On distance-balanced generalized Petersen graphs

Gang Ma[a] Jianfeng Wang[a,*] Sandi Klavžar[b,c,d]

[a] School of Mathematics and Statistics, Shandong University of Technology
Zibo, China
math_magang@163.com
jfwang@sdut.edu.cn

[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
sandi.klavzar@fmf.uni-lj.si

[c] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[d] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Abstract

A connected graph G of diameter $\mathrm{diam}(G) \geq \ell$ is ℓ-distance-balanced if $|W_{xy}| = |W_{yx}|$ for every $x, y \in V(G)$ with $d_G(x,y) = \ell$, where W_{xy} is the set of vertices of G that are closer to x than to y. We prove that the generalized Petersen graph $GP(n,k)$ is $\mathrm{diam}(GP(n,k))$-distance-balanced provided that n is large enough relative to k. This partially solves a conjecture posed by Miklavič and Šparl [20]. We also determine $\mathrm{diam}(GP(n,k))$ when n is large enough relative to k.

Key words: generalized Petersen graph; distance-balanced graph; ℓ-distance-balanced graph; diameter

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1 Introduction

If $G = (V(G), E(G))$ is a connected graph and $x, y \in V(G)$, then the *distance* $d_G(x,y)$ between x and y is the number of edges on a shortest x,y-path. The

*Corresponding author

diameter $\mathrm{diam}(G)$ of G is the maximum distance between its vertices. The set W_{xy} contains the vertices that are closer to x than to y, that is,

$$W_{xy} = \{w \in V(G) : \ d_G(w,x) < d_G(w,y)\}\,.$$

Vertices x and y are *balanced* if $|W_{xy}| = |W_{yx}|$. For an integer $\ell \in [\mathrm{diam}(G)] = \{1,2,\ldots,\mathrm{diam}(G)\}$ we say that G is *ℓ-distance-balanced* if each pair of vertices $x, y \in V(G)$ with $d_G(x,y) = \ell$ is balanced. G is said to be *highly distance-balanced* if it is ℓ-distance-balanced for every $\ell \in [\mathrm{diam}(G)]$. 1-distance-balanced graphs are simply called *distance-balanced* graphs.

Distance-balanced graphs were first considered by Handa [11] back in 1999, while the term "distance-balanced" was proposed a decade later by Jerebic et al. in [13]. The latter paper was the trigger for intensive research of distance-balanced graphs, see [1, 3–6, 8, 12, 16–19, 23]. The study of distance-balanced graphs is interesting from various purely graph-theoretic aspects where one focuses on particular properties of such graphs such as symmetry, connectivity or complexity aspects of algorithms related to such graphs. Moreover, distance-balanced graphs have motivated the introduction of the hitherto much-researched Mostar index [2, 7] and distance-unbalancedness of graphs [15, 21, 22]. In this context, distance-balanced graphs are the graphs with the Mostar index equal to 0.

In his dissertation [9], Frelih generalized distance-balanced graphs to ℓ-distance balanced graphs. The special case of $\ell = 2$ has been studied in detail in [10]. Among other results it was demonstrated that there exist 2-distance-balanced graphs that are not 1-distance-balanced. 2-distance-balanced graphs that are not 2-connected were characterized as well as 2-distance-balanced Cartesian and lexicographic products. In this direction, ℓ-distance-balanced corona products and lexicographic products were investigated in [14]. In [20], Miklavič and Šparl obtained some general results on ℓ-distance balanced graphs. They studied graphs of diameter at most 3 and investigated ℓ-distance-balancedness of cubic graphs, in particular of generalized Petersen graphs. Although generalized Petersen graphs are a family of cubic graphs but it is difficult to determine whether they are ℓ-distance-balanced or not for some ℓ. And that is what has stimulated the main interest in this article. Before we explain this in more detail, let us define these graphs.

If $n \geq 3$ and $1 \leq k < n/2$, then the *generalized Petersen graph* $GP(n,k)$ is defined by

$$\begin{aligned} V(GP(n,k)) &= \{u_i : \ i \in \mathbb{Z}_n\} \cup \{v_i : \ i \in \mathbb{Z}_n\}, \\ E(GP(n,k)) &= \{u_i u_{i+1} : \ i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} : \ i \in \mathbb{Z}_n\} \cup \{u_i v_i : \ i \in \mathbb{Z}_n\}. \end{aligned}$$

$GP(6,2)$ is shown in Figure 1.

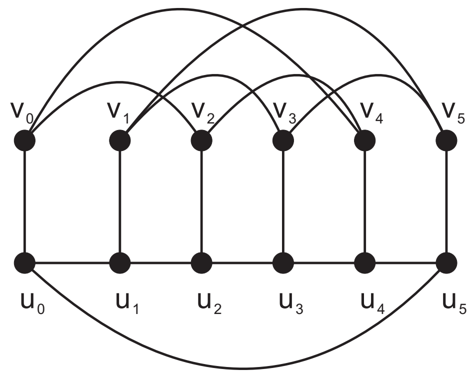

Figure 1: The generalized Petersen graph $GP(6,2)$. The cycle $u_0u_1u_2u_3u_4u_5$ contains six outer edges. v_0v_2, v_1v_3, v_2v_4, v_3v_5, v_4v_0 and v_5v_1 are six inner edges. Finally u_0v_0, u_1v_1, u_2v_2, u_3v_3, u_4v_4 and u_5v_5 are six spokes.

Now, we recall the following conjecture and result, where the conjecture was supported by an extensive computer search.

Conjecture 1. [20, Conjecture 5.2] *If $n \geq 3$, $2 \leq k < n/2$, and there exists $j \in \mathbb{Z}_n$ such that $d(u_0, v_j) = \operatorname{diam}(GP(n,k))$, then either $n = 4m$ and $k = 2m-1$ for some $m \geq 3$, or $(n,k) \in \{(5,2),(7,2),(7,3)\}$.*

Proposition 2. [20, Proposition 5.3] *If $n \geq 3$, $2 \leq k < n/2$, and if Conjecture 1 holds, then $GP(n,k)$ is $\operatorname{diam}(GP(n,k))$-distance-balanced.*

The main result of this paper reads as follows.

Theorem 3. *If n and k are integers, where $3 \leq k < n/2$ and*

$$n \geq \begin{cases} 8; & k = 3, \\ 10; & k = 4, \\ \frac{k(k+1)}{2}; & k \text{ is odd and } k \geq 5, \\ \frac{k^2}{2}; & k \text{ is even and } k \geq 6, \end{cases}$$

then $GP(n,k)$ is $\operatorname{diam}(GP(n,k))$-distance-balanced.

Theorem 3 is proved in Section 2. In view of Proposition 2, to prove Theorem 3 it suffices to verify that Conjecture 1 holds true for the cases as listed in the theorem. The difficulty in proving Conjecture 1 in general lies in the fact that the distance function on generalized Petersen graphs is very difficult to manage and depends heavily on n and k. In particular, as pointed out by Miklavič and Šparl in [20,

p. 150], the diameter of $GP(n,k)$ is not known in general. In Section 3 we then determine $\operatorname{diam}(GP(n,k))$ for the corresponding values of n and k. The rather complicated result indicates that it is indeed difficult to control the diameter of generalized Petersen graphs. Finally, in Section 4, we list some problems which are worth studying in the future.

2 Proof of Theorem 3

Consider the generalized Petersen graph $GP(n,k)$. The edges of the form $u_i u_{i+1}$ are *outer* edges, the edges of the form $v_i v_{i+k}$ are *inner* edges, and edges of the form $u_i v_i$ are *spokes*. To simplify the notation, set $D = \operatorname{diam}(GP(n,k))$ throughout this section. We will also omit the subscript in $d_{GP(n,k))}(x,y)$ as the graph $GP(n,k)$ is clear from the context.

