

Thresholds for the monochromatic clique transversal game

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Abstract

We study a recently introduced two-person combinatorial game, the (a, b) -monochromatic clique transversal game which is played by Alice and Bob on a graph G . As we observe, this game is equivalent to the (b, a) -biased Maker-Breaker game played on the clique-hypergraph of G . Our main results concern the threshold bias $a_1(G)$ that is the smallest integer a such that Alice can win in the $(a, 1)$ -monochromatic clique transversal game on G if she is the first to play. Among other results, we determine the possible values of $a_1(G)$ for the disjoint union of graphs, prove a formula for $a_1(G)$ if G is triangle-free, and obtain the exact values of $a_1(C_n \square C_m)$, $a_1(C_n \square P_m)$, and $a_1(P_n \square P_m)$ for all possible pairs (n, m) .

Keywords: clique-hypergraph; Maker-Breaker game; clique transversal game, threshold bias.

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1 Introduction

The *monochromatic transversal game* (MT game), as introduced by Mendes et al. [16], is a positional game on a hypergraph \mathcal{H} . The two players alternately choose (play) an unplayed vertex and Alice colors it red while Bob colors the chosen vertex blue. Alice wins if she obtains a red transversal (vertex cover) in \mathcal{H} and Bob wins if he colors all vertices of a hyperedge with blue. It is called Alice-start game or Bob-start game, respectively, if Alice or Bob is the first to play. The *monochromatic clique-transversal game* (MCT game) is played on the clique-hypergraph \mathcal{H}_G of a graph G . For the sake of simplicity, we will usually refer to an MCT game as being played on G instead of \mathcal{H}_G . The *(a, b)-monochromatic clique-transversal game* ((a, b)-MCT game) follows the rules of the MCT game, but Alice (resp. Bob) colors a vertices (resp. b vertices) in each of her (resp. his) turn.

The introductory paper [16] studied the (a, b)-MCT game on the clique-hypergraphs of powers of cycles and paths, observing also some basic properties of the game.¹ In this paper, besides some general results we concentrate on triangle-free graphs in general and specifically on grids, cylinder, and torus graphs.

1.1 Standard definitions

A *hypergraph* \mathcal{H} is a set system over the vertex set $V(\mathcal{H})$. The hyperedge set $E(\mathcal{H})$ of \mathcal{H} may contain any nonempty subset of $V(\mathcal{H})$ i.e., $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})} \setminus \{\emptyset\}$. It is clear that every simple graph can be considered as a (2-uniform) hypergraph. A hypergraph is *simple*, if no hyperedge $e \in \mathcal{H}$ is a subset of a different $e' \in E(\mathcal{H})$. The *maximum vertex degree* $\Delta(\mathcal{H})$ of \mathcal{H} is the maximum number of hyperedges incident to a vertex in \mathcal{H} . A *singleton* is a hyperedge of cardinality 1.

A set $S \subseteq V(\mathcal{H})$ is a *transversal* (also called vertex cover or hitting set) in the hypergraph \mathcal{H} , if S contains at least one vertex from each hyperedge $e \in E(\mathcal{H})$. Further, S is a *minimal transversal* if it contains no transversal as a proper subset. The *transversal hypergraph* $Tr(\mathcal{H})$ of \mathcal{H} is defined on the vertex set $V(\mathcal{H})$ and the hyperedges correspond to the minimal transversals of \mathcal{H} . As it has been stated already in Berge's fundamental book [2], every simple hypergraph \mathcal{H} satisfies $\mathcal{H} = Tr(Tr(\mathcal{H}))$.

In a graph G , a *clique* is a complete subgraph that is inclusion-wise maximal. The *clique-hypergraph* \mathcal{H}_G of G is defined on the same vertex set as G while its hyperedges correspond to the vertex sets of the cliques in G , cf. [4, 13]. Note that

¹The recent conference paper [15] written by Mendes et al. started the study of (a, b)-MT game played on another type of hypergraphs, namely on the biclique hypergraphs of powers of paths and cycles.

every isolated vertex in G corresponds to a singleton in \mathcal{H}_G . Moreover, if G is triangle-free and contains no isolated vertices, then $\mathcal{H}_G \cong G$.

For a non-negative integer k , a k -independent set $S \subseteq V(G)$ is a set of vertices in a graph G such that the maximum degree of the subgraph induced by S is at most k [5, 6, 14]. The k -independence number of G is the maximum cardinality of a k -independent set in G , and is denoted by $\alpha_k(G)$. The 0-independence number $\alpha_0(G)$ is the independence number $\alpha(G)$ of the graph.

The *Cartesian product* $G \square H$ of graphs G and H is defined on the vertex set $V(G) \times V(H)$ such that two vertices (g, h) and (g', h') are adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. If $h \in V(H)$, then the subgraph of $G \square H$ induced by the vertex set $\{(g, h) : g \in V(G)\}$ is isomorphic to G , called a G -layer, and denoted with G^h . Analogously the H -layers are defined and denoted with gH for a fixed vertex $g \in V(G)$. (See [12] for more details on Cartesian products.)

The *disjoint union* $G \cup H$ of (disjoint) graphs G and H has the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. The *domination number* $\gamma(G)$ of a graph G is the cardinality of a smallest dominating set, that is, a set $X \subseteq V(G)$ such that each vertex from $V(G) \setminus X$ has a neighbor in X . P_n and C_n respectively stand for the path and cycle graph on n vertices. The order of the graph G will be denoted by $n(G)$. The open neighborhood and the closed neighborhood of a vertex v in G will be respectively denoted by $N(v)$ and $N[v]$. For a set $S \subseteq V(G)$ we define $N[S] = \bigcup_{v \in S} N[v]$. For every positive integer k , $[k]$ stands for the set $\{1, \dots, k\}$.

1.2 Games

The *Maker-Breaker game*, introduced by Erdős and Selfridge [8], is played by Maker and Breaker on a hypergraph \mathcal{H} . (For different instances of the Maker-Breaker game see [7, 9, 10, 18].) The vertex set $V(\mathcal{H})$ is called the *board* and the hyperedges of \mathcal{H} are the *winning sets*. The two players alternately choose (i.e., play) an unplayed vertex. We say that a player *claims* the set $S \subseteq V(\mathcal{H})$ if he (or she) plays all vertices from S . It is a *Maker-start game* (resp. *Breaker-start game*) if Maker (resp. Breaker) is the first player. *Maker wins* the game if he claims a winning set, while *Breaker wins* if she can prevent Maker from doing this. The latter equivalently means that Breaker plays at least one vertex from each winning set, that is, she claims a minimal transversal of \mathcal{H} . The following observation was stated in [3]:

Observation 1.1 *A Maker-Breaker game on \mathcal{H} with a player A as Maker and a player B as Breaker is the same as the Maker-Breaker game on $\text{Tr}(\mathcal{H})$ where the roles of A and B are switched.*

By comparing the definitions, it is easy to see that an MT game on \mathcal{H} can always be considered as a Maker-Breaker game on $Tr(\mathcal{H})$ where Alice is Maker and Bob is Breaker. By Observation 1.1, we may infer the following statements that give us a useful approach in our study.

