# A characterization of 4- $\chi_S$ -vertex-critical graphs for packing sequences with $s_1 = 1$ and $s_2 \geq 3$

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#### Abstract

If  $S=(s_1,s_2,\ldots)$  is a non-decreasing sequence of positive integers, then the S-packing k-coloring of a graph G is a mapping  $c:V(G)\to [k]$  such that if c(u)=c(v)=i for  $u\neq v\in V(G)$ , then  $d_G(u,v)>s_i$ . The S-packing chromatic number of G is the smallest integer k such that G admits an S-packing k-coloring. A graph G is  $\chi_S$ -vertex-critical if  $\chi_S(G-u)<\chi_S(G)$  for each  $u\in V(G)$ . If G is  $\chi_S$ -vertex-critical and  $\chi_S(G)=k$ , then G is k- $\chi_S$ -vertex-critical. In this paper, 4- $\chi_S$ -vertex-critical graphs are characterized for sequences  $S=(1,s_2,s_3,\ldots)$  with  $s_2\geq 3$ . There are 28 sporadic examples and two infinite families of such graphs.

**Keywords**: graph coloring; distance in graph; S-packing coloring; S-packing chromatic number; S-packing chromatic vertex-critical graph

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## 1 Introduction

A packing k-coloring of a graph G = (V(G), E(G)) is a mapping  $c : V(G) \to [k]$  such that if  $u \neq v$  and c(u) = c(v) = i, then  $d_G(u, v) > i$ . Here and later,  $d_G(u, v)$  denotes the length of a shortest u, v-path, and  $[k] = \{1, \ldots, k\}$ . The packing chromatic number,  $\chi_{\rho}(G)$ , of G is the smallest integer k such that G admits a packing k-coloring. This concept was proposed in [13]. The seminal paper was followed by [7], where the nowadays established name and notation was proposed. The development on the packing chromatic number up to 2020 has been summarized in the substantial survey [6]. Research into this concept is still flourishing, the developments after the survey include [1, 2, 5, 8, 10].

A more general concept is the S-packing coloring. Let  $S = (s_1, s_2, ...)$  be a non-decreasing sequence of positive integers; we will refer to S as a packing sequence. An S-packing k-coloring of G is a mapping  $c: V(G) \to [k]$  such that if  $u \neq v$  and c(u) = c(v) = i, then  $d_G(u, v) > s_i$ . For example, a (1, 1, 1, ...)-packing coloring is the standard proper vertex coloring, and if S = (1, 2, 3, ...), then it is just the packing coloring. The S-packing chromatic number,  $\chi_S(G)$ , of G is the smallest integer k such that G admits an S-packing k-coloring. This concept was introduced by Goddard and Xu [14]; for more results see [4, 11, 12, 15, 18, 19, 20].

If  $S_1 = (s_1^1, s_2^1, \ldots)$  and  $S_2 = (s_1^2, s_2^2, \ldots)$  are (packing) sequences with  $|S_1| = |S_2|$ , then  $S_2 \succeq S_1$  means the coordinate order, that is,  $S_2 \succeq S_1$  if  $s_i^2 \ge s_i^1$  for every  $i \in [|S_1|]$ . If  $S_2 \succeq S_1$  and G admits an  $S_2$ -packing k-coloring, then G also admits an  $S_1$ -packing k-coloring. In [11, Theorem 3.1], Gastineau proved the following appealing dichotomy result: If S is a packing sequence with |S| = 4, then the decision problem whether a given graph G admits an S-packing coloring is polynomial-time solvable if  $S \succeq S'$ , where  $S' \in \{(2,3,3,3), (2,2,3,4), (1,4,4,4), (1,2,5,6)\}$ , and NP-complete otherwise.

We have now arrived at the central concept of interest in this paper. A graph G is packing chromatic vertex-critical if  $\chi_{\rho}(G-u) < \chi_{\rho}(G)$  holds for each  $u \in V(G)$ . When  $\chi_{\rho}(G) = k$ , we more precisely say that G is k- $\chi_{\rho}$ -vertex-critical. More generally, if S is a packing sequence, then G is S-packing chromatic vertex-critical if  $\chi_{S}(G-u) < \chi_{S}(G)$  holds for each  $u \in V(G)$ , and if  $\chi_{S}(G) = k$ , then we say that G is k- $\chi_{S}$ -vertex-critical. We also add that a closely related concept of packing chromatic critical graphs, where the packing chromatic number strictly decreases on an arbitrary proper subgraph, has been studied in [3].

Packing chromatic vertex-critical graphs were introduced in [17]. Among other results,  $3-\chi_{\rho}$ -vertex-critical graphs were characterized and a partial characterization of  $4-\chi_{\rho}$ -vertex-critical graphs was provided. The latter characterization has been completed in [9]. In [16],  $3-\chi_{S}$ -vertex-critical graphs were characterized for all possible packing sequences, while  $4-\chi_{S}$ -vertex-critical graphs were characterized for packing sequences  $(s_1, s_2, s_3, ...)$  with  $s_1 \geq 2$ .

In this article we supplement the latter result by characterizing 4- $\chi_S$ -vertex-critical graphs for packing sequences with  $s_1 = 1$  and  $s_2 \geq 3$ . The result is given in Section 3, while in the next section we introduce some additional notation and list known properties of S-packing colorings needed here.

## 2 Preliminaries

If G is a graph, then we use n(G) to denote its order, diam(G) to denote its diameter, and  $\chi(G)$  to denote its chromatic number. For  $x \in V(G)$ , let  $N_G^i(x)$  be the set of vertices which are at distance i from x in G. In particular,  $N_G(x) = N_G^1(x)$  is the neighborhood of x. The degree of x is  $d_G(x) = |N_G(x)|$ . Let  $C_n$ ,  $P_n$ , and  $K_n$  denote the cycle, the path, and the complete graph on n vertices, respectively. A set  $A \subseteq V(G)$  is k-independent if A induces a subgraph that can be properly colored by k colors. Let  $\alpha_k(G)$  be the cardinality of a largest k-independent set of G.

If in a packing sequence the term i repeats  $\ell$  times, we may abbreviate the corresponding subsequence by  $i^{\ell}$ . For example, if  $S = (1, \ldots, 1, s_{\ell+1}, \ldots)$  (where clearly 1 appears  $\ell$  times), then we may shortly write  $S = (1^{\ell}, s_{\ell+1}, \ldots)$ . If  $\varphi : V(G) \to [k]$  is an S-packing k-coloring of G, then  $\varphi^{-1}(i)$ ,  $i \in [k]$ , is the set of vertices x with  $\varphi(x) = i$ . We will also use the following convention. Consider the vertex set  $V(G) = \{v_1, \ldots, v_n\}$  of G as an ordered set, and let  $\varphi$  be an S-packing coloring of G. Then we will explicitly describe  $\varphi$  as follows:  $\varphi = "\varphi(v_1) \cdots \varphi(v_n)"$ . Typically, the order of vertices will be alphabetic. For instance, if  $V(G) = \{a, b, c, d\}$ , and  $\varphi(a) = 1$ ,  $\varphi(b) = 2$ ,  $\varphi(c) = 1$ , and  $\varphi(d) = 3$ , then  $\varphi = "1 \ 2 \ 1 \ 3$ ".

We next recall some known results that will be needed in the rest.

**Proposition 2.1** [13] Let  $n \geq 3$ . If n = 3 or n = 4k,  $k \geq 1$ , then  $\chi_{\rho}(C_n) = 3$ ; otherwise  $\chi_{\rho}(C_n) = 4$ .

**Lemma 2.2** [14] If S is a packing sequence and H is a subgraph of G, then  $\chi_S(H) \leq \chi_S(G)$ .

**Proposition 2.3** [14] Let  $S = (1^{\ell}, s_{\ell+1}, \ldots)$ , where  $\ell \geq 1$  and  $s_{\ell+1} \geq 2$ , and let G be a graph. Then  $\chi_S(G) \leq n(G) - \alpha_{\ell}(G) + \min\{\ell, \chi(G)\}$  with equality if and only if  $\operatorname{diam}(G) \leq s_{\ell+1}$ .

**Lemma 2.4** [17] If S is a packing sequence and G is a  $\chi_S$ -vertex-critical graph, then G is connected.

Finally, the following notation will be useful. Suppose we wish to consider all the packing sequences  $S = (s_1, s_2, s_3, ...)$ , for which  $s_1 = 2$ ,  $s_2 \ge 4$ , and  $s_3 = 5$  hold. We will denote the set of all such packing sequences by  $\mathcal{S}_{2,\overline{4},5}$ , that is,

$$S_{2,\overline{4},5} = \{(s_1, s_2, s_3, \ldots) : s_1 = 2, s_2 \ge 4, s_3 = 5\}.$$

Note that since  $S_{2,\overline{4},5}$  is a set of packing sequences, we have  $s_2 \in \{4,5\}$  when  $S \in S_{2,\overline{4},5}$ . The general notation should be clear from this example. For instance, using this notation we can state that  $S \succeq (s_1, s_2, s_3, \ldots)$  if and only  $S \in S_{\overline{s}_1, \overline{s}_2, \overline{s}_3, \ldots}$ .

