The geodesic cover problem for butterfly networks

Paul Manuel^a San Andre

Sandi Klavžar^{b,c,d} R. Prabha^e Andrew Arokiaraj^f

^a Department of Information Science, College of Computing Science and Engineering, Kuwait University, Kuwait pauldmanuel@gmail.com, p.manuel@ku.edu.kw

^b Faculty of Mathematics and Physics, University of Ljubljana, Slovenia sandi.klavzar@fmf.uni-lj.si

^c Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

^d Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

 e Department of Mathematics, Ethiraj College for Women, Chennai, Tamilnadu, India

prabha75@gmail.com

^f Department of Mathematics, Shraddha Children's Academy, Chennai, India andrewarokiaraj@gmail.com

Abstract

A geodesic cover, also known as an isometric path cover, of a graph is a set of geodesics which cover the vertex set of the graph. An edge geodesic cover of a graph is a set of geodesics which cover the edge set of the graph. The geodesic (edge) cover number of a graph is the cardinality of a minimum (edge) geodesic cover. The (edge) geodesic cover problem of a graph is to find the (edge) geodesic cover number of the graph. Surprisingly, only partial solutions for these problems are available for most situations. In this paper we demonstrate that the geodesic cover number of the *r*-dimensional butterfly is $\lceil (2/3)2^r \rceil$ and that its edge geodesic cover number is 2^r .

Keywords: isometric path; geodesic cover; edge geodesic cover; bipartite graph, butterfly network

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1 Introduction

A geodesic cover of a graph G = (V(G), E(G)) is a set S of geodesics such that each vertex of G belongs to at least one geodesic of S. It is popularly known as *isometric* path cover [3, 4, 7, 8, 11]. The geodesic cover problem is one of the fundamental problems in graph theory. The concept of geodesic cover is widely used in social networks, computer networks, and fixed interconnection networks [7]. Throughout this paper, Z(G) denotes the set of all geodesics of G and M(G) denotes the set of all maximal (with respect to inclusion) geodesics of G.

Given $Y \subseteq Z(G)$ and $S \subseteq V(G)$, a geodesic cover of the triple (Y, S, G) is a set of geodesics of Y that cover S. Given $Y \subseteq Z(G)$ and $S \subseteq V(G)$, the geodesic cover number of (Y, S, G), gcover(Y, S, G), is the minimum number of geodesics of Y that cover S. Note that there exist situations where gcover(Y, S, G) may not exist.

When $Y \subseteq Z(G)$ and S = V, gcover(Y, V, G) is denoted by gcover(Y, G).

When Y = Z(G) and $S \subseteq V$, gcover(Z(G), S, G) is denoted by gcover(S, G).

When Y = Z(G) and S = V, gcover(Z(G), V, G) is denoted by gcover(G).

Given $Y \subseteq Z(G)$ and $S \subseteq V(G)$, the geodesic cover problem of (Y, S) is to find gcover(Y, S, G) of G. The geodesic cover problem of G is to find gcover(G) of G. An edge geodesic cover of a graph is a set of geodesics which cover the edge set of the graph. The edge geodesic cover number of a graph G, gcover_e(G), is the cardinality of a minimum edge geodesic cover. The edge geodesic cover problem of a graph G is to find gcover_e(G).

The geodesic cover problem is known to be NP-complete [13]. Apollonio et al. [1] have studied induced path covering problems in grids. Fisher and Fitzpatrick [2] have shown that the geodesic cover number of the $(r \times r)$ -dimensional grid is $\lceil 2r/3 \rceil$. The geodesic cover number of the $(r \times s)$ -dimensional grid is s when $r \geq s(s-1)$, cf. [8]. On the other hand, the complete solution of the geodesic cover problem for two dimensional grid is still unknown, cf. [8]. There is no literature for the geodesic cover problem on multi-dimensional grids.

The geodesic cover problems for cylinder and r-dimensional grids are discussed in [8]. In particular, the isometric path cover number of the $(r \times r)$ -dimensional torus is r when r is even, and is either r or r + 1 when r is odd. In [10], the geodesic cover problem was studied on block graphs, while in [11] it was investigated on complete r-partite graphs and Cartesian products of two or three complete graphs.

Fitzpatrick et al. [3,4] have shown that the geodesic cover number of the hypercube Q_r is at least 2r/(r+1) and they have provided partial solution when r+1is a power of 2. The complete solution for the geodesic cover number of hypercubes is also not yet known, cf. [3,4,7]. Manuel [8] has proved that the geodesic cover number of the *r*-dimensional Benes network is 2^r . In [7,8] the (edge) geodesic cover problem of butterfly networks was stated as an open problem. In this paper we solve these two problems.

2 Preliminaries

The results discussed in this section will be used as tools to prove the key results of this paper.

Lemma 2.1. If G is a connected graph, then the following hold.

(i) If $S' \subseteq S'' \subseteq V(G)$ and $Y \subseteq Z(G)$, then gcover $(Y, S'', G) \ge gcover(Y, S', G)$.

(ii) If $Y' \subseteq Y'' \subseteq Z(G)$ and $S \subseteq V(G)$, then gcover $(Y'', S, G) \leq gcover(Y', S, G)$.

(*iii*) gcover(G) = gcover(M(G), G).