Observation 1.2 (i) *An MT game on a hypergraph \mathcal{H} is the same as the Maker-Breaker game on \mathcal{H} where Maker corresponds to Bob and Breaker corresponds to Alice.*

(ii) *An (a, b) -MCT game on G is the same as the (b, a) -biased Maker-Breaker game on \mathcal{H}_G where Maker is Bob and Breaker is Alice.*

For a graph G and a fixed positive integer ℓ , the *threshold bias* (or shortly threshold) $a_\ell(G)$ (resp. $a'_\ell(G)$) denotes the smallest positive integer a such that Alice can win in the Alice-start (resp. Bob-start) (a, ℓ) -MCT game. By definition, if $a < a_\ell(G)$, then Bob has a winning strategy in the Alice-start (a, ℓ) -MCT game on G . It is also known (see e.g., Exercise 3.6.2 in [11]) that for every $a \geq a_\ell(G)$, Alice wins in the Alice-start (a, ℓ) -MCT game. The same is true for the threshold $a'_\ell(G)$ that is, Alice wins in the Bob-start (a, ℓ) -MCT game on G if and only if $a \geq a'_\ell(G)$.

Structure of the paper. In the next subsection, we continue by observing some further basic properties of the (a, b) -MCT game. Section 2 is devoted to triangle-free graphs, where we prove general formulas for the thresholds $a_1(G)$ and $a'_1(G)$ if G is triangle-free. Then, in Section 3, we establish lower and upper bounds on $a_1(G_1 \cup G_2)$ in terms of $a_1(G_1)$ and $a_1(G_2)$ and afterwards, we show that all values between the lower and upper bounds are realizable. The theorems in Section 4 give a complete picture on the exact values of thresholds $a_1(C_n \square C_m)$, $a_1(C_n \square P_m)$, $a_1(P_n \square P_m)$, and $a'_1(C_n \square C_m)$, $a'_1(C_n \square P_m)$, $a'_1(P_n \square P_m)$.

1.3 Preliminaries

It can be proved via different simple approaches (see e.g., Proposition 2.1.6 in [11]) that if Maker (resp. Breaker) can win as a second player in a Maker-Breaker game on \mathcal{H} , she (resp. he) can also win as a first player. It can be checked that the analogous proofs work for the (a, b) -biased Maker-Breaker game. By Observation 1.2, we may state the same property for the (a, b) -MCT game that was already mentioned in [16] as Remark 2.1.

Observation 1.3 *If there exists a winning strategy for Alice (resp. Bob) in the Bob-start (resp. Alice-start) (a, b) -MCT game on a graph G , then there exists a winning strategy for Alice (resp. Bob) when she (resp. he) starts the game on G .*

Observation 1.3 and the definition of the threshold bias directly implies the following relation.

Observation 1.4 *It holds for every graph G that*

$$a'_1(G) \geq a_1(G). \quad (1)$$

In the rest of the paper we will also refer to the following statements.

Observation 1.5 (i) *If a hypergraph \mathcal{H} contains no singletons, then Breaker has a winning strategy in the $(1, \Delta(\mathcal{H}))$ -biased Maker-Breaker game on \mathcal{H} no matter who is the first player.*

(ii) *Let G be an isolate-free graph. If each vertex belongs to at most k cliques in G , then Alice has a winning strategy in the $(k, 1)$ -MCT game on G no matter who is the first player.*

(iii) *If G is an isolate-free graph and every vertex belongs to at most k cliques in G , then $a_1(G) \leq a'_1(G) \leq k$.*

Proof. In the $(1, \Delta(\mathcal{H}))$ -biased Maker-Breaker game on \mathcal{H} , Breaker may reply to Maker's every move v by playing an unplayed vertex from each hyperedge incident to v (and some additional vertices if necessary). This strategy ensures that whenever Maker plays a second vertex from some hyperedge, at least one vertex from this hyperedge has been already played by Breaker. Then, as \mathcal{H} contains no singletons, Maker cannot win, no matter who starts the game. This proves (i).

By Observation 1.2(ii), we may conclude statement (ii) from (i). The last part (iii) then follows by the definition of the thresholds. \square

2 Thresholds for triangle-free graphs

In this section we focus on triangle-free graphs that do not contain isolated vertices. For such a graph G , the clique hypergraph \mathcal{H}_G is isomorphic to G and therefore, by Observation 1.2(ii), Bob wins the (a, b) -MCT game on G if and only if he claims two adjacent vertices.

Theorem 2.1 *If G is a triangle-free graph that contains no isolated vertex, then*

- (i) $a_1(G) = \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\} = \min_{X \subseteq V(G)} \max\{\Delta(G - X), |X|\},$
- (ii) $a'_1(G) = \Delta(G).$

Proof. To establish (i), we are going to prove three claims.

Claim 1 *If $a \geq \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\}$, then Alice wins in the $(a, 1)$ -MCT game.*

Proof. Alice chooses a maximum k^* -independent set Y such that

$$\max\{k^*, n(G) - \alpha_{k^*}(G)\} = \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\}.$$

Observe that, by our condition, the set $X = V(G) \setminus Y$ contains at most a vertices. In the first turn, Alice plays all vertices from X and further $a - |X|$ arbitrarily chosen vertices if $a > |X|$. Since Y is a k^* -independent set, $\Delta(G[Y]) \leq k^* \leq a$; that is each vertex played by Bob in his first turn or later would have at most a unplayed neighbors. Alice thus can reply to each move v of Bob by claiming all the unplayed neighbors of v (and some further vertices if a is big enough). With this strategy, Alice ensures that Bob can never claim two adjacent vertices and Alice's moves form a vertex cover that is a clique transversal in the isolate-free graph G . \square

Claim 2 *If $a < \min_{X \subseteq V(G)} \max\{\Delta(G - X), |X|\}$, then Bob wins in the $(a, 1)$ -MCT game.*

Proof. Let A_1 be the set of the a vertices played by Alice in the first turn. Under the present condition, $a < \max\{\Delta(G - A_1), |A_1|\}$. As $|A_1| = a$, we may infer $1 \leq a < \Delta(G - A_1)$. Consequently, in his first turn, Bob can play a vertex v such that $\deg_{G-A_1}(v) > a$. Then, Alice cannot claim all the unplayed neighbors of v in her next turn and Bob wins in his second turn by claiming an unplayed neighbor of v . \square

Claim 3 *For every graph G*

$$\min_{X \subseteq V(G)} \max\{\Delta(G - X), |X|\} = \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\}.$$

Proof. Let k^* be the smallest integer such that

$$\max\{k^*, n(G) - \alpha_{k^*}(G)\} = \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\}.$$

