The general position number of the Cartesian product of two trees

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Abstract

The general position number of a connected graph is the cardinality of a largest set of vertices such that no three pairwise-distinct vertices from the set lie on a common shortest path. In this paper it is proved that the general position number is additive on the Cartesian product of two trees.

Keywords: general position set; general position number; Cartesian product; trees

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1 Introduction

Let $d_G(x,y)$ denote, as usual, the number of edges on a shortest x,y-path in G. A set S of vertices of a connected graph G is a *general position set* if $d_G(x,y) \neq d_G(x,z) + d_G(z,y)$ holds for every $\{x,y,z\} \in \binom{S}{3}$. The *general position number* $\mathrm{gp}(G)$ of G is the cardinality of a largest general position set in G. Such a set is briefly called a *gp-set* of G.

Before the general position number was introduced in [9], an equivalent concept was proposed in [14]. Much earlier, however, the general position problem has been studied by Körner [8] in the special case of hypercubes. Following [9], the graph theory general position problem has been investigated in [1, 3, 5, 6, 10, 11, 13].

The *Cartesian product* $G \,\Box\, H$ of vertex-disjoint graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g,h) and (g',h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. In this paper we are interested in ${\rm gp}(G \,\Box\, H)$, a problem earlier studied in [3, 6, 10, 13]. More precisely, we are interested in Cartesian products of two (finite) trees. (For some of the other investigations of the Cartesian product of trees see [2, 12, 15].) An important reason for this interest is the fact that the general position number of products of paths is far from being trivial. First, denoting with P_∞ the two-way infinite path, one of the main results from [10] asserts that ${\rm gp}(P_\infty \,\Box\, P_\infty) = 4$. Denoting further with G^n the n-fold Cartesian product of G, it was demonstrated in the same paper that $10 \le {\rm gp}(P_\infty^3) \le 16$. The lower bound 10 was improved to 14 in [6]. Very recently, these results were superseded in [7] by proving that if n is an arbitrary positive integer, then ${\rm gp}(P_\infty^n) = 2^{2^{n-1}}$. Denoting with $n(G)$ the order of a graph G, in this paper we prove:

Theorem 1. *If T and T^* are trees with $\min\{n(T), n(T^*)\} \ge 3$, then*

$$ {\rm gp}(T \,\Box\, T^*) = {\rm gp}(T) + {\rm gp}(T^*)\,. $$

Theorem 1 widely extends the above mentioned result ${\rm gp}(P_\infty \,\Box\, P_\infty) = 4$. Further, the equality ${\rm gp}(P_\infty^n) = 2^{2^{n-1}}$ shows that Theorem 1 has no obvious (inductive) extension to Cartesian products of more than two trees. Hence, to determine the general position number of such products remains a challenging problem.

In the next section we give further definitions, recall known results needed, and prove several auxiliary new results. Then, in Section 3, we prove Theorem 1.

2 Preliminaries

Let T be a tree. The set of leaves of T will be denoted by $L(T)$, and let $\ell(T) = |L(T)|$. If u and v are vertices of T with $\deg(u) \ge 2$ and $\deg(v) = 1$, then the unique u,v-path is a *branching path* of T. If u is not a leaf of T, then there are exactly $\ell(T)$ branching paths starting from u; we say that the u is the *root* of these branching paths and that the degree 1 vertex of a branching path P is the *leaf of* P.

Lemma 1. ([9]) *If T is a tree, then* ${\rm gp}(T) = \ell(T)$.

We next describe which vertices of a tree lie in some gp-set of the tree.

Lemma 2. *A non-leaf vertex u in a tree T belongs to a gp-set of T if and only if $T-u$ has exactly two components and at least one of them is a path.*

Proof. First, let R be a gp-set of T containing the non-leaf vertex u. Suppose that $T-u$ has at least three components, say T_1, T_2 and T_3. Since R is a gp-set containing u, R intersects with at most one of T_1, T_2 and T_3. Assume without loss of generality that $R \cap V(T_2) = \emptyset$ and $R \cap V(T_3) = \emptyset$. Choose vertices v and w in T such that $v \in V(T_2)$ and $w \in V(T_3)$. Then $(R-\{u\}) \cup \{v, w\}$ is a larger gp-set than R in T, a contradiction. Hence $T-u$ has exactly two components, say T_1 and T_2. Now suppose that neither T_1 nor T_2 is a path. Then as above, we have $R \cap V(T_1) = \emptyset$ or $R \cap V(T_2) = \emptyset$. By symmetry, we assume that $R \cap V(T_2) = \emptyset$. Since T_2 is not a path, there are at least two leaves x_1 and x_2 in T_2. Then the set $(R - \{u\}) \cup \{x_1, x_2\}$ is a larger gp-set than R, again, in T. Therefore, at least one of T_1 and T_2 is a path.

