Relating the total domination number and the annihilation number for quasi-trees and some composite graphs

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Abstract

The total domination number $\gamma_t(G)$ of a graph G is the cardinality of a smallest set $D \subseteq V(G)$ such that each vertex of G has a neighbor in D. The annihilation number a(G) of G is the largest integer k such that there exist k different vertices in G with the degree sum at most m(G). It is conjectured that $\gamma_t(G) \leq a(G) + 1$ holds for every nontrivial connected graph G. The conjecture has been proved for graphs with minimum degree at least 3, trees, certain tree-like graphs, block graphs, and cactus graphs. In the main result of this paper it is proved that the conjecture holds for quasi-trees. The conjecture is verified also for some graph constructions including bijection graphs, Mycielskians, and the newly introduced universally-identifying graphs.

Keywords: total domination number; annihilation number; quasi-trees; bijection graph; Mycielskian

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1 Introduction

Let G = (V(G), E(G)) be a simple graph. If $v \in V(G)$, then $N_G(v)$ is its neighborhood and $d_G(v) = |N_G(v)|$ its degree. $D \subseteq V(G)$ is a *total dominating set* if each vertex from V(G) has a neighbor in D. The minimum cardinality of a total dominating set is the *total domination number*, $\gamma_t(G)$, of G. For an in-depth information on the total domination see the excellent book [9].

Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the (ordered) degree sequence of a graph G. The annihilation number, a(G), of G is the largest integer k such that $\sum_{i=1}^k d_i \leq m(G)$, where m(G) is the size of G. (The order of G will be denoted by n(G).) In other words, the annihilation number is the largest integer k such that $\sum_{i=1}^k d_i \leq \sum_{i=k+1}^n d_i$. This concept was introduced by Pepper in [17]. In this paper we are interested in the following conjecture posed in a somewhat different form in Graffiti.pc [6] and later reformulated by Desormeaux, Haynes, and Henning in [7, Question 1]. (To be historically correct we emphasize that in [7] this conjecture was posed as a problem, but nowadays it is called a conjecture and we follow this naming.)

Conjecture 1.1 ([6, 7]). If G is a connected graph with $n(G) \ge 2$, then $\gamma_t(G) \le a(G) + 1$.

It immediately follows from the definition of the annihilation number that $a(G) \ge \lfloor \frac{n(G)}{2} \rfloor$. Since it was proved in [2] that if the minimum degree $\delta(G)$ of G is at least 3, then $\gamma_t(G) \le \lfloor \frac{n(G)}{2} \rfloor$ (see also [20] for this result and [10] for its generalization), Conjecture 1.1 holds for graphs G with minimum degree $\delta(G) \ge 3$. In the seminal paper [7], the conjecture was verified for trees, while recently Bujtás and Jakovac verified it for cactus graphs and for block graphs [3]. In [22], the conjecture was further verified for the so-called C-disjoint graphs and for generalized theta graphs. A graph G is a quasi-tree if there exists a vertex $x \in V(G)$ such that G - x is a tree. Clearly, any tree is also a quasi-tree since it remains to be a tree after any leaf in it is removed. We say that a quasi-tree G is non-trivial if G is not a tree. Since Conjecture 1.1 holds for all trees, we only consider non-trivial quasi-trees throughout this paper. The main result of this paper reads as follows.

Theorem 1.2. If G is a non-trivial quasi-tree, then $\gamma_t(G) \leq a(G) + 1$.

We pose an open problem to characterize the quasi-trees G satisfying the equality $\gamma_t(G) = a(G) + 1$. We add that a conjecture parallel to 1.1 has been posed also for the 2-domination number of a graph G. In [8] (see also [13]), the latter conjecture was verified for trees, and in [11] for block graphs. In [23], the conjecture was disproved by demonstrating that the 2-domination number can be arbitrarily larger than the annihilation number. However, the counterexamples presented are far from being counterexamples for Conjecture 1.1 and the authors say that they are "inclined to believe that Conjecture 1.1 holds true." The annihilation number was compared with the Roman domination number in [1] and with the locating-total domination number in [16].

We proceed as follows. In the rest of this section we list some further definitions needed. In Section 2, a proof of Theorem 1.2 is given, while in the final section we confirm the validity of Conjecture 1.1 for several graph operations which also generate graphs that have vertices of degree at most 2.