We select a maximum k^* -independent set Y^* and define $X^* = V(G) \setminus Y^*$. By the choice of k^* and Y^* , we have $|X^*| = n(G) - \alpha_{k^*}(G)$ and $\Delta(G - X^*) = k^*$. This gives

$$\max\{\Delta(G - X^*), |X^*|\} = \max\{k^*, n(G) - \alpha_{k^*}(G)\} = \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\}$$

and, as $X^* \subseteq V(G)$, we conclude

$$\min_{X \subseteq V(G)} \max\{\Delta(G - X), |X|\} \leq \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\}.$$

To prove the other direction, let X' be a set of vertices in G such that

$$\max\{\Delta(G - X'), |X'|\} = \min_{X \subseteq V(G)} \max\{\Delta(G - X), |X|\}. \quad (2)$$

From the possible candidates for the role of X' we choose one of minimum cardinality. This ensures that $Y' = V(G) \setminus X'$ is a maximum k' -independent set with $k' = \Delta(G - X')$. Note that $|X'| = n(G) - \alpha_{k'}(G)$ also follows and (2) remains true. We therefore obtain

$$\min_{X \subseteq V(G)} \max\{\Delta(G - X), |X|\} = \max\{k', n(G) - \alpha_{k'}(G)\} \geq \min_{k \geq 0} \max\{k, n(G) - \alpha_k(G)\}$$

that completes the proof of the claim. \square

Claims 1–3 directly imply that the threshold stated in (i) holds for every triangle- and isolate-free graph.

To prove (ii), we consider the Bob-start $(a, 1)$ -MCT game on G . As the graph is isolate-free, Bob cannot win in his first turn. If $a < \Delta(G)$, Bob plays a vertex v of maximum degree in the first turn. As $a < \deg(v)$, Alice cannot claim all vertices from $N(v)$ in her response and hence, Bob can win the game in his second turn by playing an unplayed vertex which is adjacent to v . This shows that $a'_1(G) \geq \Delta(G)$ holds. Under the present conditions, every clique of G is an edge and therefore, Observation 1.5(iii) implies $a'_1(G) \leq \Delta(G)$. This completes the proof of (ii). \square

Notice that, independently of Observation 1.3, Theorem 2.1 directly implies $a_1(G) \leq a'_1(G)$ for every isolate- and triangle-free graph G . Indeed, by selecting $X = \emptyset$ we obtain

$$a_1(G) \leq \max\{\Delta(G - \emptyset), |\emptyset|\} = \Delta(G) = a'_1(G).$$

Remark 2.2 *If G is triangle-free but contains $\ell \geq 1$ isolated vertices, then Bob wins the Bob-start $(a, 1)$ -MCT game by claiming a clique K_1 in his first turn. As it is true for every integer a , the threshold $a'_1(G)$ does not exist. If Alice wins the Alice-start game, she has to play all isolated vertices in her first turn. A further argument analogous to the proof of Theorem 2.1 yields the following formula:*

$$a_1(G) = \min_{k \geq 0} \max\{k, \ell + n(G') - \alpha_k(G')\},$$

where G' denotes the graph obtained from G by deleting all isolated vertices. As $n(G) = n(G') + \ell$ and $\alpha_k(G) = \alpha_k(G') + \ell$, we may conclude the following equality:

$$a_1(G) = \min_{k \geq 0} \max\{k, \ell + n(G) - \alpha_k(G)\}.$$

3 Disjoint union of graphs

Proposition 3.1 *If G_1 and G_2 are graphs, then*

$$\max\{a_1(G_1), a_1(G_2)\} \leq a_1(G_1 \cup G_2) \leq a_1(G_1) + a_1(G_2). \quad (3)$$

Proof. Let $a_1(G_1) = k_1$, $a_1(G_2) = k_2$, and set $G = G_1 \cup G_2$. Assume without loss of generality that $k_1 \geq k_2$.

Consider the $(a, 1)$ -MCT game on G . Assume first that $a < k_1$. In the first move Alice selects at most $a < k_1$ vertices from G_1 , hence Bob has the following winning strategy. He will play only on G_1 as long as it is possible, following his winning strategy on G_1 . Since $a < a_1(G_1) = k_1$, Bob indeed has such a strategy. In this way, Bob also wins on G . It follows that $a_1(G) \geq k_1$ which proves the lower bound.

To prove the upper bound, we need to show that Alice has a winning strategy if $a = k_1 + k_2$. In this case, Alice follows her winning strategies on G_1 and G_2 . More precisely, in the first move, she selects k_1 vertices from G_1 and k_2 vertices from G_2 according to her strategies on G_1 and G_2 , respectively. In the rest of the game, after each move of Bob in G_i , $i \in [2]$, she replies by selecting optimally k_i vertices in G_i and arbitrary additional k_{3-i} vertices from G_i . In this way Alice wins the game and hence $a_1(G) \leq a_1(G_1) + a_1(G_2)$. \square

Recall that a *caterpillar* is a tree in which a single path is incident to (or contains) every edge. This single path of the caterpillar is called the *spine* [19]. For $m \geq 1$ and $\ell \geq 0$ let $T_{m,\ell}$ be the caterpillar whose spine has m vertices and to each vertex of the spine, exactly ℓ leaves are attached.

In the next result we determine the threshold a_1 for the caterpillars $T_{m,\ell}$. As $T_{m,0}$ is a path of order m and $a_1(P_m)$ was already established in [16], we will assume $\ell \geq 1$ here.

Theorem 3.2 *If m and ℓ are positive integers, then*

$$a_1(T_{m,\ell}) = \begin{cases} m; & m \leq \ell, \\ \ell; & \lfloor \frac{m}{2} \rfloor \leq \ell < m, \\ \ell + 1; & \lfloor \frac{m}{3} \rfloor - 1 \leq \ell \leq \lfloor \frac{m}{2} \rfloor - 1, \\ \ell + 2; & \ell \leq \lfloor \frac{m}{3} \rfloor - 2. \end{cases}$$

Proof. Let $X = \{v_1, \dots, v_m\}$ be the set of the (consecutive) vertices in the spine of $T_{m,\ell}$. We distinguish four cases depending on the size of ℓ with respect to m .

Case 1: $m \leq \ell$.

Note that $|X| = m$ and $\Delta(T_{m,\ell} - X) = 0$, hence Theorem 2.1(i) implies that $a_1(T_{m,\ell}) \leq m$.

To prove that $a_1(T_{m,\ell}) \geq m$, we need to show that Bob wins the Alice-start $(a, 1)$ -MCT game on $T_{m,\ell}$, when $a < m$. Partition $V(T_{m,\ell})$ into sets V_i , $i \in [m]$, where V_i contains v_i and all the leaves attached to it. Let A_1 be the set of a vertices played by Alice in her first move. Then there exists $i \in [m]$ such that $A_1 \cap V_i = \emptyset$. Let Bob reply by playing the vertex v_i . Since $a < m \leq \ell$, Alice cannot claim all the neighbors of v_i in her next turn and hence Bob can win the game. We conclude that $a_1(T_{m,\ell}) \geq m$.

Case 2: $\lfloor \frac{m}{2} \rfloor \leq \ell < m$.