Proof. Assertions (i) and (ii) are straightforward, hence we consider only (iii). Since $M(G) \subseteq Z(G)$, (ii) implies gcover(G) = gcover(Z(G), V, G) ≤ gcover(M(G), V, G) = gcover(M(G), G). Since each geodesic is a subpath of some maximal geodesic, for each geodesic cover S of Z(G), there exists a geodesic cover S' of M(G) such that |S| = |S'|. Therefore, gcover(Z(G), V, G) ≥ gcover(M(G), V, G) and consecutively gcover(G) = gcover(Z(G), V, G) ≥ gcover(M(G), V, G) = gcover(M(G), G). \Box

Proposition 2.2. If $r \ge 2$, then gcover $(K_{r,r}) = \lceil (2/3)r \rceil$.

Proof. Clearly, each maximal geodesic of $K_{r,r}$ is a (diametral) path of length 2. Therefore, $gcover(K_{r,r}) \geq \lceil (2/3)r \rceil$. On the other hand, it is a simple exercise to construct a geodesic cover of cardinality $\lceil (2/3)r \rceil$.

Butterfly is considered as one of the best parallel architectures [5,6,14]. For $r \geq 3$, the *r*-dimensional butterfly network BF(r) has vertices [j, s], where $s \in \{0, 1\}^r$ and $j \in \{0, 1, \ldots, r\}$. The vertices [j, s] and [j', s'] are adjacent if |j - j'| = 1, and either s = s' or s and s' differ precisely in the jth bit. BF(r) has $(r + 1)2^r$ vertices and $r2^{r+1}$ edges. A vertex [j, s] is at level j and row s. There are two standard graphical representations for BF(r), normal representation and diamond representation, see Fig. 1.

Estimating the lower bound of gcover(BF(r)) in Section 3.1, we will use the diamond representation of BF(r), while estimating the upper bound of gcover(BF(r))in Section 3.2, the normal representation of BF(r) will be used.

Lemma 2.3. A geodesic of BF(r) contains at most two vertices of level 0 and at most two vertices of level r.

Proof. Let us assume that there exists a geodesic P which contains more than two vertices of level 0, say v_i, v_j , and v_k . See Fig. 2(b). Then one of these three vertices must be an internal vertex of P, say v_j . The deletion of the vertices at level 0 from BF(r) disconnects BF(r) into two vertex disjoint components G_1 and G_2 , where both G_1 and G_2 are isomorphic to BF(r - 1), cf. [6, 12, 15]. Since v_j is an internal



Figure 1: (a) Normal representation of BF(3) (b) Diamond representation of BF(3).

vertex of P and of degree 2, its neighbors v_{j-1} and v_{j+1} also lie in P. Moreover, one of the adjacent vertices v_{j-1} , v_{j+1} lie in G_1 and the other lie in G_2 . Since G_1 and G_2 are isomorphic, $d(v_i, v_{j-1}) = d(v_i, v_{j+1})$. This is not possible as P is a geodesic. \Box

As the butterfly network is symmetrical with respect to level 0, it is symmetrical with respect to level r, cf. [6, 12, 15]. Using the logic of the proof of Lemma 2.3, one can prove the following.

Corollary 2.4. If both end vertices of a geodesic P of BF(r) are either at level 0 or at level r, then P is maximal.

Lemma 2.5. A geodesic of BF(r) covers at most three vertices of degree 2.

Proof. Suppose a geodesic P contains four vertices a, b, c, d of degree 2. Then all the vertices a, b, c, d are at level 0 or at level r. By Lemma 2.3, three of these vertices can not be at the same level. Assume without loss of generality that a and b are at level 0 and c and d are at level r. Let P(a, b) be a subpath of P between a and b, and P(c, d) the subpath of P between c and d. By Corollary 2.4, P(a, b) and P(c, d) are maximal geodesics, a contradiction.

Corollary 2.6. If a geodesic P of BF(r) covers three vertices of degree 2, then the end vertices of P are of degree 2, and P is maximal.

3 The geodesic cover problem for BF(r)

3.1 A lower bound for gcover(BF(r))

3.1.1 Revisiting properties of BF(r)

In this section, we use the following notations. Let U and W denote the sets of vertices at level 0 and level r in BF(r), respectively. Further, let $U = U^b \bigcup U^r$, where $U^b = \{u_1^b, u_2^b, \ldots, u_{2r-1}^b\}$ and $U^r = \{u_1^r, u_2^r, \ldots, u_{2r-1}^r\}$. Similarly, $W = W^b \bigcup W^r$ where $W^b = \{w_1^b, w_2^b, \ldots, w_{2r-1}^b\}$ and $W^r = \{w_1^r, w_2^r, \ldots, w_{2r-1}^r\}$. The order of the vertices in these sets respects the diamond representation of BF(r), see Fig. 2(a).



Figure 2: (a) Vertices from U^b and W^b are blue, vertices from U^r and W^r are red. (b) v_j is an internal vertex of P and of degree 2. Its neighbors v_{j-1} and v_{j+1} lie in P. One of the adjacent vertices $\{v_{j-1}, v_{j+1}\}$ lie in G_1 and the other lie in G_2 .

In order to gain an in-depth understanding of the behavior of the geodesics of BF(r), it is necessary to enumerate all the maximal geodesics of BF(r), cf. [5,9].

Lemma 3.1. The following facts hold in BF(r).

- 1. For $u_i^b, u_j^b \in U^b$, a maximal geodesic $P(u_i^b, u_j^b)$ between u_i^b and u_j^b does not intersect W. For $u_i^r, u_j^r \in U^r$, a maximal geodesic $P(u_i^r, u_j^r)$ between u_i^r and u_j^r does not intersect W.
- 2. For $w_i^b, w_j^b \in W^b$, a maximal geodesic $P(w_i^b, w_j^b)$ between w_i^b and w_j^b does not intersect U. For $w_i^r, w_j^r \in W^r$, a maximal geodesic $P(w_i^r, w_j^r)$ between w_i^r and w_i^r does not intersect U.