Conversely, we observe that u is a non-leaf vertex on a pendant path in T. Then u belongs to a gp-set in T. □

In $G \,\square\, H$, if $h \in V(H)$, then the subgraph of $G \,\square\, H$ induced by the vertices (g, h), $g \in V(G)$, is a *G-layer*, denoted with G^h. Analogously H-layers ${}^gH$ are defined. G-layers and H-layers are isomorphic to G and to H, respectively. The distance function in Cartesian products is additive, that is, if $(g_1, h_1), (g_2, h_2) \in V(G \,\square\, H)$, then

$$d_{G \square H}((g_1, h_1), (g_2, h_2)) = d_G(g_1, g_2) + d_H(h_1, h_2). \tag{1}$$

If $u, v \in V(G)$, then the *interval* $I_G(u, v)$ between u and v in G is the set of all vertices lying on shortest u, v-paths, that is,

$$I_G(u, v) = \{w : \; d_G(u, v) = d_G(u, w) + d_G(w, u)\} .$$

In what follows, the notations $d_G(u, v)$ and $I_G(u, v)$ may be simplified to $d(u, v)$ and $I(u, v)$ if G will be clear from the context. Equality (1) implies that intervals in Cartesian products have the following nice structure, cf. [4, Proposition 12.4].

Lemma 3. *If G and H are connected graphs and $(g_1, h_1), (g_2, h_2) \in V(G \,\square\, H)$, then*

$$I_{G \square H}((g_1, h_1), (g_2, h_2)) = I_G(g_1, g_2) \times I_H(h_1, h_2) .$$

Equality (1) also easily implies the following fact (also proved in [13]).

Lemma 4. *Let G and H be connected graphs and R a general position set of $G \,\square\, H$. If $u = (g, h) \in R$, then $V({}^gH) \cap R = \{u\}$ or $V(G^h) \cap R = \{u\}$.*

For finite paths the already mentioned result $\mathrm{gp}(P_\infty \,\square\, P_\infty) = 4$ reduces to:

Lemma 5. ([10]) *If* $n_1, n_2 \geq 2$, *then*

$$\mathrm{gp}(P_{n_1} \,\square\, P_{n_2}) = \begin{cases} 4; & \min\{n_1, n_2\} \geq 3, \\ \\ 3; & \text{otherwise}. \end{cases}$$

To conclude the preliminaries we construct special maximal (with respect to inclusion) general position sets in products of trees.

Lemma 6. *Let* T *and* T^* *be two trees with* $\min\{n(T), n(T^*)\} \geq 3$, $v_i \in V(T) \setminus L(T)$, *and* $v_j^* \in V(T^*) \setminus L(T^*)$. *Then* $(L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*))$ *is a maximal general position set of* $T \,\square\, T^*$.

Proof. Set $R = (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*))$ and let $V_0 = \{u, v, w\} \subseteq R$. We first consider the case when $V_0 \subseteq L(T) \times \{v_j^*\}$ or $V_0 \subseteq \{v_i\} \times L(T^*)$. By symmetry, assume that $V_0 \subseteq L(T) \times \{v_j^*\}$. Then each vertex of V_0 is corresponding to a leaf of $L(T)$ in the layer $T^{v_j^*} \cong T$. Therefore u, v, w do not lie on a common geodesic in $T \,\square\, T^*$.

In the following, without loss of generality, we can assume that $u, w \in L(T) \times \{v_j^*\}$ with $u = (v_k, v_j^*)$, $w = (v_s, v_j^*)$ and $v = (v_i, v_\ell^*) \in \{v_i\} \times L(T^*)$. By Equality (1), we have $d(u, v) = d_T(v_k, v_i) + d_{T^*}(v_j^*, v_\ell^*)$ and $d(u, w) = d_T(v_k, v_s)$, $d(w, v) = d_T(v_s, v_i) + d_{T^*}(v_j^*, v_\ell^*)$. Note that v_k, v_s are two distinct vertices in $L(T)$ of T and $v_i \in V(T) \setminus L(T)$. Then $d_T(v_k, v_i) < d_T(v_k, v_s) + d_T(v_s, v_i)$ whenever v_i lies on the v_k, v_s-geodesic or outside v_k, v_s-geodesic of T. This implies that $d(u, v) < d(u, w) + d(w, v)$ in $T \,\square\, T^*$. Therefore w does not lie on the u, v-geodesic in $T \,\square\, T^*$. Analogously, neither u lies on the v, w-geodesic nor v lies on the u, w-geodesic of $T \,\square\, T^*$. Thus u, v, w do not lie on a common geodesic in $T \,\square\, T^*$, which implies that R is a general position set in $T \,\square\, T^*$.