To show that $a_1(T_{m,\ell}) \leq \ell$, let $Y = \{v_2, v_4, \dots, v_{2\lfloor \frac{m}{2} \rfloor}\} \cup Y'$, where Y' contains $\ell - \lfloor \frac{m}{2} \rfloor$ arbitrary additional vertices of $T_{m,\ell}$. Then $|Y| = \ell$ and $\Delta(T_{m,\ell} - Y) = \ell$, hence using Theorem 2.1(i) we get $a_1(T_{m,\ell}) \leq \ell$.

To prove $a_1(T_{m,\ell}) \geq \ell$, we need to show that Bob wins the Alice-start $(a, 1)$ -MCT game on $T_{m,\ell}$, when $a < \ell$. For this sake partition $V(T_{m,\ell})$ just as in Case 1 into the sets V_i , $i \in [m]$. If A_1 is the set of a vertices played by Alice in her first move, then since $a < \ell < m$ there exists $i \in [m]$ such that $A_1 \cap V_i = \emptyset$. Then Bob replies by playing v_i and wins the game in the next turn. We conclude that $a_1(T_{m,\ell}) \geq \ell$.

Case 3: $\lfloor \frac{m}{3} \rfloor - 1 \leq \ell \leq \lfloor \frac{m}{2} \rfloor - 1$.

Since $\ell \geq 1$, in this case we have $m \geq 4$. To show that $a_1(T_{m,\ell}) \leq \ell + 1$, let $Z = \{v_3, v_6, \dots, v_{3\lfloor \frac{m}{3} \rfloor}\}$. Then $|Z| \leq \ell + 1$ and $\Delta(T_{m,\ell} - Z) \leq \ell + 1$. Therefore, $\max\{|Z|, \Delta(T_{m,\ell} - Z)\} \leq \ell + 1$. Applying Theorem 2.1(i) again we obtain $a_1(T_{m,\ell}) \leq \ell + 1$.

Next, consider the Alice-start $(a, 1)$ -MCT game on $T_{m,\ell}$, when $a < \ell + 1$. Partition $V(T_{m,\ell})$ into sets V_i , $i \in [\lfloor \frac{m}{2} \rfloor]$, where V_i contains v_{2i-1}, v_{2i} and the leaves attached to both of them. When m is odd, $V_{\lfloor \frac{m}{2} \rfloor}$ also contains v_m and all the leaves attached to it. Let A_1 be the a vertices played by Alice in her first move. Since $a < \ell + 1 \leq (\lfloor \frac{m}{2} \rfloor - 1) + 1 = \lfloor \frac{m}{2} \rfloor$, by the Pigeonhole principle there exists $i \in [\lfloor \frac{m}{2} \rfloor]$ such that $A_1 \cap V_i = \emptyset$. Then Bob can play the vertex v_{2i-1} . Since neither the vertex v_{2i} nor the leaves attached to v_{2i-1} have been played, Bob wins the game in his next move. We conclude that $a_1(T_{m,\ell}) \geq \ell + 1$.

Case 4: $\ell \leq \lfloor \frac{m}{3} \rfloor - 2$.

Since $\ell \geq 1$, in this case we have $m \geq 9$. To prove the first direction of the statement, we refer to Theorem 2.1 which implies

$$a_1(T_{m,\ell}) \leq a'_1(T_{m,\ell}) = \Delta(T_{m,\ell}) = \ell + 2.$$

Consider now the Alice-start $(a, 1)$ -MCT game on $T_{m,\ell}$, where $a < \ell + 2$. Partition $V(T_{m,\ell})$ into sets V_i , $i \in [\lfloor \frac{m}{3} \rfloor]$, where V_i contains $v_{3i-2}, v_{3i-1}, v_{3i}$, and all their leaves. If $3 \nmid m$, then add to $V_{\lfloor \frac{m}{3} \rfloor}$ also the (one or two) vertices v_j , where $j > 3\lfloor \frac{m}{3} \rfloor$, as well as all the leaves attached to them. Let A_1 be the a vertices played by Alice in her first move. Since $a < \ell + 2 \leq \lfloor \frac{m}{3} \rfloor$, there exists $i \in [\lfloor \frac{m}{3} \rfloor]$ such that $A_1 \cap V_i = \emptyset$. Then Bob can play v_{3i-1} . Since neither v_{3i-2} nor v_{3i} nor any leaf attached to v_{3i-1} were played by now, Bob wins the game in the next turn. We conclude that $a_1(T_{m,\ell}) \geq \ell + 2$. \square

Note that the first case of Theorem 3.2, that is $a_1(T) = m$, also holds for caterpillars T which are obtained from $T_{m,m}$ by attaching some further leaves (without any restrictions) to the vertices of the spine.

Proposition 3.3 *If $1 \leq \ell \leq k$ and $0 \leq i \leq \ell$, then $a_1(T_{k,k+i} \cup T_{\ell,k+i}) = k + i$.*

Proof. For this proof set $T_1 = T_{k,k+i}$, $T_2 = T_{\ell,k+i}$, and $T = T_1 \cup T_2$. Let v_1, \dots, v_k and u_1, \dots, u_ℓ be the consecutive vertices of the spines of T_1 and T_2 , respectively.

First we set

$$X = \{v_2, v_4, \dots, v_{2\lfloor k/2 \rfloor}, u_2, u_4, \dots, u_{2\lfloor \ell/2 \rfloor}\}$$

and observe that $|X| = \lfloor k/2 \rfloor + \lfloor \ell/2 \rfloor \leq k$ and $\Delta(T - X) = k + i$ hold. Theorem 2.1 then implies $a_1(T) \leq \max\{k, k + i\} = k + i$.

Consider now the Alice-start $(a, 1)$ -MCT game on T , when $a < k + i$. Let A_1 be the set of vertices played by Alice in her first move. Then $|A_1| = a$. Because $a < k + i \leq k + \ell$, by an argument parallel to the one from the proof of Theorem 3.2 we infer that there exists a non-leaf x from T (so $x = u_i$ or $x = v_j$) such that neither x nor any leaves attached to x belong to A_1 . Bob can then play x , and Alice cannot claim all its neighbors in her second move. Thus Bob wins the game and $a_1(T) \geq k + i$ follows. \square

The following result shows that both inequalities in (3) are sharp and, moreover, every integer between the lower and the upper bound may be the value of $a_1(G_1 \cup G_2)$ for appropriately chosen graphs.

Corollary 3.4 *For every three integers ℓ , k , and p satisfying $1 \leq \ell \leq k \leq p \leq \ell+k$, there exist caterpillars G_1 and G_2 such that*

$$\max\{a_1(G_1), a_1(G_2)\} = k, \quad a_1(G_1 \cup G_2) = p, \quad \text{and} \quad a_1(G_1) + a_1(G_2) = k + \ell.$$

Proof. Let us consider the caterpillars $G_1 = T_{k,p}$ and $G_2 = T_{\ell,p}$. As $1 \leq \ell \leq k \leq p$, $a_1(G_1) = k$ and $a_1(G_2) = \ell$ hold by Theorem 3.2. This already shows $\max\{a_1(G_1), a_1(G_2)\} = k$ and $a_1(G_1) + a_1(G_2) = k + \ell$. To prove the second equality, we set $i = p - k$ and observe that $0 \leq i \leq \ell$ holds under the present conditions. It follows from Proposition 3.3 that $a_1(G_1 \cup G_2) = k + i = p$. This completes the proof. \square

4 Cartesian product of paths and cycles

Throughout this section, let $V(C_n) = [n]$ for $n \geq 3$, and let $V(P_n) = [n]$ for $n \geq 1$, where 1 and n are end vertices of P_n . Moreover, for C_n we will consider indices modulo n .