Let G be a graph. Then $S \subseteq V(G)$ is an annihilation set if $\sum_{v \in S} d_G(v) \leq m(G)$. S is an optimal annihilation set if |S| = a(G) and $\max\{d_G(v)|v \in S\} \leq \min\{d_G(u)|u \in V(G)\setminus S\}$. A total dominating set of cardinality $\gamma_t(G)$ is called a γ_t -set of G. By G[S] we denote the subgraph of G induced by all vertices in $S \subseteq V(G)$. For a subset $S \subseteq V(G)$, we define $\sum(S,G) = \sum_{v \in S} d_G(v)$. The path and the cycle of order n are respectively denoted by P_n and C_n . A generalized theta graph Θ_{s_1,\ldots,s_k} is formed by taking a pair of vertices u, v and joining them by k internally disjoint paths of lengths s_1, \ldots, s_k , where $k \geq 3$. In particular, Θ_{s_1,s_2,s_3} is said to be a theta graph. Let C be the set containing all cycles, cliques and generalized theta graphs. We say a connected graph F is C-disjoint if any two subgraphs from C in F have no edge in common. C-disjoint graphs form a natural generalization of trees, cactus graphs, and block graphs. From our perspective, the most important thing is that Yue, Zhu, and Wei [22] proved Conjecture 1.1 for all C-disjoint graphs.

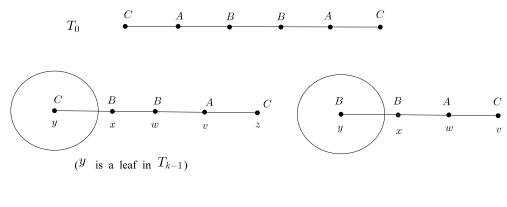
For further graph terminology and notation not defined here see [21]. Finally, for a positive integer n we use the convention $[n] = \{1, \ldots, n\}$.

2 Proof of Theorem 1.2

We start by recalling the definition of a class of labeled trees Γ due to Chen and Sohn [5]. The label of a vertex v will be called its *status* and denoted by sta(v). The labels in the definition are needed in order to be able to describe the family. But since in principle we are only interested in unlabeled graphs, we may later consider Γ (as well as its subclasses) also as a class of unlabeled trees.

Definition 2.1. Let Γ be the family of labeled trees $T = T_k$ that can be obtained as follows. Let T_0 be a P_6 in which the two leaves have status C, the two support vertices have status A and the remaining two vertices have status B. If $k \ge 1$, then T_k can be obtained recursively from T_{k-1} by one of the following operations, cf. Fig. 1.

- Operation o_1 . For any $y \in V(T_{k-1})$, if sta(y) = C and y is a leaf of T_{k-1} , then add a path xwvz and edge xy. Set sta(x) = sta(w) = B, sta(v) = A, and sta(z) = C.
- Operation o_2 . For any $y \in V(T_{k-1})$, if sta(y) = B, then add a path xwv and edge xy. Set sta(x) = B, sta(w) = A, and sta(v) = C.





Operation o_2

Figure 1: The labelled tree T_0 , and the two operations that define Γ

The class of labeled trees Γ is thus the smallest class of labeled trees that contains T_0 and can be built from it by successive applications of operations o_1 and o_2 . Desormeaux, Haynes, and Henning [7] proved that a tree T of order at least 3 satisfies $\gamma_t(T) = a(T) + 1$ if and only if T belongs to Γ (considered as unlabeled trees). Since P_2 also satisfies this equality, we thus have:

Theorem 2.2 ([7]). If T is a tree, then $\gamma_t(T) \leq a(T) + 1$. Moreover, the equality holds if and only if $T \in \Gamma \cup \{P_2\}$.

If G is a quasi-tree and $x \in V(G)$ such that G - x is a tree, then we say that x is a quasi-vertex. We now partition quasi-trees into two classes as follows.

Definition 2.3. A quasi-tree G is type-1 if it contains a quasi-vertex x, such that $G - x \in \Gamma$. Otherwise, G is type-2.

Let $\Gamma_1 \subset \Gamma$ be the class of trees from Γ for which in their construction, at the last step operation o_1 was performed. Similarly, $\Gamma_2 \subset \Gamma$ is be the class of trees from Γ for which in their construction, at the last step operation o_2 was performed. Moreover, if $T \in \Gamma_1$, then the vertices x, w, v, z will be the vertices added in the last step (cf. Fig. 1), and if $T \in \Gamma_2$, then the vertices x, w, v will be the vertices added in the last step (cf. Fig. 1). With this agreement we define the following six subclasses of type-1 quasi-trees; see Fig. 2 for their schematic presentation.

- QT_1 contains quasi-trees G obtained from a tree $T \in \Gamma_1$ and an isolated vertex h by adding $t \ge 2$ edges between h and $V(T) \setminus \{x, w, v, z\}$.
- QT_2 contains quasi-trees G obtained from a tree $T \in \Gamma_2$ and an isolated vertex h by adding $t \ge 2$ edges between h and $V(T) \setminus \{x, w, v\}$.
- QT_3 contains quasi-trees G obtained from a tree $T \in \Gamma_1$ and an isolated vertex h by adding $t \ge 2$ edges between h and $\{x, w, v, z\}$.
- QT_4 contains quasi-trees G obtained from a tree $T \in \Gamma_2$ and an isolated vertex h by adding $t \ge 2$ edges between h and $\{x, w, v\}$.
- QT_5 contains quasi-trees G obtained from a tree $T \in \Gamma_1$ and an isolated vertex h by adding $t_1 \ge 1$ edges between h and $\{x, w, v, z\}$, and $t_2 \ge 1$ edges between h and $V(T) \setminus \{x, w, v, z\}$.

