

On the Relation Between Wiener Index and Eccentricity of a Graph

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Abstract

The relation between the Wiener index $W(G)$ and the eccentricity $\varepsilon(G)$ of a graph G is studied. Lower and upper bounds on $W(G)$ in terms of $\varepsilon(G)$ are proved and extremal graphs characterized. A Nordhaus-Gaddum type result on $W(G)$ involving $\varepsilon(G)$ is given. A sharp upper bound on the Wiener index of a tree in terms of its eccentricity is proved. It is shown that in the class of trees of the same order, the difference $W(T) - \varepsilon(T)$ is minimized on caterpillars. An exact formula for $W(T) - \varepsilon(T)$ in terms of the radius of a tree T is obtained. A lower bound on the eccentricity of a tree in terms of its radius is also given. Two conjectures are proposed. The first asserts that the difference $W(G) - \varepsilon(G)$ does not increase after contracting an edge of G . The second conjecture asserts that the difference between the Wiener index of a graph and its eccentricity is largest on paths.

Key words: Wiener index; eccentricity; eccentric connectivity; tree; extremal graph

1 Introduction

All graphs in this paper are simple and connected. The order and the size of a graph G will be denoted by $n(G)$ and $m(G)$, respectively. If $G = (V(G), E(G))$ is a graph and $u, v \in V(G)$, then the distance $d_G(u, v)$ is the number of edges on a shortest u, v -path. (By a u, v -path in G we mean a path in G whose end-vertices are the vertices u and v .) The Wiener index of a graph G ,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v),$$

is the oldest graph invariant (alias *topological index*) studied in mathematical chemistry [39]. It is also one of the most studied among such indices, cf. the surveys [14–16, 29], and continues to be an active research field [1, 13, 21, 22, 26, 28, 34, 35, 37]. The *total distance of a vertex v* of a graph G is defined as $d_G(v) = \sum_{u \in V(G)} d_G(v, u)$.

If v is a vertex of a graph G , then the *eccentricity* $\varepsilon_G(v)$ of a vertex v is the distance from v to a farthest vertex from v . A vertex u is said to be an *eccentric vertex of v* if $d_G(v, u) = \varepsilon_G(v)$. The *radius* $\text{rad}(G)$ of G and the *diameter* $\text{diam}(G)$ of G are the minimum and the maximum eccentricity, respectively. The *center* $C(G)$ of G is the set of vertices with minimum eccentricity, that is, $C(G) = \{u \in V(G) : \varepsilon_G(u) = \text{rad}(G)\}$. The *eccentricity* of a graph G is

$$\varepsilon(G) = \sum_{v \in V(G)} \varepsilon_G(v).$$

The eccentricity of a graph has been earlier studied on graph operations in [12, 19], where the invariant was named *total eccentricity of a graph* but we believe that “eccentricity of a graph” suffices because (i) this term is not used elsewhere and (ii) this is also consistent with the notation and terminology from [23, 24]. The investigations of the Wiener index are in a way equivalent with the studies of the average distance. Similarly, the studies of the eccentricity are parallel with the research of the average eccentricity, the later being studied in particular in [7–9, 18, 23–25, 30, 38]. In [4], the Wiener index has been studied on strong product graphs along with the average eccentricity. For a wider picture on eccentricity based descriptors for QSAR/QSPR we refer to [32].

In this paper we are interested in the difference between the Wiener index and the eccentricity of a graph. In the next section we first give two lower bounds on $W(G)$ in terms of $\varepsilon(G)$ and then prove two related upper bounds. In all the cases we characterize the graphs that attain the bounds. In the last result of the section we prove a Nordhaus-Gaddum type result on the Wiener index of a graph involving its eccentricity. In Section 3 we concentrate on trees. First a sharp upper bound on the

Wiener index of a tree in terms of its eccentricity is given. Then we prove that the difference $W(T) - \varepsilon(T)$ on trees T of the same order is minimized on caterpillars. We next give an exact formula for $W(T) - \varepsilon(T)$ in terms of the radius of T . We also give a lower bound on the eccentricity of a tree in terms of the radius. (For results that relate the maximum Wiener index of trees with a given radius see [11].) We conclude the paper with two conjectures. The first asserts that the difference $W - \varepsilon$ does not increase after contracting an edge. We support the conjecture by proving that it holds for the case when the contracted edge is a bridge. The second conjecture asserts that the difference between the Wiener index of a graph and its eccentricity is largest on paths. Before we begin with the results, some further definitions are given.