3. If u^b ∈ U^b, u^r ∈ U^r, and w ∈ W, then there is a unique geodesic P_w(u^b, u^r) between u^b and u^r passing through w. This geodesic is the concatenation of geodesics P(u^b, w) and P(w, u^r), where u^b ∈ U^b, u^r ∈ U^r, and w ∈ W. Consequently, if u^b ∈ U^b and u^r ∈ U^r, then there are 2^r maximal u^b, u^r-geodesics. If w^b ∈ W^b, w^r ∈ W^r, and u ∈ U, then there is a unique geodesic P_u(w^b, w^r) between w^b and w^r passing through u. This geodesic is the concatenation of geodesics P(w^b, u) and P(u, w^r), where w^b ∈ W^b, w^r ∈ W^r, and u ∈ U. Hence, if w^b ∈ W^b and w^r ∈ W^r, then there are 2^r maximal w^b, w^r-geodesics.

Proof. BF(r) - W consists of two components both isomorphic to BF(r-1), cf. [5,6]. As these components are furthermore convex in BF(r), we get that if either $u_i^b, u_j^b \in U^b$, a maximal u_i^b, u_j^b -geodesic does not intersect W. Analogously the other assertions hold. The assertion (3) follows from the fact that when $u \in U$ and $w \in W$, a u, v-geodesic is unique.

Set now

 $M_{U,W}(BF(r)) = \{P : P \text{ is a maximal } x, y \text{-geodesic, either } x, y \in U \text{ or } x, y \in W\}.$

By Lemma 3.1, the set of geodesics $M_{U,W}(BF(r))$ is partitioned into six disjoint subsets as follows.

Observation 3.2. $M_{U,W}(BF(r))$ partitions into the following sets:

- (i) $\{P(u_i^b, u_j^b) : u_i^b, u_j^b \in U^b\},\$
- (*ii*) { $P(u_i^r, u_j^r) : u_i^r, u_j^r \in U^r$ },
- (*iii*) $\{P(w_i^b, w_j^b) : w_i^b, w_j^b \in W^b\},\$
- (*iv*) $\{P(w_i^r, w_j^r) : w_i^r, w_j^r \in W^r\},\$
- (v) $\{P_w(u^b, u^r) : u^b \in U^b, u^r \in U^r, w \in W\},\$
- (vi) $\{P_u(w^b, w^r) : w^b \in W^b, w^r \in W^r, u \in U\}.$

 $M_{U,W}(BF(r))$ is thus the set of all maximal x, y-geodesic, where either $x, y \in U$ or $x, y \in W$. In (i)-(iv) of Observation 3.2, given a pair of vertices $x, y \in U$ or $x, y \in W$, there are more than one maximal geodesics between x and y in $M_{U,W}(BF(r))$. Now we define $M'_{U,W}(BF(r)) \subset M_{U,W}(BF(r))$ as follows. First, $M'_{U,W}(BF(r))$ contains all the geodesics from (v) and (vi) of Observation 3.2. Second, for each pair of vertices $u_i^b, u_j^b \in U^b$ from $(i), u_i^r, u_j^r \in U^r$ from $(ii), w_i^b, w_j^b \in W^b$ from (ii), and $w_i^r, w_j^r \in W^r$ from (iv), select an arbitrary but fixed geodesic between them and add it to $M'_{U,W}(BF(r))$. In this way the set $M'_{U,W}(BF(r))$ is defined.

For the sake of clarity, we write the members of $M'_{U,W}(BF(r))$ explicitly below:

$$\begin{split} M'_{U,W}(\mathrm{BF}(r)) = & \{P'(u_i^b, u_j^b) : P'(u_i^b, u_j^b) \text{ is a fixed geodesic between } u_i^b, u_j^b \in U^b\} \\ & \cup \{P'(u_i^r, u_j^r) : P'(u_i^r, u_j^r) \text{ is a fixed geodesic between } u_i^r, u_j^r \in U^r\} \\ & \cup \{P'(w_i^b, w_j^b) : P'(w_i^b, w_j^b) \text{ is a fixed geodesic between } w_i^b, w_j^b \in W^b\} \\ & \cup \{P'(w_i^r, w_j^r) : P'(w_i^r, w_j^r) \text{ is a fixed geodesic between } w_i^r, w_j^r \in W^r\} \\ & \cup \{P_w(u^b, u^r) : u^b \in U^b, u^r \in U^r, w \in W\} \\ & \cup \{P_u(w^b, w^r) : w^b \in W^b, w^r \in W^r, u \in U\}. \end{split}$$

Note for each pair u_i, u_j , for each pair w_i, w_j , for each triple u^b, u^r, w , and for each triple w^b, w^r, u , the set $M'_{U,W}(BF(r))$ contains a unique corresponding geodesic. Note also that $M'_{U,W}(BF(r)) \subset M_{U,W}(BF(r)) \subset M(BF(r))$.

3.1.2 Estimating a lower bound for gcover(BF(r))

Lemma 3.3. If U and V are as above, then

$$gcover(BF(r)) \ge gcover(M'_{U,W}(BF(r)), U \cup W, BF(r)).$$

Proof. Set G = BF(r). To prove the lemma, we are going to show that

$$gcover(G) \ge gcover(M(G), U \cup W, G)$$

= gcover(M_{U,W}(G), U \cup W, G)
= gcover(M'_{U,W}(G), U \cup W, G).