# 3 Vertex-critical graphs for different packing sequences

As mentioned in the introduction, a characterization of 3- $\chi_S$ -vertex-critical graphs is known for all possible packing sequences, while 4- $\chi_S$ -vertex-critical graphs were by now characterized for packing sequences from  $S_{\overline{2}}$ . In this section we supplement the latter result by characterizing 4- $\chi_S$ -vertex-critical graphs for packing sequences S from  $S_{1,\overline{3}}$ . To this end note that

$$\mathcal{S}_{1,\overline{3}} = \mathcal{S}_{1,\overline{4}} \cup \mathcal{S}_{1,3,\overline{4}} \cup \mathcal{S}_{1,3,3}$$
.

In view of this fact we will solve our problem by characterizing 4- $\chi_S$ -vertex-critical graphs for packing sequences from each of the sets  $S_{1,\bar{4}}$ ,  $S_{1,3,\bar{4}}$ , and  $S_{1,3,3}$ .

In Figs. 1 and 2, several graphs are drawn that will turn out to be 4- $\chi_S$ -vertex-critical for packing sequences from  $\mathcal{S}_{1,\overline{3}}$ . Fig. 1 contains two small families of graphs, the family  $\mathcal{C}_5$  contains four graphs of order 5, while  $\mathcal{C}_6$  contains three graphs of order 6. Fig. 2 displays the family of graphs  $\mathcal{H}$  consisting of graphs  $H_i$ ,  $i \in [15]$ .

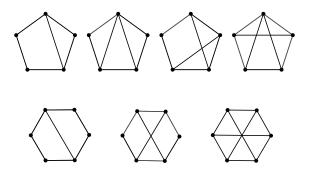


Figure 1: Family  $C_5$  (top row) and family  $C_6$  (bottom row)

In the rest we will frequently consider different subsets of  $\mathcal{H}$ . To shorten the presentation, we will specify subsets of  $\mathcal{H}$  by (ranges of) indices. For instance,  $\mathcal{H}_{1-3,7,9-11} = \{H_1, H_2, H_3, H_7, H_9, H_{10}, H_{11}\}$ .

First we detect the following critical graphs.

**Lemma 3.1** Let  $S \in \mathcal{S}_{1,\overline{3}}$ . Then each of the graphs from  $\mathcal{G} = \{K_4, C_5, C_6\} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{H}_{1-5,7}$  is  $4-\chi_S$ -vertex-critical.

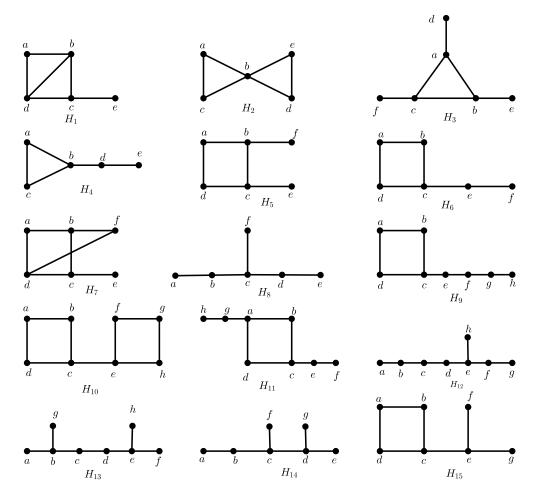


Figure 2: Family  $\mathcal{H} = \{H_i : i \in [15]\}$ 

**Proof.** Observe that for each  $G \in \mathcal{G}$ , diam $(G) \leq s_2$ . Using Proposition 2.3 it is then straightforward to check that  $\chi_S(G) = 4$  for each  $G \in \mathcal{G}$ . It remains to show that each graph  $G \in \mathcal{G}$  is 4- $\chi_S$ -vertex-critical.

By Proposition 2.3, we have  $\chi_S(K_3) = 3 - 1 + 1 = 3$ ,  $\chi_S(P_k) \le k - \alpha(P_k) + 1 \le 3$  for  $k \le 5$ ,  $\chi_S(G-x) = 4 - 2 + 1 = 3$  for any  $G \in \mathcal{C}_5$  and  $x \in V(G)$ , and  $\chi_S(G-x) = 5 - 3 + 1 = 3$  for any  $G \in \mathcal{C}_6$  and  $x \in V(G)$ . Therefore,  $K_4$ ,  $C_5$ ,  $C_6$ , each of the graphs from  $\mathcal{C}_5$ , and each of the graphs from  $\mathcal{C}_6$  are  $\chi_S$ -vertex-critical.

Now we prove that each graph in  $\mathcal{H}_{1-5,7}$  is  $4-\chi_S$ -vertex-critical where  $S \in \mathcal{S}_{1,\overline{3}}$ . First consider the case where  $S \in \mathcal{S}_{1,\overline{4}}$ . We give an S-packing 3-colorings  $\varphi$  for every G-x, where  $G \in \mathcal{H}_{1-5,7}$  and  $x \in V(G)$ . (By symmetry, we do not need to consider all the vertices.) Suppose  $G = H_1$ . Then we define  $\varphi$  as "1 2 3 1", "1 1 3 2", "1 2 3 2", "1 2 1 3" when x = a, b, c, e, respectively. Suppose  $G = H_2$ . Then we define  $\varphi$  as "2 1 1 3", when x = a

or x=b. Suppose  $G=H_3$ . Then we define  $\varphi$  as "2 3 1 1 1", "1 2 3 1 1" when x=a,d, respectively. Suppose  $G=H_4$ . Then we define  $\varphi$  as "3 1 1 2", "2 1 1 2", "2 1 3 2", "2 3 1 1" when x=a,b,d,e, respectively. Suppose  $G=H_5$ . Then we define  $\varphi$  as "1 2 1 1 3", "3 2 1 1 3", "3 1 2 1 1" when x=a,b,f, respectively. Suppose  $G=H_7$ . Then we define  $\varphi$  as "1 2 1 1 3", "1 2 3 1 1", "2 1 1 1 3", "1 2 1 3 1", when x=a,b,c,e, respectively. We have thus verified that each  $G\in\mathcal{H}_{1-5,7}$  is  $\chi_S$ -vertex-critical for  $S\in\mathcal{S}_{1\,\overline{4}}$ .

Finally suppose  $S \in \mathcal{S}_{1,3,\overline{4}} \cup \mathcal{S}_{1,3,3}$ . Let  $G \in \mathcal{H}_{1-5,7}$  and  $x \in V(G)$  be an arbitrary vertex. Since  $\chi_S(G) = 4$ , it suffices to show that  $\chi_S(G - x) = 3$ . Observe that for any packing sequence  $S \in \mathcal{S}_{1,3,\overline{4}} \cup \mathcal{S}_{1,3,3}$  there is a packing sequence  $S' \in \mathcal{S}_{1,\overline{4}}$  such that  $S' \succeq S$ . Thus, the above S'-packing 3-coloring of G - x, where  $G \in \mathcal{H}_{1-5,7}$  and  $x \in V(G)$ , yields an S-packing 3-coloring of G - x. Therefore, we are finished.

# 3.1 4- $\chi_S$ -vertex-critical graphs for $S \in \mathcal{S}_{1,\overline{4}} \cup \mathcal{S}_{1,3,\overline{4}}$

In this subsection we characterize 4- $\chi_S$ -vertex-critical graphs for  $S \in \mathcal{S}_{1,\overline{4}}$  and for  $S \in \mathcal{S}_{1,3,\overline{4}}$ . The results are given in Theorems 3.5, 3.6, and 3.7.

**Lemma 3.2**  $P_6$ ,  $H_6$ , and  $H_8$  are 4- $\chi_S$ -vertex-critical graphs for  $S \in \mathcal{S}_{1\,\overline{4}}$ .

# **Proof.** Let $S \in \mathcal{S}_{1,\overline{4}}$ .

First we prove that  $P_6$  is  $4-\chi_S$ -vertex-critical. Suppose that  $P_6 = abcdef$  has an S-packing 3-coloring  $\varphi$ . Since  $|\varphi^{-1}(1)| \leq 3$ , we have  $|\varphi^{-1}(2)| \geq 2$  or  $|\varphi^{-1}(3)| \geq 2$ . Since  $s_2 \geq 4$ , we must have  $\varphi(a) = \varphi(f) = \alpha \in \{2,3\}$ . Then at least three vertices of  $\{b,c,d,e\}$  must receive color 1, but this is impossible. The pattern "1 2 1 3 1 4" gives an S-packing 4-coloring of  $P_6$ , so  $\chi_S(P_6) = 4$ . By Proposition 2.3,  $\chi_S(P_k) \leq k - \alpha(P_k) + 1 \leq 3$  for  $k \leq 5$ . Hence,  $P_6$  is  $4-\chi_S$ -vertex-critical.

Now we prove that both  $H_6$  and  $H_8$  are  $4-\chi_S$ -vertex-critical. Observe that  $\alpha(H_6)=\alpha(H_8)=3$ . By Proposition 2.3, we have  $\chi_S(H_6)=\chi_S(H_8)=6-3+1=4$ . Now we give an S-packing 3-coloring  $\phi$  of G-x for  $G\in\{H_6,H_8\}$  and  $x\in V(G)$ . If  $G=H_6$ , then we define  $\phi$  as "1 2 1 1 3", "1 1 2 3 1", "2 1 1 1 3", "3 1 2 1 3", "3 1 2 1 1" when x=a,b,c,e,f, respectively. If  $G=H_8$ , then we define  $\phi$  as "1 2 1 3 1", "2 1 1 3 1", "2 1 1 3 1", "1 2 1 3 1" when x=a,b,c,f, respectively.