• QT_6 contains quasi-trees G obtained from a tree $T \in \Gamma_2$ and an isolated vertex h by adding $t_1 \ge 1$ edges between h and $\{x, w, v\}$, and $t_2 \ge 1$ edges between h and $V(T) \setminus \{x, w, v\}$.

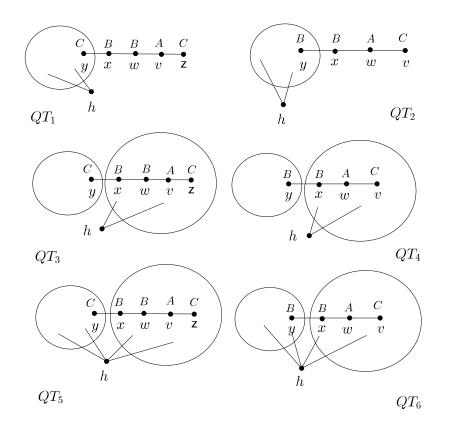


Figure 2: Type-1 quasi-trees from classes QT_1, \ldots, QT_6

In the following series of six lemmas we respectively consider type-1 quasi-trees from classes QT_1, \ldots, QT_6 .

Lemma 2.4. Let $G \in QT_1$ and let $G' = G[V(G) \setminus \{x, w, v, z\}]$. If $\gamma_t(G') \le a(G') + 1$, then $\gamma_t(G) \le a(G) + 1$.

Proof. Let D' be a γ_t -set of G'. Then $D = D' \cup \{w, v\}$ is a total dominating set of G. So, $\gamma_t(G) \le |D' \cup \{w, v\}| = |D'| + 2 = \gamma_t(G') + 2$. Next, we prove that $a(G) \ge a(G') + 2$.

Clearly, m(G) = m(G') + 4. Let S be an optimal annihilation set of G'. Then $\sum(S, G') \le m(G')$. If $y \notin S$, then $\sum(S \cup \{v, z\}, G) = \sum(S, G') + 3 \le m(G') + 3 < m(G)$. Thus, if $y \notin S$, then $a(G) \ge |S \cup \{v, z\}| = |S| + 2 = a(G') + 2$. If $y \in S$, then we set

 $S^* = (S \setminus \{y\}) \cup \{w, v, z\}. \text{ As } d_{G'}(y) \ge 1, \text{ we have } \sum(S^*, G) = \sum(S, G') - d_{G'}(y) + d_G(w) + d_G(v) + d_G(z) \le \sum(S, G') - 1 + 5 \le m(G') + 4 = m(G). \text{ Thus, also if } y \in S, \text{ we have the conclusion } a(G) \ge |S^*| = |S| + 2 = a(G') + 2.$

In conclusion, if $\gamma_t(G') \leq a(G') + 1$, then $\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1$. \Box

Lemma 2.5. Let $G \in QT_2$ and let $G' = G[V(G) \setminus \{x, w, v\}]$. If $\gamma_t(G') \le a(G') + 1$, then $\gamma_t(G) \le a(G) + 1$.

Proof. Let D' be a γ_t -set of G'. Then $D = D' \cup \{x, w\}$ is a total dominating set of G. So, $\gamma_t(G) \le |D' \cup \{x, w\}| = |D'| + 2 = \gamma_t(G') + 2$. Next, we prove that $a(G) \ge a(G') + 2$.

Clearly, m(G) = m(G') + 3. Let *S* be an optimal annihilation set of *G'*. Then $\sum(S, G') \leq m(G')$. If $y \notin S$, then $\sum(S \cup \{w, v\}, G) = \sum(S, G') + 3 \leq m(G') + 3 = m(G)$. Thus, $a(G) \geq |S \cup \{w, v\}| = |S| + 2 = a(G') + 2$. If $y \in S$, then we set $S^* = (S \setminus \{y\}) \cup \{x, w, v\}$. Since $d_{G'}(y) \geq 2$, we have $\sum(S^*, G) = \sum(S, G') - d_{G'}(y) + d_G(x) + d_G(w) + d_G(v) = \sum(S, G') - d_{G'}(y) + 5 \leq m(G') - 2 + 5 = m(G)$. Thus, $a(G) \geq |S^*| = |S| + 2 = a(G') + 2$.