4.1 Torus graphs

In this subsection we prove:

Theorem 4.1 *If $m \geq n \geq 3$, then*

$$a_1(C_n \square C_m) = \begin{cases} 1; & n = m = 3, \\ 2; & n = 3, m = 4, \\ 3; & n = 3, m \geq 5, \\ 4; & m \geq n \geq 4, \end{cases}$$

and

$$a'_1(C_n \square C_m) = \begin{cases} 1; & n = m = 3, \\ 3; & n = 3, m \geq 4, \\ 4; & m \geq n \geq 4. \end{cases}$$

Proof. Consider first the case $m \geq n \geq 4$ and recall that $V(C_n \square C_m) = \{(i, j) : i \in [n], j \in [m]\}$. Then $C_n \square C_m$ is a 4-regular, triangle-free graph, hence Theorem 2.1(ii) implies $a'_1(C_n \square C_m) = 4$. Thus by (1), $a_1(C_n \square C_m) \leq a'_1(C_n \square C_m) = 4$.

It remains to show that $a_1(C_n \square C_m) \geq 4$. For this sake consider the following winning strategy for Bob in the Alice-start $(3, 1)$ -MCT game on $C_n \square C_m$. Let A_1 be the set of three vertices played by Alice in her first move. Since we have assumed that $m \geq n \geq 4$, there exist $i \in [n]$ and $j \in [m]$ such that $A_1 \cap V((C_n)^j) = \emptyset$ and $A_1 \cap V({}^i(C_m)) = \emptyset$. Then Bob plays (i, j) in his first move. Since (i, j) is adjacent to four unplayed vertices, Alice cannot claim all these vertices in her next turn. Therefore Bob can win by playing an unplayed neighbor of (i, j) . This finishes the proof for $m \geq n \geq 4$.

In the rest of the proof we assume that $n = 3$ and consider the $(a, 1)$ -MCT game on $C_3 \square C_m$.

Case 1: $m = 3$.

Consider the Bob-start $(1, 1)$ -MCT game on $C_3 \square C_3$. By symmetry we may assume without loss of generality that the first move of Bob is $(1, 1)$. Then Alice replies by the move $(2, 1)$. Then, using symmetry again, we need to consider four cases for a possible second move of Bob: $(1, 2)$, $(2, 2)$, $(3, 2)$, and $(3, 1)$. If Bob plays $(1, 2)$, Alice replies by $(1, 3)$. Then the only layers in which Alice has not played yet are $(C_3)^2$ and ${}^3(C_3)$. Whatever Bob plays next, Alice replies in the layer $(C_3)^2$ and later she can finish the game with a move in ${}^3(C_3)$. In all the other possible second moves of Bob, Alice replies by playing $(1, 2)$ and then the only layers in which Alice has not played yet are $(C_3)^3$ and ${}^3(C_3)$. Then it is straightforward to see that in all the cases Alice can win the game. We conclude that $a'_1(C_3 \square C_3) = 1$ and by Observation 1.3 also $a_1(C_3 \square C_3) = 1$.

Case 2: $m = 4$.

Since each vertex is incident with three cliques (one 3-cycle and two edges), Observation 1.5(iii) yields $a'_1(C_3 \square C_4) \leq 3$.

Consider the Bob-start $(2, 1)$ -MCT game on $C_3 \square C_4$. Let Bob choose $(2, 2)$ in his first move. Then Alice must play the vertices $(2, 1)$ and $(2, 3)$, for otherwise Bob can claim an edge in the layer ${}^2(C_4)$ in his next move. Then Bob replies with $(1, 2)$. As Alice cannot play all the three vertices $(3, 2)$, $(1, 1)$, and $(1, 3)$ in her next move, Bob can claim a clique in his second move. Thus $a'_1(C_3 \square C_4) \geq 3$. We conclude that $a'_1(C_3 \square C_4) = 3$.

Next we consider the Alice-start $(2, 1)$ -MCT game on $C_3 \square C_4$. To show that $a_1(C_3 \square C_4) \leq 2$, we will give a winning strategy for Alice. Let her start the game by playing $(1, 1)$ and $(2, 3)$. Assume that Bob plays a vertex (i, j) . If (i, j) is in $(C_3)^1$ or in $(C_3)^3$, then Alice responds by playing two neighbors of (i, j) in ${}^i(C_4)$, and then Bob cannot claim any clique incident to (i, j) . As Alice has already played a vertex from each C_3 , in the continuation, after any move (i', j') of Bob, Alice can

reply by choosing the (at most two) unplayed neighbors in ${}^{i'}(C_4)$, and she wins the game. In the second case suppose that Bob's first move (i, j) is from $(C_3)^2$ or $(C_3)^4$. If $(i, j) \in \{(1, 2), (1, 4), (2, 2), (2, 4)\}$, then Alice replies by choosing one neighbor of (i, j) in $(C_3)^j$ and the neighbor of (i, j) in ${}^i(C_4)$ which was not yet selected. One can now check easily that Alice can win the game. The remaining subcase is $(i, j) = (3, 2)$ (or symmetrically $(i, j) = (3, 4)$). Then Alice responds by choosing $(3, 1)$ and $(3, 3)$. In each case it is now easy to see that Alice will win the game. This shows that $a_1(C_3 \square C_4) \leq 2$.

It remains to prove that $a_1(C_3 \square C_4) \geq 2$ by providing a winning strategy for Bob in the $(1, 1)$ -MCT game. Assume that Alice first plays (i, j) . Then Bob replies by playing (i', j') , where $i' \neq i$ and $j' \neq j$, and can win the game in his second move. We conclude that $a_1(C_3 \square C_4) = 2$.

Case 3: $m \geq 5$.

By Observation 1.5(iii), $a_1(C_3 \square C_m) \leq 3$.

To show that $a_1(C_3 \square C_m) \geq 3$, we will show that Bob has a winning strategy in the Alice-start $(2, 1)$ -MCT game on $C_3 \square C_m$. Assume that Alice plays two vertices in her first move. Then there are two consecutive layers $(C_3)^k$ and $(C_3)^{k+1}$ such that no vertex in these layers was played by Alice. By symmetry we may assume that the vertices $(1, k-1)$ and $(2, k-1)$ were not selected by Alice. Then Bob replies by selecting the vertex $(1, k)$. This forces Alice to reply by playing $(1, k-1)$ and $(1, k+1)$. Then Bob plays $(2, k)$ which forces Alice to play $(2, k-1)$ and $(2, k+1)$. Now Bob wins by playing $(3, k)$. Hence $a_1(C_3 \square C_m) \geq 3$ and consequently $a_1(C_3 \square C_m) = 3$.