**Lemma 3.3** Each of the graphs from  $\{P_8, C_8\} \cup \mathcal{H}_{9,11-15}$  is  $4\text{-}\chi_S\text{-}vertex\text{-}critical for } S \in \mathcal{S}_{1,3,\overline{4}}$ .

#### **Proof.** Let $S \in \mathcal{S}_{1,3,\overline{4}}$ .

Suppose that  $P_8 = abcdefgh$  has an S-packing 3-coloring  $\varphi$ . Since  $|\varphi^{-1}(1)| \leq 4$ ,  $|\varphi^{-1}(2)| \leq 2$ , and  $|\varphi^{-1}(3)| \leq 2$ , we have  $|\varphi^{-1}(1)| = 4$ ,  $|\varphi^{-1}(2)| = 2$  and  $|\varphi^{-1}(3)| = 2$ .

Without loss of generality assume  $\varphi(a) = \varphi(c) = \varphi(e) = \varphi(g) = 1$ . Then we have c(b) = c(h) = 3 because  $s_3 \geq 4$ . Thus d and f must receive color 2, a contradiction. The pattern "1 2 1 3 1 2 1 4" gives an S-packing 4-coloring of  $P_8$ , so  $\chi_S(P_8) = 4$ . Since  $\chi_S(P_8) = 4$  and the pattern "1 2 1 3 1 2 1 4" gives an S-packing 4-coloring of  $C_8$ , we have  $\chi_S(C_8) = 4$ . The first k entries in the pattern "1 2 1 3 1 2 1" gives an S-packing 3-coloring of  $P_k$  with  $k \leq 7$ , so  $P_8$  and  $C_8$  are 4- $\chi_S$ -vertex-critical.

If  $G = H_9$ , then the pattern "2 1 3 1 1 2 1 4" is an S-packing 4-coloring of  $H_9$ , so  $\chi_S(H_9) \leq 4$ . Suppose that  $H_9$  has an S-packing 3-coloring  $\varphi$ . Observe that  $\{\varphi(a), \varphi(c)\} = \{2,3\}$ , which implies that  $\varphi(e) = \varphi(f) = 1$  or  $\varphi(g) = \varphi(h) = 1$ , a contradiction. Hence  $\chi_S(H_9) = 4$ . Now we give an S-packing 3-coloring  $\varphi$  of  $H_9 - x$  for any  $x \in V(H_9)$ . We define  $\varphi$  as "1 2 1 1 3 1 2", "2 1 1 1 2 1 3", "1 2 1 3 1 2 1", "2 1 3 1 2 1 3", "2 1 3 1 1 1 3", "2 1 3 1 1 2 3", "2 1 3 1 1 2 1" when x = a, c, d, e, f, g, h, respectively.

If  $G = H_{11}$ , then the pattern "2 1 3 1 1 2 1 4" is an S-packing 4-coloring, so  $\chi_S(H_{11}) \leq 4$ . Suppose that  $H_{11}$  admits an S-packing 3-coloring  $\varphi$ . Observe that  $\{\varphi(a), \varphi(c)\} = \{2, 3\}$ . Then  $\varphi(g) = \varphi(h) = 1$  or  $\varphi(e) = \varphi(f) = 1$ , a contradiction. Hence  $\chi_S(H_{11}) = 4$ . Now we give an S-packing 3-coloring  $\varphi$  of  $H_{11} - x$  for any  $x \in V(H_{11})$ . We define  $\varphi$  as "1 3 1 1 2 1 3", "1 3 1 2 1 2 1", "2 1 3 1 1 2 3", "2 1 3 1 1 2 1" when x = a, d, g, h, respectively.

If  $G = H_{12}$ , then the pattern "4 1 2 1 3 1 2 1" is an S-packing 4-coloring of  $H_{12}$ . Hence  $\chi_S(H_{12}) \leq 4$ . Suppose  $H_{12}$  admits an S-packing 3-coloring  $\varphi$ , then  $\varphi(e) = 2$  or 3. If  $\varphi(e) = 2$ , then we have  $\{\varphi(f), \varphi(g)\} = \{1,3\}$  and  $\varphi(d) = 1$ . Thus  $\varphi(c) \in \{2,3\}$ , a contradiction. If  $\varphi(e) = 3$ , then  $\varphi(d) = 1$ ,  $\varphi(c) = 2$ ,  $\varphi(b) = 1$ . Thus  $\varphi(a) \in \{2,3\}$ , a contradiction. Therefore  $\chi_S(H_{12}) = 4$ . Now we give an S-packing 3-coloring  $\varphi$  of  $H_{12} - x$  for any  $x \in V(H_{12})$ . We define  $\varphi$  as "1 2 1 3 1 2 1", "3 2 1 3 1 2 1", "3 1 1 3 1 2 1", "3 1 2 1 1 2 1", "2 1 3 1 2 2 1", "2 1 3 1 2 1 1", "1 2 1 3 1 2 1" when x = a, b, c, d, e, f, g, h, respectively.

If  $G = H_{13}$ , then the pattern "1 2 1 3 1 2 1 4" is an S-packing 4-coloring. Hence  $\chi_S(H_{13}) \leq 4$ . Suppose that  $H_{13}$  admits an S-packing 3-coloring  $\varphi$ , then  $\{\varphi(b), \varphi(e)\} = \{2,3\}$ . Then  $\varphi(c) = \varphi(d) = 1$ , a contradiction. Therefore  $\chi_S(H_{13}) = 4$ . Now we give an S-packing 3-coloring  $\varphi$  of  $H_{13} - x$  for any  $x \in V(H_{13})$ . We define  $\varphi$  as "1 3 1 2 1 1 1", "1 3 1 2 1 1 1", "2 1 3 1 2 1 1" when x = b, c, g, respectively.

If  $G = H_{14}$ , then the pattern "2 1 3 1 2 1 4" is an S-packing 4-coloring of  $H_{14}$ . Hence  $\chi_S(H_{14}) \leq 4$ . Suppose that  $H_{14}$  admits an S-packing 3-coloring  $\varphi$ , then  $\{\varphi(c), \varphi(d)\} = \{2,3\}$ . Thus  $\varphi(a) = \varphi(c) > 1$  or  $\varphi(a) = \varphi(d) > 1$ , a contradiction. Therefore  $\chi_S(H_{14}) = 4$ . Now we give an S-packing 3-coloring  $\varphi$  of  $H_{14} - x$  for any  $x \in V(H_{14})$ . We define  $\varphi$  as "1 2 3 1 1 1", "2 2 3 1 1 1", "1 2 3 1 1 1", "2 1 3 2 1 1", "1 2 1 3 1 1", "2 1 3 1 2 1", when x = a, b, c, d, f, g, respectively.

Finally, if  $G = H_{15}$ , then the pattern "4 1 2 1 3 1 1" is an S-packing 4-coloring of

 $H_{15}$ . Hence  $\chi_S(H_{15}) \leq 4$ . Suppose that  $H_{15}$  admits an S-packing 3-coloring  $\varphi$ , then  $\{\varphi(c), \varphi(e)\} = \{2,3\}$  and  $\varphi(b) = \varphi(d) = 1$ . Thus  $\varphi(a) \in \{2,3\}$ , a contradiction. Now we give an S-packing 3-coloring  $\varphi$  of  $H_{15} - x$  for any  $x \in V(H_{15})$ . We define  $\varphi$  as "1 2 1 3 1 1", "1 1 2 3 1 1", "2 1 1 3 1 1", "2 1 3 1 1", "2 1 3 1 1", when x = a, b, c, e, f, respectively.

**Lemma 3.4** If  $S \in \mathcal{S}_{1,\overline{4}} \cup \mathcal{S}_{1,3,\overline{4}}$ , G is a 4- $\chi_S$ -vertex-critical graph with at least one cycle, and C is a longest cycle of G, then the following hold.

- (a) If n(C) = 3, then  $G \in \mathcal{H}_{2-4}$ .
- (b) If n(C) = 4 and C contains a chord, then  $G \in \{K_4, H_1\}$ .
- (c) If  $n(C) \in \{5,6\}$ , then  $G \in \{C_{n(C)}\} \cup C_{n(C)}$ .

**Proof.** Let  $S \in \mathcal{S}_{1,\overline{4}} \cup \mathcal{S}_{1,3,\overline{4}}$ . Note that the graphs from Lemma 3.1 are 4- $\chi_S$ -vertex-critical. Let now G be a 4- $\chi_S$ -vertex-critical graph with a longest cycle C.

(a) Suppose n(C)=3. Let  $V(C)=\{a,b,c\}$ . We first assume that G contains only one triangle. If  $H_3$  or  $H_4$  is a subgraph of G, then we actually have  $G=H_3$  or  $G=H_4$ , for otherwise we find another triangle in G or a cycle longer than 3. If  $d_G(v)\geq 3$  holds for each vertex of C, then  $G=H_3$  since  $H_3$  is  $4\text{-}\chi_S$ -vertex-critical. If  $d_G(v)=2$  for some  $v\in\{a,b,c\}$ , then assume without loss of generality that  $d_G(a)=2$ . If  $N_G^2(b)\setminus N_G(c)=\emptyset$  and  $N_G^2(c)\setminus N_G(b)=\emptyset$ , then  $V(G)\setminus\{b,c\}$  is an independent set in G, and so a coloring  $\varphi$  with  $\varphi(b)=2$ ,  $\varphi(c)=3$  and other vertices with color 1 is an S-packing 3-coloring of G, a contradiction. So  $N_G^2(b)\setminus N_G(c)\neq\emptyset$  or  $N_G^2(c)\setminus N_G(b)\neq\emptyset$ . Since  $H_4$  is  $4\text{-}\chi_S$ -vertex-critical,  $G=H_4$ .