As already stated at the end of the previous section, in order to prove Theorem 3, it suffices to prove that if n and k are integers, where $3 \le k < n/2$ and

$$n \ge \begin{cases} 8; & k = 3, \\ 10; & k = 4, \\ \frac{k(k+1)}{2}; & k \text{ is odd and } k \ge 5, \\ \frac{k^2}{2}; & k \text{ is even and } k \ge 6, \end{cases}$$

then for any $j \in \mathbb{Z}_n$ we have $d(u_0, v_j) < D$.

By the symmetry of $GP(n,k)$ it suffices to consider $d(u_0, v_j)$, where $0 \le j \le n/2$. Our aim is to find an index j^*, where $0 \le j^* \le n/2$, such that

$$d(u_0, v_{j^*}) = \max\{d(u_0, v_j) : \; 0 \le j \le n/2\},$$

and prove that $d(u_0, v_{j^*}) < D$.

Let j be an integer such that $1 \le j \le n/2$. Suppose $j = m_0 k + j_0$ and $n - j = m_1 k + j_1$, where $0 \le j_0, j_1 < k$. Four types of u_0, v_j-path are defined in the following.

$$\begin{aligned} P_1 &= u_0 u_1 u_2 \cdots u_{j_0} v_{j_0} v_{k+j_0} v_{2k+j_0} \cdots v_{m_0 k + j_0}, \\ P_2 &= u_0 u_{-1} u_{-2} \cdots u_{-(k-j_0)} v_{-(k-j_0)} v_{j_0} v_{k+j_0} \cdots v_{m_0 k + j_0}, \\ P_3 &= u_0 u_{-1} u_{-2} \cdots u_{-j_1} v_{-j_1} v_{-(k+j_1)} v_{-(2k+j_1)} \cdots v_{-(m_1 k + j_1)}, \\ P_4 &= u_0 u_1 u_2 \cdots u_{k-j_1} v_{k-j_1} v_{-j_1} v_{-k-j_1} \cdots v_{-m_1 k - j_1}. \end{aligned}$$

Note that $u_{-i} = u_{n-i}$, so $v_{-m_1 k - j_1} = v_{n - m_1 k - j_1} = v_j = v_{m_0 k + j_0}$. Also note that all P_1, P_2, P_3, P_4 have only one spoke. The length of P_1 is $j_0 + m_0 + 1$, the length of P_2 is $(k - j_0) + m_0 + 2$, the length of P_3 is $j_1 + m_1 + 1$, and the length of P_4 is $(k - j_1) + m_1 + 2$.

In $GP(6,2)$, the u_0, v_3-path of type P_1 is $u_0u_1v_1v_3$. The u_0, u_3-path of type P_2 is $u_0u_{-1}v_{-1}v_1v_3 = u_0u_5v_5v_1v_3$. The u_0, u_3-path of type P_3 is $u_0u_{-1}v_{-1}v_{-3} = u_0u_5v_5v_3$. The u_0, u_3-path of type P_4 is $u_0u_1v_1v_{-1}v_{-3} = u_0u_1v_1v_5v_3$.

We first prove the following lemma about the u_0, v_j-path of $GP(n,k)$.

Lemma 4. *Suppose that two integers k, n and four paths P_1, P_2, P_3, P_4 are the same as above. In $GP(n,k)$, for any integer j where $1 \le j \le n/2$, a u_0, v_j-path of minimum length contains only one spoke and belongs to one of the four types $\{P_1, P_2, P_3, P_4\}$.*

Proof. For the convenience of computing the distance of the path in $GP(n,k)$, we divide the direction of the path into *positive* and *negative* direction, and use different vertex subscripts marking method according to the direction of the path.

Suppose that there is a path from u_{i_1} via outer edges. If the path from u_{i_1} is of *positive* direction, then the path is denoted by $u_{i_1}u_{i_1+1}u_{i_1+2}\cdots$. If the path from u_{i_1} is of *negative* direction, then the path is denoted by $u_{i_1}u_{i_1-1}u_{i_1-2}\cdots$. Using the above vertex subscripts marking method, the distance of a u_{i_1}, u_{i_2}-path (via both directions) via outer edges is $|i_2 - i_1|$.

Similarly, suppose that there is a path from v_{i_1} via inner edges. If the path from v_{i_1} is of *positive* direction, then the path is denoted by $v_{i_1}v_{i_1+k}v_{i_1+2k}\cdots$. If the path from v_{i_1} is of *negative* direction, then the path is denoted by $v_{i_1}v_{i_1-k}v_{i_1-2k}\cdots$. Using the above vertex subscripts marking method, the distance of a v_{i_1}, v_{i_2}-path (via both directions) via inner edges is $|\frac{i_2-i_1}{k}|$.

Whenever considering the path that connects u_{i_1} to u_{i_2} via outer edges, it is always negative if $i_2 < i_1$, and positive otherwise. Same for the path that connects v_{i_2} to v_{i_3} via inner edges.

Claim 1. A u_0, v_j-path of minimum length contains only one spoke.

Let $J = j + rn$ where r is an integer. Note that $v_J = v_j$.

Note that a u_0, v_J-path cannot contain even number of spokes. Let $P^{(1)}$ be a u_0, v_J-path containing 3 spokes. Suppose that $P^{(1)}$ connects u_0 and u_{i_1} via outer edges, then spoke $u_{i_1}v_{i_1}$, then connects v_{i_1} and v_{i_2} via inner edges, then spoke $v_{i_2}u_{i_2}$, then connects u_{i_2} and u_{i_3} via outer edges, then spoke $u_{i_3}v_{i_3}$, and then connects v_{i_3} and v_J via inner edges.

Let $P^{(2)}$ be the u_0, v_J-path that connects u_0 and $u_{i_1+i_3-i_2}$ via outer edges, then spoke $u_{i_1+i_3-i_2}v_{i_1+i_3-i_2}$, and then connects $v_{i_1+i_3-i_2}$ and v_J via inner edges.

Let $LEN(P)$ be the length of path P. Then

$$LEN(P^{(1)}) = (|i_1| + 1 + |\frac{i_2 - i_1}{k}| + 1 + |i_3 - i_2| + 1 + |\frac{J - i_3}{k}|),$$
$$LEN(P^{(2)}) = (|i_1 + i_3 - i_2| + 1 + |\frac{J - i_1 - i_3 + i_2}{k}|).$$

Because $|a+b| \leq |a|+|b|$ for two integers, $|i_1+i_3-i_2| \leq |i_1|+|i_3-i_2|$ and $|\frac{J-i_1-i_3+i_2}{k}| \leq |\frac{J-i_3}{k}|+|\frac{i_2-i_1}{k}|$. We get $LEN(P^{(1)})-LEN(P^{(2)}) \geq 2$. So $P^{(1)}$ is a u_0, v_j-path but not of minimum length.

If a u_0, v_J-path contains 5 or more than 5 spokes, similar transformation like above can give a new u_0, v_J-path which has smaller spokes and smaller length than the original u_0, v_J-path.

Claim 2. A u_0, v_j-path of minimum length belongs to one of the four types $\{P_1, P_2, P_3, P_4\}$.

Let $P^{(3)}$ be a u_0, v_J-path with one spoke. Suppose that $P^{(3)}$ connects u_0 and u_{i_1} via outer edges, then spoke $u_{i_1}v_{i_1}$, and then connects v_{i_1} and v_J via inner edges.

Firstly we prove that $P^{(3)}$ is not a minimum u_0, v_j-path if $|i_1| \geq k$. Suppose $|i_1| \geq k$. Let $i_1 = sk+t$ such that $s \geq 1$ and $0 \leq t < k$ when $i_1 \geq k$, and $s \leq -1$ and $-k < t \leq 0$ when $i_1 \leq -k$.

Let $P^{(4)}$ be the u_0, v_J-path which connects u_0 and u_t via outer edges (with the same direction as the u_0, u_{i_1}-path via outer edges in $P^{(3)}$), then spoke u_tv_t, and then connects v_t and v_J via inner edges (with the same direction as the v_{i_1}, v_J-path via inner edges in $P^{(3)}$).

We discuss the following four cases.

(1) In $P^{(3)}$, the u_0, u_{i_1}-path via outer edges is of positive direction and the v_{i_1}, v_J-path via inner edges is of positive direction.