Let G be a graph. The degree of a vertex $u \in V(G)$ will be denoted with $\deg_G(u)$ or $\deg(u)$ for short. A *matching* of G is a set of independent edges of G , that is, a set of edges no pair of them sharing an end-vertex. A matching M is *perfect* if every vertex of G is an end-vertex of some edge from M . We will use K_n , P_n , and C_n to denote the complete graph of order n , the path of order n , and the cycle of order n , respectively. By $K_{n,m}$ we denote the complete bipartite graph with bipartition sets of order n and m ; in particular, $K_{1,m}$ is the star of order $m + 1$. In the rest of the paper we may abbreviate $d_G(u, v)$, $d_G(v)$, $\varepsilon_G(v)$, and $\deg_G(u)$ to $d(u, v)$, $d(v)$, $\varepsilon(v)$, and $\deg(u)$, respectively, when G will be clear from the context.

2 General graphs

Theorem 2.1 *If G is a graph, then*

$$W(G) \geq \varepsilon(G) + m(G) - n(G),$$

equality holding if and only if G is obtained from $K_{n(G)}$ by removing a matching.

Proof. Set $n = n(G)$, $m = m(G)$, and let S be the set of universal vertices of G , that is, $S = \{u \in V(G) : \deg(u) = n - 1\}$. Set further $k = |S|$.

Case 1. $k \geq 1$.

From the definition of the Wiener index we get that

$$2W(G) = \sum_{v \in V(G)} d(v) \geq k(n - 1) + (n - k)n = n^2 - k. \quad (1)$$

Moreover, equality holds in (1) if and only if $\deg(v) = n - 2$ for every vertex $v \in V(G) \setminus S$. This in turn holds if and only if G is obtained from K_n by removing a matching. (An easy way to verify this fact is to consider the complement of G .)

From the definition of the total eccentricity we infer that

$$\varepsilon(G) = \sum_{v \in V(G)} \varepsilon(v) = k + (n - k)2 = 2n - k. \quad (2)$$

Moreover, the Handshaking Lemma yields

$$2m = \sum_{v \in V(G)} \deg(v) \leq k(n - 1) + (n - k)(n - 2) = n^2 - 2n + k. \quad (3)$$

Again, the equality in (3) holds if and only if $\deg(v) = n - 2$ for every $v \in V(G) \setminus S$, that is, if and only if G is obtained from K_n by removing a matching.

From the above (in)equalities we get:

$$\begin{aligned} 2W(G) &\stackrel{(1)}{\geq} n^2 - k \stackrel{(3)}{\geq} (2m + 2n - k) - k = 2m + 2n - 2k \\ &\stackrel{(2)}{=} 2m + 2n - 2(2n - \varepsilon(G)) = 2\varepsilon(G) + 2m - 2n. \end{aligned}$$

Case 2. $k = 0$.

In this case $\varepsilon(v) \geq 2$ holds for every $v \in V(G)$ and consequently

$$\begin{aligned} 2W(G) = \sum_{v \in V(G)} d(v) &\geq \sum_{v \in V(G)} (\varepsilon(v) + (\varepsilon(v) - 1) + (n - 3)) \\ &= 2\varepsilon(G) + n(n - 4). \end{aligned} \quad (4)$$

As $k = 0$, the Handshaking Lemma implies $n(n - 2) \geq 2m$ and thus $n(n - 4) \geq 2m - 2n$. Combining this fact with (4) we infer that $2W(G) \geq 2\varepsilon(G) + 2m - 2n$ holds also in this case. Moreover, the equality holds if and only if $d(v) = 2\varepsilon(v) + n - 4$ for every $v \in V(G)$ and $2m = n(n - 2)$, that is, if and only if G is the graph obtained from K_n by removing a perfect matching. \square

With the help of Theorem 2.1 we can deduce the following result independent from the order and size of a graph considered.

