# A characterization of $4-\chi_{S}$-vertex-critical graphs for packing sequences with $s_{1}=1$ and $s_{2} \geq 3$ 

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#### Abstract

If $S=\left(s_{1}, s_{2}, \ldots\right)$ is a non-decreasing sequence of positive integers, then the $S$ packing $k$-coloring of a graph $G$ is a mapping $c: V(G) \rightarrow[k]$ such that if $c(u)=c(v)=i$ for $u \neq v \in V(G)$, then $d_{G}(u, v)>s_{i}$. The $S$-packing chromatic number of $G$ is the smallest integer $k$ such that $G$ admits an $S$-packing $k$-coloring. A graph $G$ is $\chi_{S^{-}}$ vertex-critical if $\chi_{S}(G-u)<\chi_{S}(G)$ for each $u \in V(G)$. If $G$ is $\chi_{S}$-vertex-critical and $\chi_{S}(G)=k$, then $G$ is $k$ - $\chi_{S}$-vertex-critical. In this paper, 4 - $\chi_{S}$-vertex-critical graphs are characterized for sequences $S=\left(1, s_{2}, s_{3}, \ldots\right)$ with $s_{2} \geq 3$. There are 28 sporadic examples and two infinite families of such graphs.


Keywords: graph coloring; distance in graph; $S$-packing coloring; $S$-packing chromatic number; $S$-packing chromatic vertex-critical graph

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## 1 Introduction

A packing $k$-coloring of a graph $G=(V(G), E(G))$ is a mapping $c: V(G) \rightarrow[k]$ such that if $u \neq v$ and $c(u)=c(v)=i$, then $d_{G}(u, v)>i$. Here and later, $d_{G}(u, v)$ denotes the length of a shortest $u, v$-path, and $[k]=\{1, \ldots, k\}$. The packing chromatic number, $\chi_{\rho}(G)$, of $G$ is the smallest integer $k$ such that $G$ admits a packing $k$-coloring. This concept was proposed in [13]. The seminal paper was followed by [7], where the nowadays established name and notation was proposed. The development on the packing chromatic number up to 2020 has been summarized in the substantial survey [6]. Research into this concept is still flourishing, the developments after the survey include [1, 2, [5, 8, 10].

A more general concept is the $S$-packing coloring. Let $S=\left(s_{1}, s_{2}, \ldots\right)$ be a nondecreasing sequence of positive integers; we will refer to $S$ as a packing sequence. An $S$ packing $k$-coloring of $G$ is a mapping $c: V(G) \rightarrow[k]$ such that if $u \neq v$ and $c(u)=c(v)=i$, then $d_{G}(u, v)>s_{i}$. For example, a $(1,1,1, \ldots)$-packing coloring is the standard proper vertex coloring, and if $S=(1,2,3, \ldots)$, then it is just the packing coloring. The $S$-packing chromatic number, $\chi_{S}(G)$, of $G$ is the smallest integer $k$ such that $G$ admits an $S$-packing $k$-coloring. This concept was introduced by Goddard and Xu [14]; for more results see (4, 11, 12, 15, 18, 19, 20.

If $S_{1}=\left(s_{1}^{1}, s_{2}^{1}, \ldots\right)$ and $S_{2}=\left(s_{1}^{2}, s_{2}^{2}, \ldots\right)$ are (packing) sequences with $\left|S_{1}\right|=\left|S_{2}\right|$, then $S_{2} \succeq S_{1}$ means the coordinate order, that is, $S_{2} \succeq S_{1}$ if $s_{i}^{2} \geq s_{i}^{1}$ for every $i \in\left[\left|S_{1}\right|\right]$. If $S_{2} \succeq S_{1}$ and $G$ admits an $S_{2}$-packing $k$-coloring, then $G$ also admits an $S_{1}$-packing $k$-coloring. In [11, Theorem 3.1], Gastineau proved the following appealing dichotomy result: If $S$ is a packing sequence with $|S|=4$, then the decision problem whether a given graph $G$ admits an $S$-packing coloring is polynomial-time solvable if $S \succeq S^{\prime}$, where $S^{\prime} \in\{(2,3,3,3),(2,2,3,4),(1,4,4,4),(1,2,5,6)\}$, and NP-complete otherwise.

We have now arrived at the central concept of interest in this paper. A graph $G$ is packing chromatic vertex-critical if $\chi_{\rho}(G-u)<\chi_{\rho}(G)$ holds for each $u \in V(G)$. When $\chi_{\rho}(G)=k$, we more precisely say that $G$ is $k$ - $\chi_{\rho}$-vertex-critical. More generally, if $S$ is a packing sequence, then $G$ is $S$-packing chromatic vertex-critical if $\chi_{S}(G-u)<\chi_{S}(G)$ holds for each $u \in V(G)$, and if $\chi_{S}(G)=k$, then we say that $G$ is $k$ - $\chi_{S}$-vertex-critical. We also add that a closely related concept of packing chromatic critical graphs, where the packing chromatic number strictly decreases on an arbitrary proper subgraph, has been studied in [3].

Packing chromatic vertex-critical graphs were introduced in [17]. Among other results, $3-\chi_{\rho}$-vertex-critical graphs were characterized and a partial characterization of $4-\chi_{\rho}$-vertexcritical graphs was provided. The latter characterization has been completed in [9]. In [16], $3-\chi_{S}$-vertex-critical graphs were characterized for all possible packing sequences, while 4 - $\chi_{S^{-}}$ vertex-critical graphs were characterized for packing sequences $\left(s_{1}, s_{2}, s_{3}, \ldots\right)$ with $s_{1} \geq 2$.

In this article we supplement the latter result by characterizing $4-\chi_{S}$-vertex-critical graphs for packing sequences with $s_{1}=1$ and $s_{2} \geq 3$. The result is given in Section 3, while in the next section we introduce some additional notation and list known properties of $S$-packing colorings needed here.

## 2 Preliminaries

If $G$ is a graph, then we use $n(G)$ to denote its order, $\operatorname{diam}(G)$ to denote its diameter, and $\chi(G)$ to denote its chromatic number. For $x \in V(G)$, let $N_{G}^{i}(x)$ be the set of vertices which are at distance $i$ from $x$ in $G$. In particular, $N_{G}(x)=N_{G}^{1}(x)$ is the neighborhood of $x$. The degree of $x$ is $d_{G}(x)=\left|N_{G}(x)\right|$. Let $C_{n}, P_{n}$, and $K_{n}$ denote the cycle, the path, and the complete graph on $n$ vertices, respectively. A set $A \subseteq V(G)$ is $k$-independent if $A$ induces a subgraph that can be properly colored by $k$ colors. Let $\alpha_{k}(G)$ be the cardinality of a largest $k$-independent set of $G$.

If in a packing sequence the term $i$ repeats $\ell$ times, we may abbreviate the corresponding subsequence by $i^{\ell}$. For example, if $S=\left(1, \ldots, 1, s_{\ell+1}, \ldots\right)$ (where clearly 1 appears $\ell$ times), then we may shortly write $S=\left(1^{\ell}, s_{\ell+1}, \ldots\right)$. If $\varphi: V(G) \rightarrow[k]$ is an $S$-packing $k$-coloring of $G$, then $\varphi^{-1}(i), i \in[k]$, is the set of vertices $x$ with $\varphi(x)=i$. We will also use the following convention. Consider the vertex set $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ of $G$ as an ordered set, and let $\varphi$ be an $S$-packing coloring of $G$. Then we will explicitly describe $\varphi$ as follows: $\varphi=" \varphi\left(v_{1}\right) \cdots \varphi\left(v_{n}\right)$ ". Typically, the order of vertices will be alphabetic. For instance, if $V(G)=\{a, b, c, d\}$, and $\varphi(a)=1, \varphi(b)=2, \varphi(c)=1$, and $\varphi(d)=3$, then $\varphi=" 1213$ ".

We next recall some known results that will be needed in the rest.
Proposition 2.1 13] Let $n \geq 3$. If $n=3$ or $n=4 k, k \geq 1$, then $\chi_{\rho}\left(C_{n}\right)=3$; otherwise $\chi_{\rho}\left(C_{n}\right)=4$.

Lemma 2.2 [14] If $S$ is a packing sequence and $H$ is a subgraph of $G$, then $\chi_{S}(H) \leq \chi_{S}(G)$.
Proposition 2.3 14] Let $S=\left(1^{\ell}, s_{\ell+1}, \ldots\right)$, where $\ell \geq 1$ and $s_{\ell+1} \geq 2$, and let $G$ be a graph. Then $\chi_{S}(G) \leq n(G)-\alpha_{\ell}(G)+\min \{\ell, \chi(G)\}$ with equality if and only if $\operatorname{diam}(G) \leq$ $s_{\ell+1}$.

Lemma 2.4 [17] If $S$ is a packing sequence and $G$ is a $\chi_{S}$-vertex-critical graph, then $G$ is connected.