Note that

$$LEN(P^{(3)}) = |i_1|+1+|\frac{J-i_1}{k}| = i_1+1+\frac{J-i_1}{k},$$
$$LEN(P^{(4)}) = |t|+1+|\frac{J-t}{k}| = t+1+\frac{J-t}{k}.$$

$LEN(P^{(3)}) - LEN(P^{(4)}) = s(k-1) > 0$. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(2) In $P^{(3)}$, the u_0, u_{i_1}-path via outer edges is of positive direction and the v_{i_1}, v_J-path via inner edges is of negative direction.

Note that

$$LEN(P^{(3)}) = |i_1|+1+|\frac{J-i_1}{k}| = i_1+1+\frac{i_1-J}{k},$$
$$LEN(P^{(4)}) = |t|+1+|\frac{J-t}{k}| = t+1+\frac{t-J}{k}.$$

$LEN(P^{(3)}) - LEN(P^{(4)}) = s(k+1) > 0$. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(3) In $P^{(3)}$, the u_0, u_{i_1}-path via outer edges is of negative direction and the v_{i_1}, v_J-path via inner edges is of positive direction.

Note that

$$LEN(P^{(3)}) = |i_1| + 1 + |\frac{J - i_1}{k}| = -i_1 + 1 + \frac{J - i_1}{k},$$
$$LEN(P^{(4)}) = |t| + 1 + |\frac{J - t}{k}| = -t + 1 + \frac{J - t}{k}.$$

$LEN(P^{(3)}) - LEN(P^{(4)}) = -s(k+1) > 0$. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(4) In $P^{(3)}$, the u_0, u_{i_1}-path via outer edges is of negative direction and the v_{i_1}, v_J-path via inner edges is of negative direction.

Note that

$$LEN(P^{(3)}) = |i_1| + 1 + |\frac{J - i_1}{k}| = -i_1 + 1 + \frac{i_1 - J}{k},$$
$$LEN(P^{(4)}) = |t| + 1 + |\frac{J - t}{k}| = -t + 1 + \frac{t - J}{k}.$$

$LEN(P^{(3)}) - LEN(P^{(4)}) = -s(k-1) > 0$. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

Secondly we prove that $P^{(3)}$ is not a minimum u_0, v_j-path if $J > n$ or $J < -n$ (that is $r \neq 0, -1$). Suppose that $|i_1| < k$ and $r \neq 0, -1$. We discuss the following four cases.

(1) $0 \leq i_1 < k$ and $r \geq 1$.

In this case, the u_0, u_{i_1}-path via outer edges is of positive direction and the v_{i_1}, v_J-path via inner edges is of positive direction. Note that $LEN(P^{(3)}) = |i_1| + 1 + |\frac{j+rn-i_1}{k}| = i_1 + 1 + \frac{j+rn-i_1}{k}$.

(1.1) When k is odd and $j_0 \leq \frac{k+1}{2}$, or k is even and $j_0 \leq \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_1)$.

If $i_1 \geq j_0$, $LEN(P^{(3)}) - LEN(P_1) = (i_1 + 1 + \frac{j+rn-i_1}{k}) - (j_0 + 1 + \frac{j-j_0}{k}) = \frac{rn-(i_1-j_0)}{k} + (i_1 - j_0) > 0$.

If $i_1 < j_0$, $1 \leq j_0 - i_1 \leq \frac{k+1}{2}$ when k is odd (or $1 \leq j_0 - i_0 \leq \frac{k}{2}$ when k is even). Recall that $n \geq \frac{k(k+1)}{2}$ when k is odd (or $n \geq \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_1) &= \frac{rn + j_0 - i_1}{k} - (j_0 - i_1) \\ &> \frac{rk(k+1)/2}{k} - \frac{k+1}{2} \\ &= \frac{(r-1)(k+1)}{2} \geq 0 \end{aligned}$$

when k is odd and

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_1) &= \frac{rn + j_0 - i_1}{k} - (j_0 - i_1) \\ &> \frac{rk^2/2}{k} - \frac{k}{2} \\ &= \frac{(r-1)k}{2} \geq 0 \end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(1.2) When k is odd and $j_0 > \frac{k+1}{2}$, or k is even and $j_0 > \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_2)$.

If $i_1 \geq j_0$, $i_1 > k - j_0$ and so $i_1 + j_0 - k > 0$. Then $LEN(P^{(3)}) - LEN(P_2) = (i_1 + 1 + \frac{j+rn-i_1}{k}) - (k - j_0 + 2 + \frac{j-j_0}{k}) = \frac{rn-(i_1-j_0)}{k} + (i_1 + j_0 - k - 1) > 0$.

If $i_1 < j_0$, $j_0 - i_1 \geq 1$. Recall that $n \geq \frac{k(k+1)}{2}$ when k is odd (or $n \geq \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_2) &= \frac{rn + j_0 - i_1}{k} - (k + 1 - i_1 - j_0) \\ &> \frac{rk(k+1)/2}{k} - \frac{k+1}{2} \\ &= \frac{(r-1)(k+1)}{2} \geq 0 \end{aligned}$$

when k is odd and

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_2) &= \frac{rn + j_0 - i_1}{k} - (k + 1 - i_1 - j_0) \\ &> \frac{rk^2/2}{k} - \frac{k}{2} \\ &= \frac{(r-1)k}{2} \geq 0 \end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(2) $0 \leq i_1 < k$ and $r \leq -2$.

In this case, the u_0, u_{i_1}-path via outer edges is of positive direction and the v_{i_1}, v_J-path via inner edges is of negative direction. Note that $LEN(P^{(3)}) = |i_1| + 1 + |\frac{j+rn-i_1}{k}| = i_1 + 1 + \frac{i_1-(j+rn)}{k}$.

(2.1) When k is odd and $j_0 \leq \frac{k+1}{2}$, or k is even and $j_0 \leq \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_1)$.

If $i_1 \geq j_0$, $LEN(P^{(3)}) - LEN(P_1) = (i_1 + 1 + \frac{i_1-(j+rn)}{k}) - (j_0 + 1 + \frac{j-j_0}{k}) = \frac{i_1+j_0-rn-2j}{k} + (i_1 - j_0) > 0$.

If $i_1 < j_0$, $1 \le j_0 - i_1 \le \frac{k+1}{2}$ when k is odd (or $1 \le j_0 - i_0 \le \frac{k}{2}$ when k is even). Recall that $n \ge \frac{k(k+1)}{2}$ when k is odd (or $n \ge \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_1) &= \frac{i_1 + j_0 - rn - 2j}{k} - (j_0 - i_1) \\ &\ge \frac{i_1 + j_0 - rn - n}{k} - \frac{k+1}{2} \\ &> \frac{(-r-1)k(k+1)/2}{k} - \frac{k+1}{2} \\ &= \frac{(-r-2)(k+1)}{2} \ge 0 \end{aligned}$$

when k is odd and

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_1) &= \frac{i_1 + j_0 - rn - 2j}{k} - (j_0 - i_1) \\ &\ge \frac{i_1 + j_0 - rn - n}{k} - \frac{k}{2} \\ &> \frac{(-r-1)k^2/2}{k} - \frac{k}{2} \\ &= \frac{(-r-2)k}{2} \ge 0 \end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(2.2) When k is odd and $j_0 > \frac{k+1}{2}$, or k is even and $j_0 > \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_2)$.

If $i_1 \ge j_0$, $i_1 > k - j_0$ and so $i_1 + j_0 - k > 0$. Then $LEN(P^{(3)}) - LEN(P_2) = (i_1 + 1 + \frac{i_1 - (j + rn)}{k}) - (k - j_0 + 2 + \frac{j - j_0}{k}) = \frac{i_1 + j_0 - rn - 2j}{k} + (i_1 + j_0 - k - 1) > 0$.