Suppose secondly that there are at least two triangles in G. Since  $H_4$  is  $\chi_S$ -vertex-critical, the triangles in G have exactly one common vertex, for otherwise there is vertex v in G such that  $H_4 \subseteq G - v$  or  $n(C) \ge 4$ . This implies that  $H_2$  is a spanning subgraph of G. Since n(C) = 3, we conclude that  $G = H_2$ .

- (b) Suppose n(C) = 4. Let C = abcda. If  $ac \in E(G)$  and  $bd \in E(G)$ , then  $G = K_4$  by Lemma 3.1. Suppose  $bd \in E(G)$ . If there is a vertex  $x \in N_G(b) \setminus V(C)$  such that  $N_G(x) \setminus V(C) \neq \emptyset$ , then  $H_4 \subseteq G a$ , a contradiction. Therefore, for any vertex  $x \in (N_G(b) \cup N_G(d)) \setminus V(C)$  we have  $N_G(x) \setminus V(C) = \emptyset$ . If  $d_G(a) = d_G(c) = 2$ , then  $V(G) \setminus \{b,d\}$  is an independent set in G, and so a mapping  $\varphi$  with  $\varphi(b) = 2$ ,  $\varphi(d) = 3$  and  $\varphi(N_G(b) \cup N_G(d) \setminus \{b,d\}) = 1$  is an S-packing 3-coloring of G, a contradiction. Thus  $d_G(a) \geq 2$  or  $d_G(c) \geq 2$ . It implies that  $H_1$  is a subgraph of G. Since n(C) = 4 and by Lemma 3.1  $H_1$  is  $4 \cdot \chi_S$ -vertex-critical, we have  $G = H_1$ .
- (c) Suppose finally that  $n(C) \in \{5,6\}$ . Since C is 4- $\chi_S$ -vertex-critical, C is a spanning subgraph of G. If n(C) = 5, then since  $K_4$  and all the four graphs from  $C_5$  are 4- $\chi_S$ -vertex-critical,  $C_5$  is the family of 4- $\chi_S$ -vertex-critical graphs that contain  $C_5$  as a proper spanning

subgraph. If n(C) = 6, then since  $C_5$  is  $4-\chi_S$ -vertex-critical, any two vertex at distance 2 are not adjacent in  $C_6$ . Hence  $C_6$  is the family of  $4-\chi_S$ -vertex-critical graphs that contain  $C_6$  as a proper spanning subgraph by Lemma 3.1. Therefore if  $n(C) \in \{5,6\}$  and G is  $4-\chi_S$ -vertex-critical, then  $G \in \{C_{n(C)}\} \cup C_{n(C)}$ .

We can now state out first characterization.

**Theorem 3.5** Let  $S \in \mathcal{S}_{1,\overline{4}}$ . Then a graph G is  $4-\chi_S$ -vertex-critical if and only if

$$G \in \{K_4, C_5, C_6, P_6\} \cup C_5 \cup C_6 \cup \mathcal{H}_{1-8}$$
.

**Proof.** Let  $S \in \mathcal{S}_{1,\overline{4}}$  and let G be 4- $\chi_S$ -vertex-critical. First suppose that G contains a cycle, and let C be a longest cycle of G. Since  $P_6$  is 4- $\chi_S$ -vertex-critical, Lemma 3.2 implies  $n(C) \leq 6$ . By Lemma 3.4 and the fact that  $\chi_S(C_4) = 3$ , it remains to consider the case in which n(C) = 4,  $n(G) \geq 5$ , and there is no chord in C. Let C = abcda. Since  $\chi_S(C) \leq 3$ , there is a vertex  $w \in V(C)$  such that  $N_G(w) \setminus V(C) \neq \emptyset$ . Let  $w_1 \in N_G(w) \setminus V(C)$ .

First suppose  $N_G(w_1) \setminus V(C) \neq \emptyset$ . We may assume that w = c and  $w_1 = e$ . Let  $f \in N_G(e) \setminus V(C)$ . Then  $H_6$  is subgraph of G. By Lemma 3.2,  $H_6$  is a spanning subgraph of G. Since G is  $C_k$ -free for  $k \geq 5$ , at most one of the edges  $\{ae, cf\}$  can be possibly contained in G. If  $ae \notin E(G)$  and  $cf \notin E(G)$ , then  $G = H_6$  by Lemma 3.2. If  $ae \in E(G)$ , then  $G = H_7$  by Lemma 3.1. If  $cf \in E(G)$ , then  $H_4 \subseteq G - b$ , a contradiction.

Thus we may assume that  $N_G(w_1) \setminus V(C) = \emptyset$  for each  $w_1 \in N_G(w) \setminus V(C)$ . It implies that  $N_G(u) \setminus V(C)$  is an independent set for any  $u \in V(C)$ . If  $N_G(b) \cup N_G(d) \setminus V(C) = \emptyset$ , then  $V(G) \setminus \{a,c\} = N(a) \cup N(c)$  is an independent set in G, and so a mapping  $\varphi$  with  $\varphi(a) = 2$ ,  $\varphi(c) = 3$  and  $\varphi(N(a) \cup N(c)) = 1$  is an S-packing 3-coloring of G, a contradiction. Thus  $N_G(b) \cup N_G(d) \setminus V(C) \neq \emptyset$  and  $N_G(a) \cup N_G(c) \setminus V(C) \neq \emptyset$ , and so  $H_5$  is a spanning subgraph of G by Lemma 3.1. If some edge from  $\{af, de\}$  or from  $\{ef, cf, be\}$  is contained in G, then  $H_4 \subseteq G - y$  for some  $y \in V(G)$  or  $C_k \subseteq G$  with  $k \geq 5$ , a contradiction. Therefore, only one of df and de can be contained in G, and so  $G \in \mathcal{H}_{5,7}$  by Lemma 3.1.

Suppose now that G is acyclic. If P is a longest path in G, then  $n(P) \leq 6$  by Lemma 3.2. If n(P) = 6, then  $G = P_6$ . If n(P) = 5, then let  $P_5 = abcde$ . If  $d_G(c) = 2$ , then the mapping  $\varphi$  with  $\varphi(b) = 2$ ,  $\varphi(d) = 3$  and  $\varphi(N_G(b) \cup N_G(d)) = 1$  is an S-packing 3-coloring of G which implies that  $\chi_S(G) \leq 3$ , a contradiction. Therefore  $d_G(c) \geq 3$ . But then  $G = H_8$  by Lemma 3.2. If  $n(P) \leq 4$ , then we have that  $\chi_S(G) \leq 3$ , so we get no new graph.

**Theorem 3.6** Let  $S \in \mathcal{S}_{1,3,\overline{4}}$  and let G be a graph with a cycle. Then G is  $4-\chi_S$ -vertex-critical if and only if

$$G \in \{K_4, C_5, C_6, C_8\} \cup C_5 \cup C_6 \cup \mathcal{H}_{1-5,7,9,11,15}$$
.

**Proof.** Let  $S \in \mathcal{S}_{1,3,\overline{4}}$  and let C be a longest cycle of G. Since  $P_8$  is  $4\text{-}\chi_S$ -vertex-critical,  $n \leq 8$ . If n(C) = 8, then C is a spanning subgraph of G by Lemma 3.3. Since  $\chi_S(C_5) = \chi_S(C_6) = 4$ , and  $\chi_S(C_7) = 7 - \alpha(C_7) + 1 > 4$  by Proposition 2.3, there is no chord in C. Therefore G = C when n(C) = 8. Since  $\chi_S(C_7) = 5$ , we have  $n(C) \neq 7$ . By Lemma 3.4 and the fact  $\chi_S(C_4) = 3$  it remains to consider the case that n(C) = 4,  $n(G) \geq 5$ , and there is no chord in C.

Let C = abcda. First suppose that there is an edge in E(C), say bc, such that  $d_G(b) \geq 3$  and  $d_G(c) \geq 3$ . Then  $N_G(b) \cap N_G(c) = \emptyset$  for otherwise G has a cycle of length at least 5. It follows that  $H_5$  is a spanning subgraph of G by Lemma 3.1 because there is no chord in G. If af or de is contained in G, then  $H_4 \subseteq G - y$  for some  $y \in V(G)$ . Hence at most one of df and df are can be added to G. Therefore  $G \in \mathcal{H}_{5,7}$  by Lemma 3.1.

Now consider the case in which  $d_G(s) = 2$  or  $d_G(t) = 2$  for each edge  $st \in E(C)$ . Without loss of generality, suppose that  $d_G(b) = 2$  and  $d_G(d) = 2$ . Let P be a longest path with endpoint c, such that  $a, b, d \notin V(P)$ , and let P' be a longest path with endpoint a, such that  $c, b, d \notin V(P')$ . Without loss of generality assume that  $n(P) \ge n(P')$ . If  $n(P) \ge 3$  and  $V(P) \cap V(P') \ne \emptyset$ , then  $G = H_7$ . Indeed, for otherwise by the definition of P and P' we have  $a, c \notin V(P) \cap V(P')$ , and then for some  $k \ge 5$  we have  $C_k \subseteq G - b$ , a contradiction. In the rest of the proof we may thus assume that if  $n(P) \ge 3$ , then  $V(P) \cap V(P') = \emptyset$ . Since  $P_8$  is  $4-\chi_S$ -vertex-critical,  $n(P) + n(P') \le 6$ .