Theorem 2.2 *If G is a graph with $n(G) \geq 4$, then $W(G) \geq \varepsilon(G)$. Moreover, equality holds if and only if $G \in \{P_4, C_4\}$.*

Proof. Again set $n = n(G)$ and $m = m(G)$. If $m \geq n + 1$, then Theorem 2.1 immediately yields $W(G) > \varepsilon(G)$. Suppose next that $m = n$. Then Theorem 2.1 gives $W(G) \geq \varepsilon(G)$ with the equality if and only if G is K_n minus a matching, where $m = n$. If $n \geq 5$, then this is clearly not possible. Hence the only graph with $m = n$

that attains the equality is C_4 . Since G is connected, the only remaining case to consider is $m = n - 1$ which means that G is a tree.

So let G be a tree. If $G \cong K_{1, n-1}$, then $W(G) > \varepsilon(G)$ as $n \geq 4$. Assume hence in the rest that $\varepsilon(v) \geq 2$ for every $v \in V(G)$. Then

$$\begin{aligned} d(v) &= \sum_{u \in V(G)} d(u, v) \geq (1 + 2 + \cdots + \varepsilon(v)) + (n - 1 - \varepsilon(v)) \\ &= n - 1 + \frac{1}{2} \left(\varepsilon(v)^2 - \varepsilon(v) \right). \end{aligned}$$

Since $f(x) = 2(n - 1) + x^2 - 5x$ is an increasing function on $[3, n - 1]$, for $x \geq 2$ (x is an integer) we have

$$f(x) \geq \min\{f(2), f(3)\} \geq 0 \quad \text{as } n \geq 4.$$

From the above results we deduce

$$2(n - 1) + \varepsilon(v)^2 - 5\varepsilon(v) \geq 0, \quad \text{that is, } d(v) \geq n - 1 + \frac{1}{2} \left(\varepsilon(v)^2 - \varepsilon(v) \right) \geq 2\varepsilon(v).$$

Therefore

$$2W(G) = \sum_{v \in V(G)} d(v) \geq 2 \sum_{v \in V(G)} \varepsilon(v) = 2\varepsilon(G).$$

Moreover, the equality holds if and only if $n = 4$ and $\varepsilon(v) \in \{2, 3\}$ for any $v \in V(G)$, that is, if and only if $G \cong P_4$. \square

Note that Theorem 2.2 implies that if $n(G) \geq 5$, then $W(G) > \varepsilon(G)$.

In Theorems 2.1 and 2.2 we have bounded $W(G)$ from below using $\varepsilon(G)$. On the other hand, in [5] it was observed (in terms of the average distance and average eccentricity) that $W(G) \leq \frac{n(G)-1}{2} \varepsilon(G)$ holds for any graph G . We add here that equality holds if and only if $G \cong K_{n(G)}$. To give further upper bounds on $W(G)$ using $\varepsilon(G)$, we need to recall a couple of concepts. First, the *eccentric connectivity* $\xi^c(G)$ of G is

$$\xi^c(G) = \sum_{v \in V(G)} \deg(v) \varepsilon(v).$$

This graph invariant has been already well investigated, see the selection of related papers [6, 7, 10, 17, 20, 36, 40, 41]. Second, G is a *self-centered graph* if $\text{rad}(G) = \text{diam}(G)$, cf. [3, 27].

Theorem 2.3 (i) *If G is a graph, then $2W(G) \leq (n(G) - 1) \varepsilon(G) - \xi^c(G) + 2m(G)$, equality holding if and only if $\text{diam}(G) \leq 2$.*

(ii) If G is a self-centered graph, then

$$W(G) \leq \varepsilon(G) + \begin{cases} \frac{n(n-2)^2}{8}; & n \text{ even}, \\ \frac{n[(n-2)^2-1]}{8}; & n \text{ odd}. \end{cases}$$

Moreover, equality holds if and only if G is an odd cycle.

Proof. Throughout the proof let $n = n(G)$ and $m = m(G)$.