If $i_1 < j_0$, $j_0 - i_1 \ge 1$. Recall that $n \ge \frac{k(k+1)}{2}$ when k is odd (or $n \ge \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_2) &= \frac{i_1 + j_0 - rn - 2j}{k} - (k + 1 - i_1 - j_0) \\ &> \frac{i_1 + j_0 - rn - n}{k} - \frac{k+1}{2} \\ &> \frac{(-r-1)k(k+1)/2}{k} - \frac{k+1}{2} \\ &= \frac{(-r-2)(k+1)}{2} \ge 0 \end{aligned}$$

when k is odd and

$$\begin{aligned}LEN(P^{(3)}) - LEN(P_2) &= \frac{i_1 + j_0 - rn - 2j}{k} - (k + 1 - i_1 - j_0)\\ &\geq \frac{i_1 + j_0 - rn - n}{k} - \frac{k}{2}\\ &> \frac{(-r-1)k^2/2}{k} - \frac{k}{2}\\ &= \frac{(-r-2)k}{2} \geq 0\end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(3) $-k < i_1 \leq 0$ and $r \geq 1$.

In this case, the u_0, u_{i_1}-path via outer edges is of negative direction and the v_{i_1}, v_J-path via inner edges is of positive direction. Note that $LEN(P^{(3)}) = |i_1| + 1 + |\frac{j+rn-i_1}{k}| = -i_1 + 1 + \frac{j+rn-i_1}{k}$.

(3.1) When k is odd and $j_0 \leq \frac{k+1}{2}$, or k is even and $j_0 \leq \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_1)$.

If $-i_1 \geq j_0$, $LEN(P^{(3)}) - LEN(P_1) = (-i_1 + 1 + \frac{j+rn-i_1}{k}) - (j_0 + 1 + \frac{j-j_0}{k}) = \frac{rn-i_1+j_0}{k} + (-i_1 - j_0) > 0$.

If $-i_1 < j_0$, $1 \leq j_0 + i_1 \leq \frac{k+1}{2}$ when k is odd (or $1 \leq j_0 + i_0 \leq \frac{k}{2}$ when k is even). Recall that $n \geq \frac{k(k+1)}{2}$ when k is odd (or $n \geq \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned}LEN(P^{(3)}) - LEN(P_1) &= \frac{rn + j_0 - i_1}{k} - (j_0 + i_1)\\ &> \frac{rk(k+1)/2}{k} - \frac{k+1}{2}\\ &= \frac{(r-1)(k+1)}{2} \geq 0\end{aligned}$$

when k is odd and

$$\begin{aligned}LEN(P^{(3)}) - LEN(P_1) &= \frac{rn + j_0 - i_1}{k} - (j_0 + i_1)\\ &> \frac{rk^2/2}{k} - \frac{k}{2}\\ &= \frac{(r-1)k}{2} \geq 0\end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(3.2) When k is odd and $j_0 > \frac{k+1}{2}$, or k is even and $j_0 > \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_2)$.

If $-i_1 \geq j_0$, $-i_1 > k - j_0$ and so $-i_1 + j_0 - k > 0$. Then $LEN(P^{(3)}) - LEN(P_2) = (-i_1 + 1 + \frac{j+rn-i_1}{k}) - (k - j_0 + 2 + \frac{j-j_0}{k}) = \frac{rn-i_1+j_0}{k} + (-i_1 + j_0 - k - 1) > 0$.

If $-i_1 < j_0$, $j_0 + i_1 \geq 1$. Recall that $n \geq \frac{k(k+1)}{2}$ when k is odd (or $n \geq \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_2) &= \frac{rn + j_0 - i_1}{k} - (k + 1 + i_1 - j_0) \\ &> \frac{rk(k+1)/2}{k} - \frac{k+1}{2} \\ &= \frac{(r-1)(k+1)}{2} \geq 0 \end{aligned}$$

when k is odd and

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_2) &= \frac{rn + j_0 - i_1}{k} - (k + 1 + i_1 - j_0) \\ &> \frac{rk^2/2}{k} - \frac{k}{2} \\ &= \frac{(r-1)k}{2} \geq 0 \end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(4) $-k < i_1 \leq 0$ and $r \leq -2$.

In this case, the u_0, u_{i_1}-path via outer edges is of negative direction and the v_{i_1}, v_J-path via inner edges is of negative direction. Note that $LEN(P^{(3)}) = |i_1| + 1 + |\frac{j+rn-i_1}{k}| = -i_1 + 1 + \frac{i_1-(j+rn)}{k}$.

(4.1) When k is odd and $j_0 \leq \frac{k+1}{2}$, or k is even and $j_0 \leq \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_1)$.

If $-i_1 \geq j_0$, $LEN(P^{(3)}) - LEN(P_1) = (-i_1 + 1 + \frac{i_1-(j+rn)}{k}) - (j_0 + 1 + \frac{j-j_0}{k}) = \frac{i_1+j_0-rn-2j}{k} + (-i_1 - j_0) > 0$.

If $-i_1 < j_0$, $1 \leq j_0 + i_1 \leq \frac{k+1}{2}$ when k is odd (or $1 \leq j_0 + i_0 \leq \frac{k}{2}$ when k is even). Recall that $n \geq \frac{k(k+1)}{2}$ when k is odd (or $n \geq \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_1) &= \frac{i_1 + j_0 - rn - 2j}{k} - (j_0 + i_1) \\ &\geq \frac{i_1 + j_0 - rn - n}{k} - \frac{k+1}{2} \\ &> \frac{(-r-1)k(k+1)/2}{k} - \frac{k+1}{2} \\ &= \frac{(-r-2)(k+1)}{2} \geq 0 \end{aligned}$$

when k is odd and

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_1) &= \frac{i_1 + j_0 - rn - 2j}{k} - (j_0 + i_1) \\ &\geq \frac{i_1 + j_0 - rn - n}{k} - \frac{k}{2} \\ &> \frac{(-r-1)k^2/2}{k} - \frac{k}{2} \\ &= \frac{(-r-2)k}{2} \geq 0 \end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

(4.2) When k is odd and $j_0 > \frac{k+1}{2}$, or k is even and $j_0 > \frac{k}{2}$.

We compare $LEN(P^{(3)})$ with $LEN(P_2)$.

If $-i_1 \geq j_0$, $-i_1 > k - j_0$ and so $-i_1 + j_0 - k > 0$. Then $LEN(P^{(3)}) - LEN(P_2) = (-i_1 + 1 + \frac{i_1 - (j+rn)}{k}) - (k - j_0 + 2 + \frac{j - j_0}{k}) = \frac{i_1 + j_0 - rn - 2j}{k} + (-i_1 + j_0 - k - 1) > 0$.

If $-i_1 < j_0$, $j_0 + i_1 \geq 1$. Recall that $n \geq \frac{k(k+1)}{2}$ when k is odd (or $n \geq \frac{k^2}{2}$ when k is even). Then

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_2) &= \frac{i_1 + j_0 - rn - 2j}{k} - (k + 1 + i_1 - j_0) \\ &> \frac{i_1 + j_0 - rn - n}{k} - \frac{k+1}{2} \\ &> \frac{(-r-1)k(k+1)/2}{k} - \frac{k+1}{2} \\ &= \frac{(-r-2)(k+1)}{2} \geq 0 \end{aligned}$$

when k is odd and

$$\begin{aligned} LEN(P^{(3)}) - LEN(P_2) &= \frac{i_1 + j_0 - rn - 2j}{k} - (k + 1 + i_1 - j_0) \\ &\geq \frac{i_1 + j_0 - rn - n}{k} - \frac{k}{2} \\ &> \frac{(-r-1)k^2/2}{k} - \frac{k}{2} \\ &= \frac{(-r-2)k}{2} \geq 0 \end{aligned}$$

when k is even. So $P^{(3)}$ is a u_0, v_j-path but not of minimum length.

The proof of the lemma completes.