By Lemma 2.1, we get $gcover(G) \ge gcover(M(G), U \cup W, G)$. By Observation 3.2 and the definition of $M'_{U,W}(G)$, $gcover(M_{U,W}(G), U \cup W, G) = gcover(M'_{U,W}(G), U \cup W, G)$. Next we prove that $gcover(M(G), U \cup W, G) = gcover(M_{U,W}(G), U \cup W, G)$.

Since $M_{U,W}(G) \subseteq M(G)$, by Lemma 2.1, we get $gcover(M(G), U \cup W, G) \leq gcover(M_{U,W}(G), U \cup W, G)$. Now it is enough to prove that $gcover(M(G), U \cup W, G) \geq gcover(M_{U,W}(G), U \cup W, G)$. By Lemma 2.5, a geodesic covers at most three vertices of $U \cup W$. If P is a member of M(G) such that P covers three vertices of $U \cup W$ in BF(r), then by Corollary 2.6, $P \in M_{U,W}(G)$. On the other hand, if P is a member of M(G) covering two vertices v_1 and v_2 of $U \cup W$, then by Observation 3.2 there exists a geodesic Q of $M_{U,W}(G)$ such that Q covers both vertices v_1 and v_2 . Hence, $gcover(M(G), U \cup W, G) \geq gcover(M_{U,W}(G), U \cup W, G)$.

Let us consider two sets X and Y where $X = X^b \cup X^r$, $Y = Y^b \cup Y^r$, $X^b = \{x_1^b, x_2^b, \ldots, x_{2^{r-1}}^b\}$, $X^r = \{x_1^r, x_2^r, \ldots, x_{2^{r-1}}^r\}$, $Y^b = \{y_1^b, y_2^b, \ldots, y_{2^{r-1}}^b\}$, and $Y^r = \{y_1^r, y_2^r, \ldots, y_{2^{r-1}}^r\}$. Now we define a complete bipartite graph G' with the bipartition X, Y. Let us further define another complete bipartite graph G'' with the bipartition $X_0 = X \cup \{x_0\}, Y_0 = Y \cup \{y_0\}$. The graphs G' and G'' are presented in Fig. 3.



Figure 3: (a) The complete bipartite graph G'. (b) The complete bipartite graph G''.

Lemma 3.4. If U and V are as above, then

gcover $(M'_{U,W}(BF(r)), U \cup W, BF(r)) \ge \lceil (2/3)2^r \rceil$.

Proof. Set G = BF(r) and let G' and G'' be the complete bipartite graphs as defined above. To prove the lemma we claim that the following holds:

$$gcover(M'_{U,W}(G), U \cup W, G) \ge gcover(M(G''), X \cup Y, G'')$$
$$= gcover(M(G'), X \cup Y, G')$$
$$= gcover(G')$$
$$\ge \lceil (2/3)2^r \rceil.$$

In order to prove the inequality gcover $(M'_{U,W}(G), U \cup W, G) \ge \text{gcover}(M(G''), X \cup Y, G'')$, we define an 1-1 mapping $f \colon M'_{U,W}(G) \to M(G'')$. This mapping $f \colon P \mapsto f_P$ is defined as follows.

- 1. Each $P'(u_i^b, u_j^b)$ of $M'_{U,W}(G)$, where $u_i^b, u_j^b \in U^b$, is mapped to geodesic $x_i^b y_0 x_j^b \in M(G'')$, where $x_i^b, x_j^b \in X^b$.
- 2. Each $P'(u_i^r, u_j^r)$ of $M'_{U,W}(G)$, where $u_i^r, u_j^r \in U^r$, is mapped to geodesic $x_i^r y_0 x_j^r \in M(G'')$, where $x_i^r, x_j^r \in X^r$.
- 3. Each $P'(w_i^b, w_j^b)$ of $M'_{U,W}(G)$, where $w_i^b, w_j^b \in W^b$, is mapped to geodesic $y_i^b x_0 y_j^b \in M(G'')$, where $y_i^b, y_j^b \in Y^b$.
- 4. Each $P'(w_i^r, w_j^r)$ of $M'_{U,W}(G)$, where $w_i^r, w_j^r \in W^r$, is mapped to geodesic $y_i^r x_0 y_j^r \in M(G'')$, where $y_i^r, y_j^r \in Y^r$.

- 5. Each $P_{w_k}(u_i^b, u_j^r)$ of $M'_{U,W}(G)$, where $u_i^b \in U^b, u_j^r \in U^r, w_k \in W$, is mapped to geodesic $x_i^b y_k x_j^r \in M(G'')$ where $x_i^b \in X^b, x_j^r \in X^r, y_k \in Y$.
- 6. Each $P_{u_k}(w_i^b, w_j^r)$ of $M'_{U,W}(G)$, where $w_i^b \in W^b, w_j^r \in W^r, u_k \in U$, is mapped to geodesic $y_i^b x_k y_j^r \in M(G'')$ where $y_i^b \in Y^b, y_j^r \in Y^r, x_k \in X$.

If S is a subset set of $M'_{U,W}(G)$ in BF(r), then let $S(P \leftarrow f_P)$ be a subset of M(G'') defined by $S(P \leftarrow f_P) = \{f_P : P \in S \text{ is replaced by } f_P\}$. By the mapping defined above, if the geodesics of S cover $U \cup W$ of BF(r), then the geodesics of $S(P \leftarrow f_P)$ cover $X \cup Y$ of G''. See Fig. 4. Since $|S| = |S(P \leftarrow f_P)|$, by applying Lemma 2.1, we get the inequality gcover $(M'_{U,W}(G), U \cup W, G) \geq \operatorname{gcover}(M(G''), X \cup Y, G'')$.