Next we prove the maximality of $(L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*))$ as a general position set in $T \,\square\, T^*$. Otherwise, there is a general position set R' in $T \,\square\, T^*$ of order greater than $\ell(T) + \ell(T^*)$ such that $R \subset R'$. Then there exists a vertex $z \in R' \setminus R$, say $z = (v_p, v_q^*)$. If $p = i$, then there exist two vertices $(v_i, v_s^*), (v_i, v_t^*) \in R$ such that $z \in I_{T \,\square\, T^*}((v_i, v_s^*), (v_i, v_t^*))$ (since ${}^{v_i}T^* \cong T^*$). This is a contradiction showing that $p \neq i$. Similarly, we have $q \neq j$. Now we consider the positions of v_p in T and v_q^* in T^*. Suppose first that $v_p \in L(T)$, $v_q^* \in L(T^*)$. Then there are two vertices $(v_p, v_j^*), (v_i, v_q^*)$ in R such that $z \in I_{T \,\square\, T^*}((v_p, v_j^*), (v_i, v_q^*))$, contracting that $R \cup \{z\}$ is a general position set of $T \,\square\, T^*$. If $v_p \in L(T)$ and $v_q^* \notin L(T^*)$, then we select a vertex $v_{q'}^* \in L(T^*)$ such that $v_{q'}^*$ is closer to the leaf of the corresponding branching path than v_q^* in T^*. Then $z \in I_{T \,\square\, T^*}((v_p, v_j^*), (v_i, v_{q'}^*))$, a contradiction. Similarly, $v_p \notin L(T)$ and $v_q^* \in L(T^*)$ cannot occur. Finally we assume that $v_p \notin L(T)$, $v_q^* \notin L(T^*)$. Now we select two vertices $v_{p'} \in L(T)$ and $v_{q'}^* \in L(T^*)$ such that $v_{p'}$ is closer to the leaf of the branching path than v_p in T and $v_{q'}^*$ is closer to the leaf of the branching path than v_q^* in T^*. But then $(v_p, v_q^*) \in I_{T \,\square\, T^*}((v_{p'}, v_j^*), (v_i, v_{q'}^*))$, a final contradiction. $\square$

3 Proof of Theorem 1

If T and T^* are both paths, then Theorem 1 holds by Lemma 5. In the following we may thus without loss of generality assume that T^* is not a path. Lemma 6 implies that $\mathrm{gp}(T\,\Box\,T^*) \geq \mathrm{gp}(T) + \mathrm{gp}(T^*)$, hence it remains to prove that $\mathrm{gp}(T\,\Box\,T^*) \leq \mathrm{gp}(T) + \mathrm{gp}(T^*)$. Set $n = n(T)$, $n^* = n(T^*)$, $V(T) = \{v_1, \ldots, v_n\}$, and $V(T^*) = \{v_1^*, \ldots, v_{n^*}^*\}$.

Assume on the contrary that there exists a general position set R of T such that $|R| > \mathrm{gp}(T) + \mathrm{gp}(T^*)$. Since the restriction of R to a T-layer of $T\,\Box\,T^*$ is a general position set of the layer (which is in turn isomorphic to T), the restriction contains at most $\mathrm{gp}(T) = \ell(T)$ elements. Similarly, the restriction of R to a T^*-layer contains at most $\mathrm{gp}(T^*) = \ell(T^*)$ elements. We now distinguish the following cases.

Case 1. There exists a T-layer $T^{v_j^*}$ with $|V(T^{v_j^*}) \cap R| = \mathrm{gp}(T)$, or a T^*-layer ${}^{v_i}T^*$ with $|V({}^{v_i}T^*) \cap R| = \mathrm{gp}(T^*)$.

By the commutativity of the Cartesian product, we may without loss of generality assume that there is a layer ${}^{v_i}T^*$ with $|R \cap V({}^{v_i}T^*)| = \mathrm{gp}(T^*)$. Let $R = R_1 \cup R_2$, where $R_1 = R \cap V({}^{v_i}T^*)$ and $R_2 = R \setminus R_1$, that is, $R_2 = \bigcup\limits_{t\in[n]\setminus\{i\}} \Big(V({}^{v_t}T^*) \cap R\Big)$. Let further S^* be the projection of $R \cap V({}^{v_i}T^*)$ on T^*, that is, $S^* = \{v_j^* :\ (v_i, v_j^*) \in R_1\}$. Since $|R_1| = \mathrm{gp}(T^*)$, our assumption implies $|R_2| \geq \mathrm{gp}(T) + 1$. Then, as $\mathrm{gp}(T) = \ell(T)$, there exist two different vertices $w = (v_p, v_q^*)$ and $w' = (v_{p'}, v_{q'}^*)$ from R_2 such that v_p and $v_{p'}$ lie on a same branching path P of T. (Note that it is possible that $v_p = v_{p'}$.) We may assume that $d_T(v_{p'}, x) \leq d_T(v_p, x)$, where x is the leaf of P. We proceed by distinguishing two subcases based on the position of v_q^* and $v_{q'}^*$ in T^*.