□

Let $d_{12}(u_0, v_j)$ be the distance between u_0 and v_j in $GP(n,k)$ via paths of type P_1 or P_2. Let $d_{34}(u_0, v_j)$ be the distance between u_0 and v_j in $GP(n,k)$ via paths of type P_3 or P_4. Then

$$d(u_0, v_j) = \min\{d_{12}(u_0, v_j), d_{34}(u_0, v_j)\}$$

and

$$d(u_0, v_{j^*}) = \max\{d(u_0, v_j) : \ 0 \le j \le n/2\}.$$

We first find j^1 such that $d_{12}(u_0, v_{j^1}) = \max\{d_{12}(u_0, v_j) : \ 0 \le j \le n/2\}$. If $d_{34}(u_0, v_{j^1}) \ge d_{12}(u_0, v_{j^1})$, then $j^* = j^1$. If $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$, we can find j^* around j^1 such that $|d_{12}(u_0, v_{j^*}) - d_{34}(u_0, v_{j^*})| \le 1$ and $\min\{d_{12}(u_0, v_{j^*}), d_{34}(u_0, v_{j^*})\}$ is as large as possible. Note that j^* is not unique.

The following discussions are organized using a tree with depth 3. At depth 1, the discussions are according to the parity of k. At depth 2, the discussions are according to the parity of n. At depth 3, the discussions are according to parameters contained in the small cases.

Case 1: k is odd.
Notice that $n \ge 3k-1$ in this case. We will prove that there exist a j^* such that $d(u_0, v_{j^*}) = \max\{d(u_0, v_j) : \ 0 \le j \le n/2\}$ and $k < j^* \le n/2$.

Case 1.1: n is even.
Suppose $n/2 = m_2k + j_2$ where $0 \le j_2 < k$. From $n \ge 3k-1$, we know that $m_2 \ge 2$, or $m_2 = 1$ and $j_2 \ge \frac{k-1}{2}$.

Case 1.1.1: $j_2 \ge \frac{k-1}{2}$.
If $0 \le j \le \frac{k+1}{2}$, then $d_{12}(u_0, v_{m_2k+j}) = m_2 + 1 + j$. If $\frac{k+1}{2} < j \le j_2$, then $d_{12}(u_0, v_{m_2k+j}) = m_2 + k + 2 - j$. Observe that $d_{12}(u_0, v_{(m_2-1)k+\frac{k+1}{2}}) = m_2 + \frac{k+1}{2}$. Because $j_2 \ge \frac{k-1}{2}$, we infer that $m_2 + 1 + j \ge m_2 + \frac{k+1}{2}$ when $j = \frac{k-1}{2}$ or $j = \frac{k+1}{2}$. So we just need to consider the distance between u_0 and v_{m_2k+j}, where $0 \le j \le j_2$. Note that $d_{12}(u_0, v_{m_2k+\frac{k+1}{2}}) = m_2 + \frac{k+1}{2} + 1$, $d_{12}(u_0, v_{m_2k+\frac{k+1}{2}-1}) = d_{12}(u_0, v_{m_2k+\frac{k+1}{2}+1}) = m_2 + \frac{k+1}{2}$, and so on.

Note that $v_{-(m_2+1)k} = v_{n-(m_2+1)k} = v_{m_2k+2j_2-k}$. Observe that $d_{34}(u_0, v_{m_2k+2j_2-k}) = m_2 + 2$, $d_{34}(u_0, v_{m_2k+2j_2-k+1}) = d_{34}(u_0, v_{m_2k+2j_2-k-1}) = m_2 + 3$, and so on.

If $j_2 = \frac{k-1}{2}$, then $j^1 = m_2k + \frac{k-1}{2}$. Because $d_{34}(u_0, v_{j^1}) \ge d_{12}(u_0, v_{j^1})$, we get $j^* = j^1 = m_2k + \frac{k-1}{2}$.

If $j_2 = \frac{k+1}{2}$, then $j^1 = m_2k + \frac{k+1}{2}$. Because $d_{34}(u_0, v_{j^1}) \ge d_{12}(u_0, v_{j^1})$, we get $j^* = j^1 = m_2k + \frac{k+1}{2}$.

If $3 \le 2j_2 - k \le \frac{k+1}{2}$, then $j^1 = m_2k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_2k + j_2$.

v_j	v_{20}	v_{21}	v_{22}	v_{23}	v_{24}	v_{25}
$d_{12}(u_0, v_j)$	5	6	7	8	7	6
$d_{34}(u_0, v_j)$	7	6	5	4	5	6

Table 1: In $GP(50, 9)$, the search of j^*. Because $j^1 = 23$, $d_{12}(u_0, v_{23}) = 8$, $n-(m_2+1)k = 23$ and $d_{34}(u_0, v_{23}) = 4$, so $j^* = 21$ or $j^* = 25$.

v_j	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
$d_{12}(u_0, v_j)$	5	6	7	6	5	4
$d_{34}(u_0, v_j)$	7	6	5	4	5	6

Table 2: In $GP(42, 9)$, the search of j^*. Because $j^1 = 14$, $d_{12}(u_0, v_{14}) = 7$, $n-(m_2+1)k = 15$ and $d_{34}(u_0, v_{15}) = 4$, so $j^* = 13$.

If $2j_2 - k > \frac{k+1}{2}$, then $j^1 = m_2k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_2k + j_2 - \frac{k-1}{2}$.

Table 1 shows that how to find j^* in $GP(50, 9)$.

Case 1.1.2: $j_2 < \frac{k-1}{2}$.

If $0 \le j \le j_2$, then $d_{12}(u_0, v_{m_2k+j}) = m_2+1+j$. Note that $d_{12}(u_0, v_{(m_2-1)k+\frac{k+1}{2}}) = m_2 + \frac{k+1}{2}$. Because $j_2 < \frac{k-1}{2}$, we have $m_2 + 1 + j < m_2 + \frac{k+1}{2}$ when $0 \le j \le j_2$. So we just need to consider the distance between u_0 and $v_{(m_2-1)k+j}$, where $0 \le j \le k$. Note that $d_{12}(u_0, v_{(m_2-1)k+\frac{k+1}{2}}) = m_2 + \frac{k+1}{2}$, $d_{12}(u_0, v_{(m_2-1)k+\frac{k+1}{2}-1}) = d_{12}(u_0, v_{(m_2-1)k+\frac{k+1}{2}+1}) = m_2 + \frac{k+1}{2} - 1$, and so on.

Note that $v_{-(m_2+1)k} = v_{n-(m_2+1)k} = v_{(m_2-1)k+2j_2}$ and $2j_2 < k - 1$. Moreover, $d_{34}(u_0, v_{(m_2-1)k+2j_2}) = m_2 + 2$, $d_{34}(u_0, v_{(m_2-1)k+2j_2+1}) = d_{34}(u_0, v_{(m_2-1)k+2j_2-1}) = m_2 + 3$, and so on.

If $j_2 = 0$ or $j_2 = 1$, we set $j^1 = (m_2-1)k+\frac{k+1}{2}$. Because $d_{34}(u_0, v_{j^1}) \ge d_{12}(u_0, v_{j^1})$, we have $j^* = j^1 = (m_2 - 1)k + \frac{k+1}{2}$.

If $4 \le 2j_2 \le \frac{k+1}{2}$, we have $j^1 = (m_2 - 1)k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_2 - 1)k + \frac{k+1}{2} + j_2 - 1$.

If $2j_2 > \frac{k+1}{2}$, then $j^1 = (m_2 - 1)k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_2 - 1)k + j_2 + 1$.

Table 2 shows that how to find j^* in $GP(42, 9)$.

Case 1.2: n is odd.

Suppose $(n - 1)/2 = m_3k + j_3$ where $0 \le j_3 < k$. From $n \ge 3k - 1$, we know that $m_3 \ge 2$, or $m_3 = 1$ and $j_3 \ge \frac{k-2}{2}$.

v_j	v_{20}	v_{21}	v_{22}	v_{23}	v_{24}	v_{25}
$d_{12}(u_0, v_j)$	5	6	7	8	7	6
$d_{34}(u_0, v_j)$	8	7	6	5	4	5

Table 3: In $GP(51, 9)$, the search of j^*. Because $j^1 = 23$, $d_{12}(u_0, v_{23}) = 8$, $n-(m_3+1)k = 24$ and $d_{34}(u_0, v_{24}) = 4$, so $j^* = 21$ or $j^* = 22$.