Claim. If  $n(P) \leq 4$  and  $n(P') \leq 2$ , then  $G = H_{15}$ .

**Proof.** Since  $n(P') \leq 2$ , we infer that if  $x \in N_G(a) \setminus N_G(c)$  and  $y \in N_G(a) \cap N_G(c)$ , then  $d_G(x) = 1$  and  $d_G(y) = 2$ . Hence  $N_G(a)$  is an independent set in G. If there is a vertex  $x \in N_G(c) \setminus N_G(a)$  such that  $d_G(x) \geq 3$ , then  $H_{15} \subseteq G$ . Since  $H_{15}$  is  $4\text{-}\chi_S$ -vertex-critical by Lemma 3.3,  $H_{15}$  is a spanning subgraph of G. Since  $n(P') \leq 2$  and  $d_G(b) = d_G(d) = 2$ , only edges from  $\{fg, cf\}$  are possibly contained in G. If an edge from fg or cf is contained in G, then there is a vertex  $v \in G$  such that  $H_4 \subseteq G - v$ , a contradiction. Hence  $G = H_{15}$ . It remains to consider the case in which  $d_G(x) \leq 2$  holds for each  $x \in N_G(c)$ . Then  $N_G(c)$  is an independent set in G, for otherwise  $H_4 \subseteq G - b$ , a contradiction. Since  $n(P) \leq 4$  and  $d_G(x) \leq 2$  for every  $x \in N_G(c)$ , the second neighborhood  $N_G^2(c)$  is an independent set and  $d_G(y) = 1$  for each  $y \in N_G^3(c)$ . (It is possible that  $N_G^3(c) = \emptyset$ .) Then a mapping  $\varphi$  with  $\varphi(c) = 3$ ,  $\varphi(N_G(c) \cup N_G^3(c)) = 1$ , and  $\varphi(N_G^2(c)) = 2$  is an S-packing 3-coloring of G. This contradiction proves the claim.

It remains to consider the following two cases: (i) n(P) = 5, n(P') = 1, and (ii) n(P) = n(P') = 3. If n(P) = 5, then  $H_9$  is a spanning subgraph of G. (The vertices of  $H_9$  are denoted as in Fig. 2.) If some edge from  $\{cf, eg, fh\}$  or ch or cg is contained in G, then  $H_4 \subseteq G - b$  or  $C_5 \subseteq G - b$  or  $H_5 \subseteq G - b$ , respectively, a contradiction. Now we only need to check that whether eh can be added to  $H_9$ . The graph obtained from  $H_9$  by adding the

edge eh is  $H_{10}$ , cf. Fig. 2 again. Then  $H_{15} \subseteq H_{10} - g$ , a contradiction. Hence  $G = H_9$ . If n(P) = 3 and n(P') = 3, then  $H_{11} \subseteq G$ . By symmetry, if some edge from  $\{af, ge, gf\}$  or ah or ae is contained in G, then  $C_k \subseteq G - b$  with  $k \ge 5$  or  $H_4 \subseteq G - b$  or  $H_5 \subseteq G - b$ , respectively, a contradiction. Since  $H_{11}$  is  $4-\chi_S$ -vertex-critical, no additional edge can be added to  $H_{11}$ . We conclude that  $G = H_{11}$ .

It remains to consider acyclic graphs for  $S \in \mathcal{S}_{1,3,\overline{4}}$ .

**Theorem 3.7** Let  $S \in \mathcal{S}_{1,3,\overline{4}}$  and let G be an acyclic graph. Then G is 4- $\chi_S$ -vertex-critical if and only if  $G \in \{P_8\} \cup \mathcal{H}_{12-14}$ .

**Proof.** Let G be  $4-\chi_S$ -vertex-critical and acyclic. Denote by P a longest path in G. If n(P)=8, then Lemma 3.3 implies that  $G=P_8$ . Since  $\chi_S(G)\leq 3$  when  $n(P)\leq 4$ , it remains to consider the cases  $5\leq n(P)\leq 7$ .

Suppose n(P)=5 and let P=abcde. If  $d_G(c)=2$ , then a coloring  $\varphi$  with  $\varphi(\{c\}\cup N_G^2(c))=1$ ,  $\varphi(b)=2$ , and  $\varphi(d)=3$  is an S-packing 3-coloring of G, contradicting the fact that  $\chi_S(G)=4$ , hence  $d_G(c)\geq 3$ . If  $d_G(x)\leq 2$  for any  $x\in N_G(c)$ , then the coloring  $\varphi$  with  $\varphi(N_G(c))=1$ ,  $\varphi(N_G^2(c))=2$ , and  $\varphi(c)=3$  is an S-packing 3-coloring of G, a contradiction. So  $G=H_{14}$  by Lemma 3.3.

Suppose n(P)=6 and let P=abcdef. Then either  $d_G(s)=2$  or  $d_G(t)=2$  for  $st\in E(P)\setminus\{ab,ef\}$ , otherwise there is a vertex  $x\in V(G)$  such that  $H_{14}\subseteq G-x$ . If  $d_G(c)\geq 3$ , then a mapping  $\varphi$  with  $\varphi(N_G(c)\cup N_G^3(c))=1$ ,  $\varphi(N_G^2(c))=2$ , and  $\varphi(c)=3$  is an S-packing 3-coloring of G, a contradiction. Thus  $d_G(c)=d_G(d)=2$ . If  $d_G(b)=2$ , then a mapping  $\varphi$  with  $\varphi(N_G(e)\cup N_G^3(e))=1$ ,  $\varphi(a)=\varphi(e)=2$ , and  $\varphi(c)=3$  is an S-packing 3-coloring of G. Thus  $d_G(b)\geq 3$  and  $d_G(e)\geq 3$ . Hence  $G=H_{13}$  by Lemma 3.3.

Let finally P = abcdefg. If  $d_G(x) = 2$  for any  $x \in N_G(d)$ , a mapping  $\varphi$  with  $\varphi(N_G(d) \cup N_G^3(d)) = 1$ ,  $\varphi(N_G^2(d)) = 2$ , and  $\varphi(d) = 3$  is an S-packing 3-coloring of G contradicting the fact  $\chi_S(G) = 4$ . Hence  $G = H_{12}$  by Lemma 3.3.

Combining Theorem 3.7 with Theorem 3.6 we get:

Corollary 3.8 Let  $S \in \mathcal{S}_{1,3,\overline{4}}$  and let G be a graph. Then G is 4- $\chi_S$ -vertex-critical if and only if

$$G \in \{K_4, C_5, C_6, C_8, P_8\} \cup C_5 \cup C_6 \cup \mathcal{H}_{1-5,7,9,11-15}$$
.

#### 3.2 4- $\chi_S$ -vertex-critical graphs for $S \in \mathcal{S}_{1,3,3}$

In this subsection we consider packing sequences  $S \in \mathcal{S}_{1,3,3}$ . In Lemmas 3.9, 3.10, and 3.11, we present some graphs that are 4- $\chi_S$ -vertex-critical. After that we characterize 4- $\chi_S$ -vertex-critical graphs by distinguishing the distance between vertices of degree at least 3.

**Lemma 3.9** If  $S \in S_{1,3,3}$ , then the following hold.

- (a) If  $n \geq 4$ , then  $\chi_S(P_n) = 3$ .
- (b) Let  $n \geq 3$ . If n = 3 or  $n \equiv 0 \mod 4$ , then  $\chi_S(C_n) = 3$ . If  $n \equiv 1, 2 \mod 4$ , or  $n \equiv 3 \mod 4$  and  $s_4 < \lfloor n/2 \rfloor$ , then  $\chi_S(C_n) = 4$ ; otherwise,  $\chi_S(C_n) = 5$ . Moreover,  $C_n$  is  $\chi_S$ -vertex-critical when  $n \not\equiv 0 \mod 4$  and  $n \geq 5$ .
- **Proof.** (a) Note that  $\chi_S(P_n) \geq 3$  for  $n \geq 4$ . The pattern "1 2 1 3 1 2 1 3..." is an S-packing 3-coloring of  $P_n$ . Thus  $\chi_S(P_n) = 3$  for  $n \geq 4$ .
- (b) First,  $\chi_S(C_n) \geq 3$  for  $n \geq 3$ . The pattern "1 2 3" gives an S-packing 3-coloring of  $C_3$  and the pattern "1 2 1 3 1 2 1 3 . . . 1 2 1 3" gives an S-packing 3-coloring of  $C_n$  when  $n \equiv 0 \mod 4$ . Thus  $\chi_S(C_n) = 3$  when  $n \equiv 0 \mod 4$ .

Next, if  $n \geq 4$  and  $n \not\equiv 0 \mod 4$ , then since  $(1,3,3) \succeq (1,2,3)$ , we have  $\chi_S(C_n) \geq 4$  by Proposition 2.1. The pattern "1 2 1 3 1 2 1 3 . . . 1 2 1 3 4" gives an S-packing 4-coloring of  $C_n$  when  $n \equiv 1 \mod 4$  and the pattern "1 2 1 3 1 2 1 3 . . . 1 2 1 3 1 4" gives an S-packing 4-coloring of  $C_n$  when  $n \equiv 2 \mod 4$ . Thus  $\chi_S(C_n) = 4$  when  $n \equiv 1, 2 \mod 4$ .