Hence, when $\gamma_t(G') \leq a(G') + 1$, we have $\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1$. \Box

Lemma 2.6. If $G \in QT_3$, then $\gamma_t(G) \leq a(G) + 1$.

Proof. Let $G' = G[V(G) \setminus \{h, x, w, v, z\}]$. Then $G' \in \Gamma$. By Theorem 2.2, we have $\gamma_t(G') = a(G') + 1$. Let D' be a γ_t -set of G' and let $d_G(h) = t$. Then m(G) = m(G') + t + 4, where $2 \le t \le 4$.

If t = 2, then G is a unicyclic graph. So G is a cactus graph and by the validity of Conjecture 1.1 for cacti due to Bujtás and Jakovac [3] we have $\gamma_t(G) \leq a(G) + 1$. Assume in the rest that $t \in \{3, 4\}$. Since $D = D' \cup \{w, v\}$ is a total dominating set of G, we have $\gamma_t(G) \leq |D' \cup \{w, v\}| = |D'| + 2 = \gamma_t(G') + 2$. It remains to prove that $a(G) \geq a(G') + 2$.

Let S be an optimal annihilation set of G'. Then $\sum(S, G') \leq m(G')$. Since y is a pendent vertex in G', we must have $y \in S$ by the definition of optimal annihilation sets. Let $S^* = (S \setminus \{y\}) \cup \{z, w, v\}$. Since $d_{G'}(y) = 1$, we have $\sum(S^*, G) = \sum(S, G') - d_{G'}(y) + d_G(z) + d_G(w) + d_G(v) = \sum(S, G') - 1 + 5 + t \leq m(G') + 4 + t = m(G)$. Thus $a(G) \geq |S^*| = |S| + 2 = a(G') + 2$ and we are done.

Lemma 2.7. If $G \in QT_4$, then $\gamma_t(G) \leq a(G) + 1$.

Proof. Let $G' = G[V(G) \setminus \{h, x, w, v\}]$. Then $G' \in \Gamma$. By Theorem 2.2 we have $\gamma_t(G') = a(G') + 1$. Let D' be a γ_t -set of G'. Since $t \ge 2$, $D = D' \cup \{x, w\}$ is a total dominating set of G. So, $\gamma_t(G) \le |D' \cup \{x, w\}| = |D'| + 2 = \gamma_t(G') + 2$. Hence the lemma will be proved by showing that $a(G) \ge a(G') + 2$.

If $d_G(h) = t$, then m(G) = m(G') + t + 3, where $t \in \{2,3\}$. Let S be an optimal annihilation set of G'. Then $\sum(S, G') \leq m(G')$. If $y \notin S$, then $\sum(S \cup \{w, v\}, G) = \sum(S, G') + d_G(w) + d_G(v) \leq m(G') + 3 + t = m(G)$. Thus, $a(G) \geq |S \cup \{w, v\}| = |S| + 2 = a(G') + 2$. If $y \in S$, then set $S^* = (S \setminus \{y\}) \cup \{x, w, v\}$. Since $d_{G'}(y) \geq 2$, we have $\sum(S^*, G) = \sum(S, G') - d_{G'}(y) + d_G(x) + d_G(w) + d_G(v) = \sum(S, G') - d_{G'}(y) + 5 + t \leq m(G') - 2 + 5 + t = m(G)$. Thus, $a(G) \geq |S^*| = |S| + 2 = a(G') + 2$.

Lemma 2.8. Let $G \in QT_5$ and let $G' = G[V(G) \setminus \{x, w, v, z\}]$. If $\gamma_t(G') \le a(G') + 1$, then $\gamma_t(G) \le a(G) + 1$.

Proof. Let D' be a γ_t -set of G'. Then $D = D' \cup \{w, v\}$ is a total dominating set of G. So, $\gamma_t(G) \le |D' \cup \{w, v\}| = |D'| + 2 = \gamma_t(G') + 2$.

Let $d_G(h) = t$ and recall that $t = t_1 + t_2$ by the definition of QT_5 . Then $m(G) = m(G') + t_1 + 4$, where $t_1 \in [t - 1]$. Let S be an optimal annihilation set of G'. Then $\sum (S, G') \leq m(G')$.

If $h \notin S$ and $y \notin S$, then $\sum (S \cup \{v, z\}, G) = \sum (S, G') + d_G(v) + d_G(z) \leq m(G') + 3 + t_1 < m(G)$. Thus, $a(G) \geq |S \cup \{v, z\}| = |S| + 2 = a(G') + 2$. If $h \notin S$ and $y \in S$, then set $S^* = (S \setminus \{y\}) \cup \{z, w, v\}$. As $d_{G'}(y) \geq 1$, we have $\sum (S^*, G) = \sum (S, G') - d_{G'}(y) + d_G(z) + d_G(w) + d_G(v) \leq \sum (S, G') - 1 + 5 + t_1 \leq m(G') + 4 + t_1 = m(G)$. Thus, $a(G) \geq |S^*| = |S| + 2 = a(G') + 2$. By our assumption that $\gamma_t(G') \leq a(G') + 1$, $\gamma_t(G) \leq a(G) + 1$.