Case 1.2.1: $j_3 \geq \frac{k-2}{2}$.

Because k is odd, $\frac{k-2}{2}$ is not an integer and hence $j_3 \geq \frac{k-1}{2}$. It suffices to consider $d(u_0, v_{m_3k+j})$, where $0 \leq j \leq j_3$. Note that $d_{12}(u_0, v_{m_3k+\frac{k+1}{2}}) = m_3 + \frac{k+1}{2} + 1$, $d_{12}(u_0, v_{m_3k+\frac{k+1}{2}-1}) = d_{12}(u_0, v_{m_3k+\frac{k+1}{2}+1}) = m_3 + \frac{k+1}{2}$, and so on.

Note that $v_{-(m_3+1)k} = v_{n-(m_3+1)k} = v_{m_3k+2j_3+1-k}$. Moreover, $d_{34}(u_0, v_{m_3k+2j_3+1-k}) = m_3+2$, $d_{34}(u_0, v_{m_3k+2j_3+1-k+1}) = d_{34}(u_0, v_{m_3k+2j_3+1-k-1}) = m_3+3$, $d_{34}(u_0, v_{m_3k+2j_3+1-k+2}) = d_{34}(u_0, v_{m_3k+2j_3+1-k-2}) = m_3 + 4$, and so on.

If $j_3 = \frac{k-1}{2}$, then select $j^1 = m_3k + \frac{k-1}{2}$. Because $d_{34}(u_0, v_{j^1}) \geq d_{12}(u_0, v_{j^1})$, we have $j^* = j^1 = m_3k + \frac{k-1}{2}$.

If $2 \leq 2j_3 + 1 - k \leq \frac{k+1}{2}$, then $j^1 = m_3k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_3k + j_3$.

If $2j_3 + 1 - k > \frac{k+1}{2}$, then $j^1 = m_3k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_3k + j_3 - \frac{k-3}{2}$ or $j^* = m_3k + j_3 - \frac{k-1}{2}$.

Table 3 shows that how to find j^* in $GP(51, 9)$.

Case 1.2.2: $j_3 < \frac{k-2}{2}$.

It suffices to consider the distances $d(u_0, v_{(m_3-1)k+j})$, where $0 \leq j \leq k$. We infer that $d_{12}(u_0, v_{(m_3-1)k+\frac{k+1}{2}}) = m_3+\frac{k+1}{2}$, $d_{12}(u_0, v_{(m_3-1)k+\frac{k+1}{2}-1}) = d_{12}(u_0, v_{(m_3-1)k+\frac{k+1}{2}+1}) = m_3 + \frac{k+1}{2} - 1$, and so on.

Note that $v_{-(m_3+1)k} = v_{n-(m_3+1)k} = v_{(m_3-1)k+2j_3+1}$ and $2j_3+1 < k-1$. Moreover, $d_{34}(u_0, v_{(m_3-1)k+2j_3+1}) = m_3+2$, $d_{34}(u_0, v_{(m_3-1)k+2j_3+1+1}) = d_{34}(u_0, v_{(m_3-1)k+2j_3+1-1}) = m_3 + 3$, and so on.

If $j_3 = 0$, then let $j^1 = (m_3 - 1)k + \frac{k+1}{2}$. Because $d_{34}(u_0, v_{j^1}) \geq d_{12}(u_0, v_{j^1})$, we have $j^* = j^1 = (m_3 - 1)k + \frac{k+1}{2}$.

If $3 \leq 2j_3 + 1 \leq \frac{k+1}{2}$, then $j^1 = (m_3 - 1)k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_3 - 1)k + \frac{k+1}{2} + j_3 - 1$ or $j^* = (m_3 - 1)k + \frac{k+1}{2} + j_3$.

If $2j_3 + 1 > \frac{k+1}{2}$, then $j^1 = (m_3 - 1)k + \frac{k+1}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_3 - 1)k + j_3 + 2$ or $j^* = (m_3 - 1)k + j_3 + 1$.

Table 4 shows that how to find j^* in $GP(43, 9)$.

Case 2: k is even.

v_j	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
$d_{12}(u_0, v_j)$	5	6	7	6	5	4
$d_{34}(u_0, v_j)$	8	7	6	5	4	5

Table 4: In $GP(43, 9)$, the search of j^*. Because $j^1 = 14$, $d_{12}(u_0, v_{14}) = 7$, $n-(m_3+1)k = 16$ and $d_{34}(u_0, v_{16}) = 4$, so $j^* = 13$ or $j^* = 14$.

v_j	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}
$d_{12}(u_0, v_j)$	5	6	7	8	8	7	6
$d_{34}(u_0, v_j)$	8	7	6	5	4	5	6

Table 5: In $GP(56, 10)$, the search of j^*. Because $j^1 = 25$, $d_{12}(u_0, v_{25}) = 8$, $n-(m_4+1)k = 26$ and $d_{34}(u_0, v_{26}) = 4$, so $j^* = 23$, $j^* = 24$ or $j^* = 28$.

Notice that $n \geq 3k - 2$ in this case. We will prove that there exists j^* such that $d(u_0, v_{j^*}) = \max\{d(u_0, v_j) : \ 0 \leq j \leq n/2\}$ and $k < j^* \leq n/2$.

Case 2.1: n is even.
Suppose $n/2 = m_4k + j_4$ where $0 \leq j_4 < k$. From $n \geq 3k-2$, we know that $m_4 \geq 2$, or $m_4 = 1$ and $j_4 \geq \frac{k-2}{2}$.

Case 2.1.1: $j_4 \geq \frac{k-2}{2}$.

It suffices to consider the distances $d(u_0, v_{m_4k+j})$, where $0 \leq j \leq j_4$. Note that $d_{12}(u_0, v_{m_4k+\frac{k}{2}}) = d_{12}(u_0, v_{m_4k+\frac{k+2}{2}}) = m_4+\frac{k}{2}+1$, $d_{12}(u_0, v_{m_4k+\frac{k}{2}-1}) = d_{12}(u_0, v_{m_4k+\frac{k+2}{2}+1}) = m_4 + \frac{k}{2}$, and so on.

Observe that $v_{-(m_4+1)k} = v_{n-(m_4+1)k} = v_{m_4k+2j_4-k}$. Moreover, $d_{34}(u_0, v_{m_4k+2j_4-k}) = m_4+2$, $d_{34}(u_0, v_{m_4k+2j_4-k+1}) = d_{34}(u_0, v_{m_4k+2j_4-k-1}) = m_4+3$, $d_{34}(u_0, v_{m_4k+2j_4-k+2}) = d_{34}(u_0, v_{m_4k+2j_4-k-2}) = m_4 + 4$, and so on.

If $j_4 = \frac{k-2}{2}$, then $j^1 = m_4k + \frac{k-2}{2}$. Because $d_{34}(u_0, v_{j^1}) \geq d_{12}(u_0, v_{j^1})$, we have $j^* = j^1 = m_4k + \frac{k-2}{2}$.

If $j_4 = \frac{k}{2}$, then $j^1 = m_4k + \frac{k}{2}$. Because $d_{34}(u_0, v_{j^1}) \geq d_{12}(u_0, v_{j^1})$, we have $j^* = j^1 = m_4k + \frac{k}{2}$.

If $2 \leq 2j_4 - k \leq \frac{k}{2}$, then $j^1 = m_4k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_4k + j_4$.

If $2j_4 - k \geq \frac{k+2}{2}$, then $j^1 = m_4k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_4k + j_4 - \frac{k}{2} + 1$ or $j^* = m_4k + j_4 - \frac{k}{2}$.

Table 5 shows that how to find j^* in $GP(56, 10)$.