Consider now the case  $n \equiv 3 \mod 4$ . When n = 4k + 3,  $n \ge 7$ , and  $s_4 < \lfloor n/2 \rfloor$ , we give an S-packing 4-coloring  $\varphi$  of  $C_n = v_0 v_1 \dots v_{n-1} v_0$  as:

$$\varphi(v_i) = \begin{cases} 1; & (i \equiv 0 \bmod 4) \text{ or } (i \equiv 2 \bmod 4 \text{ and } i \neq 4k+2), \\ 2; & (i \equiv 3 \bmod 4 \text{ and } i < 2k+1) \text{ or } (i \equiv 1 \bmod 4 \text{ and } i > 2k+1), \\ 3; & (i \equiv 1 \bmod 4 \text{ and } i < 2k+1) \text{ or } (i \equiv 3 \bmod 4 \text{ and } i > 2k+1), \\ 4; & i \in \{2k+1, 4k+2\}. \end{cases}$$

Hence  $\chi_S(C_n) = 4$  when  $n \equiv 3 \mod 4$ ,  $n \ge 7$ , and  $s_4 < \lfloor n/2 \rfloor$ .

When n=4k+3,  $n\geq 7$ , and  $s_4\geq \lfloor n/2\rfloor$ , the pattern "1 2 1 3 1 2 1 3...1 2 1 3 1 4 5" is an S-packing 5-coloring of  $C_n$ . Hence  $4\leq \chi_S(C_n)\leq 5$ . Now suppose that there is an S-packing 4-coloring  $\varphi$  of  $C_n$ . Since  $s_4\geq \lfloor n/2\rfloor$ , we have  $|\varphi^{-1}(4)|=1$ . Without loss of generality we may assume that  $\varphi(v_0)=4$ . We claim that for any edge in  $C_n$  which is not incident with  $v_0$ , one of its endpoints receives color 1. Suppose on the contrary that there is an edge  $v_iv_{i+1}\in E(C_n)$  such that  $\{\varphi(v_i),\varphi(v_{i+1})\}=\{2,3\}$ , where  $1\leq i\leq n-2$ . Since  $n\geq 7$ , one of  $v_{i-2}$  and  $v_{i+3}$  (indices taken modulo n) cannot be colored under  $\varphi$ . This contradiction proves the claim. Since  $s_2=s_3=3$ , we only need to consider two cases:  $\varphi(v_1)=2$  and  $\varphi(v_1)=1$ . If  $\varphi(v_1)=2$ , then the colors of  $v_0,v_1,\ldots,v_{4k+2}$  under  $\varphi$  can be described as the pattern "4 2 1 3 1 2 1 3 1 ... 2 1 3 1 2 1". We have  $\varphi(v_1)=\varphi(v_{4k+1})=2$  with  $d_{C_n}(v_1,v_{4k+1})=3\leq s_2$ , a contradiction. If  $\varphi(v_1)=1$ , then we may without loss of generality assume  $\varphi(v_2)=2$ . Then the colors of  $v_0,v_1,\ldots,v_{4k+2}$  under  $\varphi$  can be described as the pattern "4 1 2 1 3 1 2 1 3 ... 1 2 1 3 1 2". However, we have  $\varphi(v_2)=\varphi(v_{4k+2})=2$ 

with  $d_{C_n}(v_2, v_{4k+2}) = 3 \le s_2$ , a contradiction. Therefore  $\chi_S(C_n) = 5$  when  $n \equiv 3 \mod 4$ ,  $n \ge 7$ , and  $s_4 \ge \lfloor n/2 \rfloor$ .

If  $n \not\equiv 0 \mod 4$ , then  $C_n$  is  $\chi_S$ -vertex-critical because  $\chi_S(P_n) \leq 3$  and  $\chi_S(C_n) \geq 4$ .

Let  $G_{2k}$ ,  $k \geq 3$ , be the graph obtained from the path  $P_{2k}$  by attaching a pendent vertex to each of the two support vertices of  $P_{2k}$ . Equivalently,  $G_{2k}$  is obtained from  $P_{2k-2}$  by attaching two pendant vertices to each of the two leaves of  $P_{2k-2}$ .

**Lemma 3.10** If  $S \in S_{1,3,3}$  and  $k \geq 3$ , then  $G_{2k}$  is 4- $\chi_S$ -vertex-critical.

**Proof.** Let  $P_{2k} = v_1 v_2 \dots v_{2k}$ , and let  $v_2'$  and  $v_{2k-1}'$  be the pendent vertices attached to  $v_2$  and  $v_{2k-1}$ , respectively. Coloring the vertices of  $P_{2k}$  with the pattern "1 2 1 3 1 2 1 3..." and the vertices  $v_2'$  and  $v_{2k-1}'$  with 1 and 4, respectively, we get  $\chi_S(G_{2k}) \leq 4$ .

Suppose now that  $G_{2k}$  admits an S-packing 3-coloring  $\varphi$ . Observe that  $\varphi(v_2) \in \{2,3\}$ , without loss of generality assume that  $\varphi(v_2) = 2$ . Then we have  $\varphi(v_1) = 1$  and  $\varphi(v_3) = 1$ , for otherwise  $\varphi(v_3) = \varphi(v_4) = 1$  or  $\varphi(v_4) = \varphi(v_5) = 1$ . If  $2 \le i \le 2k - 2$ , then at least one of  $v_i$  and  $v_{i+1}$  must be colored 1. Indeed, if we would have  $\varphi(v_i) = 2$  and  $\varphi(v_{i+1}) = 3$ , then  $v_{i-2}$  or  $v_{i+3}$  can not be colored under  $\varphi$ . Thus we have  $\varphi(v_{2k-2}) = 2$  and  $\varphi(v_{2k-1}) = 1$ , or  $\varphi(v_{2k-2}) = 3$  and  $\varphi(v_{2k-1}) = 1$ . However, this implies that  $v'_{2k-1}$  or  $v_{2k}$  can not be colored under  $\varphi$ , a contradiction. Hence,  $\chi_S(G_{2k}) = 4$ .

If  $v \in G_{2k}$ , then an S-packing 3-coloring of  $G_{2k} - v$  can be given by coloring a longest path of each component of  $G_{2k} - v$  with either the pattern "1 2 1 3 ..." or the pattern "2 1 3 1 ..." and coloring the pendent vertices with 1. Therefore  $G_{2k}$  is 4- $\chi_S$ -vertex-critical.

**Lemma 3.11** If  $S \in S_{1,3,3}$ , then the graphs  $H_{14}$  and  $H_{15}$  are 4- $\chi_S$ -vertex-critical.

**Proof.** Since  $H_{14}$  and  $H_{15}$  are  $4-\chi_{S'}$ -vertex-critical, where  $S' \in \mathcal{S}_{1,3,\overline{4}}$ , and  $(1,3,4) \succeq (1,3,3)$ , by Corollary 3.8 it suffices to show that  $\chi_S(H_{14}) = \chi_S(H_{15}) = 4$ . The pattern "4 1 2 3 1 1" is an S-packing 4-coloring of  $H_{14}$ , so  $\chi_S(H_{14}) \leq 4$ . Suppose that  $H_{14}$  admits an S-packing 3-coloring  $\varphi$ . Then  $\{\varphi(c), \varphi(d)\} = \{2,3\}$ , and so the vertex a cannot be colored under  $\varphi$ . It follows that  $\chi_S(H_{14}) = 4$ . The pattern "4 1 2 1 3 1 1" is an S-packing 4-coloring of  $H_{15}$ . Moreover, since  $H_{14} \subseteq H_{15}$ , by Lemma 2.2, we conclude that  $\chi_S(H_{15}) = 4$ .

Our next result, Theorem 3.14, follows from the following lemma and theorem.

**Lemma 3.12** Let  $S \in \mathcal{S}_{1,3,3}$  and let  $n \not\equiv 0 \mod 4$ , n > 3. If a graph G contains a cycle  $C_n$  and  $V(G) - V(C_n) \neq \emptyset$ , then G is not 4- $\chi_S$ -vertex-critical.

**Proof.** Since  $V(G) - V(C_n) \neq \emptyset$ , there exists a vertex  $x \in V(G)$  such that  $C_n \subseteq G - x$ . By Lemma 2.2, we have  $\chi_S(G - x) \geq \chi_S(C_n) \geq 4$ , and so G is not 4- $\chi_S$ -vertex-critical.

**Theorem 3.13** [17, Theorem 4.3] If G is a graph that contains a cycle of length  $n \geq 5$ , where  $n \not\equiv 0 \mod 4$ , then G is  $4-\chi_{\rho}$ -vertex-critical if and only if one of the following holds.

- $n = 5 \text{ and } G \in \{C_5\} \cup C_5$ ,
- n = 6 and  $G \in \{C_6\} \cup C_6$ ,
- n > 7 and G is isomorphic to  $C_n$ .

**Theorem 3.14** Let  $S \in \mathcal{S}_{1,3,3}$ . If G is a graph that contains a cycle of length  $n \geq 5$ , where  $n \not\equiv 0 \mod 4$ , then G is  $4-\chi_S$ -vertex-critical if and only if one of the following holds.