Finally, the following notation will be useful. Suppose we wish to consider all the packing sequences $S=\left(s_{1}, s_{2}, s_{3}, \ldots\right)$, for which $s_{1}=2, s_{2} \geq 4$, and $s_{3}=5$ hold. We will denote the set of all such packing sequences by $\mathcal{S}_{2, \overline{4}, 5}$, that is,

$$
\mathcal{S}_{2, \overline{4}, 5}=\left\{\left(s_{1}, s_{2}, s_{3}, \ldots\right): s_{1}=2, s_{2} \geq 4, s_{3}=5\right\}
$$

Note that since $\mathcal{S}_{2, \overline{4}, 5}$ is a set of packing sequences，we have $s_{2} \in\{4,5\}$ when $S \in \mathcal{S}_{2, \overline{4}, 5}$ ． The general notation should be clear from this example．For instance，using this notation we can state that $S \succeq\left(s_{1}, s_{2}, s_{3}, \ldots\right)$ if and only $S \in \mathcal{S}_{\bar{s}_{1}, \bar{s}_{2}, \bar{s}_{3}}, \ldots$ ．

## 3 Vertex－critical graphs for different packing sequences

As mentioned in the introduction，a characterization of 3 －$\chi_{S}$－vertex－critical graphs is known for all possible packing sequences，while $4-\chi_{S}$－vertex－critical graphs were by now character－ ized for packing sequences from $\mathcal{S}_{\overline{2}}$ ．In this section we supplement the latter result by characterizing 4 －$\chi_{S}$－vertex－critical graphs for packing sequences $S$ from $\mathcal{S}_{1, \overline{3}}$ ．To this end note that

$$
\mathcal{S}_{1, \overline{3}}=\mathcal{S}_{1, \overline{4}} \cup \mathcal{S}_{1,3, \overline{4}} \cup \mathcal{S}_{1,3,3} .
$$

In view of this fact we will solve our problem by characterizing $4-\chi_{S}$－vertex－critical graphs for packing sequences from each of the sets $\mathcal{S}_{1, \overline{4}}, \mathcal{S}_{1,3, \overline{4}}$ ，and $\mathcal{S}_{1,3,3}$ ．

In Figs． 1 and 2，several graphs are drawn that will turn out to be 4 －$\chi_{S}$－vertex－critical for packing sequences from $\mathcal{S}_{1, \overline{3}}$ ．Fig．$⿴ 囗 十$ contains two small families of graphs，the family $\mathcal{C}_{5}$ contains four graphs of order 5 ，while $\mathcal{C}_{6}$ contains three graphs of order 6 ．Fig． 2 displays the family of graphs $\mathcal{H}$ consisting of graphs $H_{i}, i \in[15]$ ．


Figure 1：Family $\mathcal{C}_{5}$（top row）and family $\mathcal{C}_{6}$（bottom row）

In the rest we will frequently consider different subsets of $\mathcal{H}$ ．To shorten the presen－ tation，we will specify subsets of $\mathcal{H}$ by（ranges of）indices．For instance， $\mathcal{H}_{1-3,7,9-11}=$ $\left\{H_{1}, H_{2}, H_{3}, H_{7}, H_{9}, H_{10}, H_{11}\right\}$ ．

First we detect the following critical graphs．
Lemma 3．1 Let $S \in \mathcal{S}_{1, \overline{3}}$ ．Then each of the graphs from $\mathcal{G}=\left\{K_{4}, C_{5}, C_{6}\right\} \cup \mathcal{C}_{5} \cup \mathcal{C}_{6} \cup \mathcal{H}_{1-5,7}$ is 4 －$\chi_{S}$－vertex－critical．


Figure 2: Family $\mathcal{H}=\left\{H_{i}: \quad i \in[15]\right\}$

Proof. Observe that for each $G \in \mathcal{G}$, $\operatorname{diam}(G) \leq s_{2}$. Using Proposition 2.3 it is then straightforward to check that $\chi_{S}(G)=4$ for each $G \in \mathcal{G}$. It remains to show that each graph $G \in \mathcal{G}$ is 4 - $\chi_{S}$-vertex-critical.

By Proposition [2.3, we have $\chi_{S}\left(K_{3}\right)=3-1+1=3, \chi_{S}\left(P_{k}\right) \leq k-\alpha\left(P_{k}\right)+1 \leq 3$ for $k \leq 5, \chi_{S}(G-x)=4-2+1=3$ for any $G \in \mathcal{C}_{5}$ and $x \in V(G)$, and $\chi_{S}(G-x)=5-3+1=3$ for any $G \in \mathcal{C}_{6}$ and $x \in V(G)$. Therefore, $K_{4}, C_{5}, C_{6}$, each of the graphs from $\mathcal{C}_{5}$, and each of the graphs from $\mathcal{C}_{6}$ are $\chi_{S}$-vertex-critical.

Now we prove that each graph in $\mathcal{H}_{1-5,7}$ is 4 - $\chi_{S}$-vertex-critical where $S \in \mathcal{S}_{1, \overline{3}}$. First consider the case where $S \in \mathcal{S}_{1, \overline{4}}$. We give an $S$-packing 3-colorings $\varphi$ for every $G-x$, where $G \in \mathcal{H}_{1-5,7}$ and $x \in V(G)$. (By symmetry, we do not need to consider all the vertices.)
 $x=a, b, c, e$, respectively. Suppose $G=H_{2}$. Then we define $\varphi$ as "2 113 ", when $x=a$
or $x=b$. Suppose $G=H_{3}$. Then we define $\varphi$ as "2 3111 ", "1 2311 " when $x=a, d$, respectively. Suppose $G=H_{4}$. Then we define $\varphi$ as "3 112 ", "2 112 ", "2 132 ", "2 31 1 " when $x=a, b, d, e$, respectively. Suppose $G=H_{5}$. Then we define $\varphi$ as "1 2113 ", " 3 2113 ", "31211" when $x=a, b, f$, respectively. Suppose $G=H_{7}$. Then we define $\varphi$ as "12113", "12311", "2 1113 ", "12131", when $x=a, b, c, e$, respectively. We have thus verified that each $G \in \mathcal{H}_{1-5,7}$ is $\chi_{S}$-vertex-critical for $S \in \mathcal{S}_{1, \overline{4}}$.

Finally suppose $S \in \mathcal{S}_{1,3, \overline{4}} \cup \mathcal{S}_{1,3,3}$. Let $G \in \mathcal{H}_{1-5,7}$ and $x \in V(G)$ be an arbitrary vertex. Since $\chi_{S}(G)=4$, it suffices to show that $\chi_{S}(G-x)=3$. Observe that for any packing sequence $S \in \mathcal{S}_{1,3, \overline{4}} \cup \mathcal{S}_{1,3,3}$ there is a packing sequence $S^{\prime} \in \mathcal{S}_{1, \overline{4}}$ such that $S^{\prime} \succeq S$. Thus, the above $S^{\prime}$-packing 3-coloring of $G-x$, where $G \in \mathcal{H}_{1-5,7}$ and $x \in V(G)$, yields an $S$-packing 3 -coloring of $G-x$. Therefore, we are finished.

### 3.1 4- $\chi_{S}$-vertex-critical graphs for $S \in \mathcal{S}_{1, \overline{4}} \cup \mathcal{S}_{1,3, \overline{4}}$

In this subsection we characterize 4 - $\chi_{S}$-vertex-critical graphs for $S \in \mathcal{S}_{1, \overline{4}}$ and for $S \in \mathcal{S}_{1,3, \overline{4}}$. The results are given in Theorems 3.5, 3.6, and 3.7,

Lemma 3.2 $P_{6}, H_{6}$, and $H_{8}$ are 4-$\chi_{S}$-vertex-critical graphs for $S \in \mathcal{S}_{1, \overline{4}}$.
Proof. Let $S \in \mathcal{S}_{1, \overline{4}}$.
First we prove that $P_{6}$ is 4 - $\chi_{S}$-vertex-critical. Suppose that $P_{6}=a b c d e f$ has an $S$ packing 3 -coloring $\varphi$. Since $\left|\varphi^{-1}(1)\right| \leq 3$, we have $\left|\varphi^{-1}(2)\right| \geq 2$ or $\left|\varphi^{-1}(3)\right| \geq 2$. Since $s_{2} \geq 4$, we must have $\varphi(a)=\varphi(f)=\alpha \in\{2,3\}$. Then at least three vertices of $\{b, c, d, e\}$ must receive color 1, but this is impossible. The pattern "121314" gives an $S$-packing 4-coloring of $P_{6}$, so $\chi_{S}\left(P_{6}\right)=4$. By Proposition 2.3, $\chi_{S}\left(P_{k}\right) \leq k-\alpha\left(P_{k}\right)+1 \leq 3$ for $k \leq 5$. Hence, $P_{6}$ is 4 - $\chi_{S}$-vertex-critical.

Now we prove that both $H_{6}$ and $H_{8}$ are $4-\chi_{S}$-vertex-critical. Observe that $\alpha\left(H_{6}\right)=$ $\alpha\left(H_{8}\right)=3$. By Proposition 2.3, we have $\chi_{S}\left(H_{6}\right)=\chi_{S}\left(H_{8}\right)=6-3+1=4$. Now we give an $S$-packing 3 -coloring $\phi$ of $G-x$ for $G \in\left\{H_{6}, H_{8}\right\}$ and $x \in V(G)$. If $G=H_{6}$, then we define $\phi$ as "12113", "11231", "2113", "31213", "31211" when $x=a, b, c, e, f$, respectively. If $G=H_{8}$, then we define $\phi$ as "12131", "2 2131 ", "2 1131 ", "1 213 1 " when $x=a, b, c, f$, respectively.