Case 1.1. There exists a branching path P^* of T^* that contains both v_q^* and $v_{q'}^*$.
Recall that T^* is not a path. Lemma 2 implies that a vertex of a tree belongs to a gp-set if and only if it lies on a pendant path and has degree 1 or 2. Therefore, we can select P^* with the root of degree at least 3. Assume that $d_{T^*}(v_{q'}^*, y) \leq d_{T^*}(v_q^*, y)$, where y is the leaf of P^*. (The reverse case can be treated analogously.) Since S^* is a gp-set of T^* which is not isomorphic to a path, there is a vertex $v_k^* \in S^*$ lying on P^*. So we may consider that P^* is a branching path that contains v_q^*, $v_{q'}^*$ and a vertex $v_k^* \in S^*$. (It is possible that some of these vertices are the same.) Let $z = (v_i, v_k^*)$. Then $z \in R_1$. We proceed by distinguishing the following subcases based on the position of v_p, $v_{p'}$ and v_i in T.

Subcase 1.1.1. $v_{p'} \in I(v_i, v_p)$.
In this subcase, if v_k^* is closer than v_q^*, $v_{q'}^*$ to the leaf y of P^*, then, by Lemma 3, $w' \in I_{T\,\Box\,T^*}(w, z)$, a contradiction.

If $v_k^* \in I(v_q^*, v_{q'}^*)$, then since $\ell(T^*) \geq 3$, there exists $z' = (v_i, v_{k'}^*) \in \{v_i\} \times S^*$ such that v_k^*,$v_q^* \in I(v_{q'}^*, v_{k'}^*)$ in T^*. Then we have

$$\begin{aligned} d(w', z') &= d_T(v_{p'}, v_i) + d_{T^*}(v_{q'}^*, v_{k'}^*) \\ &= d_T(v_{p'}, v_i) + d_{T^*}(v_{q'}^*, v_k^*) + d_{T^*}(v_k^*, v_{k'}^*) \\ &= d(w', z) + d(z, z'), \end{aligned}$$

which implies that $z \in I_{T \,\square\, T^*}(w', z')$, a contradiction.

Subcase 1.1.2. $v_i \in I(v_p, v_{p'})$.
In this subcase, if $v_k^* \in I(v_q^*, v_{q'}^*)$ in P^*, then $z \in I_{T \,\square\, T^*}(w, w')$ by Lemma 3, a contradiction.

Assume that v_k^* is closer than v_q^*, $v_{q'}^*$ to the leaf of P^*. Since $|S^*| = \ell(T^*) \geq 3$, there is a vertex $z' = (v_i, v_{k'}^*) \in \{v_i\} \times S^*$ such that v_q^*, $v_{q'}^* \in I(v_k^*, v_{k'}^*)$ in T^*. Let $v_{k'}^*$ be on a branching path P'^* in T^* where $P'^* \neq P^*$. Note that $\ell(T) + 1 \geq 3$. There exists at least one vertex $a = (v_x, v_y^*) \in R_2 \setminus \{w, w'\}$. Next we consider the positions of v_x, v_y^* in T, T^*, respectively.

Suppose first that $v_y^* \in V(P^* \cup P'^*)$. If v_x, v_p, $v_{p'}$ and v_i lie on a path in T, then there are five vertices w, w', z, z' and a in R_2, three of which lie on a common geodesic in $T \,\square\, T^*$, a contradiction. Note that if T is a path, then we are done as above. Therefore, assume that T is not isomorphic to a path in the following and the root of P has degree at least 3. Otherwise, $v_x \notin P$ and v_x, v_p lie on a common branching path in T. Let V_s be the set of vertices of T but not contained in $T_{ip'}$ where $T_{ip'}$ is the subtree of $T - v_p$ containing v_i and $v_{p'}$. If there is a vertex $a' = (v_s, v_l^*) \in R_2$ with $v_s \in V_s$, then R_2 contains w, w', z, z' and a', three of which are on a common geodesic, a contradiction. Therefore, the first coordinate of any vertex in R_2 cannot be in V_s. Assume that $P' \neq P$ is any branching path containing v_p and a leaf both in $T_{ip'}$ and T. Then, besides w, $P' \,\square\, T^*$ contains at most one vertex in R_2 of $T \,\square\, T^*$. Otherwise, $P' \,\square\, T^*$ contain two vertices h, h' in R_2. Then there exist two vertices $h_0, h_0' \in \{v_i\} \times S^*$ such that three vertices from $\{h, h', h_0, h_0', w\}$ lie on some geodesic in $T \,\square\, T^*$, a contradiction. (Here h_0 may be equal to h_0'.) Note that V_s contains at least two leaves of T since the root of P (just in V_s) has degree at least 3. Then $T_{ip'}$ has at most $\ell(T) - 2$ leaves in T. Since $P \,\square\, T^*$ contains two vertices w and w' in R_2, we have $|R_2| \leq \ell(T) - 2 + 1 < \ell(T) = \mathrm{gp}(T)$, a contradiction with the assumption.