Figure 4: (a) G = BF(3). (b) Complete bipartite graph G''. If a set S of geodesics of BF(r) cover $U \cup W$ of BF(r), then the geodesics of $S(P \leftarrow f_P)$ cover $X \cup Y$ of G''.

Next we shall prove that $gcover(M(G''), X \cup Y, G'') = gcover(M(G'), X \cup Y, G')$. By Lemma 2.1, we get the inequality $gcover(M(G''), X \cup Y, G'') \leq gcover(M(G'), X \cup Y, G')$ because M(G'') is superset of M(G'). Now we prove the reverse inequality. If $P \in M(G'')$ and $V(P) \subseteq X \cup Y$, then $P \in M(G')$. In other words, if a subset S of M(G'') covers $X \cup Y$, there exists a subset S' of M(G') such that S' covers $X \cup Y$ and |S| = |S'|. Thus, $gcover(M(G'), X \cup Y, G') \leq gcover(M(G''), X \cup Y, G'')$.

Since G'(U, V, E') is a complete bipartite graph $K_{2^r,2^r}$, by Lemma 2.1 and Proposition 2.2, we infer that $gcover(M(G'), X \cup Y, G') = gcover(G') \ge \lceil (2/3)2^r \rceil$. \Box

Combining Lemma 3.3 with Lemma 3.4, we have:

Lemma 3.5. If $r \ge 2$, then gcover(BF(r)) $\ge \lfloor (2/3)2^r \rfloor$.

3.2 An upper bound for the geodesic cover number of butterfly networks

In this section, our aim is to construct a geodesic cover of cardinality $\lceil (2/3)2^r \rceil$ for BF(r). In BF(r), there are 2^r rows and r + 1 levels. The set of vertices at level 0 is $U = \{u_1, \ldots, u_{2^r}\}$, the set of vertices at level r in $W = \{w_1, \ldots, w_{2^r}\}$. In this section, the order of vertices in U and W are with respect to normal representation of BF(r). (In the previous section, the order was with respect to diamond representation.) Refer to Fig. 5. The set U is further partitioned into A and B, and W is partitioned into C and D, see Fig. 5. These sets are formally defined as follows:

$$A = \{[0, 1], [0, 2], \dots, [0, 2^{r-1}]\},\$$

$$B = \{[0, 2^{r-1} + 1], [0, 2^{r-1} + 2], \dots, [0, 2^r]\},\$$

$$C = \{[r, 1], [r, 2], \dots, [r, 2^{r-1}]\},\$$

$$D = \{[r, 2^{r-1} + 1], [r, 2^{r-1} + 2], \dots, [r, 2^r]\}.$$

The next important step is to color the vertices of BF(r) in two colors—red and blue. In U, the vertex [0, i] is colored in red if i is even, and is colored in blue otherwise. In W, the vertices of C are colored in red and the vertices of D are colored in blue. See Fig. 5 again.

We concentrate only on diametrals of BF(r) because we shall a construct geodesic cover of BF(r) in terms of diametrals. Thus, it is necessary to study the properties of diametrals of BF(r). Throughout this section, $P_v(u, w)$ denotes a diametral in BF(r) such that u and w are the end vertices of $P_v(u, w)$ and v is the middle vertex of $P_v(u, w)$. Now onward, we only consider BF(r) with colored vertices as described before, see Fig. 1.

Property 3.6. If a vertex v is at level 0 (level r) and vertices u, w at level r (level 0) are in opposite colors, then there exists a unique diametral $P_v(u, w)$ in BF(r).

Proof. The structural details of two different representations of BF(r) which are illustrated in Fig. 1 are explained in [9]. By Lemma 3.1, a geodesic P(x, y) between a vertex x at level 0 and a vertex y at level r is unique in BF(r) and the length of P(x, y) is r. From the diamond representation of BF(r) in Fig. 1 (b), whenever the vertex v at level 0 (level r) is in any color and vertices u, w at level r (level 0) are in opposite colors, there exists a diametral $P_v(u, w)$ of BF(r) between u and w passing through v. Since P(u, v) and P(v, w) are unique, $P_v(u, w)$ is also unique. \Box



Figure 5: In Stage 1, geodesics $P_{u_1}(w_1, w_9)$, $P_{u_2}(w_2, w_{10})$ and $P_{u_{16}}(w_{16}, w_8)$, $P_{u_{15}}(w_{15}, w_7)$ are constructed (pink). In Stage 2, geodesics $P_{w_3}(u_3, u_4)$, $P_{w_5}(u_5, u_6)$ and $P_{w_{14}}(u_{14}, u_{13})$, $P_{w_{12}}(u_{12}, u_{11})$ are constructed (green). The vertices which are not covered in previous stages are circled. They are then covered in Stage 3.

Thus, by Property 3.6, in order to construct a diametral path $P_v(u, w)$ in BF(r), it is enough to identify the middle vertex v at level 0 (level r) in any color and end vertices $\{u, w\}$ at level r (level 0) in opposite colors.

The construction of a geodesic cover of BF(r) is carried out in three stages, cf. Fig. 5.

Stage 1

In the first stage, the following diametrals are constructed:

- 1. $P_{u_i}(w_i, w_{i+2^{r-1}})$, where $i \in [2^{r-3}]$ and $u_i \in A$.
- 2. $P_{u_i}(w_i, w_{i-2^{r-1}})$, where $i \in \{2^r, 2^r 1, \dots, 7 \cdot 2^{r-3} + 1\}$ and $u_i \in B$.