Lemma 3.3 Each of the graphs from $\left\{P_{8}, C_{8}\right\} \cup \mathcal{H}_{9,11-15}$ is 4 - $\chi_{S}$-vertex-critical for $S \in$ $\mathcal{S}_{1,3, \overline{4}}$.

Proof. Let $S \in \mathcal{S}_{1,3, \overline{4}}$.
Suppose that $P_{8}=a b c d e f g h$ has an $S$-packing 3 -coloring $\varphi$. Since $\left|\varphi^{-1}(1)\right| \leq 4$, $\left|\varphi^{-1}(2)\right| \leq 2$, and $\left|\varphi^{-1}(3)\right| \leq 2$, we have $\left|\varphi^{-1}(1)\right|=4,\left|\varphi^{-1}(2)\right|=2$ and $\left|\varphi^{-1}(3)\right|=2$.

Without loss of generality assume $\varphi(a)=\varphi(c)=\varphi(e)=\varphi(g)=1$. Then we have $c(b)=c(h)=3$ because $s_{3} \geq 4$. Thus $d$ and $f$ must receive color 2 , a contradiction. The pattern "12 1 31214 " gives an $S$-packing 4-coloring of $P_{8}$, so $\chi_{S}\left(P_{8}\right)=4$. Since $\chi_{S}\left(P_{8}\right)=4$ and the pattern "12131214" gives an $S$-packing 4-coloring of $C_{8}$, we have $\chi_{S}\left(C_{8}\right)=4$. The first $k$ entries in the pattern "1213121" gives an $S$-packing 3-coloring of $P_{k}$ with $k \leq 7$, so $P_{8}$ and $C_{8}$ are 4- $\chi_{S}$-vertex-critical.

If $G=H_{9}$, then the pattern "2 1311214 " is an $S$-packing 4-coloring of $H_{9}$, so $\chi_{S}\left(H_{9}\right) \leq 4$. Suppose that $H_{9}$ has an $S$-packing 3-coloring $\varphi$. Observe that $\{\varphi(a), \varphi(c)\}=$ $\{2,3\}$, which implies that $\varphi(e)=\varphi(f)=1$ or $\varphi(g)=\varphi(h)=1$, a contradiction. Hence $\chi_{S}\left(H_{9}\right)=4$. Now we give an $S$-packing 3 -coloring $\phi$ of $H_{9}-x$ for any $x \in V\left(H_{9}\right)$. We define $\phi$ as "1211312", "2111213", "1213121", "2131213", "2 131113 ", "2131123", "2 131121 " when $x=a, c, d, e, f, g$, $h$, respectively.

If $G=H_{11}$, then the pattern " 21311214 " is an $S$-packing 4-coloring, so $\chi_{S}\left(H_{11}\right) \leq 4$. Suppose that $H_{11}$ admits an $S$-packing 3 -coloring $\varphi$. Observe that $\{\varphi(a), \varphi(c)\}=\{2,3\}$. Then $\varphi(g)=\varphi(h)=1$ or $\varphi(e)=\varphi(f)=1$, a contradiction. Hence $\chi_{S}\left(H_{11}\right)=4$. Now we give an $S$-packing 3 -coloring $\phi$ of $H_{11}-x$ for any $x \in V\left(H_{11}\right)$. We define $\phi$ as "1 31121 3", "1 31212 1", "2 131123 ", "2 131121 " when $x=a, d, g$, $h$, respectively.

If $G=H_{12}$, then the pattern "41213121" is an $S$-packing 4-coloring of $H_{12}$. Hence $\chi_{S}\left(H_{12}\right) \leq 4$. Suppose $H_{12}$ admits an $S$-packing 3-coloring $\varphi$, then $\varphi(e)=2$ or 3. If $\varphi(e)=2$, then we have $\{\varphi(f), \varphi(g)\}=\{1,3\}$ and $\varphi(d)=1$. Thus $\varphi(c) \in\{2,3\}$, a contradiction. If $\varphi(e)=3$, then $\varphi(d)=1, \varphi(c)=2, \varphi(b)=1$. Thus $\varphi(a) \in\{2,3\}$, a contradiction. Therefore $\chi_{S}\left(H_{12}\right)=4$. Now we give an $S$-packing 3 -coloring $\phi$ of $H_{12}-x$ for any $x \in V\left(H_{12}\right)$. We define $\phi$ as "1213121", "3 2113121 ", "3113121", "3123121", "3121121", "2131221", "2131211", "1213121" when $x=a, b, c, d, e, f, g, h$, respectively.

If $G=H_{13}$, then the pattern "12131214" is an S-packing 4-coloring. Hence $\chi_{S}\left(H_{13}\right) \leq 4$. Suppose that $H_{13}$ admits an $S$-packing 3-coloring $\varphi$, then $\{\varphi(b), \varphi(e)\}=$ $\{2,3\}$. Then $\varphi(c)=\varphi(d)=1$, a contradiction. Therefore $\chi_{S}\left(H_{13}\right)=4$. Now we give an $S$-packing 3-coloring $\phi$ of $H_{13}-x$ for any $x \in V\left(H_{13}\right)$. We define $\phi$ as "1 312111 ", "1 3 12111 ", "2 131211 " when $x=b, c, g$, respectively.

If $G=H_{14}$, then the pattern "2 131214 " is an $S$-packing 4-coloring of $H_{14}$. Hence $\chi_{S}\left(H_{14}\right) \leq 4$. Suppose that $H_{14}$ admits an $S$-packing 3 -coloring $\varphi$, then $\{\varphi(c), \varphi(d)\}=$ $\{2,3\}$. Thus $\varphi(a)=\varphi(c)>1$ or $\varphi(a)=\varphi(d)>1$, a contradiction. Therefore $\chi_{S}\left(H_{14}\right)=4$. Now we give an $S$-packing 3 -coloring $\phi$ of $H_{14}-x$ for any $x \in V\left(H_{14}\right)$. We define $\phi$ as " 1 $23111 ", " 22311 ", " 123111 ", " 213211 ", " 121311 ", " 213121 "$, when $x=a, b, c, d, f, g$, respectively.

Finally, if $G=H_{15}$, then the pattern "4121311" is an $S$-packing 4-coloring of
$H_{15}$. Hence $\chi_{S}\left(H_{15}\right) \leq 4$. Suppose that $H_{15}$ admits an $S$-packing 3 -coloring $\varphi$, then $\{\varphi(c), \varphi(e)\}=\{2,3\}$ and $\varphi(b)=\varphi(d)=1$. Thus $\varphi(a) \in\{2,3\}$, a contradiction. Now we give an $S$-packing 3 -coloring $\phi$ of $H_{15}-x$ for any $x \in V\left(H_{15}\right)$. We define $\phi$ as "1 21311 ", "112311", "211311", "131211", "213112", when $x=a, b, c, e, f$, respectively.

Lemma 3.4 If $S \in \mathcal{S}_{1, \overline{4}} \cup \mathcal{S}_{1,3, \overline{4}}, G$ is a 4 - $\chi_{S}$-vertex-critical graph with at least one cycle, and $C$ is a longest cycle of $G$, then the following hold.
(a) If $n(C)=3$, then $G \in \mathcal{H}_{2-4}$.
(b) If $n(C)=4$ and $C$ contains a chord, then $G \in\left\{K_{4}, H_{1}\right\}$.
(c) If $n(C) \in\{5,6\}$, then $G \in\left\{C_{n(C)}\right\} \cup \mathcal{C}_{n(C)}$.

Proof. Let $S \in \mathcal{S}_{1, \overline{4}} \cup \mathcal{S}_{1,3, \overline{4}}$. Note that the graphs from Lemma3.1 are 4-$\chi_{S}$-vertex-critical. Let now $G$ be a 4 - $\chi_{S}$-vertex-critical graph with a longest cycle $C$.
(a) Suppose $n(C)=3$. Let $V(C)=\{a, b, c\}$. We first assume that $G$ contains only one triangle. If $H_{3}$ or $H_{4}$ is a subgraph of $G$, then we actually have $G=H_{3}$ or $G=H_{4}$, for otherwise we find another triangle in $G$ or a cycle longer than 3. If $d_{G}(v) \geq 3$ holds for each vertex of $C$, then $G=H_{3}$ since $H_{3}$ is 4 - $\chi_{S}$-vertex-critical. If $d_{G}(v)=2$ for some $v \in\{a, b, c\}$, then assume without loss of generality that $d_{G}(a)=2$. If $N_{G}^{2}(b) \backslash N_{G}(c)=\emptyset$ and $N_{G}^{2}(c) \backslash N_{G}(b)=\emptyset$, then $V(G) \backslash\{b, c\}$ is an independent set in $G$, and so a coloring $\varphi$ with $\varphi(b)=2, \varphi(c)=3$ and other vertices with color 1 is an $S$-packing 3-coloring of $G$, a contradiction. So $N_{G}^{2}(b) \backslash N_{G}(c) \neq \emptyset$ or $N_{G}^{2}(c) \backslash N_{G}(b) \neq \emptyset$. Since $H_{4}$ is 4-$\chi_{S}$-vertex-critical, $G=H_{4}$.

