k$ is presented more than once on $C_k$.*

For the main result of this section we also need:

**Lemma 3.2.** *Let G be a labeled graph fulfilling Conditions B and C and let $u, v$ be vertices of G with $d_G(u, v) \geq 2$. Then, if labels i and j appear on every induced $u, v$-path, they appear on every $u, v$-path.*

***Proof.*** Suppose that labels $i$ and $j$ appear on every induced $u, v$-path. Let $P = x_1x_2 \cdots x_r$, $x_1 = u$, $x_r = v$, be a $u, v$-path of minimal length that does not contain both labels $i$ and $j$. Then $P$ is not induced, hence we have an edge $e = x_kx_\ell$ with $\ell - k > 1$. We may assume that $e$ is selected such that $\ell - k$ is as small as possible. By the minimality of $P$, the path $x_1x_2 \cdots x_kx_\ell x_{\ell+1} \cdots x_r$ contains both labels $i$ and $j$. Hence, the label of the edge $x_kx_l$ is either $i$ or $j$. Assume without loss of generality it is $i$. Then, using minimality again, label $j$ appears on the path $x_1x_2 \cdots x_k$ or on $x_\ell x_{\ell+1} \cdots x_r$. It follows that $i$ does not appear on the path $x_kx_{k+1} \cdots x_\ell$. But then the label $i$ appears only once on the cycle $C = x_kx_{k+1} \cdots x_\ell x_k$. If $C$ is a triangle, we have a contradiction with Condition B, otherwise with Lemma 3.1. ■

**Theorem 3.3.** *Let G be a connected graph. Then G is an induced subgraph of a Hamming graph if and only if there exists a labeling of G that fulfills Conditions B and C.*

***Proof.*** Let $G$ be an induced subgraph of $H = K_{n_1} \,\square\, K_{n_2} \,\square \cdots \square\, K_{n_k}$. Denote $p_i = p_{K_{n_i}}$ and consider the labeling of $E(G)$ induced by the color map $c$ of $H$.

Condition B is clear. Indeed, if $u$, $v$, and $w$ induce a triangle, then they all lie in the same layer of $H$ and so the edges $uv$, $uw$, and $vw$ receive the same label. We next show Condition C. Let $u$ and $v$ be two vertices of $G$ with $d_G(u, v) \geq 2$. Suppose that there is no label that appears on all induced $u, v$-paths. Then

$p_i(u) = p_i(v)$ for all $i$, contrary to $d_G(u, v) \geq 2$. Suppose now that all induced $u, v$-paths have exactly one label in common, say $i$. We have $p_j(u) = p_j(v)$ for all $j \neq i$ and $p_i(u) \neq p_i(v)$. Vertices $p_i(u)$ and $p_i(v)$ are adjacent in $K_{n_i}$, since $K_{n_i}$ is a complete graph. Hence, $u$ and $v$ are adjacent in $H$ and therefore also in $G$ which is impossible.

Conversely, let $\ell$ be a labeling of $G$ that fulfills Conditions B and C. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be the partition of $E(G)$ induced by $\ell$ and let $f$ be the quotient map of $G$ with respect to $\mathcal{F}$. We claim that $f$ embeds $G$ as an induced subgraph into $G/F_1 \,\square\, G/F_2 \,\square \cdots \square\, G/F_k$.

We show first that $f$ is one-to-one. Suppose that vertices $x$ and $y$ are not adjacent in $G$. Then by Condition C and Lemma 3.2, there exist labels $i$ and $j$ such that on every $x, y$-path we find labels $i$ and $j$. So $x$ and $y$ are in different components in both $G \backslash F_i$ and $G \backslash F_j$. Already the first fact assures that $f(x) \neq f(y)$. Let next $x$ and $y$ be adjacent vertices of $G$ and let $\ell(xy) = i$. Suppose that there exists an $x, y$-path $P = x_1 x_2 \cdots x_r$ in $G \backslash F_i$, where $x_1 = x$ and $x_r = y$. We can assume that $P$ is shortest among all $x, y$-paths in $G \backslash F_i$. If $P$ is induced in $G - xy$ we have a contradiction with Condition B when $r = 3$ and a contradiction with Lemma 3.1 when $r > 3$. Thus $P$ is not induced in $G - xy$, $r > 3$, and there are adjacent vertices $x_j$ and $x_k$ with $k > j + 1$. By the minimality of $P$ we have $\ell(x_j x_k) = i$. We can select $j$ and $k$ such that $k - j$ is minimal among all such vertices $x_j$ and $x_k$. Then the cycle $C = x_j x_{j+1} \cdots x_{k-1} x_k x_j$ is induced. If $C$ is a triangle we have a contradiction with Condition B, otherwise we have a contradiction with Lemma 3.1. Hence, we have shown that $f$ is one-to-one.

Let $xy$ be an edge with $\ell(xy) = i$. Then, by the above, $x$ and $y$ are in different components of $G \backslash F_i$. Moreover, they belong to the same component in any of the graphs $G \backslash F_j$, $j \neq i$. It follows that $f$ maps edges to edges and the claim is proved.