Now, we consider the case when $h \in S$.

If $t_2 \ge 2$, since $d_{G'}(h) = t_2$ and y may belong to S, then $\sum ((S \setminus \{h\}) \cup \{w, v, z\}, G) \le \sum (S, G') - d_{G'}(h) + 1 + d_G(w) + d_G(v) + d_G(z) \le m(G') - 2 + 1 + 5 + t_1 = m(G)$. Thus, $a(G) \ge |(S \setminus \{h\}) \cup \{w, v, z\}| = |S| + 2 = a(G') + 2$. By our assumption that $\gamma_t(G') \le a(G') + 1, \gamma_t(G) \le a(G) + 1$.

Now, we suppose that $t_2 = 1$. If $t_1 = 1$, then G is a unicyclic graph. So G is a cactus graph and by the validity of Conjecture 1.1 for cacti due to Bujtás and Jakovac [3] we have $\gamma_t(G) \leq a(G) + 1$. If $t_1 = 2$, then G is a C-disjoint graph, since it can be obtained by planting trees to vertices of a theta graph. Hence, by the validity of Conjecture 1.1 for *C*-disjoint graphs due to Yue et al. [22], we have $\gamma_t(G) \leq a(G) + 1$. So, we may assume that $t_1 \geq 3$.

Let $G'' = G - \{h, x, w, v, z\}$. Then $m(G'') + 5 + t_1 = m(G)$. Let S'' be an optimal annihilation set of G''. Then $\sum(S'', G'') \leq m(G'')$. Obviously, $G'' \in \Gamma$. By Theorem 2.2, we have $\gamma_t(G'') = a(G'') + 1$. Let D'' be a γ_t -set of G''. Since $t_1 \geq 3$, then $D'' \cup \{w, v\}$ is a total dominating set of G. Then $\gamma_t(G) \leq |D'' \cup \{w, v\}| = \gamma_t(G'') + 2$. Now, we prove that $a(G) \geq a(G'') + 2$. Since y and the unique neighbor of h in G'' may belong to S'', we have $\sum(S'' \cup \{v, z\}, G) \leq \sum(S'', G'') + 2 + d_G(v) + d_G(z) \leq m(G'') + 2 + t_1 + 3 = m(G)$. Thus, $a(G) \geq |S'' \cup \{v, z\}| = |S''| + 2 = a(G'') + 2$. Therefore, $\gamma_t(G) \leq a(G) + 1$.

Lemma 2.9. Let $G \in QT_6$. Then $\gamma_t(G) \leq a(G) + 1$.

Proof. Let $d_G(h) = t$ and recall that $t = t_1 + t_2$ by the definition of QT_6 . Let $G' = G - \{h, x, w, v\}$. Then $G' \in \Gamma$. By Theorem 2.2, we have $\gamma_t(G') = a(G') + 1$. Let D' be a γ_t -set of G'. Then either $D = D' \cup \{x, w\}$ or $D = D' \cup \{w, v\}$ is a total dominating set of G. So, $\gamma_t(G) \leq |D' \cup \{x, w\}| = |D'| + 2 = \gamma_t(G') + 2$ or $\gamma_t(G) \leq |D' \cup \{w, v\}| = |D'| + 2 = \gamma_t(G') + 2$. To complete the argument we next again prove that $a(G) \geq a(G') + 2$.

Obviously, $m(G) = m(G') + t_1 + t_2 + 3$, where $t_1, t_2 \in [t-1]$. Let S be an optimal annihilation set of G'. Then $\sum (S, G') \leq m(G')$.

If $y \notin S$, then $\sum (S \cup \{w, v\}, G) \leq \sum (S, G') + t_2 + d_G(w) + d_G(v) \leq m(G') + t_2 + t_1 + 3 = m(G)$. Thus, $a(G) \geq |S \cup \{w, v\}| = |S| + 2 = a(G') + 2$. If $y \in S$, since $d_{G'}(y) \geq 2$, then $\sum ((S \setminus \{y\}) \cup \{x, w, v\}, G) \leq \sum (S, G') - d_{G'}(y) + t_2 + d_G(x) + d_G(w) + d_G(v) \leq m(G') - 2 + t_2 + t_1 + 5 = m(G') + t_1 + t_2 + 3 = m(G)$. Thus, $a(G) \geq |(S \setminus \{y\}) \cup \{x, w, v\}| = |S| + 2 = a(G') + 2$, and we are done.