(i) If $v \in V(G)$, then

$$\begin{aligned} d(v) &= \sum_{u \in V(G)} d(v, u) \leq \deg(v) + (n-1 - \deg(v)) \varepsilon(v) \\ &= (n-1) \varepsilon(v) - \deg(v) (\varepsilon(v) - 1). \end{aligned}$$

Summing over the vertices of G we get

$$2W(G) \leq (n-1) \varepsilon(G) - \xi^c(G) + 2m.$$

Moreover, equality holds if and only if $\varepsilon(v) \in \{1, 2\}$, in other words, if and only if $\text{diam}(G) \leq 2$.

(ii) Note first that considering a possible cut-vertex of a self-centered graph G we infer that G is 2-connected. This means that $n \geq 3$. Whitney's theorem (which characterizes 2-connected graphs) asserts that for every distinct vertices v and u , there exist two internally disjoint u, v -paths. Hence, if $v \in V(G)$, then, observing a vertex w at distance $d = \text{diam}(G) = \text{rad}(G)$ from v , we infer that

$$\begin{aligned} \frac{d(v)}{2} - \varepsilon(v) &\leq [1 + 2 + \cdots + (d-1)] + \frac{d(n-2d+1)}{2} - d \\ &= \frac{d(n-d-2)}{2}. \end{aligned}$$

Since $g(x) = x(n-x-2)$ is an increasing function on $x \leq \frac{n-2}{2}$, and a decreasing function on $x \geq \frac{n-2}{2}$, we have

$$g(x) \leq \begin{cases} \frac{(n-2)^2}{4}; & n \text{ even}, \\ \frac{(n-2)^2-1}{4}; & n \text{ odd}. \end{cases}$$

Hence

$$\frac{d(v)}{2} - \varepsilon(v) \leq \begin{cases} \frac{(n-2)^2}{8}; & n \text{ even}, \\ \frac{(n-2)^2-1}{8}; & n \text{ odd}. \end{cases}$$

Summing over all vertices we get

$$W(G) - \varepsilon(G) \leq \begin{cases} \frac{n(n-2)^2}{8} & \text{when } n \text{ is even,} \\ \frac{n[(n-2)^2-1]}{8} & \text{when } n \text{ is odd.} \end{cases}$$

Moreover, it is easily seen that equality holds if and only if G is an odd cycle. \square

As a consequence of Theorem 2.3 we have the following Nordhaus-Gaddum type result (cf. [31, 33]).

Corollary 2.4 *If both G and its complement \bar{G} are connected, then*

$$W(G) + W(\bar{G}) \leq \binom{n(G)}{2} + \frac{1}{2} [(n(G) - 1)(\varepsilon(G) + \varepsilon(\bar{G})) - \xi^c(\bar{G}) - \xi^c(G)] .$$

Proof. Summing the inequalities for G and \bar{G} expressed by Theorem 2.3 and using the facts $n(\bar{G}) = n(G)$ and $m(G) + m(\bar{G}) = \binom{n(G)}{2}$, the result follows. \square

3 Trees

An upper bound on the Wiener index in terms of eccentric connectivity has been reported in [6]. Here we determine a sharp upper bound on the Wiener index of a tree in terms of its eccentricity.

Theorem 3.1 *If T is a tree of order n , then*

$$W(T) \leq \frac{1}{4} (2n - 3)\varepsilon(T) + \frac{1}{4}$$

with equality holding if and only if T is the star $K_{1,n-1}$.

Proof. If $T = K_{1,n-1}$, then

$$W(K_{1,n-1}) = (n - 1)^2 = \frac{1}{4} (2n - 3)\varepsilon(K_{1,n-1}) + \frac{1}{4},$$

hence the equality holds. If T is not a star, then $\varepsilon(v) \geq 2$ for every vertex $v \in V(T)$. Hence,

$$\left(n - \frac{3}{2}\right) \varepsilon(v) = (n - 2) \varepsilon(v) + \frac{1}{2} \varepsilon(v) \geq (n - 2) \varepsilon(v) + 1 \geq \sum_{u \in V(T)} d(v, u) = d(v),$$

that is,

$$(2n - 3) \varepsilon(v) \geq 2d(v).$$

Summing over all vertices of T we get $4W(T) \leq (2n - 3) \varepsilon(T)$, that is, $4W(T) < (2n - 3) \varepsilon(T) + 1$. \square

We next show that the minimum difference between the Wiener index and the eccentricity of a tree is achieved on caterpillars. Recall that a tree is a *caterpillar* if it contains a (diametrical) path, such that any vertex not on the path is at distance 1 from it.