- n = 5 and  $G \in \{C_5\} \cup C_5$ ,
- n = 6 and  $G \in \{C_6\} \cup C_6$ ,
- $n \ge 7$  and  $G = C_n$  except when  $n \equiv 3 \mod 4$  and  $s_4 \ge \lfloor n/2 \rfloor$ .

In order to characterize  $4-\chi_S$ -vertex-critical graphs, where  $S \in \mathcal{S}_{1,3,3}$ , we need to distinguish whether there are two vertices of degree at least 3 that are at odd distance. For this sake we need the following classes of cycles that depend on a positive integer  $s_4$  (this  $s_4$  will, of course, be the fourth component of a packing sequence S):

$$C_{s_4} = \{C_n, n \ge 5 : (n \equiv 1, 2 \mod 4) \text{ or } (n \equiv 3 \mod 4 \text{ and } s_4 < \lfloor n/2 \rfloor \}.$$

**Theorem 3.15** Let  $S \in \mathcal{S}_{1,3,3}$  and let G be a 4- $\chi_S$ -vertex-critical graph. If all the vertices of G of degree at least 3 are pairwise at even distances in G, then  $G \in \{H_2, H_4\} \cup \mathcal{C}_{s_4}$ .

**Proof.** By Lemmas 3.9 and 3.1, every graph from  $\{H_2, H_4\} \cup \mathcal{C}_{s_4}$  is  $4\text{-}\chi_S$ -vertex-critical. If  $\Delta(G) \leq 2$ , then  $G \in \{P_n, C_n\}$ , hence  $G \in \mathcal{C}_{s_4}$ . Suppose now that  $\Delta(G) \geq 3$  and that all the vertices of degree at least 3 are pairwise at even distances in G. Let  $u \in V(G)$  be an arbitrary vertex of degree at least 3. Then define  $\varphi : V(G) \to [3]$  by:

$$\varphi(v) = \begin{cases} 1; & d_G(u, v) \equiv 1, 3 \mod 4, \\ 2; & d_G(u, v) \equiv 0 \mod 4, \\ 3; & d_G(u, v) \equiv 2 \mod 4. \end{cases}$$

By Lemma 2.4, G is a connected graph, and so  $\varphi$  is well-defined. Since G is  $4-\chi_S$ -vertex-critical, there are two vertices  $x,y \in V(G) \setminus \{u\}$  such that  $\varphi(x) = \varphi(y) = i$  and  $d_G(x,y) \le s_i$  for some  $i \in [3]$ . Let P and P' be arbitrary shortest u,x-path and u,y-path in G, respectively. Let  $w \in V(P) \cap V(P')$  such that  $d_G(u,w)$  is as large as possible. Then we have  $w \neq x,y$  and  $d_G(u,x) = d_G(u,y)$ . If w = x or  $d_G(u,x) < d_G(u,y)$ , then  $d_G(u,y) \le d_G(u,y)$ , then  $d_G(u,y) \le d_G(u,y)$ .

 $d_G(u,x) + s_i < d_G(u,x) + 4$ , and so  $\varphi(x) \neq \varphi(y)$  by the definition of  $\varphi$ , which leads to a contradiction. Thus we have  $d_G(w) \geq 3$ , and so  $\varphi(w) \in \{2,3\}$ .

**Claim.** G contains a cycle consisting of wPx, wP'y, and a shortest x, y-path.

**Proof.** Let P'' be a shortest x, y-path. By the choice of w, it suffices to show that  $(V(P'') \cap V(P)) \setminus \{x\} = (V(P'') \cap V(P')) \setminus \{y\} = \emptyset$ .

Suppose that  $(V(P'') \cap V(P)) \setminus \{x\} \neq \emptyset$ . Note that this can only happen when  $\varphi(x) = \varphi(y) \in \{2,3\}$  and  $d_G(x,y) \in \{2,3\}$ . Without loss of generality, let  $\varphi(x) = \varphi(y) = 3$ . If  $d_G(x,y) = 2$ , let P'' = xzy. Then  $d_G(z) \geq 3$  and  $d_G(u,z) = d_G(u,x) - 1 \equiv 1 \mod 4$ , contradicting the fact that the vertices of degree at least 3 are at even distance. For  $d_G(x,y) = 3$ , let  $P'' = xz_1z_2y$ . Then  $xz_2 \notin E(G)$ . If  $z_2 \in V(P'') \cap V(P)$ , then  $d_G(u,y) \leq d_G(u,z_2) + d_G(z_2,y) = d_G(u,x) - 2 + 1 < d_G(u,x)$ , contradicting the fact that  $d_G(u,x) = d_G(u,y)$ . Therefore  $z_2 \notin V(P'') \cap V(P)$ . However, if  $z_1 \in V(P'') \cap V(P)$ , then  $d_G(z_1) \geq 3$  and  $d_G(u,z_1) = d_G(u,x) - 1 \equiv 1 \mod 4$ , a contradiction. Therefore G contains a cycle, say  $C_n$ , consisting of wPx, wP'y and a shortest x,y-path, and so  $n = 2d_G(u,x) - 2d_G(u,w) + d_G(x,y)$ .

Suppose  $\varphi(x) = \varphi(y) = 1$  and  $d_G(x,y) = 1$ . Then we have  $n \equiv 3 \mod 4$ . If  $n \equiv 3 \mod 4$  and n > 3, then  $G = C_n$  with  $s_4 < \lfloor n/2 \rfloor$  by Theorem 3.14. If n = 3, then wxy is a triangle with  $d_G(w) \ge 3$  and  $d_G(x) = d_G(y) = 2$ . If  $N_G^2(w) \ne \emptyset$ , then  $G = H_4$ . If  $N_G^2(w) = \emptyset$  and  $N_G(w) \setminus \{x,y\}$  is an independent set, then a mapping  $\varphi$  with  $\varphi(N_G(w) \setminus \{x,y\}) = \varphi(x) = 1$ ,  $\varphi(w) = 2$  and  $\varphi(y) = 3$  is an S-packing 3-coloring of G. Moreover, it is easy to see that no edge can be added to  $H_2$  and to  $H_4$ . Hence  $G \in \{H_2, H_4\}$  when n = 3. Suppose  $\varphi(x) = \varphi(y) = 2$  or  $\varphi(x) = \varphi(y) = 3$ . If  $d_G(x,y) = i$ ,  $i \in [3]$ , then  $n \equiv i \mod 4 \ge 5$  because  $d_G(u,x)$  and  $d_G(u,w)$  are even. Therefore  $G \in \mathcal{C}_{s_4}$  by Theorem 3.14.

**Theorem 3.16** Let  $S \in S_{1,3,3}$  and let G be a 4- $\chi_S$ -vertex-critical graph in which there exist two vertices of degree at least 3 that are at odd distance. Then

$$G \in \{K_4\} \cup \mathcal{H}_{1,3,5,7,14,15} \cup \{G_{2k}: k \ge 3\} \cup \mathcal{C}_5 \cup \mathcal{C}_6$$
.

**Proof.** Let  $u, v \in V(G)$  with  $d_G(u), d_G(v) \geq 3$  such that  $d_G(u, v) = \ell$  is odd and as small as possible. We consider the following two cases.

#### Case 1: $\ell \geq 3$ .

By the choice of u and v,  $N_G(u) \cap N_G(v) = \emptyset$  and each vertex on a shortest u, v-path has degree 2 in G. Therefore  $G_{\ell+3} = G_{2k}$  is a spanning subgraph of G, where  $k = \frac{\ell+3}{2}$  by Lemma 3.10. Let the vertices of  $G_{2k}$  be denoted as in Lemma 3.10 with  $u = v_2$  and  $v = v_{2k-1}$ . By symmetry, only some of the edges  $v_1v_2'$ ,  $v_1v_{2k}$ , and  $v_1v_{2k-1}$  can possibly be added to  $G_{2k}$ . If  $v_1v_2' \in E(G)$ , then  $H_4 \subseteq G - v_{2k}$ , a contradiction. If  $v_1v_{2k} \in E(G)$ , then

 $C_{2k} \subseteq G$ . Further, we have  $2k \equiv 0 \mod 4 \geq 8$ , for otherwise  $\chi_S(C_{2k}) = 4$  by Theorem 3.14 and  $C_{2k} \subseteq G - v_2'$ . Then we can find a copy of  $G_6$  consisting of  $v_3v_2v_1v_2kv_2k-1v_2k-2$  and the pendent vertices  $v_2'$  and  $v_{2k-1}'$  contained in  $G - v_4$ , which also leads to a contradiction. If  $v_1v_{2k-1} \in E(G)$ , then there is a cycle  $C_{2k-1} \subseteq G - v_2'$  with  $2k-1 \not\equiv 0 \pmod 4 > 3$ , again a contradiction. We conclude that in Case 1,  $G = G_{2k}$ .

#### Case 2: $\ell = 1$ .

We claim that in this case,  $G \in \{K_4\} \cup \mathcal{H}_{1,3,5,7,14,15} \cup \mathcal{C}_5 \cup \mathcal{C}_6$ .