Assume now that $v_y^* \notin V(P^* \cup P'^*)$. Then there exists a vertex $z'' = (v_i, v_{k''}^*) \in \{v_i\} \times S^*$ such that $v_y^*, v_{k''}^*$ lie on a common branching path in T^*. If v_y^* is closer to the leaf of the branching path than $v_{k''}^*$ in T^*, then $v_i \in I(v_x, v_i)$ and $v_{k''}^* \in I(v_y^*, v_k^*)$. Therefore, by Lemma 3, we get $z'' \in I_{T \,\square\, T^*}(a, z)$, a contradiction. In the case that $v_{k''}^*$ is closer to the leaf of the branching path than v_y^* in T^*, we consider the positions of v_x, v_p, $v_{p'}$ and v_i in T. Let $V_1 = \{z, z', w, w', a, z''\}$. Then $V_1 \subseteq R_2$. If v_x, v_p, $v_{p'}$ and v_i lie on a path in T, then there exist three vertices in V_1 lying on a common geodesic in

$T \,\square\, T^*$, a contradiction again. Otherwise, $v_x \notin P$ and v_x, v_p lie on a common branching path in T. Similarly as above, a contradiction occurs.

Subcase 1.1.3. $v_p \in I(v_i, v_{p'})$.
In this subcase, since $\ell(T^*) \geq 3$, there exists a vertex $z' = (v_i, v^*_{k'}) \in \{v_i\} \times S^*$ such that $v^*_{k'} \notin P^*$ and $v^*_q \in I(v^*_{k'}, v^*_{q'})$ in T^*. Since

$$\begin{aligned} d(z', w') &= d_T(v_i, v_{p'}) + d_{T^*}(v^*_{k'}, v^*_{q'}) \\ &= d_T(v_i, v_p) + d_{T^*}(v^*_{k'}, v^*_q) + d_T(v_p, v_{p'}) + d_{T^*}(v^*_q, v^*_{q'}) \\ &= d(z', w) + d(w, w'), \end{aligned}$$

we have $w \in I_{T \,\square\, T^*}(z', w')$, a contradiction.

Subcase 1.1.4. $v_i \notin V(P)$ such that v_i, v_p lie on a same branching path in T.
In this subcase, since $\ell(T^*) \geq 3$, there is a vertex $z' = (v_i, v^*_{k'}) \in \{v_i\} \times S^*$ such that $v^*_q \in I(v^*_{k'}, v^*_k)$ in T^*. If $v^*_k \in I(v^*_q, v^*_{q'})$, then obviously $v^*_k \in I(v^*_q, v^*_{k'})$ and therefore,

$$\begin{aligned} d(w', z') &= d_T(v_{p'}, v_i) + d_{T^*}(v^*_{q'}, v^*_{k'}) \\ &= d_T(v_{p'}, v_i) + d_{T^*}(v^*_{q'}, v^*_k) + d_{T^*}(v^*_k, v^*_{k'}) \\ &= d(w', z) + d(z, z') \,. \end{aligned}$$

We conclude that $z \in I_{T \,\square\, T^*}(w', z')$, a contradiction.

If v^*_k is closer to the leaf of P^* than $v^*_q, v^*_{q'}$, then we get a contradiction similarly as in Subcase 1.1.2.

Case 1.2. v^*_q and $v^*_{q'}$ do not lie on a same branching path in T^*.
In this subcase, we may assume that v^*_q and $v^*_{q'}$ lie on distinct branching paths P^* and P'^* in T^*, respectively. Since $\ell(T^*) \geq 3$ and T^* is not isomorphic to a path, there exist two vertices $z = (v_i, v^*_k)$ and $z' = (v_i, v^*_{k'})$ from $\{v_i\} \times S^*$, such that $v^*_k \in P^*$ and $v^*_{k'} \in P'^*$. We consider the following subcases based on the positions of v_p, $v_{p'}$ and v_i in T.