By Observation 1.5(iii), we have $3 = a_1(C_3 \square C_m) \leq a'_1(C_3 \square C_m) \leq 3$. We conclude that $a'_1(C_3 \square C_m) = 3$. \square

4.2 Cylinder graphs

In this subsection we consider cylinder graphs, that is, Cartesian products of paths by cycles. The result reads as follows.

$n \setminus m$	2	3	4	5	≥ 6
3	2	2	2	2	3
4	2	3	3	3	4
5	3	3	3	4	4
6	3	3	4	4	4
7	3	3	4	4	4
8	3	3	4	4	4
9	3	3	4	4	4
≥ 10	3	4	4	4	4

$n \setminus m$	2	3	4	5	≥ 6
3	2	3	3	3	3
4	3	4	4	4	4
5	3	4	4	4	4
6	3	4	4	4	4
7	3	4	4	4	4
8	3	4	4	4	4
9	3	4	4	4	4
≥ 10	3	4	4	4	4

Table 1: $a_1(C_n \square P_m)$ (left) and $a'_1(C_n \square P_m)$ (right)

Theorem 4.2 *If $n \geq 3$ and $m \geq 2$, then*

$$a_1(C_n \square P_m) = \begin{cases} 2; & n = 3, 2 \leq m \leq 5, \text{ or} \\ & n = 4, m = 2, \\ 3; & n = 3, m \geq 6, \text{ or} \\ & n = 4, 3 \leq m \leq 5, \text{ or} \\ & n = 5, 2 \leq m \leq 4, \text{ or} \\ & 6 \leq n \leq 9, m = 2, 3, \text{ or} \\ & n \geq 10, m = 2, \\ 4; & \text{otherwise.} \end{cases}$$

and

$$a'_1(C_n \square P_m) = \begin{cases} 2; & n = 3, m = 2, \\ 3; & n = 3, m \geq 3, \text{ or} \\ & n \geq 4, m = 2, \\ 4; & n \geq 4, m \geq 3. \end{cases}$$

The values for $a_1(C_n \square P_m)$ and $a'_1(C_n \square P_m)$ are summarized in Table 1.

In the rest of the subsection we prove Theorem 4.2. We begin with the following lemma that allows us to use the results from the previous subsection.

Lemma 4.3 *If $n \geq 3$ and $m \geq 4$, then*

$$a_1(C_n \square P_m) \leq a_1(C_n \square C_m) \quad \text{and} \quad a'_1(C_n \square P_m) \leq a'_1(C_n \square C_m).$$

Proof. Let P_m be obtained from C_m after the removal of the edge $1m$. The clique hypergraph of $C_n \square P_m$ is then obtained from the clique hypergraph of $C_n \square C_m$ by removing the cliques $\{(i, m), (i, 1)\}$ for every $i \in [n]$.

Suppose now that Alice can win in the $(a, 1)$ -MCT game on $C_n \square C_m$ and consider one of her optimal strategies. If she plays according to the same strategy in the $(a, 1)$ -MCT game on $C_n \square P_m$, Bob cannot claim a clique of $C_n \square C_m$ and therefore, he cannot claim a clique in $C_n \square P_m$. It follows that Alice can also win the $(a, 1)$ -MCT game on $C_n \square P_m$. This argumentation is valid no matter Alice or Bob starts the games and implies that both $a_1(C_n \square P_m) \leq a_1(C_n \square C_m)$ and $a'_1(C_n \square P_m) \leq a'_1(C_n \square C_m)$ are true. \square

Proposition 4.4 *If $m \geq 2$, then*

$$a_1(C_3 \square P_m) = \begin{cases} 2; & 2 \leq m \leq 5, \\ 3; & \text{otherwise,} \end{cases}$$

and

$$a'_1(C_3 \square P_m) = \begin{cases} 2; & m = 2, \\ 3; & \text{otherwise.} \end{cases}$$

Proof. Assume that $m \geq 2$ and consider the $(a, 1)$ -MCT game on $C_3 \square P_m$. We first suppose that Alice starts the game.

Case 1: $2 \leq m \leq 5$.

To show that $a_1(C_3 \square P_m) \geq 2$, we will provide a winning strategy for Bob in the Alice-start $(1, 1)$ -MCT game. Assume that Alice plays (i, j) . Since $m \geq 2$, there exists a vertex (i', j') with $i' \neq i$ and $j' \neq j$. Then Bob replies by playing (i', j') and he can win the game within his next two moves. We infer that $a_1(C_3 \square P_m) \geq 2$.

To show that $a_1(C_3 \square P_m) \leq 2$, we will provide a winning strategy for Alice in the Alice-start $(2, 1)$ -MCT game. We distinguish several subcases depending on m .

Case 1.1: $m = 2$.

In $C_3 \square P_2$, each vertex is incident with exactly two cliques. Hence Observation 1.5(ii) yields the result.

Case 1.2: $m = 3$.

Alice plays $(1, 2)$ and $(2, 1)$ in her first move. If Bob plays (i, j) in $(C_3)^1$ or in $(C_3)^2$, then Alice replies by playing all unplayed neighbors of (i, j) in ${}^i(P_3)$ (and a further arbitrary unplayed vertex if it is needed). If Bob plays $(i, 3)$, where $i \in [3]$, then

Alice plays one vertex in the layer $(C_3)^3$ and $(i, 2)$ if it is possible. Otherwise, Alice can play an arbitrary unplayed vertex. Now, every vertex is incident to at most two cliques which have not been played by Alice. Therefore, if Alice plays optimally, Bob cannot claim any clique and Alice wins the game.

Case 1.3: $m = 4$.

By Theorem 4.1 and Lemma 4.3, we have $a_1(C_3 \square P_4) \leq a_1(C_3 \square C_4) = 2$.

Case 1.4: $m = 5$.

Alice plays $(1, 4)$ and $(2, 2)$ in her first move. Assume that Bob replies with the vertex (i, j) . If $(i, j) \in X = \{(1, 2), (2, 4), (3, 2), (3, 3), (3, 4)\}$, then Alice chooses two neighbors of (i, j) in the layer ${}^i(P_m)$. When $(i, j) \notin X$ then, if it is possible, Alice replies by choosing two neighbors of (i, j) , one in $(C_3)^j$ and the other in ${}^i(P_m)$. Otherwise, Alice can play any unplayed vertex. If she continues playing according to this strategy, Alice prevents Bob from claiming a clique and she wins the game.

We conclude that $a_1(C_3 \square P_m) \leq 2$ for $2 \leq m \leq 5$.

Case 2: $m \geq 6$.

By Lemma 4.3 and Theorem 4.1 we have $a_1(C_3 \square P_m) \leq a_1(C_3 \square C_m) = 3$.