Hence, $G = f(G)$ is an induced subgraph of $G/F_1 \,\square\, G/F_2 \,\square \cdots \square\, G/F_k$. To complete the proof we show that $G$ is also an induced subgraph of the Hamming graph

$$K_{|G/F_1|} \,\square\, K_{|G/F_2|} \,\square \cdots \square\, K_{|G/F_k|}\,.$$

Let $x$ and $y$ be non-adjacent vertices of $G$. Then, by the same reasoning as above, $x$ and $y$ are in different components of at least two graphs $G \backslash F_i$. It follows that $f(x)$ and $f(y)$ differ in at least two coordinates which remains valid after adding edges to the factor graphs. ■

Note that the quotient graphs obtained in the proof of Theorem 3.3 need not be complete. For instance, consider the path $P_4$ together with the labeling 1, 2, 1.

Theorem 3.3 (and its proof) are illustrated in Figure 2, where an admissible labeling is assigned to $C_7$, that is in turn embedded into $K_2 \,\square\, K_3 \,\square\, K_2$.

## 4. ISOMETRIC SUBGRAPHS OF HAMMING GRAPHS

We have already mentioned that isometric subgraphs are induced. It is well known that $C_7$ is an induced but not an isometric subgraph of a Hamming graph.

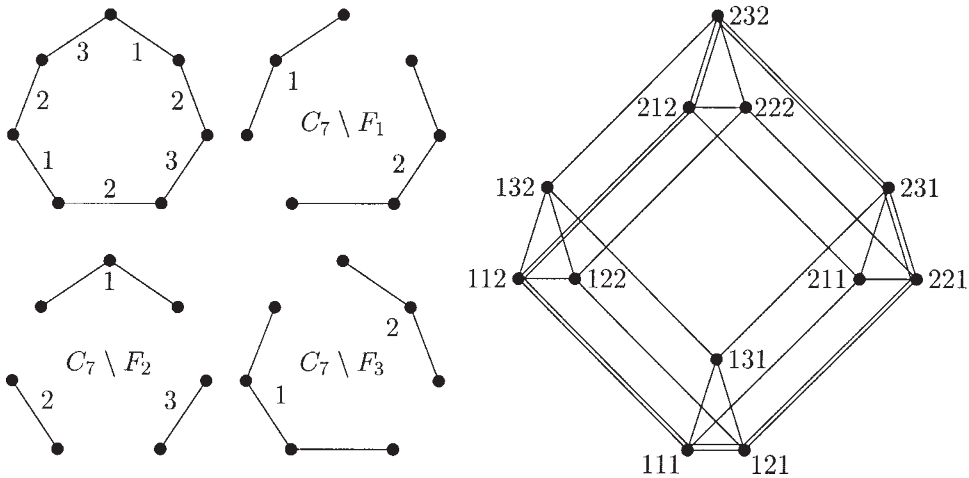


FIGURE 2. $C_7$ as an induced subgraph of $K_2 \square K_3 \square K_2$.

Hence, in order to characterize partial Hamming graphs, we add another labeling condition.

**Condition D.** *Let G be a labeled graph. Then the labels of any shortest path are pairwise different.*

**Theorem 4.1.** *Let G be a connected graph. Then G is a partial Hamming graph if and only if there exists a labeling of G that fulfills Conditions B, C, and D.*

***Proof.*** Let $G$ be an isometric subgraph of a Hamming graph $H$. Consider the labeling of $E(G)$ induced by the color map $c$ of $H$. By the proof of Theorem 3.3, $c$ fulfills Conditions B and C. Moreover, as $G$ is isometric in $H$, any shortest path of $G$ is a shortest path of $H$, hence $c$ fulfills Condition D as well.

Conversely, let $\ell$ be a labeling of $G$ that satisfies Conditions B, C, and D. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be the partition of $E(G)$ induced by $\ell$ and let $f$ be the corresponding embedding into $H = G/F_1 \square G/F_2 \square \cdots \square G/F_k$. By Conditions B and C and the proof of Theorem 3.3 we know that $G$ is an induced subgraph of $H$. We claim it is also isometric. Let $u$ and $v$ be any vertices of $G$ and let $P = x_1x_2 \cdots x_r$ ($x_1 = u$, $x_r = v$) be a shortest $u, v$-path in $G$. Then as the embedding is induced, $f(x_i)f(x_{i+1})$ is an edge of $H$ and hence $d_H(f(u), f(v)) \leq d_G(u, v)$. Moreover, by Condition D, edges of $P$ receive pairwise different labels which implies that $f(u)$ and $f(v)$ differ in at least $d_G(u, v)$ coordinates. Hence, $d_H(f(u)f(v)) \geq d_G(u, v)$ and so $d_H(f(u)f(v)) = d_G(u, v)$.

To complete the proof we show that $G/F_i$ is a complete graph for $i = 1, 2, \ldots, k$. Let $C$ and $C'$ be connected components of $G \setminus F_i$ and assume there is no edge in $F_i$ connecting a vertex of $C$ with a vertex of $C'$. Then a shortest path between a vertex of $C$ and a vertex of $C'$ contains at least two edges of label $i$, a contradiction with Condition D. ■

An isometric embedding $\beta : G \to H_1 \Box H_2 \Box \cdots \Box H_n$ is called *irredundant* if $|H_i| \geq 2$ for all $i$ and if every vertex $u \in \cup_{i=1}^{n} H_i$ occurs as a coordinate value of the image of some $w \in G$. In [14] (*cf.* also [7,17]) it is proved that any isometric irredundant embedding of a graph $G$ into a product of complete graphs is the canonical isometric embedding. Hence,

**Corollary 4.2.** *Let G be a connected graph equipped with a labeling that fulfills Conditions B, C, and D. Then this labeling coincides with the partition induced by* $\Theta^*$.

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