Suppose secondly that there are at least two triangles in $G$. Since $H_{4}$ is $\chi_{S}$-vertexcritical, the triangles in $G$ have exactly one common vertex, for otherwise there is vertex $v$ in $G$ such that $H_{4} \subseteq G-v$ or $n(C) \geq 4$. This implies that $H_{2}$ is a spanning subgraph of $G$. Since $n(C)=3$, we conclude that $G=H_{2}$.
(b) Suppose $n(C)=4$. Let $C=a b c d a$. If $a c \in E(G)$ and $b d \in E(G)$, then $G=$ $K_{4}$ by Lemma 3.1. Suppose $b d \in E(G)$. If there is a vertex $x \in N_{G}(b) \backslash V(C)$ such that $N_{G}(x) \backslash V(C) \neq \emptyset$, then $H_{4} \subseteq G-a$, a contradiction. Therefore, for any vertex $x \in\left(N_{G}(b) \cup N_{G}(d)\right) \backslash V(C)$ we have $N_{G}(x) \backslash V(C)=\emptyset$. If $d_{G}(a)=d_{G}(c)=2$, then $V(G) \backslash\{b, d\}$ is an independent set in $G$, and so a mapping $\varphi$ with $\varphi(b)=2, \varphi(d)=3$ and $\varphi\left(N_{G}(b) \cup N_{G}(d) \backslash\{b, d\}\right)=1$ is an $S$-packing 3-coloring of $G$, a contradiction. Thus $d_{G}(a) \geq 2$ or $d_{G}(c) \geq 2$. It implies that $H_{1}$ is a subgraph of $G$. Since $n(C)=4$ and by Lemma $3.1 H_{1}$ is $4-\chi_{S}$-vertex-critical, we have $G=H_{1}$.
(c) Suppose finally that $n(C) \in\{5,6\}$. Since $C$ is 4 - $\chi_{S}$-vertex-critical, $C$ is a spanning subgraph of $G$. If $n(C)=5$, then since $K_{4}$ and all the four graphs from $\mathcal{C}_{5}$ are 4 - $\chi_{S}$-vertexcritical, $\mathcal{C}_{5}$ is the family of $4-\chi_{S}$-vertex-critical graphs that contain $C_{5}$ as a proper spanning
subgraph. If $n(C)=6$, then since $C_{5}$ is $4-\chi_{S}$-vertex-critical, any two vertex at distance 2 are not adjacent in $C_{6}$. Hence $\mathcal{C}_{6}$ is the family of 4 - $\chi_{S}$-vertex-critical graphs that contain $C_{6}$ as a proper spanning subgraph by Lemma 3.1. Therefore if $n(C) \in\{5,6\}$ and $G$ is 4- $\chi_{S}$-vertex-critical, then $G \in\left\{C_{n(C)}\right\} \cup \mathcal{C}_{n(C)}$.

We can now state out first characterization.
Theorem 3.5 Let $S \in \mathcal{S}_{1, \overline{4}}$. Then a graph $G$ is 4 - $\chi_{S}$-vertex-critical if and only if

$$
G \in\left\{K_{4}, C_{5}, C_{6}, P_{6}\right\} \cup \mathcal{C}_{5} \cup \mathcal{C}_{6} \cup \mathcal{H}_{1-8} .
$$

Proof. Let $S \in \mathcal{S}_{1, \overline{4}}$ and let $G$ be 4 - $\chi_{S}$-vertex-critical. First suppose that $G$ contains a cycle, and let $C$ be a longest cycle of $G$. Since $P_{6}$ is $4-\chi_{S}$-vertex-critical, Lemma 3.2 implies $n(C) \leq 6$. By Lemma 3.4 and the fact that $\chi_{S}\left(C_{4}\right)=3$, it remains to consider the case in which $n(C)=4, n(G) \geq 5$, and there is no chord in $C$. Let $C=a b c d a$. Since $\chi_{S}(C) \leq 3$, there is a vertex $w \in V(C)$ such that $N_{G}(w) \backslash V(C) \neq \emptyset$. Let $w_{1} \in N_{G}(w) \backslash V(C)$.

First suppose $N_{G}\left(w_{1}\right) \backslash V(C) \neq \emptyset$. We may assume that $w=c$ and $w_{1}=e$. Let $f \in N_{G}(e) \backslash V(C)$. Then $H_{6}$ is subgraph of $G$. By Lemma 3.2, $H_{6}$ is a spanning subgraph of $G$. Since $G$ is $C_{k}$-free for $k \geq 5$, at most one of the edges $\{a e, c f\}$ can be possibly contained in $G$. If $a e \notin E(G)$ and $c f \notin E(G)$, then $G=H_{6}$ by Lemma 3.2. If $a e \in E(G)$, then $G=H_{7}$ by Lemma 3.1. If $c f \in E(G)$, then $H_{4} \subseteq G-b$, a contradiction.

Thus we may assume that $N_{G}\left(w_{1}\right) \backslash V(C)=\emptyset$ for each $w_{1} \in N_{G}(w) \backslash V(C)$. It implies that $N_{G}(u) \backslash V(C)$ is an independent set for any $u \in V(C)$. If $N_{G}(b) \cup N_{G}(d) \backslash V(C)=\emptyset$, then $V(G) \backslash\{a, c\}=N(a) \cup N(c)$ is an independent set in $G$, and so a mapping $\varphi$ with $\varphi(a)=2, \varphi(c)=3$ and $\varphi(N(a) \cup N(c))=1$ is an $S$-packing 3-coloring of $G$, a contradiction. Thus $N_{G}(b) \cup N_{G}(d) \backslash V(C) \neq \emptyset$ and $N_{G}(a) \cup N_{G}(c) \backslash V(C) \neq \emptyset$, and so $H_{5}$ is a spanning subgraph of $G$ by Lemma 3.1. If some edge from $\{a f, d e\}$ or from $\{e f, c f, b e\}$ is contained in $G$, then $H_{4} \subseteq G-y$ for some $y \in V(G)$ or $C_{k} \subseteq G$ with $k \geq 5$, a contradiction. Therefore, only one of $d f$ and ae can be contained in $G$, and so $G \in \mathcal{H}_{5,7}$ by Lemma 3.1.

Suppose now that $G$ is acyclic. If $P$ is a longest path in $G$, then $n(P) \leq 6$ by Lemma3.2. If $n(P)=6$, then $G=P_{6}$. If $n(P)=5$, then let $P_{5}=a b c d e$. If $d_{G}(c)=2$, then the mapping $\varphi$ with $\varphi(b)=2, \varphi(d)=3$ and $\varphi\left(N_{G}(b) \cup N_{G}(d)\right)=1$ is an $S$-packing 3-coloring of $G$ which implies that $\chi_{S}(G) \leq 3$, a contradiction. Therefore $d_{G}(c) \geq 3$. But then $G=H_{8}$ by Lemma 3.2. If $n(P) \leq 4$, then we have that $\chi_{S}(G) \leq 3$, so we get no new graph.

Theorem 3.6 Let $S \in \mathcal{S}_{1,3, \overline{4}}$ and let $G$ be a graph with a cycle. Then $G$ is 4 - $\chi_{S}$-vertexcritical if and only if

$$
G \in\left\{K_{4}, C_{5}, C_{6}, C_{8}\right\} \cup \mathcal{C}_{5} \cup \mathcal{C}_{6} \cup \mathcal{H}_{1-5,7,9,11,15}
$$

Proof. Let $S \in \mathcal{S}_{1,3, \overline{4}}$ and let $C$ be a longest cycle of $G$. Since $P_{8}$ is 4 - $\chi_{S}$-vertex-critical, $n \leq 8$. If $n(C)=8$, then $C$ is a spanning subgraph of $G$ by Lemma 3.3. Since $\chi_{S}\left(C_{5}\right)=$ $\chi_{S}\left(C_{6}\right)=4$, and $\chi_{S}\left(C_{7}\right)=7-\alpha\left(C_{7}\right)+1>4$ by Proposition 2.3, there is no chord in $C$. Therefore $G=C$ when $n(C)=8$. Since $\chi_{S}\left(C_{7}\right)=5$, we have $n(C) \neq 7$. By Lemma 3.4 and the fact $\chi_{S}\left(C_{4}\right)=3$ it remains to consider the case that $n(C)=4, n(G) \geq 5$, and there is no chord in $C$.

Let $C=a b c d a$. First suppose that there is an edge in $E(C)$, say $b c$, such that $d_{G}(b) \geq 3$ and $d_{G}(c) \geq 3$. Then $N_{G}(b) \cap N_{G}(c)=\emptyset$ for otherwise $G$ has a cycle of length at least 5 . It follows that $H_{5}$ is a spanning subgraph of $G$ by Lemma 3.1 because there is no chord in $C$. If $a f$ or $d e$ is contained in $G$, then $H_{4} \subseteq G-y$ for some $y \in V(G)$. Hence at most one of $d f$ and ae can be added to $G$. Therefore $G \in \mathcal{H}_{5,7}$ by Lemma 3.1.