Case 2.1.2: $j_4 < \frac{k-2}{2}$.

v_j	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}
$d_{12}(u_0, v_j)$	4	5	6	7	7	6	5
$d_{34}(u_0, v_j)$	6	5	4	5	6	7	8

Table 6: In $GP(44, 10)$, the search of j^*. Because $j^1 = 15$, $d_{12}(u_0, v_{15}) = 7$, $n-(m_4+1)k = 14$ and $d_{34}(u_0, v_{14}) = 4$, so $j^* = 16$ or $j^* = 17$.

It is enough to consider the distances $d(u_0, v_{(m_4-1)k+j})$, where $0 \le j \le k$. Note that $d_{12}(u_0, v_{(m_4-1)k+\frac{k}{2}}) = d_{12}(u_0, v_{(m_4-1)k+\frac{k+2}{2}}) = m_4 + \frac{k}{2}$, $d_{12}(u_0, v_{(m_4-1)k+\frac{k}{2}-1}) = d_{12}(u_0, v_{(m_4-1)k+\frac{k+2}{2}+1}) = m_4 + \frac{k}{2} - 1$, and so on.

Note that $v_{-(m_4+1)k} = v_{n-(m_4+1)k} = v_{(m_4-1)k+2j_2}$ and $2j_2 < k - 2$. We also have $d_{34}(u_0, v_{(m_4-1)k+2j_2}) = m_4 + 2$, $d_{34}(u_0, v_{(m_4-1)k+2j_2+1}) = d_{34}(u_0, v_{(m_4-1)k+2j_2-1}) = m_4 + 3$, $d_{34}(u_0, v_{(m_4-1)k+2j_2+2}) = d_{34}(u_0, v_{(m_4-1)k+2j_2-2}) = m_4 + 4$, and so on.

If $j_4 = 0$ or $j_4 = 1$, then $j^1 = (m_4 - 1)k + \frac{k}{2}$. Because $d_{34}(u_0, v_{j^1}) \ge d_{12}(u_0, v_{j^1})$, we have $j^* = j^1 = (m_4 - 1)k + \frac{k}{2}$.

If $4 \le 2j_4 \le \frac{k}{2}$, then $j^1 = (m_4 - 1)k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_4 - 1)k + \frac{k}{2} + j_4 - 1$ or $j^* = (m_4 - 1)k + \frac{k}{2} + j_4$.

If $2j_4 \ge \frac{k+2}{2}$, then $j^1 = (m_4 - 1)k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_4 - 1)k + j_4 + 1$.

Table 6 shows that how to find j^* in $GP(44, 10)$.

Case 2.2: n is odd.
Suppose $(n - 1)/2 = m_5k + j_5$, where $0 \le j_5 < k$. From $n \ge 3k - 2$, we know that $m_5 \ge 2$, or $m_5 = 1$ and $j_5 \ge \frac{k-3}{2}$.

Case 2.2.1: $j_5 \ge \frac{k-3}{2}$.

Because k is even, $\frac{k-3}{2}$ is not an integer and hence $j_5 \ge \frac{k-2}{2}$. Again it suffices to consider the distances $d(u_0, v_{m_5k+j})$, where $0 \le j \le j_5$. Note that $d_{12}(u_0, v_{m_5k+\frac{k}{2}}) = d_{12}(u_0, v_{m_5k+\frac{k+2}{2}}) = m_5 + \frac{k}{2} + 1$, $d_{12}(u_0, v_{m_5k+\frac{k}{2}-1}) = d_{12}(u_0, v_{m_5k+\frac{k+2}{2}+1}) = m_5 + \frac{k}{2}$, and so on.

Observe that $v_{-(m_5+1)k} = v_{n-(m_5+1)k} = v_{m_5k+2j_5+1-k}$. Also, $d_{34}(u_0, v_{m_5k+2j_5+1-k}) = m_5+2$, $d_{34}(u_0, v_{m_5k+2j_5+1-k+1}) = d_{34}(u_0, v_{m_5k+2j_5+1-k-1}) = m_5+3$, $d_{34}(u_0, v_{m_5k+2j_5+1-k+2}) = d_{34}(u_0, v_{m_5k+2j_5+1-k-2}) = m_5 + 4$, and so on.

If $j_5 = \frac{k-2}{2}$, then $j^1 = m_5k + \frac{k-2}{2}$. Because $d_{34}(u_0, v_{j^1}) \ge d_{12}(u_0, v_{j^1})$, we have $j^* = j^1 = m_5k + \frac{k-2}{2}$.

If $j_5 = \frac{k}{2}$, then $j^1 = m_5k + \frac{k}{2}$. Because $d_{34}(u_0, v_{j^1}) \ge d_{12}(u_0, v_{j^1})$, we infer that $j^* = j^1 = m_5k + \frac{k}{2}$.

If $3 \le 2j_5 + 1 - k \le \frac{k}{2}$, then $j^1 = m_5k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_5k + j_5$.

v_j	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}
$d_{12}(u_0, v_j)$	6	7	8	8	7	6
$d_{34}(u_0, v_j)$	8	7	6	5	4	5

Table 7: In $GP(57, 10)$, the search of j^*. Because $j^1 = 25$, $d_{12}(u_0, v_{25}) = 8$, $n-(m_5+1)k = 27$ and $d_{34}(u_0, v_{27}) = 4$, so $j^* = 24$.

v_j	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}
$d_{12}(u_0, v_j)$	4	5	6	7	7	6	5
$d_{34}(u_0, v_j)$	7	6	5	4	5	6	7

Table 8: In $GP(45, 10)$, the search of j^*. Because $j^1 = 15$, $d_{12}(u_0, v_{15}) = 7$, $n-(m_5+1)k = 15$ and $d_{34}(u_0, v_{15}) = 4$, so $j^* = 17$.

If $2j_5 + 1 - k \geq \frac{k+2}{2}$, then $j^1 = m_5k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = m_5k + j_5 + 1 - \frac{k}{2}$.

Table 7 shows that how to find j^* in $GP(57, 10)$.

Case 2.2.2: $j_5 < \frac{k-3}{2}$.

It suffices to consider the distances $d(u_0, v_{(m_5-1)k+j})$, where $0 \leq j \leq k$. Note that $d_{12}(u_0, v_{(m_5-1)k+\frac{k}{2}}) = d_{12}(u_0, v_{(m_5-1)k+\frac{k+2}{2}}) = m_5 + \frac{k}{2}$, $d_{12}(u_0, v_{(m_5-1)k+\frac{k}{2}-1}) = d_{12}(u_0, v_{(m_5-1)k+\frac{k+2}{2}+1}) = m_5 + \frac{k}{2} - 1$, and so on.

Note that $v_{-(m_5+1)k} = v_{n-(m_5+1)k} = v_{(m_5-1)k+2j_5+1}$ and $2j_5 + 1 < k - 2$. We have $d_{34}(u_0, v_{(m_5-1)k+2j_5+1}) = m_5+2$, $d_{34}(u_0, v_{(m_5-1)k+2j_5+1+1}) = d_{34}(u_0, v_{(m_5-1)k+2j_5+1-1}) = m_5 + 3$, $d_{34}(u_0, v_{(m_5-1)k+2j_5+1+2}) = d_{34}(u_0, v_{(m_5-1)k+2j_5+1-2}) = m_5 + 4$, and so on.

If $j_5 = 0$, then $j^1 = (m_5 - 1)k + \frac{k}{2}$. Because $d_{34}(u_0, v_{j^1}) \geq d_{12}(u_0, v_{j^1})$, we infer that $j^* = j^1 = (m_5 - 1)k + \frac{k}{2}$.

If $3 \leq 2j_5 + 1 \leq \frac{k}{2}$, then $j^1 = (m_5 - 1)k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_5 - 1)k + \frac{k}{2} + j_5$.

If $2j_5 + 1 \geq \frac{k+2}{2}$, then $j^1 = (m_5 - 1)k + \frac{k}{2}$ and $d_{34}(u_0, v_{j^1}) < d_{12}(u_0, v_{j^1})$. We get $j^* = (m_5 - 1)k + j_5 + 2$ or $j^* = (m_5 - 1)k + j_5 + 1$.