If G contains a cycle C from  $C_{s_4}$ , then by Theorem 3.14, C is a spanning subgraph of G. Since there are two vertices of degree at least 3 which are of distance 1 in G, we have  $G \in C_5 \cup C_6$ . Suppose G contains  $H_4$  as a subgraph. Then  $H_4$  is a spanning subgraph of G. Since G has two vertices of degree at least 3,  $G \in C_5 \cup \{H_1\}$ . By the same argument, if G contains  $H_2$  as a spanning subgraph, then  $G \in C_5$ . Therefore we may assume G does not contain a graph from  $C_{s_4} \cup \{H_2, H_4\}$  as a subgraph. Let G = u and G = v.

Suppose that  $|N_G(a) \cap N_G(c)| \geq 2$ . If  $d_G(x) = 2$  for any  $x \in N_G(a) \cap N_G(c)$ , then  $V(G) \setminus \{a, c\}$  is an independent set in G since G does not contain  $H_2$  or  $H_4$  as a subgraph, and so a coloring  $\varphi$  of G with  $\varphi(V(G) \setminus \{a, c\}) = 1$ ,  $\varphi(a) = 2$  and  $\varphi(c) = 3$  is an S-packing 3-coloring, a contradiction. Then either  $bd \in E(G)$  for some  $b, d \in N_G(a) \cap N_G(c)$  and so  $G = K_4$ , or there is a vertex  $x \in N_G(a) \cap N_G(c)$  such that  $N_G(x) \setminus (\{a, c\} \cup V(N_G(a) \cap N_G(c))) \neq \emptyset$  and so  $H_1 \subseteq G$ . Since G contains no cycle from  $C_{s_4}$ , we infer that no more edges can be added to  $H_1$ . Hence  $G = H_1$ .

Suppose that  $|N_G(a) \cap N_G(c)| = 1$ , and let  $b \in N_G(a) \cap N_G(c)$ . If  $d_G(b) = 2$ , then since G does not contain  $H_2$  and  $H_4$  as a subgraph, a coloring  $\varphi$  of G with  $\varphi(V(G) \setminus \{a,c\}) = 1$ ,  $\varphi(a) = 2$ , and  $\varphi(c) = 3$  is an S-packing 3-coloring, a contradiction. Therefore,  $d_G(b) \geq 3$ . If  $|N_G(b) \cap N_G(a)| \geq 2$  or  $|N_G(b) \cap N_G(c)| \geq 2$ , then  $G \in \{K_4, H_1\}$  because  $d_G(a) \geq 3$ ,  $d_G(c) \geq 3$ , and  $ac \in E(G)$ . If  $N_G(b) \cap N_G(a) = c$  and  $N_G(b) \cap N_G(c) = a$ , then  $H_3 \subseteq G$  since  $d_G(z) \geq 3$  for  $z \in \{a,b,c\}$ . Let d,e,f be the three vertices of  $H_3$  as shown in Fig. 2. If  $af \in E(G)$ , then  $H_1 \subseteq G - d$ , a contradiction. If  $df \in E(G)$ , then  $C_5 \subseteq G - e$ , again a contradiction. Therefore  $G = H_3$ .

Lastly, consider the case when  $N_G(a) \cap N_G(c) = \emptyset$ . If  $d_G(w) = 1$  for each  $w \in (N_G(a) \cup N_G(c)) \setminus \{a,c\}$ , then a mapping  $\varphi$  with  $\varphi(V(G) \setminus \{a,c\}) = 1$ ,  $\varphi(a) = 2$ , and  $\varphi(c) = 3$  is an S-packing 3-coloring of G, contradicting the fact that  $\chi_S(G) = 4$ . Let  $x_1 \neq x_2 \in N_G(a) \setminus \{c\}$  and  $y_1 \neq y_2 \in N_G(c) \setminus \{a\}$ . Without loss of generality assume that  $N_G(x_1) \setminus \{a\} \neq \emptyset$ . If  $x_1x_2 \in E(G)$ , then  $H_4 \subseteq G - y_2$ , a contradiction. Hence  $x_1x_2 \notin E(G)$  and  $y_1y_2 \notin E(G)$  by symmetry. If  $x_1y_1 \in E(G)$ , then  $H_5 \subseteq G$ . Moreover,  $H_5$  is a spanning subgraph of G by Lemma 3.1. If  $x_1y_1 \in E(G)$  and  $x_2y_2 \in E(G)$ , then  $C_6$  is a proper subgraph of G, again a contradiction. If  $x_1y_1 \in E(G)$  and only one of  $x_1y_2$  and  $x_2y_1$  is contained in G, then  $H_7 \subseteq G$ . Since there is no more edge which can be added to  $H_7$ , we get  $G \in \mathcal{H}_{5,7}$ 

when  $x_1y_1 \in E(G)$ . If there is a vertex  $x_1' \in N(x_1) \setminus \{a, x_2, y_1, y_2\}$ , then  $H_{14}$  is a spanning subgraph of G by Lemma 3.11. If one of the edges  $x_1'a$ ,  $x_1x_2$ , and  $y_1y_2$  is contained in G, there is a vertex  $y \in V(G)$  such that  $H_4 \subseteq G - y$ , a contradiction. If one of the edges  $x_1'c$ ,  $x_1y_1$ ,  $x_1y_2$ ,  $y_1x_2$ , and  $x_2y_2$  is contained in G, there is a vertex  $y \in V(G)$  such that  $H_5 \subseteq G - y$ , a contradiction. If  $x_1'y_i \in E(G)$ , then  $C_5 \subseteq G - x_2$ , a contradiction. Since  $ay_i, cx_i \notin E(G)$  for  $i \in [2]$ , only  $x_1'x_2$  can be possibly contained in G, and  $H_{15} \subseteq G$  when  $x_1'x_2 \in E(G)$ . Moreover, since  $H_{15}$  is  $4-\chi_S$ -vertex-critical, and there is no more edge can be added to G, we conclude that  $G \in \mathcal{H}_{14,15}$  when  $N(x_1) \setminus \{a, x_2, y_1, y_2\} \neq \emptyset$ .

Theorems 3.16 and 3.15 are combined into the following final result of this paper.

Corollary 3.17 Let  $S \in S_{1,3,3}$ . Then a graph G is 4- $\chi_S$ -vertex-critical if and only if

$$G \in \{K_4\} \cup \mathcal{H}_{1-5,7,14,15} \cup \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_{s_4} \cup \{G_{2k} : k \ge 3\}.$$

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# References

- [1] E. Bidine, T. Gadi, M. Kchikech, Independence number and packing coloring of generalized Mycielski graphs, Discuss. Math. Graph Theory 41 (2021) 725–747.
- [2] D. Božović, I. Peterin, A note on the packing chromatic number of lexicographic products, Discrete Appl. Math. 293 (2021) 34–37.
- [3] B. Brešar, J. Ferme, Graphs that are critical for the packing chromatic number, Discuss. Math. Graph Theory 42 (2022) 569–589.
- [4] B. Brešar, J. Ferme, K. Kamenická, S-packing colorings of distance graphs  $G(\mathbb{Z}, \{2, t\})$ , Discrete Appl. Math. 298 (2021) 143–154.
- [5] B. Brešar, N. Gastineau, O. Togni, Packing colorings of subcubic outerplanar graphs, Aequationes Math. 94 (2020) 945–967.
- [6] B. Brešar, J. Ferme, S. Klavžar, D.F. Rall, A survey on packing colorings, Discuss. Math. Graph Theory 40 (2020) 923–970.
- [7] B. Brešar, S. Klavžar, D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, Discrete Appl. Math. 155 (2007) 2303–2311.
- [8] F. Deng, Z. Shao, A. Vesel, On the packing coloring of base-3 Sierpiński graphs and *H*-graphs, Aequationes Math. 95 (2021) 329–341.
- [9] J. Ferme, A characterization of  $4-\chi_{\rho}$ -(vertex-)critical graphs, Filomat 36 (2022) 6481–6501.
- [10] J. Fresán-Figueroa, D. González-Moreno, M. Olsen, On the packing chromatic number of Moore graphs, Discrete Appl. Math. 289 (2021) 185–193.

- [11] N. Gastineau, Dichotomies properties on computational complexity of S-packing coloring problems, Discrete Math. f338 (2015) 1029–1041.
- [12] N. Gastineau, O. Togni, On S-packing edge-colorings of cubic graphs, Discrete Appl. Math. 259 (2019) 63–75.
- [13] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, Broadcast chromatic numbers of graphs, Ars Combin. 86 (2008) 33–49.
- [14] W. Goddard, H. Xu, The S-packing chromatic number of a graph, Discuss. Math. Graph Theory 32 (2012) 795–806.
- [15] W. Goddard, H. Xu, A note on S-packing colorings of lattices, Discrete Appl. Math. 166 (2014) 255–262.
- [16] P. Holub, M. Jakovac, S. Klavžar, S-packing chromatic vertex-critical graphs, Discrete Appl. Math. 285 (2020) 119–127.
- [17] S. Klavžar, D.F. Rall, Packing chromatic vertex-critical graphs, Discrete Math. Theor. Comput. Sci. 21(3) (2019) paper #8, 18 pp.
- [18] A. Kostochka, X. Liu, Packing (1,1,2,4)-coloring of subcubic outerplanar graphs, Discrete Appl. Math. 302 (2021) 8–15.
- [19] R. Liu, X. Liu, M. Rolek, G. Yu, Packing (1,1,2,2)-coloring of some subcubic graphs, Discrete Appl. Math. 283 (2020) 626–630.
- [20] W. Yang, B. Wu, On packing S-colorings of subcubic graphs, Discrete Appl. Math. 334~(2023)~1-14.