Now consider the case in which $d_{G}(s)=2$ or $d_{G}(t)=2$ for each edge st $\in E(C)$. Without loss of generality, suppose that $d_{G}(b)=2$ and $d_{G}(d)=2$. Let $P$ be a longest path with endpoint $c$, such that $a, b, d \notin V(P)$, and let $P^{\prime}$ be a longest path with endpoint $a$, such that $c, b, d \notin V\left(P^{\prime}\right)$. Without loss of generality assume that $n(P) \geq n\left(P^{\prime}\right)$. If $n(P) \geq 3$ and $V(P) \cap V\left(P^{\prime}\right) \neq \emptyset$, then $G=H_{7}$. Indeed, for otherwise by the definition of $P$ and $P^{\prime}$ we have $a, c \notin V(P) \cap V\left(P^{\prime}\right)$, and then for some $k \geq 5$ we have $C_{k} \subseteq G-b$, a contradiction. In the rest of the proof we may thus assume that if $n(P) \geq 3$, then $V(P) \cap V\left(P^{\prime}\right)=\emptyset$. Since $P_{8}$ is $4-\chi_{S}$-vertex-critical, $n(P)+n\left(P^{\prime}\right) \leq 6$.

Claim. If $n(P) \leq 4$ and $n\left(P^{\prime}\right) \leq 2$, then $G=H_{15}$.
Proof. Since $n\left(P^{\prime}\right) \leq 2$, we infer that if $x \in N_{G}(a) \backslash N_{G}(c)$ and $y \in N_{G}(a) \cap N_{G}(c)$, then $d_{G}(x)=1$ and $d_{G}(y)=2$. Hence $N_{G}(a)$ is an independent set in $G$. If there is a vertex $x \in N_{G}(c) \backslash N_{G}(a)$ such that $d_{G}(x) \geq 3$, then $H_{15} \subseteq G$. Since $H_{15}$ is 4 - $\chi_{S}$-vertex-critical by Lemma 3.3, $H_{15}$ is a spanning subgraph of $G$. Since $n\left(P^{\prime}\right) \leq 2$ and $d_{G}(b)=d_{G}(d)=2$, only edges from $\{f g, c f\}$ are possibly contained in $G$. If an edge from $f g$ or $c f$ is contained in $G$, then there is a vertex $v \in G$ such that $H_{4} \subseteq G-v$, a contradiction. Hence $G=H_{15}$. It remains to consider the case in which $d_{G}(x) \leq 2$ holds for each $x \in N_{G}(c)$. Then $N_{G}(c)$ is an independent set in $G$, for otherwise $H_{4} \subseteq G-b$, a contradiction. Since $n(P) \leq 4$ and $d_{G}(x) \leq 2$ for every $x \in N_{G}(c)$, the second neighborhood $N_{G}^{2}(c)$ is an independent set and $d_{G}(y)=1$ for each $y \in N_{G}^{3}(c)$. (It is possible that $N_{G}^{3}(c)=\emptyset$.) Then a mapping $\varphi$ with $\varphi(c)=3, \varphi\left(N_{G}(c) \cup N_{G}^{3}(c)\right)=1$, and $\varphi\left(N_{G}^{2}(c)\right)=2$ is an $S$-packing 3-coloring of $G$. This contradiction proves the claim.

It remains to consider the following two cases: (i) $n(P)=5, n\left(P^{\prime}\right)=1$, and (ii) $n(P)=n\left(P^{\prime}\right)=3$. If $n(P)=5$, then $H_{9}$ is a spanning subgraph of $G$. (The vertices of $H_{9}$ are denoted as in Fig. 2) If some edge from $\{c f, e g, f h\}$ or $c h$ or $c g$ is contained in $G$, then $H_{4} \subseteq G-b$ or $C_{5} \subseteq G-b$ or $H_{5} \subseteq G-b$, respectively, a contradiction. Now we only need to check that whether eh can be added to $H_{9}$. The graph obtained from $H_{9}$ by adding the
edge $e h$ is $H_{10}$, cf. Fig. 2 again. Then $H_{15} \subseteq H_{10}-g$, a contradiction. Hence $G=H_{9}$. If $n(P)=3$ and $n\left(P^{\prime}\right)=3$, then $H_{11} \subseteq G$. By symmetry, if some edge from $\{a f, g e, g f\}$ or $a h$ or $a e$ is contained in $G$, then $C_{k} \subseteq G-b$ with $k \geq 5$ or $H_{4} \subseteq G-b$ or $H_{5} \subseteq G-b$, respectively, a contradiction. Since $H_{11}$ is $4-\chi_{S}$-vertex-critical, no additional edge can be added to $H_{11}$. We conclude that $G=H_{11}$.

It remains to consider acyclic graphs for $S \in \mathcal{S}_{1,3, \overline{4}}$.
Theorem 3.7 Let $S \in \mathcal{S}_{1,3, \overline{4}}$ and let $G$ be an acyclic graph. Then $G$ is 4 - $\chi_{S}$-vertex-critical if and only if $G \in\left\{P_{8}\right\} \cup \mathcal{H}_{12-14}$.

Proof. Let $G$ be 4 - $\chi_{S}$-vertex-critical and acyclic. Denote by $P$ a longest path in $G$. If $n(P)=8$, then Lemma 3.3 implies that $G=P_{8}$. Since $\chi_{S}(G) \leq 3$ when $n(P) \leq 4$, it remains to consider the cases $5 \leq n(P) \leq 7$.

Suppose $n(P)=5$ and let $P=a b c d e$. If $d_{G}(c)=2$, then a coloring $\varphi$ with $\varphi(\{c\} \cup$ $\left.N_{G}^{2}(c)\right)=1, \varphi(b)=2$, and $\varphi(d)=3$ is an $S$-packing 3-coloring of $G$, contradicting the fact that $\chi_{S}(G)=4$, hence $d_{G}(c) \geq 3$. If $d_{G}(x) \leq 2$ for any $x \in N_{G}(c)$, then the coloring $\varphi$ with $\varphi\left(N_{G}(c)\right)=1, \varphi\left(N_{G}^{2}(c)\right)=2$, and $\varphi(c)=3$ is an $S$-packing 3 -coloring of $G$, a contradiction. So $G=H_{14}$ by Lemma 3.3,

Suppose $n(P)=6$ and let $P=a b c d e f$. Then either $d_{G}(s)=2$ or $d_{G}(t)=2$ for st $\in E(P) \backslash\{a b, e f\}$, otherwise there is a vertex $x \in V(G)$ such that $H_{14} \subseteq G-x$. If $d_{G}(c) \geq 3$, then a mapping $\varphi$ with $\varphi\left(N_{G}(c) \cup N_{G}^{3}(c)\right)=1, \varphi\left(N_{G}^{2}(c)\right)=2$, and $\varphi(c)=3$ is an $S$-packing 3 -coloring of $G$, a contradiction. Thus $d_{G}(c)=d_{G}(d)=2$. If $d_{G}(b)=2$, then a mapping $\varphi$ with $\varphi\left(N_{G}(e) \cup N_{G}^{3}(e)\right)=1, \varphi(a)=\varphi(e)=2$, and $\varphi(c)=3$ is an $S$-packing 3 -coloring of $G$. Thus $d_{G}(b) \geq 3$ and $d_{G}(e) \geq 3$. Hence $G=H_{13}$ by Lemma 3.3,

Let finally $P=a b c d e f g$. If $d_{G}(x)=2$ for any $x \in N_{G}(d)$, a mapping $\varphi$ with $\varphi\left(N_{G}(d) \cup\right.$ $\left.N_{G}^{3}(d)\right)=1, \varphi\left(N_{G}^{2}(d)\right)=2$, and $\varphi(d)=3$ is an $S$-packing 3 -coloring of $G$ contradicting the fact $\chi_{S}(G)=4$. Hence $G=H_{12}$ by Lemma 3.3,

Combining Theorem 3.7 with Theorem 3.6 we get:
Corollary 3.8 Let $S \in \mathcal{S}_{1,3, \overline{4}}$ and let $G$ be a graph. Then $G$ is 4 - $\chi_{S}$-vertex-critical if and only if

$$
G \in\left\{K_{4}, C_{5}, C_{6}, C_{8}, P_{8}\right\} \cup \mathcal{C}_{5} \cup \mathcal{C}_{6} \cup \mathcal{H}_{1-5,7,9,11-15}
$$

### 3.2 4- $\chi_{S}$-vertex-critical graphs for $S \in \mathcal{S}_{1,3,3}$

In this subsection we consider packing sequences $S \in \mathcal{S}_{1,3,3}$. In Lemmas 3.9, 3.10, and 3.11, we present some graphs that are $4-\chi_{S}$-vertex-critical. After that we characterize 4-$\chi_{S}$-vertex-critical graphs by distinguishing the distance between vertices of degree at least 3.

Lemma 3.9 If $S \in \mathcal{S}_{1,3,3}$, then the following hold.
(a) If $n \geq 4$, then $\chi_{S}\left(P_{n}\right)=3$.
(b) Let $n \geq 3$. If $n=3$ or $n \equiv 0 \bmod 4$, then $\chi_{S}\left(C_{n}\right)=3$. If $n \equiv 1,2 \bmod 4$, or $n \equiv 3 \bmod 4$ and $s_{4}<\lfloor n / 2\rfloor$, then $\chi_{S}\left(C_{n}\right)=4$; otherwise, $\chi_{S}\left(C_{n}\right)=5$. Moreover, $C_{n}$ is $\chi_{S}$-vertex-critical when $n \not \equiv 0 \bmod 4$ and $n \geq 5$.