Stage 2

In the second stage, the following diametrals are constructed:

- 1. $P_{w_i}(u_i, u_{i+1})$, where $i \in \{2^{r-3} + 1, 2^{r-3} + 3, \dots, 3 \cdot 2^{r-3}\}$ and $w_i \in C$.
- 2. $P_{w_i}(u_i, u_{i-1})$, where $i \in \{7 \cdot 2^{r-3}, 7 \cdot 2^{r-3} 2, \dots, 5 \cdot 2^{r-3} + 1\}$ and $w_i \in D$.

The vertices not covered by the diametrals during the first two stages are:

1. $A' = \{u_i \in A : i \in \{3 \cdot 2^{r-3} + 1, 3 \cdot 2^{r-3} + 2, \dots, 2^{r-1}\}\}.$ 2. $B' = \{u_i \in B : i \in \{2^{r-1} + 1, 2^{r-1} + 2, \dots, 5 \cdot 2^{r-3}\}\}.$ 3. $C' = \{w_i \in C : i \in \{2^{r-3} + 2, 2^{r-3} + 4, \dots, 3 \cdot 2^{r-3}\}\}.$ 4. $D' = \{w_i \in D : i \in 5 \cdot 2^{r-3} + 1, 5 \cdot 2^{r-3} + 3, \dots, 7 \cdot 2^{r-3}\}\}.$

Note that A' has equal number of red and blue vertices and that the same holds B'. Also, C' has only red vertices, while D' has only blue vertices. Refer to Fig. 5. There are 2^{r-3} red vertices in $A' \cup B'$, 2^{r-3} blue vertices in $A' \cup B'$, 2^{r-3} red vertices in C', and 2^{r-3} blue vertices in D'.

Stage 3

We first regroup and rename the vertices of A' and B' into the following sets:

- 1. $U^r = \{u_i^r : i \in [2^{r-3}]\}$ the red vertices of A' and B'.
- 2. $U^b = \{u_i^b : i \in [2^{r-3}]\}$ the blue vertices of A' and B'.
- 3. $W^r = \{w_i^r : i \in [2^{r-3}]\}$ the vertices of C'.
- 4. $W^b = \{w_i^b : i \in [2^{r-3}]\}$ the vertices of D'.

(The sets W^r and W^b are thus obtained by renaming the vertices of C' and D'.) The next step in Stage 3 is to partition the vertices of U^r , U^b , W^r , and W^b into four subsets. For a fixed $r \ge 5$, let $\ell = \lfloor \frac{2^{r-3}}{3} \rfloor$. Then $2^{r-3} = 3 \cdot \ell + 1$ or $2^{r-3} = 3 \cdot \ell + 2$.

Case 1: $2^{r-3} = 3 \cdot \ell + 1$.

Recall that U^r contains 2^{r-3} red vertices. The set U^r is further partitioned into three subsets each subset containing ℓ vertices and one subset containing the remaining vertex u_x^r of U^r . The other three sets U^b , W^r , and W^b are also partitioned similarly. The partition of U^r , U^b , W^r , and W^b and their subpartitions are:

U^r	W^r
$egin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{matrix} W_1^r & w_1^r, & w_2^r, & \dots, & w_\ell^r \end{matrix}$
$\begin{bmatrix} U_2^r & u_{\ell+1}^r, & u_{\ell+2}^r, & \dots, & u_{2\ell}^r \end{bmatrix}$	$\begin{bmatrix} W_2^r & w_{\ell+1}^r, w_{\ell+2}^r, \dots, w_{2\ell}^r \end{bmatrix}$
$\begin{bmatrix} U_3^r & u_{2\ell+1}^r, u_{2\ell+2}^r, \dots, & u_{3\ell}^r \end{bmatrix}$	$\begin{bmatrix} W_3^r & w_{2\ell+1}^r, w_{2\ell+2}^r, \dots, & w_{3\ell}^r \end{bmatrix}$
$egin{array}{c c} U_4^r & u_x^r \end{array}$	$W_4^r \mid w_x^r$
U^b	W^b
$\begin{array}{c c} U^b \\ \hline U^b_1 & u^b_1, & u^b_2, & \dots, & u^b_\ell \end{array}$	$egin{array}{c c} W^b \ W^b_1 & w^b_1, & w^b_2, & \dots, & w^b_\ell \end{array}$
$\begin{array}{c c} U^b \\ \hline U^b_1 & u^b_1, & u^b_2, & \dots, & u^b_\ell \\ \hline U^b_2 & u^b_{\ell+1}, & u^b_{\ell+2}, & \dots, & u^b_{2\ell} \end{array}$	$\begin{array}{c c c} & & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & &$
$\begin{array}{c c c} & U^b \\ \hline U^b_1 & u^b_1, & u^b_2, & \dots, & u^b_\ell \\ \hline U^b_2 & u^b_{\ell+1}, & u^b_{\ell+2}, & \dots, & u^b_{2\ell} \\ \hline U^b_3 & u^b_{2\ell+1}, & u^b_{2\ell+2}, & \dots, & u^b_{3\ell} \\ \hline \end{array}$	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $

The motivation to partition U^r , U^b , W^r , and W^b into three subsets of equal cardinality ℓ is illustrated in Fig. 6.



Figure 6: Here $U = \{u_1^r, u_2^r, u_3^r, u_1^b, u_2^b, u_3^b\}$ and $W = \{w_1^r, w_2^r, w_3^r, w_1^b, w_2^b, w_3^b\}$, where $u_1^r, u_2^r, u_3^r, w_1^r, w_2^r, w_3^r$ are red vertices, and $u_1^b, u_2^b, u_3^b, w_1^b, w_2^b, w_3^b$ are blue vertices. How to cover the vertices by geodesics of length 3 with end vertices in opposite colors?