With the above lemmas in hand we are now in a position to prove Theorem 1.2. If G is itself a tree, then the result follows from Theorem 2.2. Assume hence that G contains at least one cycle. If n(G) = 3, then $G = C_3$ for which $\gamma_t(C_3) = 2 = a(C_3) + 1$ holds. We may thus assume in the rest that $n(G) \ge 4$. We consider the following two cases.

Case 1: G is a type-2 quasi-tree.

Let h be a quasi-vertex of G and let $d_G(h) = t$. Since G has at least one cycle, we have $t \ge 2$. Since G is a type-2 quasi-tree, $G - h \notin \Gamma$. Let S be an optimal annihilation set of

G-h. As

$$\sum(S, G) \le \sum(S, G - h) + t \le m(G - h) + t = m(G)$$

we have $a(G) \ge |S| = a(G - h)$. We further consider the following two subcases.

Case 1.1: There exists a γ_t -set D of G - h such that $N_G(h) \cap D \neq \emptyset$.

In this case, D is also a total dominating set of G. So, $\gamma_t(G) \leq |D| = \gamma_t(G-h)$. Since G-h is a tree, by Theorem 2.2, we have

$$\gamma_t(G) \le \gamma_t(G-h) \le a(G-h) + 1 \le a(G) + 1.$$

Case 1.2: For each γ_t -set D of G - h we have $N_G(h) \cap D = \emptyset$.

Let $\mathcal{S}(G-h)$ and $\mathcal{L}(G-h)$ be the set of support vertices and the set of pendent vertices of G-h, respectively. Since G-h is a tree, each vertex $u \in \mathcal{S}(G-h)$ belongs to every γ_t -set of G-h. By our assumption that $N_G(h) \cap D = \emptyset$ for each γ_t -set D of G-h, we have $\mathcal{S}(G-h) \cap N_G(h) = \emptyset$. We now claim that $\gamma_t(G) \leq \gamma_t(G-h) + 1$.

First, assume that there exists a pendent vertex x in $\mathcal{L}(G-h)$ such that $x \in N_G(h)$. Let D be a γ_t -set of G-h. Then $D \cup \{x\}$ is a total dominating set of G, as the unique neighbor of x in G-h is a support vertex belonging to D. So, $\gamma_t(G) \leq |D \cup \{x\}| = \gamma_t(G-h) + 1$. Second, assume that for any pendent vertex x of G-h we have $x \notin N_G(h)$. Then there exists a vertex y in $V(G-h) \setminus (\mathcal{S}(G-h) \cup \mathcal{L}(G-h))$ such that $y \in N_G(h)$, as $t \geq 2$. Since D is a γ_t -set of G-h, there exists a neighbor of y, say z, such that $z \in D$. Then $D \cup \{y\}$ is a total dominating set of G. Hence $\gamma_t(G) \leq |D \cup \{y\}| = \gamma_t(G-h) + 1$ and the claim is proved.

Since $G - h \notin \Gamma$ and $G - h \not\cong P_2$, Theorem 2.2 implies that $\gamma_t(G - h) \leq a(G - h)$. So,

$$\gamma_t(G) \le \gamma_t(G-h) + 1 \le a(G-h) + 1 \le a(G) + 1$$
,

which completes the argument for type-2 quasi-trees.

Case 2: G is a type-1 quasi-tree.

Let h be a quasi-vertex of G, such that $G - h \in \Gamma$. To prove that $\gamma_t(G) \leq a(G) + 1$ we use induction on $f(G) = n(G) + m(G) + n_1(G)$, where $n_1(G)$ is the number of leaves of G.

Let $d_G(h) = t$. Since G has at least one cycle, $t \ge 2$. Since $G - h \in \Gamma$, we have $f(G) = n(G) + m(G) + n_1(G) \ge (1+6) + (5+t) + 0 \ge (1+6) + (5+2) + 0 = 14$

with equality only if n(G) = m(G) = 7 and $n_1(G) = 0$. (Recall that a quasi-tree G is connected.) So the base case of our induction is f(G) = 14, in which case we have $G \cong C_7$. As $\gamma_t(C_7) = 4 = 3 + 1 = a(C_7) + 1$, the desired result holds for the base case.

Suppose now that G is a type-1 quasi-tree with $f(G) \ge 15$. Since $G - h \in \Gamma$, we must have $G \in QT_i$ for some $i \in [6]$.

If $G \in QT_1$, set $G' = G[V(G) \setminus \{x, w, v, z\}]$. Then G' is a type-1 quasi-tree, as $G' - h \in \Gamma$. Obviously, f(G') < f(G). So, by the induction hypothesis, $\gamma_t(G') \le a(G') + 1$. Thus, by Lemma 2.4, we have $\gamma_t(G) \le a(G) + 1$.