Theorem 3.2 *If n is a positive integer, then $\min\{W(T) - \varepsilon(T) : T \text{ tree, } n(T) = n\}$ is achieved on a caterpillar.*

Proof. Let x and y be diametrical vertices of T and let P the x, y -path in T . Since P contains the center of T , for each vertex $w \in T$, the eccentricity of w is equal to $d(w, x)$ or $d(w, y)$.

Suppose that T is not a caterpillar. This is equivalent to the fact that in $V(T) \setminus V(P)$ there exists at least one vertex which is not of degree 1. Among all such vertices select a vertex u which is farthest from P . That is, u is a vertex from $V(T) \setminus V(P)$ of degree at least 2, such that $d_T(u, P) = \min\{d(u, v) : v \in V(P)\}$ is largest. Let $z \in V(P)$ be the unique vertex of P which is closest to u , that is, $d_T(u, z) = d_T(u, P)$. Let $d_T(u, z) = \ell$ and note that since T is not a caterpillar, $\ell \geq 1$. Let v be the neighbor of u on the u, z -path, and let T_z be the maximal subtree of T that contains z and no other vertex of P . Clearly, $\varepsilon_{T_z}(z) = \ell + 1 \geq 2$. If $\ell = 1$, then vertices v and z are the same vertex.

Consider the following transformation. Let u and v be two adjacent vertices of T_z which $d(z, u) = \ell$ and $d(z, v) = \ell - 1$. Let $S = \{w : uw \in E(T)\} \setminus \{v\}$ and $|S| = s \geq 1$. Assume that T' is a tree obtained from T by removing the edges between u and the vertices of S and then connecting vertex v to the vertices of S . It is clear that the distances between the vertices of $V(G) \setminus S$ are the same in T and T' . Also the distances of vertices of S from the other vertices, except u , decrease by 1, and for each vertex $w \in S$ we have $d_{T'}(u, w) = 2, d_T(u, w) = 1$. Therefore,

$$\begin{aligned} W(T) - W(T') &= \sum_{\substack{j \in S \\ w \in V(T) - S}} [d_T(j, w) - d_{T'}(j, w)] \\ &= \sum_{j \in S} [d_T(j, u) - d_{T'}(j, u)] + \sum_{\substack{j \in S \\ u \neq w \in V(T) - S}} [d_T(j, w) - d_{T'}(j, w)] \\ &= -s + s(n - s - 1) = sn - s^2 - 2s. \end{aligned}$$

By the above transformation, each vertex $w \in V(T) \setminus S$ has the same eccentricity in T and in T' , and the eccentricity of the vertices from S decreases by 1. Hence $\varepsilon(T) - \varepsilon(T') = s$. It follows that

$$W(T) - W(T') = (sn - s^2 - 2s) > s = \varepsilon(T) - \varepsilon(T'),$$

and we are done. \square

We next give a formula for $W(T) - \varepsilon(T)$ for a tree T of a given radius. For this sake recall that the *line graph*, $L(G)$, of a graph G has the vertex set $V(L(G)) = E(G)$ and two distinct vertices of $L(G)$ are adjacent if the corresponding edges of G share a common end-vertex. Buckley [2] observed the following simple relation between the Wiener index of a tree T and of its line graph:

$$W(T) = W(L(T)) + \binom{n}{2}. \quad (5)$$

For the rest of the section we recall that if T is a tree, then its center $C(T)$ consists either of a single vertex or of two adjacent vertices. This fact can be, for instance, deduced by iteratively removing all the leaves of a tree considered, until the center is found. Using (5) we next prove the following result.