To prove that $a_1(C_3 \square P_m) \geq 3$ we show that Bob can win in the Alice-start $(2, 1)$ -MCT game. After Alice plays two vertices in her first move, there exist three consecutive layers $(C_3)^{j-1}$, $(C_3)^j$, and $(C_3)^{j+1}$, such that at most one vertex from these three layers were played by Alice and the possibly played vertex is not from $(C_3)^j$. Let $(1, j-1)$ be the possibly played vertex. No matter whether it was played or not, Bob replies by $(2, j)$. Then Alice is forced to select $(2, j-1)$ and $(2, j+1)$. After that, Bob plays $(3, j)$ and will win in his next move.

It remains to consider the Bob-start game on $C_3 \square P_m$. If $m = 2$, each vertex is contained in two cliques and Observation 1.5(iii) implies $a'_1(C_3 \square P_2) \leq 2$. By (1) and the first part of this proof we have $a'_1(C_3 \square P_2) \geq a_1(C_3 \square P_2) = 2$. Therefore $a'_1(C_3 \square P_2) = 2$.

Suppose that $m \geq 3$. Then $a'_1(C_3 \square P_m) \leq 3$ by Observation 1.5(iii). To see that $a'_1(C_3 \square P_m) \geq 3$ consider the following strategy of Bob in the Bob-start $(2, 1)$ -MCT game. In his first move, he plays $(1, 2)$. Then Alice is forced to play $(1, 1)$ and $(1, 3)$. Afterwards, Bob replies by the move $(2, 2)$ and since Alice cannot play all of its neighbors, Bob can win in the next turn.

□

Next, we consider the thresholds for $C_n \square P_2$ where $n \geq 4$.

Proposition 4.5 *If $n \geq 4$, then*

$$a_1(C_n \square P_2) = \begin{cases} 2; & n = 4, \\ 3; & n \geq 5. \end{cases}$$

Proof. Assume that $n \geq 4$ and consider the Alice-start $(a, 1)$ -MCT game on $C_n \square P_2$.

Case 1: $n = 4$.

Let $X = \{(1, 1), (3, 2)\}$. Then $\Delta((C_4 \square P_2) - X) = 2$. By Theorem 2.1(i), we have that $a_1(C_4 \square P_2) \leq \max\{\Delta((C_4 \square P_2) - X), |X|\} = 2$. It suffices to show that Bob has a winning strategy in the $(1, 1)$ -MCT game. Assume Alice plays (i, j) in her first move. Then Bob replies by choosing a vertex (i', j') such that $i' \neq i$ and $j' \neq j$, and he will win the game in his second move. Hence $a_1(C_4 \square P_2) = 2$.

Case 2: $n \geq 5$.

Since each vertex is incident with three cliques, Observation 1.5(iii) implies that $a_1(C_n \square P_2) \leq 3$. It remains to show that $a_1(C_n \square P_2) \geq 3$ by giving a winning strategy for Bob in the $(2, 1)$ -MCT game. Let A_1 be the set of two vertices played by Alice in her first move. There exists a vertex (i, j) such that $(i, j) \notin N[A_1]$. Then Bob plays (i, j) in his first move and can win the game in his next turn. Thus $a_1(C_n \square P_2) = 3$. \square

For $n \geq 4$, the Cartesian product $C_n \square P_m$ is triangle-free with no isolated vertices. By Theorem 2.1, $a'_1(C_n \square P_m) = \Delta(C_n \square P_m)$ which implies the following:

Observation 4.6 *If $n \geq 4$, then*

$$a'_1(C_n \square P_m) = \begin{cases} 3; & m = 2, \\ 4; & m \geq 3. \end{cases}$$

Lemma 4.7 *Assume that $n \geq 4$ and $m \geq 3$. Then $a_1(C_n \square P_m) = 3$ if and only if $\gamma(C_n \square P_{m-2}) \leq 3$.*

Proof. Let G be the induced subgraph of $C_n \square P_m$ which remains after the deletion of the vertices from $V((C_n)^1) \cup V((C_n)^m)$. Then G is isomorphic to $C_n \square P_{m-2}$. It suffices to show that $a_1(C_n \square P_m) = 3$ if and only if $\gamma(G) \leq 3$.

(\Leftarrow) Let X be a minimum dominating set of G and assume that $|X| \leq 3$. We will show that $a_1(C_n \square P_m) = 3$. Since each vertex in $V((C_n)^1) \cup V((C_n)^m)$ has

degree 3 in $C_n \square P_m$, and X is a dominating set of G , $\Delta((C_n \square P_m) - X) \leq 3$. By Theorem 2.1(i), $a_1(C_n \square P_m) \leq \max\{\Delta((C_n \square P_m) - X), |X|\} \leq 3$. To show that $a_1(C_n \square P_m) \geq 3$, we will show that Bob has a winning strategy in the Alice-start $(2, 1)$ -MCT game. After Alice plays two vertices in her first move, there is a vertex (i, j) such that (i, j) is adjacent to at least three unplayed vertices. Then Bob plays (i, j) and wins the game in his next move. Therefore $a_1(C_n \square P_m) = 3$.

(\Rightarrow) Assume that $a_1(C_n \square P_m) = 3$. By Theorem 2.1(i), there exists a set X such that $\Delta((C_n \square P_m) - X) \leq 3$ and $|X| \leq 3$. Then all vertices of degree 4 in $C_n \square P_m$ belong to $N_{C_n \square P_m}[X]$ which implies that X dominates all vertices from $V(G)$. If there exists a vertex $(i, j) \in X - V(G)$, then (i, j) is adjacent to exactly one vertex (i', j') from $V(G)$ and we can replace (i, j) with (i', j') in X . By repeating this process for the remaining vertices of X if necessary, we find a dominating set of G of order at most 3. \square

In [17], the domination number of $\gamma(C_n \square P_m)$ was determined for each pair of parameters with $n \geq 4$ and $2 \leq m \leq 7$. It is well-known that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ and that $\gamma(C_n \square P_m) > 3$ holds whenever $n \geq 4$ and $m \geq 8$. From these facts we can deduce the following statement:

Proposition 4.8 *If $n \geq 4$ and $m \geq 1$, then $\gamma(C_n \square P_m) \leq 3$ if and only if one of the following conditions holds:*

- (i) $n = 4$ and $m \in [3]$;
- (ii) $n = 5$ and $m \in [2]$;
- (iii) $6 \leq n \leq 9$ and $m = 1$.

We are now in a position to complete the proof of Theorem 4.2. Since the other cases have already been covered, it suffices to consider cylinder graphs $C_n \square P_m$ with $n \geq 4$.

From Lemma 4.7 and Proposition 4.8 we deduce that if $n \geq 4$, then $a_1(C_n \square P_m) = 3$ if and only if either $n = 4$ and $3 \leq m \leq 5$, or $n = 5$ and $3 \leq m \leq 4$, or $6 \leq n \leq 9$ and $m = 3$. It remains to show that $a_1(C_n \square P_m) = 4$ in each of the following cases:

- (i) $n = 4$ and $m \geq 6$,
- (ii) $n = 5$ and $m \geq 5$,
- (iii) $n \geq 6$ and $m \geq 4$,
- (iv) $n \geq 10$ and $m = 3$.