Subcase 1.2.1. $v_{p'} \in I(v_i, v_p)$.
In this subcase, if $v^*_{k'}$ is closer than $v^*_{q'}$ to the leaf of P'^*, then $v_{p'} \in I(v_p, v_i)$ and $v^*_{q'} \in I(v^*_q, v^*_{k'})$. Lemma 3 gives $w' \in I_{T \,\square\, T^*}(w, z')$, a contradiction. On the other hand, if $v^*_{q'}$ is closer than $v^*_{k'}$ to the leaf of P'^*, then $v_i \in I(v_i, v_{p'})$ and $v^*_{k'} \in I(v^*_k, v^*_{q'})$, hence Lemma 3 gives $z' \in I_{T \,\square\, T^*}(w', z)$, a contradiction again.

Subcase 1.2.2. $v_i \in I(v_p, v_{p'})$.
In this subcase, we first assume that $v^*_{q'}$ is closer than $v^*_{k'}$ to the leaf of P'^*. Then $v_i \in I(v_i, v_{p'})$ and $v^*_{k'} \in I(v^*_k, v^*_{q'})$. Therefore, by Lemma 3, we get $z' \in I_{T \,\square\, T^*}(z, w')$ as a contradiction. Otherwise we suppose that $v^*_{k'}$ is closer than $v^*_{q'}$ to the leaf of P'^*. If v^*_q

is closer than v_k^* to the leaf of P^*, then $v_i \in I(v_p, v_i)$ and $v_k^* \in I(v_q^*, v_{k'}^*)$. Therefore, by Lemma 3, we get $z \in I_{T \,\square\, T^*}(w, z')$, a contradiction. In the case that v_k^* is closer than v_q^* to the leaf of P^*, we find a contradiction similarly as the proof of Subcase 1.1.2.

Subcase 1.2.3. $v_p \in I(v_i, v_{p'})$.
In this subcase, if v_k^* is closer than v_q^* to the leaf of P^*, then $v_p \in I(v_i, v_{p'})$ and $v_q^* \in I(v_k^*, v_{q'}^*)$. So Lemma 3 gives $w \in I_{T \,\square\, T^*}(z, w')$, a contradiction. And if v_q^* is closer than v_k^* to the leaf of P^*, then $v_i \in I(v_i, v_p)$ and $v_k^* \in I(v_{k'}^*, v_q^*)$, hence we get $z \in I_{T \,\square\, T^*}(z', w)$.

Subcase 1.2.4. $v_i \notin V(P)$ such that v_i, v_p lie on a same branching path in T.
First suppose that v_q^* is closer to the leaf than v_k^* in P^*, then $v_i \in I(v_i, v_p)$ and $v_k^* \in I(v_q^*, v_{k'}^*)$. Thus, by Lemma 3, we get $z \in I_{T \,\square\, T^*}(w, z')$.

Assume that v_k^* is closer than v_q^* to the leaf of P^*. If $v_{q'}^*$ is closer to the leaf than $v_{k'}^*$, then $v_i \in I(v_i, v_{p'})$ and $v_{k'}^* \in I(v_k^*, v_{q'}^*)$, which gives $z' \in I_{T \,\square\, T^*}(z, w')$. If $v_{k'}^*$ is closer than $v_{q'}^*$ to the leaf of P'^*, we can proceed similarly as in Subcase 1.1.4.

Now we turn to the second case.

Case 2. $|R \cap V({}^{v_k}T^*)| < \ell(T^*)$ for any $k \in [n]$, and $|R \cap V(T^{v_t^*})| < \ell(T)$ for any $t \in [n^*]$.
In this case, let ${}^{v_i}T^*$ be a layer with $|R \cap V({}^{v_i}T^*)| = \max\{|R \cap V({}^{v_k}T^*)| : k \in [n]\}$. Let $R = R_1 \cup R_2$ where $R_1 = R \cap V({}^{v_i}T^*)$ and $R_2 = R \setminus R_1$, that is, $R_2 = \bigcup\limits_{k \in [n] \setminus \{i\}} \Big(V({}^{v_k}T^*) \cap R \Big)$. Set further $S^* = \{v_j^* : \ (v_i, v_j^*) \in R_1\}$. Then $1 \le |S^*| \le \ell(T^*) - 1$.