Theorem 3.3 *If T is a tree, $r = \text{rad}(T)$, and $n = n(T)$, then*

$$W(T) - \varepsilon(T) = W(L(T)) - \varepsilon(L(T)) + \frac{n(n-3)}{2} - r + 1.$$

Proof. First we are going to find a relation between $\varepsilon(T)$ and $\varepsilon(L(T))$. For a vertex v , denote by $d(v, C(T))$ the minimum distance between v and central vertices of T . For each non-central vertex v , there is a unique adjacent vertex w such that $d(v, C(T)) = d(w, C(T)) + 1$. Consider the bijection $f : V(T) \setminus C(T) \rightarrow E(T) \setminus E(C(T))$, where $f(v) = vw$. It is not difficult to see that $\varepsilon_T(v) = \varepsilon_{L(T)}(f(v)) + 1$. We consider two cases.

Suppose first that $C(T) = \{p\}$. Then

$$\begin{aligned} \varepsilon(T) &= r + \sum_{v \in V(T) - C(T)} \varepsilon_T(v) \\ &= r + \sum_{v \in V(T) - C(T)} \left[\varepsilon_{L(T)}(vw) + 1 \right] \\ &= r + n - 1 + \varepsilon(L(T)). \end{aligned}$$

In the second case assume that $C(T) = \{p, q\}$. Then

$$\begin{aligned}
\varepsilon(T) &= 2r + \sum_{v \in V(T), v \neq p, q} \varepsilon(v) \\
&= 2r + \sum_{v \in V(T), v \neq p, q} [\varepsilon(f(v)) + 1] \\
&= 2r + \varepsilon(L(T)) - (r - 1) + n - 2 \\
&= \varepsilon(L(T)) + n + r - 1.
\end{aligned}$$

We have thus seen that in each of the cases $\varepsilon(T) = \varepsilon(L(T)) + n + r - 1$ holds. The assertion of the theorem follows by combining this relation with (5). \square

From [5, Theorem 2], which is stated in terms of the average distance and average eccentricity, we extract that if T is a tree, $n = n(T)$, and $r = \text{rad}(T)$, then

$$\varepsilon(T) = \begin{cases} d_T(p) + nr; & C(T) = \{p\}, \\ \frac{d_T(p) + d_T(q) - n}{2} + nr; & C(T) = \{p, q\}. \end{cases} \quad (6)$$

In the last result of the section we apply (6) to give a lower bound on the eccentricity of a tree in terms of its radius.

Theorem 3.4 *If T is a tree, $n = n(T)$, and $r = \text{rad}(T)$, then*

$$\varepsilon(T) \geq \begin{cases} r(n + r + 1); & |C(T)| = 1, \\ \frac{2r(n + r) - n}{2}; & |C(T)| = 2. \end{cases}$$

Moreover, the equality holds if and only if T is a path.

Proof. We again consider two cases based on the cardinality of the center of T .

Case 1: $C(T) = \{p\}$.

In this case, for any integer t , where $1 \leq t \leq r$, there are at least two vertices at distance t from p . Thus $d_T(p) \geq 2(1 + 2 + \cdots + r) = r(r + 1)$.

Case 2: $C(T) = \{p, q\}$.

Analogously, for any integer $1 \leq t \leq r - 1$, there are at least two vertices of distance t and a vertex of distance r from central vertex p . Then $d_T(p) \geq 2(1 + 2 + \cdots + r - 1) + r = r^2$.

Note that in any of the two cases, the equality holds if and only if T is a path. The result now follows from (6). \square

4 Two conjectures

If e is an edge of a graph G , then let $G.e$ denote the graph obtained from G by contracting the edge e . Our first conjecture asserts the following.

Conjecture 4.1 *If e is an edge of a graph G with $n(G) \geq 3$, then*

$$W(G.e) - \varepsilon(G.e) \leq W(G) - \varepsilon(G).$$

The next result is a partial support for the conjecture.