Using the technique of Fig. 6, the geodesics are formally constructed as follows:

$$\begin{split} &1. \ \big\{P_{u_i^r}(w_i^r, w_i^b): \ u_i^r \in U_1^r, w_i^r \in W_1^r, w_i^b \in W_1^b, i \in [\ell]\big\}. \\ &2. \ \big\{P_{w_{\ell+i}^r}(u_{\ell+i}^r, u_i^b): w_{\ell+i}^r \in W_2^r, u_{\ell+i}^r \in U_2^r, u_i^b \in U_1^b, i[\in \ell]\big\}. \\ &3. \ \big\{P_{w_{2\ell+i}^b}(u_{2\ell+i}^b, u_{2\ell+i}^r): w_{2\ell+i}^b \in W_3^b, u_{2\ell+i}^b \in U_3^b, u_{2\ell+i}^r \in U_3^r, i \in [\ell]\big\}. \\ &4. \ \big\{P_{u_{\ell+i}^b}(w_{\ell+i}^b, w_{2\ell+i}^r): u_{\ell+i}^b \in U_2^b, w_{\ell+i}^b \in W_2^b, w_{2\ell+i}^r \in W_3^r, i \in [\ell]\big\}. \end{split}$$

- 5. $P_{u_x^r}(w_x^r, w_x^b)$.
- 6. Any geodesic covering of u_x^b .

In Case 1 of Stage 3 we have thus constructed $4\ell + 2 = \lceil \frac{2^{r-1}}{3} \rceil$ geodesics covering all the vertices of $U^r \cup U^b \cup W^r \cup W^b = A' \cup B' \cup C' \cup D'$.

Case 2: $2^{r-3} = 3 \cdot \ell + 2$.

The sets U_1^r , U_2^r , U_3^r of U^r , U_1^b , U_2^b , U_3^b of U^b , W_1^r , W_2^r , W_3^r of W^r , and W_1^b , W_2^b , W_3^b of W^b are the same as in Case 1. The only changes are that $U_4^r = \{u_x^r, u_y^r\}$, $U_4^b = \{u_x^b, u_y^b\}$, $W_4^r = \{w_x^r, w_y^b\}$, $W_4^r = \{w_x^b, w_y^b\}$. The partition of U^r , U^b , W^r , and W^b and their subpartitions are now:

	U^r	
U_1^r	$u_1^r, u_2^r, \dots, \; u_\ell^r$	
U_2^r	$u_{\ell+1}^r, u_{\ell+2}^r, \dots, u_{2\ell}^r$	
U_3^r	$u_{2\ell+1}^r, u_{2\ell+2}^r, \dots, u_{3\ell}^r$	
U_4^r	u_x^r, u_y^r	
	U^b	
U_1^b	$u_1^b, u_2^b, \dots, u_\ell^b$	
U_2^b	$u_{\ell+1}^b, u_{\ell+2}^b, \dots, u_{2\ell}^b$	
U_3^b	$u_{2\ell+1}^b, u_{2\ell+2}^b, \dots, u_{3\ell}^b$	
U_4^b	u_x^b, u_y^b	

	W^r	
W_1^r	$w_1^r, w_2^r, \dots, \; w_\ell^r$	
W_2^r	$w_{\ell+1}^r, w_{\ell+2}^r, \ldots, w_{2\ell}^r$	
W_3^r	$w_{2\ell+1}^r, w_{2\ell+2}^r, \dots, w_{3\ell}^r$	
W_4^r	w^r_x, w^r_y	
W^b		
W_1^b	$w_1^b, w_2^b, \ldots, w_\ell^b$	
W_2^b	$w_{\ell+1}^b, w_{\ell+2}^b, \ldots, w_{2\ell}^b$	
W_3^b	$w_{2\ell+1}^b, w_{2\ell+2}^b, \dots, w_{3\ell}^b$	
W_4^b	w_x^b, w_y^b	

As in Case 1, we we can construct $4\ell + 3$ geodesics to cover all the vertices of U^r , U^b , W^r , and W^b . We have thus constructed $4\ell + 3 = \lceil \frac{2^{r-1}}{3} \rceil$ geodesics covering all the vertices of $U^r \cup U^b \cup W^r \cup W^b = A' \cup B' \cup C' \cup D'$.

Stage 1 constructs $2^{r-3} + 2^{r-3} = 2^{r-2}$ geodesics, Stage 2 constructs $2^{r-3} + 2^{r-3} = 2^{r-2}$ geodesics, and Stage 3 constructs $\lceil \frac{2^{r-1}}{3} \rceil$ geodesics, in total $2^{r-2} + 2^{r-2} + \lceil \frac{2^{r-1}}{3} \rceil = \lceil (2/3)2^r \rceil$ geodesics. Together with Lemma 3.5 this gives our main result:

Theorem 3.7. If $r \ge 5$, then gcover $(BF(r) = \lfloor (2/3)2^r \rfloor$.

In Theorem 3.7 we require $r \ge 5$ because $\ell = \lfloor \frac{2^{r-3}}{3} \rfloor$ is well-defined only when $r \ge 5$. It can be checked by hand that the theorem is not true for r = 2.