Proof. (a) Note that $\chi_{S}\left(P_{n}\right) \geq 3$ for $n \geq 4$. The pattern "12131213..." is an $S$-packing 3 -coloring of $P_{n}$. Thus $\chi_{S}\left(P_{n}\right)=3$ for $n \geq 4$.
(b) First, $\chi_{S}\left(C_{n}\right) \geq 3$ for $n \geq 3$. The pattern " 123 " gives an $S$-packing 3 -coloring of $C_{3}$ and the pattern "12131213_1213" gives an $S$-packing 3 -coloring of $C_{n}$ when $n \equiv 0 \bmod 4$. Thus $\chi_{S}\left(C_{n}\right)=3$ when $n=3$ or $n \equiv 0 \bmod 4$.

Next, if $n \geq 4$ and $n \not \equiv 0 \bmod 4$, then since $(1,3,3) \succeq(1,2,3)$, we have $\chi_{S}\left(C_{n}\right) \geq 4$ by Proposition 2.1. The pattern "12131213...12134" gives an $S$-packing 4-coloring of $C_{n}$ when $n \equiv 1 \bmod 4$ and the pattern " $12131213 \ldots 121314$ " gives an $S$-packing 4 -coloring of $C_{n}$ when $n \equiv 2 \bmod 4$. Thus $\chi_{S}\left(C_{n}\right)=4$ when $n \equiv 1,2 \bmod 4$.

Consider now the case $n \equiv 3 \bmod 4$. When $n=4 k+3, n \geq 7$, and $s_{4}<\lfloor n / 2\rfloor$, we give an $S$-packing 4 -coloring $\varphi$ of $C_{n}=v_{0} v_{1} \ldots v_{n-1} v_{0}$ as:

$$
\varphi\left(v_{i}\right)= \begin{cases}1 ; & (i \equiv 0 \bmod 4) \text { or }(i \equiv 2 \bmod 4 \text { and } i \neq 4 k+2), \\ 2 ; & (i \equiv 3 \bmod 4 \text { and } i<2 k+1) \text { or }(i \equiv 1 \bmod 4 \text { and } i>2 k+1), \\ 3 ; & (i \equiv 1 \bmod 4 \text { and } i<2 k+1) \text { or }(i \equiv 3 \bmod 4 \text { and } i>2 k+1), \\ 4 ; & i \in\{2 k+1,4 k+2\} .\end{cases}
$$

Hence $\chi_{S}\left(C_{n}\right)=4$ when $n \equiv 3 \bmod 4, n \geq 7$, and $s_{4}<\lfloor n / 2\rfloor$.
When $n=4 k+3, n \geq 7$, and $s_{4} \geq\lfloor n / 2\rfloor$, the pattern " $12131213 \ldots 121314$ 5 " is an $S$-packing 5 -coloring of $C_{n}$. Hence $4 \leq \chi_{S}\left(C_{n}\right) \leq 5$. Now suppose that there is an $S$-packing 4 -coloring $\varphi$ of $C_{n}$. Since $s_{4} \geq\lfloor n / 2\rfloor$, we have $\left|\varphi^{-1}(4)\right|=1$. Without loss of generality we may assume that $\varphi\left(v_{0}\right)=4$. We claim that for any edge in $C_{n}$ which is not incident with $v_{0}$, one of its endpoints receives color 1 . Suppose on the contrary that there is an edge $v_{i} v_{i+1} \in E\left(C_{n}\right)$ such that $\left\{\varphi\left(v_{i}\right), \varphi\left(v_{i+1}\right)\right\}=\{2,3\}$, where $1 \leq i \leq n-2$. Since $n \geq 7$, one of $v_{i-2}$ and $v_{i+3}$ (indices taken modulo $n$ ) cannot be colored under $\varphi$. This contradiction proves the claim. Since $s_{2}=s_{3}=3$, we only need to consider two cases: $\varphi\left(v_{1}\right)=2$ and $\varphi\left(v_{1}\right)=1$. If $\varphi\left(v_{1}\right)=2$, then the colors of $v_{0}, v_{1}, \ldots, v_{4 k+2}$ under $\varphi$ can be described as the pattern "421312131...213121". We have $\varphi\left(v_{1}\right)=\varphi\left(v_{4 k+1}\right)=2$ with $d_{C_{n}}\left(v_{1}, v_{4 k+1}\right)=3 \leq s_{2}$, a contradiction. If $\varphi\left(v_{1}\right)=1$, then we may without loss of generality assume $\varphi\left(v_{2}\right)=2$. Then the colors of $v_{0}, v_{1}, \ldots, v_{4 k+2}$ under $\varphi$ can be described as the pattern " $412131213 \ldots 121312$ ". However, we have $\varphi\left(v_{2}\right)=\varphi\left(v_{4 k+2}\right)=2$
with $d_{C_{n}}\left(v_{2}, v_{4 k+2}\right)=3 \leq s_{2}$, a contradiction. Therefore $\chi_{S}\left(C_{n}\right)=5$ when $n \equiv 3 \bmod 4$, $n \geq 7$, and $s_{4} \geq\lfloor n / 2\rfloor$.

If $n \not \equiv 0 \bmod 4$, then $C_{n}$ is $\chi_{S}$-vertex-critical because $\chi_{S}\left(P_{n}\right) \leq 3$ and $\chi_{S}\left(C_{n}\right) \geq 4$.
Let $G_{2 k}, k \geq 3$, be the graph obtained from the path $P_{2 k}$ by attaching a pendent vertex to each of the two support vertices of $P_{2 k}$. Equivalently, $G_{2 k}$ is obtained from $P_{2 k-2}$ by attaching two pendant vertices to each of the two leaves of $P_{2 k-2}$.

Lemma 3.10 If $S \in \mathcal{S}_{1,3,3}$ and $k \geq 3$, then $G_{2 k}$ is 4 - $\chi_{S}$-vertex-critical.
Proof. Let $P_{2 k}=v_{1} v_{2} \ldots v_{2 k}$, and let $v_{2}^{\prime}$ and $v_{2 k-1}^{\prime}$ be the pendent vertices attached to $v_{2}$ and $v_{2 k-1}$, respectively. Coloring the vertices of $P_{2 k}$ with the pattern "12131213..." and the vertices $v_{2}^{\prime}$ and $v_{2 k-1}^{\prime}$ with 1 and 4 , respectively, we get $\chi_{S}\left(G_{2 k}\right) \leq 4$.

Suppose now that $G_{2 k}$ admits an $S$-packing 3 -coloring $\varphi$. Observe that $\varphi\left(v_{2}\right) \in\{2,3\}$, without loss of generality assume that $\varphi\left(v_{2}\right)=2$. Then we have $\varphi\left(v_{1}\right)=1$ and $\varphi\left(v_{3}\right)=1$, for otherwise $\varphi\left(v_{3}\right)=\varphi\left(v_{4}\right)=1$ or $\varphi\left(v_{4}\right)=\varphi\left(v_{5}\right)=1$. If $2 \leq i \leq 2 k-2$, then at least one of $v_{i}$ and $v_{i+1}$ must be colored 1 . Indeed, if we would have $\varphi\left(v_{i}\right)=2$ and $\varphi\left(v_{i+1}\right)=3$, then $v_{i-2}$ or $v_{i+3}$ can not be colored under $\varphi$. Thus we have $\varphi\left(v_{2 k-2}\right)=2$ and $\varphi\left(v_{2 k-1}\right)=1$, or $\varphi\left(v_{2 k-2}\right)=3$ and $\varphi\left(v_{2 k-1}\right)=1$. However, this implies that $v_{2 k-1}^{\prime}$ or $v_{2 k}$ can not be colored under $\varphi$, a contradiction. Hence, $\chi_{S}\left(G_{2 k}\right)=4$.

If $v \in G_{2 k}$, then an $S$-packing 3 -coloring of $G_{2 k}-v$ can be given by coloring a longest path of each component of $G_{2 k}-v$ with either the pattern "1213 $\ldots$ " or the pattern "2 $131 \ldots$ " and coloring the pendent vertices with 1 . Therefore $G_{2 k}$ is 4 - $\chi_{S}$-vertexcritical.

Lemma 3.11 If $S \in \mathcal{S}_{1,3,3}$, then the graphs $H_{14}$ and $H_{15}$ are 4-$\chi_{S}$-vertex-critical.
Proof. Since $H_{14}$ and $H_{15}$ are $4-\chi_{S^{\prime}}$-vertex-critical, where $S^{\prime} \in \mathcal{S}_{1,3, \overline{4}}$, and $(1,3,4) \succeq$ $(1,3,3)$, by Corollary 3.8 it suffices to show that $\chi_{S}\left(H_{14}\right)=\chi_{S}\left(H_{15}\right)=4$. The pattern "4 1 23111 " is an $S$-packing 4-coloring of $H_{14}$, so $\chi_{S}\left(H_{14}\right) \leq 4$. Suppose that $H_{14}$ admits an $S$-packing 3-coloring $\varphi$. Then $\{\varphi(c), \varphi(d)\}=\{2,3\}$, and so the vertex $a$ cannot be colored under $\varphi$. It follows that $\chi_{S}\left(H_{14}\right)=4$. The pattern "4121311" is an $S$-packing 4-coloring of $H_{15}$. Moreover, since $H_{14} \subseteq H_{15}$, by Lemma 2.2, we conclude that $\chi_{S}\left(H_{15}\right)=4$.

Our next result, Theorem 3.14, follows from the following lemma and theorem.
Lemma 3.12 Let $S \in \mathcal{S}_{1,3,3}$ and let $n \not \equiv 0 \bmod 4, n>3$. If a graph $G$ contains a cycle $C_{n}$ and $V(G)-V\left(C_{n}\right) \neq \emptyset$, then $G$ is not 4 - $\chi_{S}$-vertex-critical.