If $G \in QT_2$, we set $G' = G[V(G) \setminus \{x, w, v\}]$. Then G' is a type-1 quasi-tree as $G' - h \in \Gamma$. Clearly, f(G') < f(G) and hence by the induction hypothesis, $\gamma_t(G') \leq a(G') + 1$. Lemma 2.5 implies that $\gamma_t(G) \leq a(G) + 1$.

If $G \in QT_3$, then $\gamma_t(G) \leq a(G) + 1$ holds by Lemma 2.6, and if $G \in QT_4$, then the same conclusion follows from Lemma 2.7.

If $G \in QT_5$, set $G' = G[V(G) \setminus \{x, w, v, z\}]$. Then G' is a tree or a type-1 quasitree with f(G') < f(G). If G' is a tree, then $\gamma_t(G') \leq a(G') + 1$ by Theorem 2.2 and consequently $\gamma_t(G) \leq a(G) + 1$ by Lemma 2.8. If G' is a type-1 quasi-tree, then since f(G') < f(G), the induction hypothesis yields $\gamma_t(G') \leq a(G') + 1$. Thus $\gamma_t(G) \leq a(G) + 1$ by Lemma 2.8.

Finally, if $G \in QT_6$, then $\gamma_t(G) \leq a(G) + 1$ by Lemma 2.9.

We have completed the argument for Case 2 which completes the proof of Theorem 1.2.

3 Composition graphs

In this section, we prove that Conjecture 1.1 holds for some composition graphs, the first four of which can have minimum degree equal to 2, and the last of which can also have minimum degree 1.

3.1 Triangulations of graphs

The triangulation, $\tau(G)$, of a graph G, is the graph obtained from G by adding, for each edge e = uv of G, a new vertex x_e and the two edges $x_e u$ and $x_e v$, cf. [18].

Proposition 3.1. If G is a connected graph, then $\gamma_t(\tau(G)) \leq a(\tau(G)) + 1$.

Proof. Note first that $m(\tau(G)) = 3m(G)$. Let $S = V(\tau(G)) \setminus V(G)$. Then $\sum(S, \tau(G)) = 2m(G) < 3m(G) = m(\tau(G))$. Thus, $a(\tau(G)) \ge |S| = m(G)$. Let D be a vertex subset composed of arbitrary n(G) - 1 vertices of G. Since G is connected, we infer that D is a total dominating set of $\tau(G)$. So, $\gamma_t(\tau(G)) \le |D| = n(G) - 1 \le m(G) < a(\tau(G)) + 1$. \Box

3.2 Double graphs

The double graph, G^* , of a graph G is constructed as follows. Let G_1 and G_2 be disjoint copies of G, where for every $u \in V(G)$ its copy in G_i , $i \in [2]$, is denoted by u_i . Then G^* is obtained from the disjoint union of G_1 and G_2 by adding, for each edge uv of G, the edges u_1v_2 and u_2v_1 , cf. [14].

Proposition 3.2. If G is a connected graph with $\gamma_t(G) \leq a(G) + 1$, then $\gamma_t(G^*) \leq a(G^*) + 1$.

Proof. Clearly, $m(G^*) = 4m(G)$. Let S be an optimal annihilation set of G. Then $\sum(S, G) \le m(G)$. So, $\sum(S, G^*) = 2\sum(S, G) \le 2m(G) < 4m(G)$. Thus, $a(G^*) \ge |S| = a(G)$.

Let D be a γ_t -set of G. Then it is straightforward to see that the copy of D in G_1 (or in G_2 for that matter) is a total dominating set of G^* . It follows that $\gamma_t(G^*) \leq |D| = \gamma_t(G)$. Hence, if $\gamma_t(G) \leq a(G) + 1$, then $\gamma_t(G^*) \leq \gamma_t(G) \leq a(G) + 1 \leq a(G^*) + 1$.

3.3 Bijection graphs

Let G and H be disjoint graphs with n(G) = n(H) and let $f : V(G) \to V(H)$ be a bijection. The bijection graph B(G, H, f) is obtained from the disjoint union of G and H by adding the edges $uf(u), u \in V(G)$, cf. [19]. If $G \cong H$, then the bijection graph B(G, H, f) is also known as permutation graph.

Proposition 3.3. If G and H are connected graphs with n(G) = n(H) and $f : V(G) \rightarrow V(H)$ is a bijection, then $\gamma_t(B(G, H, f)) \leq a(B(G, H, f)) + 1$.

Proof. Let G, H and f be as stated, and set B = B(G, H, f). We may without loss of generality assume that $m(G) \ge m(H)$ (otherwise consider $B(G, H, f^{-1})$). Set S = V(H) and note that $\sum(S, B) = 2m(H) + n(G) \le m(G) + m(H) + n(G) = m(B)$. Thus,

 $a(B) \ge |S| = n(H)$. Further, since H is connected, V(H) is a total dominating set of B. Thus, $\gamma_t(B) \le n(H) < n(H) + 1 \le a(B) + 1$.