4 The edge geodesic cover problem

In this section we turn our attention to the edge geodesic cover problem for BF(r). An edge uv of BF(r) is called a (2, 4)-edge if $\{\deg(u), \deg(v)\} = \{2, 4\}$. The number of (2, 4)-edges of BF(r) is 2^{r+2} [6,9,12]. **Lemma 4.1.** If $r \ge 3$, then E(BF(r)) can be partitioned by a set S(r) of edgedisjoint isometric cycles of length 4r, where $|S(r)| = 2^{r-1}$ and each isometric cycle of S(r) has two vertices at level 0.

Proof. The proof is by induction on the dimension r of BF(r). The base case is r = 3 and is ellaborated in Fig. 7, where E(BF(3)) is partitioned by a set S(3) of edge-disjoint isometric cycles of length $4 \cdot 3$ and each isometric cycle of S(3) has two vertices at level 0.



Figure 7: The base case is BF(3) in which E(BF(3)) is partitioned by a set S of edge-disjoint isometric cycles of length $4 \cdot 3$ where $|S| = 2^{3-1} = 4$.

Assume now that the edge set of BF(k-1) can be partitioned by a set S(k-1) of edge-disjoint isometric cycles of length 4(k-1), where $|S| = 2^{k-2}$ and each isometric cycle C of S(k-1) has two vertices u and w at level 0. A cycle C in S(k-1) is represented by u - P - w - Q - u where u and w are the two vertices of C at level 0, P is the path segment in C between u and w and Q is the path segment in C between u and w. Refer to Fig. 8.

Recall that BF(r) has two copies of BF(k-1), BF'(k-1) and BF''(k-1), where $V(BF'(k-1)) = \{v': v \in BF(k-1)\}$ and $V(BF''(k-1)) = \{v'': v \in BF(k-1)\}.$ Also, there are two copies of S(k-1) in BF(k), S'(k-1) and S''(k-1), where $S'(k-1) = \{C' = u' - P' - w' - Q' - u': C = u - P - w - Q - u \in S(k-1)\}$ and $S''(k-1) = \{C'' = u'' - P'' - w'' - Q'' - u'': C = u - P - w - Q - u \in S(k-1)\}.$ The length of cycles $C \in S(k-1)$, $C' \in S'(k-1)$, and $C'' \in S''(k-1)$ is 4(k-1). In BF(r), there are two vertices a and b at level 0 which are adjacent to u' of C'and u'' of C''. In the same way, there are two vertices x and y at level 0 which are adjacent to w' of C' and w'' of C''. Refer to Fig. 8. Now define the cycles $C_1 = a - u' - P' - w' - x - w'' - P'' - u'' - a$ and $C_2 = b - u' - Q' - w' - y - w'' - Q'' - u'' - b$. It is easy to observe that the length of C_1 and C_2 is 4k. Let us define $S(k) = \{C_1, C_2 :$ $C \in S(k-1)$. Each cycle C_1 and C_2 have two vertices at level 0. It is easy to observe that the cycles in S(k) are isometric and they are mutually edge-disjoint. Moreover, the cardinality of S(k) is 2^{k-1} . Thus, the edge set of BF(k) is partitioned by the edge-disjoint isometric cycles of S(k) such that $|S(k)| = 2^{k-1}$.



Figure 8: (a) An isometric cycle C = u - P - w - Q - u in BF(k-1). (b) Two copies of C are C' and C''. The two isometric cycles C' and C'' are connected by vertices a, b, x and y which are at level 0. (c) The two isometric cycles C' and C'' generate two different isometric cycles C_1 and C_2 (marked in different color) in BF(k).

Theorem 4.2. If $r \geq 3$, then gcover_e(BF(r)) = gpart_e(BF(r)) = 2^r .

Proof. In order to derive the claimed lower bound for $\operatorname{gpart}_{e}(\operatorname{BF}(r))$, let us consider all (2, 4)-edges of $\operatorname{BF}(r)$. As already mentioned, there are 2^{r+2} (2, 4)-edges in $\operatorname{BF}(r)$. Since a geodesic can cover a maximum of four (2, 4)-edges of $\operatorname{BF}(r)$, gcover_e $(\operatorname{BF}(r)) \geq 2^{r+2}/4 = 2^r$. Thus, $\operatorname{gpart}_{e}(\operatorname{BF}(r)) \geq \operatorname{gcover}_{e}(\operatorname{BF}(r)) \geq 2^r$.

To prove $\operatorname{gpart}_{e}(\operatorname{BF}(r)) \leq 2^{r}$, it is enough to construct an edge geodesic partition of cardinality 2^{r} for $\operatorname{BF}(r)$. By Lemma 4.1, the edge set of $\operatorname{BF}(r)$ can be partitioned by a set S of edge-disjoint isometric cycles of length 4r, where $|S| = 2^{r-1}$. In other words, the edge set of $\operatorname{BF}(r)$ can be partitioned by a set R of edge-disjoint diametrals of length 2r such that $|R| = 2^{r}$. Thus, $\operatorname{gpart}_{e}(\operatorname{BF}(r)) \leq 2^{r}$.

5 Conclusion

The geodesic cover problem is one of the fundamental problems in graph theory, but only partial solutions are available for most situations. The geodesic cover number in both vertex and edge version was unknown for butterfly networks, in this paper we provide a complete solutions for both versions.

Even though the geodesic cover and geodesic partition are frequently used in fixed interconnection networks, the exact values of geodesic cover number and geodesic partition number are unknown for popular architectures such as shuffle-exchange, de Bruijn, Kautz, star, pancake, circulant, wrapped butterfly, CCC networks. The geodesic cover problem and the geodesic partition problem (their edge versions) are wide open for researcher.

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