Proof. Since $V(G)-V\left(C_{n}\right) \neq \emptyset$, there exists a vertex $x \in V(G)$ such that $C_{n} \subseteq G-x$. By Lemma 2.2, we have $\chi_{S}(G-x) \geq \chi_{S}\left(C_{n}\right) \geq 4$, and so $G$ is not 4 - $\chi_{S}$-vertex-critical.

Theorem 3.13 [17, Theorem 4.3] If $G$ is a graph that contains a cycle of length $n \geq 5$, where $n \not \equiv 0 \bmod 4$, then $G$ is $4-\chi_{\rho}$-vertex-critical if and only if one of the following holds.

- $n=5$ and $G \in\left\{C_{5}\right\} \cup \mathcal{C}_{5}$,
- $n=6$ and $G \in\left\{C_{6}\right\} \cup \mathcal{C}_{6}$,
- $n \geq 7$ and $G$ is isomorphic to $C_{n}$.

Theorem 3.14 Let $S \in \mathcal{S}_{1,3,3}$. If $G$ is a graph that contains a cycle of length $n \geq 5$, where $n \not \equiv 0 \bmod 4$, then $G$ is 4 - $\chi_{S}$-vertex-critical if and only if one of the following holds.

- $n=5$ and $G \in\left\{C_{5}\right\} \cup \mathcal{C}_{5}$,
- $n=6$ and $G \in\left\{C_{6}\right\} \cup \mathcal{C}_{6}$,
- $n \geq 7$ and $G=C_{n}$ except when $n \equiv 3 \bmod 4$ and $s_{4} \geq\lfloor n / 2\rfloor$.

In order to characterize 4 - $\chi_{S}$-vertex-critical graphs, where $S \in \mathcal{S}_{1,3,3}$, we need to distinguish whether there are two vertices of degree at least 3 that are at odd distance. For this sake we need the following classes of cycles that depend on a positive integer $s_{4}$ (this $s_{4}$ will, of course, be the fourth component of a packing sequence $S$ ):

$$
\mathcal{C}_{s_{4}}=\left\{C_{n}, n \geq 5:(n \equiv 1,2 \bmod 4) \text { or }\left(n \equiv 3 \bmod 4 \text { and } s_{4}<\lfloor n / 2\rfloor\right\} .\right.
$$

Theorem 3.15 Let $S \in \mathcal{S}_{1,3,3}$ and let $G$ be a 4 - $\chi_{S}$-vertex-critical graph. If all the vertices of $G$ of degree at least 3 are pairwise at even distances in $G$, then $G \in\left\{H_{2}, H_{4}\right\} \cup \mathcal{C}_{s_{4}}$.

Proof. By Lemmas 3.9 and 3.1, every graph from $\left\{H_{2}, H_{4}\right\} \cup \mathcal{C}_{s_{4}}$ is $4-\chi_{S}$-vertex-critical. If $\Delta(G) \leq 2$, then $G \in\left\{P_{n}, C_{n}\right\}$, hence $G \in \mathcal{C}_{s_{4}}$. Suppose now that $\Delta(G) \geq 3$ and that all the vertices of degree at least 3 are pairwise at even distances in $G$. Let $u \in V(G)$ be an arbitrary vertex of degree at least 3 . Then define $\varphi: V(G) \rightarrow[3]$ by:

$$
\varphi(v)= \begin{cases}1 ; & d_{G}(u, v) \equiv 1,3 \bmod 4 \\ 2 ; & d_{G}(u, v) \equiv 0 \bmod 4 \\ 3 ; & d_{G}(u, v) \equiv 2 \bmod 4\end{cases}
$$

By Lemma 2.4 $G$ is a connected graph, and so $\varphi$ is well-defined. Since $G$ is 4 - $\chi_{S}$-vertexcritical, there are two vertices $x, y \in V(G) \backslash\{u\}$ such that $\varphi(x)=\varphi(y)=i$ and $d_{G}(x, y) \leq$ $s_{i}$ for some $i \in[3]$. Let $P$ and $P^{\prime}$ be arbitrary shortest $u, x$-path and $u, y$-path in $G$, respectively. Let $w \in V(P) \cap V\left(P^{\prime}\right)$ such that $d_{G}(u, w)$ is as large as possible. Then we have $w \neq x, y$ and $d_{G}(u, x)=d_{G}(u, y)$. If $w=x$ or $d_{G}(u, x)<d_{G}(u, y)$, then $d_{G}(u, y) \leq$
$d_{G}(u, x)+s_{i}<d_{G}(u, x)+4$, and so $\varphi(x) \neq \varphi(y)$ by the definition of $\varphi$, which leads to a contradiction. Thus we have $d_{G}(w) \geq 3$, and so $\varphi(w) \in\{2,3\}$.
Claim. $G$ contains a cycle consisting of $w P x, w P^{\prime} y$, and a shortest $x, y$-path.
Proof. Let $P^{\prime \prime}$ be a shortest $x, y$-path. By the choice of $w$, it suffices to show that $\left(V\left(P^{\prime \prime}\right) \cap\right.$ $V(P)) \backslash\{x\}=\left(V\left(P^{\prime \prime}\right) \cap V\left(P^{\prime}\right)\right) \backslash\{y\}=\emptyset$.

Suppose that $\left(V\left(P^{\prime \prime}\right) \cap V(P)\right) \backslash\{x\} \neq \emptyset$. Note that this can only happen when $\varphi(x)=$ $\varphi(y) \in\{2,3\}$ and $d_{G}(x, y) \in\{2,3\}$. Without loss of generality, let $\varphi(x)=\varphi(y)=3$. If $d_{G}(x, y)=2$, let $P^{\prime \prime}=x z y$. Then $d_{G}(z) \geq 3$ and $d_{G}(u, z)=d_{G}(u, x)-1 \equiv 1 \bmod 4$, contradicting the fact that the vertices of degree at least 3 are at even distance. For $d_{G}(x, y)=3$, let $P^{\prime \prime}=x z_{1} z_{2} y$. Then $x z_{2} \notin E(G)$. If $z_{2} \in V\left(P^{\prime \prime}\right) \cap V(P)$, then $d_{G}(u, y) \leq$ $d_{G}\left(u, z_{2}\right)+d_{G}\left(z_{2}, y\right)=d_{G}(u, x)-2+1<d_{G}(u, x)$, contradicting the fact that $d_{G}(u, x)=$ $d_{G}(u, y)$. Therefore $z_{2} \notin V\left(P^{\prime \prime}\right) \cap V(P)$. However, if $z_{1} \in V\left(P^{\prime \prime}\right) \cap V(P)$, then $d_{G}\left(z_{1}\right) \geq 3$ and $d_{G}\left(u, z_{1}\right)=d_{G}(u, x)-1 \equiv 1 \bmod 4$, a contradiction. Therefore $G$ contains a cycle, say $C_{n}$, consisting of $w P x, w P^{\prime} y$ and a shortest $x, y$-path, and so $n=2 d_{G}(u, x)-2 d_{G}(u, w)+$ $d_{G}(x, y)$.

Suppose $\varphi(x)=\varphi(y)=1$ and $d_{G}(x, y)=1$. Then we have $n \equiv 3 \bmod 4$. If $n \equiv 3 \bmod 4$ and $n>3$, then $G=C_{n}$ with $s_{4}<\lfloor n / 2\rfloor$ by Theorem 3.14. If $n=3$, then $w x y$ is a triangle with $d_{G}(w) \geq 3$ and $d_{G}(x)=d_{G}(y)=2$. If $N_{G}^{2}(w) \neq \emptyset$, then $G=H_{4}$. If $N_{G}^{2}(w)=\emptyset$ and $N_{G}(w) \backslash\{x, y\}$ is an independent set, then a mapping $\varphi$ with $\varphi\left(N_{G}(w) \backslash\{x, y\}\right)=\varphi(x)=1$, $\varphi(w)=2$ and $\varphi(y)=3$ is an $S$-packing 3 -coloring of $G$. Moreover, it is easy to see that no edge can be added to $H_{2}$ and to $H_{4}$. Hence $G \in\left\{H_{2}, H_{4}\right\}$ when $n=3$. Suppose $\varphi(x)=\varphi(y)=2$ or $\varphi(x)=\varphi(y)=3$. If $d_{G}(x, y)=i, i \in[3]$, then $n \equiv i \bmod 4 \geq 5$ because $d_{G}(u, x)$ and $d_{G}(u, w)$ are even. Therefore $G \in \mathcal{C}_{s_{4}}$ by Theorem 3.14.

Theorem 3.16 Let $S \in \mathcal{S}_{1,3,3}$ and let $G$ be a 4 - $\chi_{S}$-vertex-critical graph in which there exist two vertices of degree at least 3 that are at odd distance. Then

$$
G \in\left\{K_{4}\right\} \cup \mathcal{H}_{1,3,5,7,14,15} \cup\left\{G_{2 k}: k \geq 3\right\} \cup \mathcal{C}_{5} \cup \mathcal{C}_{6} .
$$

Proof. Let $u, v \in V(G)$ with $d_{G}(u), d_{G}(v) \geq 3$ such that $d_{G}(u, v)=\ell$ is odd and as small as possible. We consider the following two cases.