3.4 The Mycielskian

The famous construction of Mycielski from [15], which is especially important in chromatic graph theory, can be described as follows. The *Mycielski graph* $\mu(G)$ of a graph G contains G itself as an isomorphic subgraph, together with n + 1 additional vertices: to each vertex v_i of G, a vertex u_i is added, and there is another vertex w. The vertex w is adjacent to all the vertices u_i , and each edge $v_i v_j$ of G yields edges $u_i v_j$ and $v_i u_j$.

Next, we prove that if a graph G satisfies Conjecture 1.1, then so does $\mu(G)$. For other results on different kinds of domination in Mycielski graphs see [4, 12].

Proposition 3.4. If G is a connected graph with $\gamma_t(G) \leq a(G) + 1$, then $\gamma_t(\mu(G)) \leq a(\mu(G)) + 1$.

Proof. Let S be an optimal annihilation set in G. Then $\sum(S, G) \leq m(G)$, and therefore $\sum(S, \mu(G)) = 2\sum(S, G) \leq 2m(G)$. Then $\sum(S \cup \{w\}, \mu(G)) = \sum(S, \mu(G)) + d_{\mu(G)}(w) \leq 2m(G) + n(G) < 3m(G) + n(G) = m(\mu(G))$. Hence $a(\mu(G)) \geq a(G) + 1$. On the other hand, from [4] we know that $\gamma_t(\mu(G)) = \gamma_t(G) + 1$. Using these facts together with the assumption $\gamma_t(G) \leq a(G) + 1$ we get

$$\gamma_t(\mu(G)) = \gamma_t(G) + 1 \le a(G) + 2 \le (a(\mu(G)) - 1) + 2 = a(\mu(G)) + 1$$

and we are done.

3.5 Universally-identifying graphs

A universal vertex of a graph G is a vertex adjacent to all other vertices of V(G). For a connected graph G with $v \in V(G)$ and another graph H containing a universal vertex, we define a new graph, named universally-identifying graph, denoted by $G_v * H$, which is obtained by identifying the vertex v of G with the universal vertex of H.

In this subsection, we prove that Conjecture 1.1 holds for universally-identifying graphs.

Proposition 3.5. Let G and H be connected graphs. If H is a graph with $n(H) \ge \lfloor \frac{n(G)}{3} + 2 \rfloor$ and a universal vertex, and $v \in V(G)$, then $\gamma_t(G_v * H) \le a(G_v * H) + 1$.

Proof. If D is a γ_t -set of G, then $D \cup \{v\}$ is a total dominating set of $G_v * H$. Hence $\gamma_t(G_v * H) \leq \gamma_t(G) + 1$. Since $\gamma_t(G) \leq \frac{2n(G)}{3}$ holds for any connected graph G, see [10], we have

$$\gamma_t(G_v * H) \le \gamma_t(G) + 1 \le \frac{2n(G)}{3} + 1 \le \left\lfloor \frac{n(H) + n(G) - 1}{2} \right\rfloor + 1 \le a(G_v * H) + 1$$

If we are done.

and

If G in Proposition 3.5 has minimum degree 1 or 2, then $G_v * H$ may also have minimum degree 1 or 2, so long as G - v has a vertex in $V(G) \setminus \{v\}$ of minimum degree 1 or 2.

In our final result we provide another class of graphs, each of its members has minimum degree 1 or 2 and satisfies Conjecture 1.1.

Proposition 3.6. Let G be a connected graph with a cut-vertex $v \in V(G)$ such that $d_G(v) = k \ge \left\lfloor \frac{n(G)}{4} + 4 \right\rfloor$. If G - v has a component of order n(G) - k and this component contains exactly one neighbor of v, then $\gamma_t(G) \leq a(G) + 1$.

Proof. Set n = n(G), and let $N_G(v) = \{v_1, \ldots, v_k\}$. By our assumption, there is a vertex, say $v_1 \in N_G(v)$, such that the component G_1 of G - v containing v_1 has $n(G_1) = n - k$. Let $V_0 = \{v_2, v_3, \dots, v_k\}, V' = V(G_1) \cup \{v\}, G'_v = G[V'], \text{ and } G_0 = G[V_0 \cup \{v\}].$ Then v is a universal vertex of G_0 , and $G = G'_v * G_0$ can be seen as an universally-identifying graph. Since $k \ge \lfloor \frac{n}{4} + 4 \rfloor$, we have $\frac{n}{2} \ge \frac{2(n-k+1)}{3} + 1$. By a similar reasoning as that in the proof of Proposition 3.5, we have

$$\gamma_t(G) \le \gamma_t(G') + 1 \le \frac{2(n-k+1)}{3} + 1 \le \lfloor \frac{n}{2} \rfloor + 1 \le a(G) + 1.$$

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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