$n \setminus m$	2	3	4	5	6	7	8	9	10	11	≥ 12
2	2	2	2	2	3	3	3	3	3	3	3
3		2	2	3	3	3	3	3	3	3	4
4			3	3	3	3	4	4	4	4	4
5				3	4	4	4	4	4	4	4
≥ 6					4	4	4	4	4	4	4

Table 2: $a_1(P_n \square P_m)$ for $2 \leq n \leq m$

For these cases, $a_1(C_n \square P_m) \neq 3$ and Theorem 2.1(i) implies $a_1(C_n \square P_m) > 2$. From Observation 1.5(iii) we deduce $a_1(C_n \square P_m) \leq 4$. Therefore, $a_1(C_n \square P_m) = 4$ holds under the conditions (i)-(iv). This finishes the proof of Theorem 4.2.

4.3 Grid graphs

Theorem 4.9 *If $m \geq n \geq 2$, then*

$$a_1(P_n \square P_m) = \begin{cases} 2; & n = 2, 2 \leq m \leq 5, \text{ or} \\ & n = 3, m = 3, 4, \\ 3; & n = 2, m \geq 6, \text{ or} \\ & n = 3, 5 \leq m \leq 11, \text{ or} \\ & n = 4, 4 \leq m \leq 7, \text{ or} \\ & n = m = 5, \\ 4; & \text{otherwise.} \end{cases}$$

and

$$a'_1(P_n \square P_m) = \begin{cases} 2; & n = m = 2, \\ 3; & n = 2, m \geq 3, \\ 4; & m \geq n \geq 3. \end{cases}$$

Table 2 summarizes the values of $a_1(P_n \square P_m)$.

In the rest of the subsection we prove Theorem 4.9. Observe that the Cartesian product $P_n \square P_m$ is triangle-free with no isolated vertices for all $n, m \geq 2$. By Theorem 2.1(ii), $a'_1(P_n \square P_m) = \Delta(P_n \square P_m)$ and thus, we can conclude that

$$a'_1(P_n \square P_m) = \begin{cases} 2; & n = m = 2, \\ 3; & n = 2, m \geq 3, \\ 4; & m \geq n \geq 3. \end{cases}$$

Next, we consider the threshold for the Alice-start game on $P_2 \square P_m$ where $m \geq 2$.

Proposition 4.10 *If $m \geq 2$, then*

$$a_1(P_2 \square P_m) = \begin{cases} 2; & 2 \leq m \leq 5, \\ 3; & m \geq 6. \end{cases}$$

Proof. Assume that $m \geq 2$ and consider the Alice-start $(a, 1)$ -MCT game on $P_2 \square P_m$.

Case 1: $2 \leq m \leq 5$.

To show that $a_1(P_2 \square P_m) \geq 2$, we will provide a winning strategy for Bob in the Alice-start $(1, 1)$ -MCT game. Assume that Alice plays (i, j) . Then Bob replies by playing a vertex (i', j') such that $i' \neq i$ and $j' \neq j$, and he can win the game with his next move. Thus $a_1(P_2 \square P_m) \geq 2$.

It remains to show that $a_1(P_2 \square P_m) \leq 2$. The assertion is clear for $m = 2$. Let $X_i = \{(1, 2), (2, i - 1)\}$, $i \in \{3, 4, 5\}$. Then $\max\{\Delta((P_2 \square P_i) - X_i), |X_i|\} = 2$ holds for each $i \in \{3, 4, 5\}$. Hence Theorem 2.1(i) implies the assertion.

Case 2: $m \geq 6$.

It is straightforward to verify that by deleting any two vertices of $P_2 \square P_m$, the maximum degree of the rest of the graph remains 3. Hence Theorem 2.1(i) yields $a_1(P_2 \square P_m) \geq 3$. On the other hand, since each vertex in $P_2 \square P_m$ is incident with at most three cliques, Observation 1.5(iii) implies $a_1(P_2 \square P_m) \leq 3$. \square

Lemma 4.11 *Assume that $m \geq n \geq 3$. Then $a_1(P_n \square P_m) \leq 3$ holds if and only if $\gamma(P_{n-2} \square P_{m-2}) \leq 3$.*

Proof. The proof proceeds along the same lines as the corresponding inequality part of the proof of Lemma 4.7. \square

Obviously, $\gamma(P_n) = \gamma(P_n \square P_1) = \lceil \frac{n}{3} \rceil$. In [1], $\gamma(P_n \square P_m)$ was calculated for $m, n \leq 29$ by using a dynamic programming algorithm. According to these results we may conclude the following:

Proposition 4.12 *If $m \geq n \geq 1$, then $\gamma(P_n \square P_m) \leq 3$ if and only if one of the following conditions holds:*

- $n = 1$ and $1 \leq m \leq 9$;

- $n = 2$ and $2 \leq m \leq 5$;
- $n = m = 3$.

Lemma 4.13 *Assume that $m \geq n \geq 3$. Then $a_1(P_n \square P_m) = 2$ if and only if $n = 3$ and $m \in \{3, 4\}$.*

Proof. If $n = 3$ and $m \geq 5$, or $m \geq n \geq 4$, then by Theorem 2.1(i) we get $a_1(P_n \square P_m) > 2$. Assume next that $n = 3$ and $m \in \{3, 4\}$. Let $X = \{(2, 2), (2, 3)\}$. Then $\Delta((P_n \square P_m) - X) = 2$, so $a_1(P_n \square P_m) \leq 2$ by Theorem 2.1(i). It remains to show that $a_1(P_n \square P_m) \geq 2$ by providing a winning strategy for Bob in the Alice-start $(2, 1)$ -MCT game. After Alice plays (i, j) in her first move, it is easy to see that there is (i', j') which is $i' \neq i$ and $j' \neq j$. Then Bob replies by playing (i', j') and wins the game in his next move. Hence $a_1(P_n \square P_m) = 2$. \square

For $m \geq n \geq 3$, according to Lemmas 4.11 and 4.13, and to Proposition 4.12, we can conclude that $a_1(P_n \square P_m) = 3$ if m and n satisfy one of the following conditions:

- $n = 3$ and $5 \geq m \geq 11$,
- $n = 4$ and $4 \geq m \geq 7$,
- $n = m = 5$.

To complete the proof of Theorem 4.9, it remains to detect when $m \geq n \geq 3$ and $a_1(C_n \square P_m) > 3$ hold. By Observation 1.5(iii) we have $a_1(P_n \square P_m) \leq 4$, hence in these cases $a_1(P_n \square P_m) = 4$ holds. By Lemma 4.11 and Proposition 4.12, these are the following cases:

- $n = 3$ and $m \geq 12$,
- $n = 4$ and $m \geq 8$,
- $n = 5$ and $m \geq 6$,
- $m \geq n \geq 6$.

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