Case 1: $\ell \geq 3$.
By the choice of $u$ and $v, N_{G}(u) \cap N_{G}(v)=\emptyset$ and each vertex on a shortest $u, v$-path has degree 2 in $G$. Therefore $G_{\ell+3}=G_{2 k}$ is a spanning subgraph of $G$, where $k=\frac{\ell+3}{2}$ by Lemma 3.10. Let the vertices of $G_{2 k}$ be denoted as in Lemma 3.10 with $u=v_{2}$ and $v=v_{2 k-1}$. By symmetry, only some of the edges $v_{1} v_{2}^{\prime}, v_{1} v_{2 k}$, and $v_{1} v_{2 k-1}$ can possibly be added to $G_{2 k}$. If $v_{1} v_{2}^{\prime} \in E(G)$, then $H_{4} \subseteq G-v_{2 k}$, a contradiction. If $v_{1} v_{2 k} \in E(G)$, then
$C_{2 k} \subseteq G$. Further, we have $2 k \equiv 0 \bmod 4 \geq 8$, for otherwise $\chi_{S}\left(C_{2 k}\right)=4$ by Theorem 3.14 and $C_{2 k} \subseteq G-v_{2}^{\prime}$. Then we can find a copy of $G_{6}$ consisting of $v_{3} v_{2} v_{1} v_{2 k} v_{2 k-1} v_{2 k-2}$ and the pendent vertices $v_{2}^{\prime}$ and $v_{2 k-1}^{\prime}$ contained in $G-v_{4}$, which also leads to a contradiction. If $v_{1} v_{2 k-1} \in E(G)$, then there is a cycle $C_{2 k-1} \subseteq G-v_{2}^{\prime}$ with $2 k-1 \not \equiv 0(\bmod 4)>3$, again a contradiction. We conclude that in Case $1, G=G_{2 k}$.

Case 2: $\ell=1$.
We claim that in this case, $G \in\left\{K_{4}\right\} \cup \mathcal{H}_{1,3,5,7,14,15} \cup \mathcal{C}_{5} \cup \mathcal{C}_{6}$.
If $G$ contains a cycle $C$ from $\mathcal{C}_{s_{4}}$, then by Theorem [3.14, $C$ is a spanning subgraph of $G$. Since there are two vertices of degree at least 3 which are of distance 1 in $G$, we have $G \in \mathcal{C}_{5} \cup \mathcal{C}_{6}$. Suppose $G$ contains $H_{4}$ as a subgraph. Then $H_{4}$ is a spanning subgraph of $G$. Since $G$ has two vertices of degree at least $3, G \in \mathcal{C}_{5} \cup\left\{H_{1}\right\}$. By the same argument, if $G$ contains $H_{2}$ as a spanning subgraph, then $G \in \mathcal{C}_{5}$. Therefore we may assume $G$ does not contain a graph from $\mathcal{C}_{s_{4}} \cup\left\{H_{2}, H_{4}\right\}$ as a subgraph. Let $a=u$ and $c=v$.

Suppose that $\left|N_{G}(a) \cap N_{G}(c)\right| \geq 2$. If $d_{G}(x)=2$ for any $x \in N_{G}(a) \cap N_{G}(c)$, then $V(G) \backslash\{a, c\}$ is an independent set in $G$ since $G$ does not contain $H_{2}$ or $H_{4}$ as a subgraph, and so a coloring $\varphi$ of $G$ with $\varphi(V(G) \backslash\{a, c\})=1, \varphi(a)=2$ and $\varphi(c)=3$ is an $S$-packing 3coloring, a contradiction. Then either $b d \in E(G)$ for some $b, d \in N_{G}(a) \cap N_{G}(c)$ and so $G=$ $K_{4}$, or there is a vertex $x \in N_{G}(a) \cap N_{G}(c)$ such that $N_{G}(x) \backslash\left(\{a, c\} \cup V\left(N_{G}(a) \cap N_{G}(c)\right)\right) \neq \emptyset$ and so $H_{1} \subseteq G$. Since $G$ contains no cycle from $\mathcal{C}_{s_{4}}$, we infer that no more edges can be added to $H_{1}$. Hence $G=H_{1}$.

Suppose that $\left|N_{G}(a) \cap N_{G}(c)\right|=1$, and let $b \in N_{G}(a) \cap N_{G}(c)$. If $d_{G}(b)=2$, then since $G$ does not contain $H_{2}$ and $H_{4}$ as a subgraph, a coloring $\varphi$ of $G$ with $\varphi(V(G) \backslash\{a, c\})=1$, $\varphi(a)=2$, and $\varphi(c)=3$ is an $S$-packing 3-coloring, a contradiction. Therefore, $d_{G}(b) \geq 3$. If $\left|N_{G}(b) \cap N_{G}(a)\right| \geq 2$ or $\left|N_{G}(b) \cap N_{G}(c)\right| \geq 2$, then $G \in\left\{K_{4}, H_{1}\right\}$ because $d_{G}(a) \geq 3$, $d_{G}(c) \geq 3$, and $a c \in E(G)$. If $N_{G}(b) \cap N_{G}(a)=c$ and $N_{G}(b) \cap N_{G}(c)=a$, then $H_{3} \subseteq G$ since $d_{G}(z) \geq 3$ for $z \in\{a, b, c\}$. Let $d, e, f$ be the three vertices of $H_{3}$ as shown in Fig. 2, If $a f \in E(G)$, then $H_{1} \subseteq G-d$, a contradiction. If $d f \in E(G)$, then $C_{5} \subseteq G-e$, again a contradiction. Therefore $G=H_{3}$.

Lastly, consider the case when $N_{G}(a) \cap N_{G}(c)=\emptyset$. If $d_{G}(w)=1$ for each $w \in\left(N_{G}(a) \cup\right.$ $\left.N_{G}(c)\right) \backslash\{a, c\}$, then a mapping $\varphi$ with $\varphi(V(G) \backslash\{a, c\})=1, \varphi(a)=2$, and $\varphi(c)=3$ is an $S$-packing 3 -coloring of $G$, contradicting the fact that $\chi_{S}(G)=4$. Let $x_{1} \neq x_{2} \in N_{G}(a) \backslash\{c\}$ and $y_{1} \neq y_{2} \in N_{G}(c) \backslash\{a\}$. Without loss of generality assume that $N_{G}\left(x_{1}\right) \backslash\{a\} \neq \emptyset$. If $x_{1} x_{2} \in E(G)$, then $H_{4} \subseteq G-y_{2}$, a contradiction. Hence $x_{1} x_{2} \notin E(G)$ and $y_{1} y_{2} \notin E(G)$ by symmetry. If $x_{1} y_{1} \in E(G)$, then $H_{5} \subseteq G$. Moreover, $H_{5}$ is a spanning subgraph of $G$ by Lemma 3.1. If $x_{1} y_{1} \in E(G)$ and $x_{2} y_{2} \in E(G)$, then $C_{6}$ is a proper subgraph of $G$, again a contradiction. If $x_{1} y_{1} \in E(G)$ and only one of $x_{1} y_{2}$ and $x_{2} y_{1}$ is contained in $G$, then $H_{7} \subseteq G$. Since there is no more edge which can be added to $H_{7}$, we get $G \in \mathcal{H}_{5,7}$
when $x_{1} y_{1} \in E(G)$. If there is a vertex $x_{1}^{\prime} \in N\left(x_{1}\right) \backslash\left\{a, x_{2}, y_{1}, y_{2}\right\}$, then $H_{14}$ is a spanning subgraph of $G$ by Lemma 3.11. If one of the edges $x_{1}^{\prime} a, x_{1} x_{2}$, and $y_{1} y_{2}$ is contained in $G$, there is a vertex $y \in V(G)$ such that $H_{4} \subseteq G-y$, a contradiction. If one of the edges $x_{1}^{\prime} c, x_{1} y_{1}, x_{1} y_{2}, y_{1} x_{2}$, and $x_{2} y_{2}$ is contained in $G$, there is a vertex $y \in V(G)$ such that $H_{5} \subseteq G-y$, a contradiction. If $x_{1}^{\prime} y_{i} \in E(G)$, then $C_{5} \subseteq G-x_{2}$, a contradiction. Since $a y_{i}, c x_{i} \notin E(G)$ for $i \in[2]$, only $x_{1}^{\prime} x_{2}$ can be possibly contained in $G$, and $H_{15} \subseteq G$ when $x_{1}^{\prime} x_{2} \in E(G)$. Moreover, since $H_{15}$ is $4-\chi_{S}$-vertex-critical, and there is no more edge can be added to $G$, we conclude that $G \in \mathcal{H}_{14,15}$ when $N\left(x_{1}\right) \backslash\left\{a, x_{2}, y_{1}, y_{2}\right\} \neq \emptyset$.

Theorems 3.16 and 3.15 are combined into the following final result of this paper.
Corollary 3.17 Let $S \in \mathcal{S}_{1,3,3}$. Then a graph $G$ is 4 - $\chi_{S}$-vertex-critical if and only if

$$
G \in\left\{K_{4}\right\} \cup \mathcal{H}_{1-5,7,14,15} \cup \mathcal{C}_{5} \cup \mathcal{C}_{6} \cup \mathcal{C}_{s_{4}} \cup\left\{G_{2 k}: k \geq 3\right\} .
$$

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