Assume first $|S^*| = 1$. Therefore $|R \cap V({}^{v_k}T^*)| \le 1$ for any $k \in [n]$. Next we only need to consider $|R \cap V(T^{v_j^*})| \le 1$ for any $j \in [n^*]$. (If $|R \cap V(T^{v_j^*})| \ge 2$ for some $j \in [n^*]$, by commutativity of $T \,\square\, T^*$, the proof is similar to the subcase in which $2 \le |S^*| \le \ell(T^*) - 1$.) Therefore, suppose that $|R \cap V(T^{v_j^*})| \le 1$ for any $j \in [n^*]$. Then $|R| \le \min\{n, n^*\}$. We now claim that $|R| \le \ell(T) + \ell(T^*)$. If not, then since $|R| \ge \ell(T) + \ell(T^*) + 1 \ge 6$, there exist three vertices $u = (v_p, v_j^*)$, $v = (v_{p'}, v_q^*)$ and $w = (v_s, v_\ell^*)$ from R such that $v_p, v_{p'}$ lie on a same branching path in T, and v_j^*, v_ℓ^* lie on a common branching path in T^*. Note that there may be $p' = s, q = \ell$. But we can always select a vertex $h \in R \setminus \{u, v, w\}$ such that u, v, h or u, w, h lie on a same geodesic in $T \,\square\, T^*$, which is a contradiction. So our result holds when $|S^*| = 1$.

Suppose second that $2 \le |S^*| \le \ell(T^*) - 1$. As $|R_1| = |S^*|$, we need to prove that $|R_2| \le \ell(T) + \ell(T^*) - |S^*|$. Assume on the contrary that $|R_2| \ge \ell(T) + \ell(T^*) - |S^*| + 1$. Since $|S^*| \ge 2$, there are two distinct vertices $w = (v_i, v_j^*)$ and $w' = (v_i, v_{j'}^*)$ from $\{v_i\} \times S^*$. We distinguish the following cases based on the positions of v_j^*, $v_{j'}^*$ in T^*.

Case 2.1. v_j^* and $v_{j'}^*$ lie on a same branching path P^* of T^*.
In this subcase, we may without loss of generality assume that $v_{j'}^*$ is closer than v_j^*

to the leaf of P^*. Let $T^*_{v^*_{j'}}$ be the maximal subtree of $T^* - v^*_j$ containing $v^*_{j'}$ and let $V_{s^*} = V(T^*) \setminus V(T^*_{v^*_{j'}})$. Let further $S^*_1 = \{v^*_q : \ v^*_q \in I(v^*_j, v^*_\ell), v^*_\ell \in S^* \cap V(T^*_{v^*_{j'}})\}$. Now we prove the following claim.

Claim 1. If $z = (v_p, v^*_t) \in R_2$, then $v^*_t \in S^*_1$.

Proof of Claim 1. If not, suppose first that $v^*_t \in V(P^*)$ is closer than $v^*_{j'}$ to the leaf of P^*. Then $v_i \in I(v_i, v_p)$ and $v^*_{j'} \in I(v^*_t, v^*_j)$. Hence, $w' \in I_{T\Box T^*}(w, z)$. And if $v^*_t \in V_{s^*}$, then $v^*_j \in I(v^*_t, v^*_{j'})$. Combining this fact with $v_i \in I(v_i, v_p)$, we have $w \in I_{T\Box T^*}(w', z)$. This proves Claim 1.

By Claim 1, we have $|\bigcup_{v^*_t \in S^*_1} \big(V(T^{v^*_t}) \cap R\big)| \geq \ell(T) + \ell(T^*) - |S^*| + 1 \geq \ell(T) + 1$. Then there exist two vertices $z = (v_p, v^*_\ell)$ and $z' = (v_{p'}, v^*_{\ell'})$ from $\cup_{v^*_t \in S^*_1} \big(V(T^{v^*_t}) \cap R\big)$ such that $v^*_\ell, v^*_{\ell'} \in S^*_1$ and $v_p, v_{p'}$ lie on a same branching path P in T. Without loss of generality, let $v_{p'}$ be closer than v_p to the leaf of P, and let $v^*_\ell, v^*_{\ell'} \in I(v^*_j, v^*_{j'})$ (by the definition of S^*_1). We consider the following subcases according to the positions of $v_i, v_p, v_{p'}$ in T.

Subcase 2.1.1. $v_{p'} \in I(v_i, v_p)$.
If $v^*_{\ell'}$ is closer than v^*_ℓ to $v^*_{j'}$ in P^*, then we have $v_{p'} \in I(v_i, v_p)$ and $v^*_{\ell'} \in I(v^*_\ell, v^*_{j'})$. Therefore, $z' \in I_{T\Box T^*}(z, w')$. And if v^*_ℓ is closer than $v^*_{\ell'}$ to $v^*_{j'}$ in P^*, then we have $v_{p'} \in I(v_i, v_p)$ and $v^*_{\ell'} \in I(v^*_\ell, v^*_j)$ and so $z' \in I_{T\Box T^*}(z, w)$.