Table 8 shows that how to find j^* in $GP(45, 10)$.

Suppose that P^* is a shortest u_0, v_{j^*}-path. Let $P^* + v_{j^*}u_{j^*}$ be the path obtained from P^* by appending the edge $v_{j^*}u_{j^*}$ at v_{j^*}. Because $k \geq 3$ and $k < j^* \leq n/2$, the path $P^* + v_{j^*}u_{j^*}$ is a shortest u_0, u_{j^*}-path which contains v_{j^*}. We conclude that $d(u_0, v_{j^*}) < d(u_0, u_{j^*}) \leq D$.

3 On the diameter of $GP(n,k)$

In the previous section we found a j^*, where $0 \le j^* \le n/2$, such that $d(u_0, v_{j^*}) = \max\{d(u_0, v_j) : \ 0 \le j < n\}$. In fact, the proof also reveals that

$$\operatorname{diam}(GP(n,k)) = d(u_0, u_{j^*}) = d(u_0, v_{j^*}) + 1\,,$$

which in turn enables us to state the following theorem.

Theorem 5. *If n and k are integers, where $3 \le k < n/2$ and*

$$n \ge \begin{cases} 8; & k = 3, \\ 10; & k = 4, \\ \frac{k(k+1)}{2}; & k \text{ is odd and } k \ge 5, \\ \frac{k^2}{2}; & k \text{ is even and } k \ge 6, \end{cases}$$

then the following hold.

1. *If $k \ge 3$, k is odd, n is even, and $\frac{n}{2} = mk + j$, where $\frac{k-1}{2} \le j < k$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m + 2 + j; & j = \frac{k-1}{2} \text{ or } j = \frac{k+1}{2}, \\ m + 3 + k - j; & 3 \le 2j - k \le \frac{k+1}{2}, \\ m + 2 + j - \frac{k-1}{2}; & 2j - k > \frac{k+1}{2}. \end{cases}$$

2. *If $k \ge 3$, k is odd, n is even, and $\frac{n}{2} = mk + j$, where $0 \le j < \frac{k-1}{2}$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m + 1 + \frac{k+1}{2}; & j = 0 \text{ or } j = 1, \\ m + 3 + \frac{k-1}{2} - j; & 4 \le 2j \le \frac{k+1}{2}, \\ m + 2 + j; & 2j > \frac{k+1}{2}. \end{cases}$$

3. *If $k \ge 3$, k is odd, n is odd, and $\frac{n-1}{2} = mk + j$, where $\frac{k-2}{2} \le j < k$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m + 2 + \frac{k-1}{2}; & j = \frac{k-1}{2}, \\ m + 2 + k - j; & 2 \le 2j + 1 - k \le \frac{k+1}{2}, \\ m + 2 + j - \frac{k-1}{2}; & 2j + 1 - k > \frac{k+1}{2}. \end{cases}$$

4. *If $k \ge 3$, k is odd, n is odd, and $\frac{n-1}{2} = mk + j$, where $0 \le j < \frac{k-2}{2}$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m + 2 + \frac{k-1}{2} - j; & 1 \le 2j + 1 \le \frac{k+1}{2}, \\ m + 2 + j; & 2j + 1 > \frac{k+1}{2}. \end{cases}$$

5. *If $k \geq 4$, k is even, n is even, and $\frac{n}{2} = mk + j$, where $\frac{k-2}{2} \leq j < k$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m+2+j; & j = \frac{k-2}{2} \text{ or } j = \frac{k}{2}, \\ m+3+k-j; & 2 \leq 2j-k \leq \frac{k}{2}, \\ m+2+j-\frac{k}{2}; & 2j-k \geq \frac{k+2}{2}. \end{cases}$$

6. *If $k \geq 4$, k is even, n is even, and $\frac{n}{2} = mk + j$, where $0 \leq j < \frac{k-2}{2}$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m+1+\frac{k}{2}; & j = 0, \\ m+2+\frac{k}{2}-j; & 2 \leq 2j \leq \frac{k}{2}, \\ m+2+j; & 2j \geq \frac{k+2}{2}. \end{cases}$$

7. *If $k \geq 4$, k is even, n is odd, and $\frac{n-1}{2} = mk + j$, where $\frac{k-3}{2} \leq j < k$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m+2+\frac{k-2}{2}; & j = \frac{k-2}{2}, \\ m+2+k-j; & 1 \leq 2j+1-k \leq \frac{k}{2}, \\ m+3+j-\frac{k}{2}; & 2j+1-k \geq \frac{k+2}{2}. \end{cases}$$

8. *If $k \geq 4$, k is even, n is odd, and $\frac{n-1}{2} = mk + j$, where $0 \leq j < \frac{k-3}{2}$, then*

$$\operatorname{diam}(GP(n,k)) = \begin{cases} m+1+\frac{k}{2}; & j = 0, \\ m+2+\frac{k}{2}-j; & 3 \leq 2j+1 \leq \frac{k}{2}, \\ m+2+j; & 2j+1 \geq \frac{k+2}{2}. \end{cases}$$

4 Concluding remarks

In this paper we proved that $GP(n,k)$ is $\operatorname{diam}(GP(n,k))$-distance-balanced provided that n is large enough relative to k. In these cases we also determined $\operatorname{diam}(GP(n,k))$. For small values of k, we can strengthen these results as follows.

From [20] we know that $GP(n,2)$, $n \geq 5$, is $diam(GP(n,2))$-distance-balanced. For $k = 2$ and $n \geq 5$, $\operatorname{diam}(GP(n,2))$ can also be computed. First, $\operatorname{diam}(GP(5,2)) = 2$, $\operatorname{diam}(GP(6,2)) = 4$, and $\operatorname{diam}(GP(7,2)) = 3$. Moreover, if $n = 4m$ or $n = 4m+1$, then $\operatorname{diam}(GP(n,2)) = m+2$, and if $n = 4m+2$ or $n = 4m+3$, then $\operatorname{diam}(GP(n,2)) = m+3$.

It is straightforward to check that $\operatorname{diam}(GP(7,3)) = 3$ and that $GP(7,3)$ is highly distance-balanced. Similarly, $\operatorname{diam}(GP(9,4)) = 4$ and $GP(9,4)$ is 4-distance-balanced. In addition, from [20] we recall that $\operatorname{diam}(GP(11,5)) = \operatorname{diam}(GP(14,5)) = 5$, $\operatorname{diam}(GP(12,5)) = \operatorname{diam}(GP(13,5)) = 4$, and that $GP(n,5)$ is $\operatorname{diam}(GP(n,5))$-distance-balanced for $11 \leq n \leq 14$. Moreover, $\operatorname{diam}(GP(n,6)) = 5$ and $GP(n,6)$ is 5-distance-balanced for $13 \leq n \leq 17$.

Combining the above results with Theorems 3 and 5, the following result can be stated.

Proposition 6. *If k and n are integers, where $2 \leq k \leq 6$ and $n \geq 2k+1$, then $GP(n,k)$ is* $\mathrm{diam}(GP(n,k))$*-distance-balanced. Moreover,* $\mathrm{diam}(GP(n,k))$ *can be computed.*

For $k \geq 7$ the remaining cases to be solved are collected as follows.

Problem 7. *Let k and n be two integers, where $k \geq 7$. Moreover, if k is odd, then $2k+1 \leq n < \frac{k(k+1)}{2}$ and if k is even, then $k \geq 8$ and $2k+1 \leq n < \frac{k^2}{2}$.*

1. *Is $GP(n,k)$* $\mathrm{diam}(GP(n,k))$*-distance balanced?*
2. *Compute* $\mathrm{diam}(GP(n,k))$.

Moreover, the ℓ-distance-balancedness of $GP(n,k)$, where $\ell < \mathrm{diam}(GP(n,k))$, is widely open.

Problem 8. *Let n and k be integers, where $n \geq 5$ and $2 \leq k < n/2$. For $1 \leq \ell <$* $\mathrm{diam}(GP(n,k))$ *determine whether $GP(n,k)$ is ℓ-distance-balanced or not.*

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Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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