Theorem 4.2 *If e is a bridge of a graph G with $n(G) \geq 3$, then*

$$W(G.e) - \varepsilon(G.e) \leq W(G) - \varepsilon(G).$$

Proof. Let $e = uv$ and let $G - uv = G_u \cup G_v$ where G_u and G_v are the components of $G - uv$ containing u and v respectively. Then

$$\begin{aligned} W(G) &= W(G_u) + W(G_v) + \sum_{x \in G_u} \sum_{y \in G_v} d(x, y) \\ &= W(G_u) + W(G_v) + \sum_{x \in G_u} \sum_{y \in G_v} d(x, u) + 1 + d(v, y) \\ &= W(G_u) + W(G_v) + n(G_v)d_{G_u}(u) + n(G_u)n(G_v) + n(G_u)d_{G_v}(v). \end{aligned}$$

Since $e = uv$ is a bridge, for every $x \in V(G_u)$ and every $y \in V(G_v)$ we have $d_G(x, y) = d_G(x, u) + 1 + d_G(v, y)$. This in turn implies that $d_{G.e}(x, y) = d_G(x, y) - 1$. Therefore, in $G.e$ we have

$$\begin{aligned} W(G.e) &= W(G_u) + W(G_v) + \sum_{x \in G_u} \sum_{y \in G_v} d(x, y) \\ &= W(G_u) + W(G_v) + (n(G_v) - 1)d_{G_u}(u) + (n(G_u) - 1)d_{G_v}(v). \end{aligned}$$

Hence

$$W(G) - W(G.e) = d_{G_u}(u) + d_{G_v}(v) + n(G_u)n(G_v)$$

Let $n(G_v) \leq n(G_u)$. If $n(G_v) = 1$, then the eccentricity of the vertices from $G_u - u$ decreases by at most 1. Note further that the eccentricity of u does not change. Hence

$$\varepsilon(G) - \varepsilon(G.e) \leq n(G_u) - 1 + n(G_u) = 2n(G_u) - 1.$$

Then

$$W(G) - W(G.e) \geq n(G_u) - 1 + n(G_v) - 1 + n(G_u) \geq \varepsilon(G) - \varepsilon(G.e).$$

In the case when $n(G_v) \geq 2$, then we get

$$\begin{aligned} W(G) - W(G.e) &\geq n(G_u) - 1 + n(G_v) - 1 + n(G_u)n(G_v) \\ &\geq n(G_u) + n(G_v) - 2 + n(G_u) + n(G_v) = 2n(G) - 2. \end{aligned}$$

The eccentricity of the vertices from $V(G.e)$ decreases by at most 1 and the eccentricity of the removed vertex by at most $n(G) - 1$. Then $\varepsilon(G) - \varepsilon(G.e) \leq 2n(G) - 2 \leq W(G) - W(G.e)$. Therefore, the difference of the Wiener index of two graphs is greater than or equal to the difference of their total eccentricities. \square

In case Conjecture 4.1 holds true, it cannot be extended to all minors H of G , because the same property does not hold for removing edges. For a simple example consider the paw graph G (a graph obtained by adding a pendant vertex to a triangle), and let $H = G - e$, where e is the edge of G with both end-vertices of degree 2, that is $H = K_{1,3}$. Then $W(H) - \varepsilon(H) = 9 - 7 = 2 > 1 = 8 - 7 = W(G) - \varepsilon(G)$.

Our second conjecture asserts that the difference between the Wiener index of a graph and its eccentricity is largest possible on paths. More precisely:

Conjecture 4.3 *If G is a graph of order n with $\text{rad}(G) \geq 4$, then*

$$W(G) - \varepsilon(G) \leq \left\lfloor \frac{1}{6}n^3 - \frac{3}{4}n^2 + \frac{1}{3}n + \frac{1}{4} \right\rfloor$$

with equality holding if and only if G is a path.

The condition $\text{rad}(G) \geq 4$ is posed because otherwise the equality case is achieved also by graphs different from paths. For instance, let T_7 be the tree obtained from P_6 by adding one new vertex and connecting it with an edge with the second vertex of P_6 . Then $\text{rad}(T_7) = 3$ and $W(T_7) - \varepsilon(T_7) = 52 - 29 = 23$, which is the equality case in the above expression. Similarly, if T_8 is the tree obtained from P_7 by adding an extra vertex adjacent with the second vertex of P_7 , then $\text{rad}(T_8) = 3$ and $W(T_8) - \varepsilon(T_8) = 79 - 39 = 40$, again the equality case in the above expression.

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