Subcase 2.1.2. $v_i \in I(v_p, v_{p'})$.
Note that $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$. Then there exists at least a vertex $a = (v_x, v^*_y) \in \cup_{v^*_t \in S^*_1} \big(V(T^{v^*_t}) \cap R\big)$ different from z and z'. Based on the position of v^*_y ($v^*_y \in P^*$ or $v^*_y \notin P^*$) in T^*, and the positions of v_x, v_i, v_p and $v_{p'}$ in T, we get contradictions using a similar proof as in Subcase 1.1.2.

Subcase 2.1.3. $v_p \in I(v_i, v_{p'})$.
If $v^*_{\ell'}$ is closer than v^*_ℓ to $v^*_{j'}$ in T^*, then $v_p \in I(v_i, v_{p'})$ and $v^*_\ell \in I(v^*_j, v^*_{\ell'})$, therefore $z \in I_{T\Box T^*}(w, z')$. And if v^*_ℓ is closer than $v^*_{\ell'}$ to $v^*_{j'}$ in T^*, then $v_p \in I(v_i, v_{p'})$ and $v^*_\ell \in I(v^*_{j'}, v^*_{\ell'})$, hence $z \in I_{T\Box T^*}(w, z')$.

Subcase 2.1.4. $v_i \notin V(P)$ such that v_i, v_p lie on a same branching path in T.
Since $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$, there exists a vertex $(v_x, v^*_y) \in \cup_{v^*_t \in S^*_1} \big(V(T^{v^*_t}) \cap R\big)$. Proceeding similarly as in Subcase 1.1.4, we get required contradictions. But then $|\cup_{v^*_t \in S^*_1} \big(V(T^{v^*_t}) \cap R\big)| \leq \ell(T) + \ell(T^*) - |S^*|$, a contradiction with the assumption.

Case 2.2. v^*_j,$v^*_{j'}$ lie on different branching paths P^*, P'^* in T^*, respectively.
In this subcase, let S^*_2 be a set of vertices of ${}^{v_i}T^*$ closer to the leaf of a branching path than v^*_g for any $v^*_g \in S^*$. Note that $S^* \cap S^*_2 = \emptyset$. We prove the following claim.

Claim 2. If (v_p, v^*_t) in R_2, then $v^*_t \in V(T^*) \setminus (S^* \cup S^*_2)$.

Proof of Claim 2. Lemma 4 implies $v_t^* \notin S^*$. Assume that $v_t^* \in S_2^*$ lies on a same branching path for some v_g^* in T^*. Note that $|S^*| \geq 2$. Then there exists another vertex $v_{g'}^*$ such that $v_g^* \in I(v_t^*, v_{g'}^*)$. Combining this fact with $v_i \in I(v_i, v_p)$, we arrive at a contradiction $w \in I_{T \,\square\, T^*}(z, w')$. This proves Claim 2.

Let now $S_{1'}^* = \{v_q^* :\ v_q^* \in I(v_g^*, v_{g'}^*), v_g^*, v_{g'}^* \in S^*\}$. By a parallel reasoning as in Subcase 2.1 and with Claim 2 in hands we infer that $| \cup_{v_t^* \in S_{1'}^*} \big(V(T^{v_t^*}) \cap R\big)| \leq \ell(T)$.

Let $S = \{v_k :\ (v_k, v_t^*) \in \cup_{v_t^* \in S_{1'}^*} \big(V(T^{v_t^*}) \cap R\big)\}$ and set $S^{**} = V(T^*) \setminus (S^* \cup S_{1'}^*)$. From the assumption we have $| \cup_{v_t^* \in S^{**}} \big(V(T^{v_t^*}) \cap R\big)| \geq \ell(T) + \ell(T^*) - |S| - |S^*| + 1$. So there exists a vertex $z = (v_p, v_\ell^*) \in \cup_{v_t^* \in S^{**}} \big(V(T^{v_t^*}) \cap R\big)$, and we can always select two distinct vertices $u = (v_h, v_g^*)$ and $v = (v_{h'}, v_{g'}^*)$ from R such that v_p and v_h lie on a same branching path in T, while v_ℓ^* and $v_{g'}^*$ lie on a common branching path in T^*. But we can choose another vertex $w \in R$ such that either u, w, z or u, v, z lie on a same geodesic in $T \,\square\, T^*$ as a contradiction. Therefore,

$$| \bigcup_{v_t^* \in S^{**}} \Big(V(T^{v_t^*}) \cap R\Big)| \leq \ell(T) + \ell(T^*) - |S| - |S^*|.$$

and we are done.

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