Geodesic packing in graphs

Paul Manuel^a Boštjan Brešar b,c Sandi Klavžar b,c,d

 $^{\it a}$ Department of Information Science, College of Life Sciences, Kuwait University, Kuwait

pauldmanuel@gmail.com

- ^b Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia bostjan.bresar@um.si
 - c Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
 - d Faculty of Mathematics and Physics, University of Ljubljana, Slovenia ${\tt sandi.klavzar@fmf.uni-lj.si}$

Abstract

A geodesic packing of a graph G is a set of vertex-disjoint maximal geodesics. The maximum cardinality of a geodesic packing is the geodesic packing number gpack(G). It is proved that the decision version of the geodesic packing number is NP-complete. We also consider the geodesic transversal number, $\operatorname{gt}(G)$, which is the minimum cardinality of a set of vertices that hit all maximal geodesics in G. While $\operatorname{gt}(G) \geq \operatorname{gpack}(G)$ in every graph G, the quotient $\operatorname{gt}(G)/\operatorname{gpack}(G)$ is investigated. By using the rook's graph, it is proved that there does not exist a constant C < 3 such that $\operatorname{gt}(G) \leq C$ would hold for all graphs G. If G is a tree, then it is proved that $\operatorname{gpack}(G) \leq C$ would hold for all graphs G. If G is a tree, then it is proved that $\operatorname{gpack}(G) = \operatorname{gt}(G)$, and a linear algorithm for determining $\operatorname{gpack}(G)$ is derived. The geodesic packing number is also determined for the strong product of paths.

Keywords: geodesic packing; geodesic transversal; computational complexity; rook's graph; diagonal grid

AMS Subj. Class.: 05C69; 05C12; 05C85

1 Introduction

Pairs of covering-packing problems, known also as dual min-max invariant problems [2], are important topics in graph theory and in combinatorics. The max independent set problem and the min vertex cover problem is an appealing example [3]. Another well-known example is the max matching problem versus the min

edge cover problem [9]. Examples from combinatorial optimization are the min set cover problem & the max set packing problem, and the bin covering & bin packing problem [11]. In this paper, we identify a new dual min-max pair: the geodesic transversal problem and the geodesic packing problem. The first one was recently independently investigated in [16, 18], here we complement these studies by considering the geodesic packing problem.

A geodesic (i.e., a shortest path) in a graph G is maximal if it is not contained (as a subpath) in any other geodesic of G. A set S of vertices of G is a geodesic transversal of G if every maximal geodesic of G contains at least one vertex of S. When $s \in S$ is contained in a maximal geodesic P we say that vertex s hits or covers P. The geodesic transversal number of G, gt(G), is the minimum cardinality of a geodesic transversal of G. A geodesic packing of a graph G is a set of vertex-disjoint maximal geodesics in G. The geodesic packing number, gpack(G), of G is the maximum cardinality of a geodesic packing of G, and the geodesic packing problem of G is to determine gpack(G). By a gpack-set of G we mean a geodesic packing of size gpack(G).

Let us mention some related concepts. A packing of a graph often means a set of vertex-disjoint (edge-disjoint) isomorphic subgraphs, that is, the H-packing problem for an input graph G is to find the largest number of its disjoint subgraphs that are isomorphic to H. In particular, the problem has been investigated for different types of paths. For instance, Akiyama and Chvátal [1] considered the problem from algorithmic point of view when H is a path of fixed length. A survey on efficient algorithms for vertex-disjoint (as well as edge-disjoint) Steiner trees and paths packing problems in planar graphs was given in [19]. Dreier et al. [6] have studied the complexity of packing edge-disjoint paths where the paths are restricted to lengths 2 and 3. In [14] edge-disjoint packing by stars and edge-disjoint packing by cycles were studied.

In the rest of this section we first recall some notions needed in the rest of the paper. In the next section it is first proved that the geodesic packing problem is NP-complete. After that we investigate the quotient $\operatorname{gt}(G)/\operatorname{gpack}(G)$. We first prove that $\operatorname{gt}(K_n \square K_n) = n^2 - 2n + 2$ and use this result to demonstrate that there does not exist a constant C < 3 such that $\frac{\operatorname{gt}(G)}{\operatorname{gpack}(G)} \leq C$ would hold for all graphs G. In Section 3 we consider the geodesic packing number of trees and prove that for a tree T we have $\operatorname{gpack}(T) = \operatorname{gt}(T)$. A linear algorithm for determining $\operatorname{gpack}(T)$ is also derived. In the subsequent section the geodesic packing number is determined for the strong product of paths, while the paper is concluded with some closing remarks.

Let G = (V(G), E(G)) be a graph. The order of G will be denoted by n(G). A path on consecutive vertices a_1, a_2, \ldots, a_k will be denoted by $a_1 a_2 \ldots a_k$. If n is a positive integer, then let $[n] = \{1, \ldots, n\}$. The Cartesian product $G \square H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ and edges (g, h)(g', h'), where either g = g' and $hh' \in E(H)$, or h = h' and $gg' \in E(G)$. The strong product $G \boxtimes H$ is obtained from $G \square H$ by adding, for every edge $gg' \in E(G)$ and every edge $hh' \in E(H)$, an edge between the vertices (g,h) and (g',h') and another edge between the vertices (g,h') and (g',h).

2 Preliminary results and NP-completeness

We start by showing NP-completeness of the geodesic packing problem which is formally defined as follows.

GEODESIC PACKING PROBLEM

Input: A graph G and a positive integer k.

Question: Does there exist a set of k vertex-disjoint maximal geodesics in G?

For our reduction we use the concept of induced path partition. Computationally, given a graph G and a positive integer k, the MAXINDUCED P_k PACKING PROBLEM seeks for a maximum number of vertex-disjoint induced paths P_k . Saying that a set of vertex-disjoint induced paths on k vertices is an induced P_k -packing of G, the problem is thus to maximize the cardinality of an induced P_k -packing. By [17, Theorem 3.1] we know that MAXINDUCED P_3 PACKING PROBLEM is NP-hard on bipartite graphs with maximum degree 3.

Let G be a graph with $V(G) = \{x_1, \ldots, x_n\}$. Then the derived graph G' is defined as follows: $V(G') = V(G) \cup \{x, y, z\}$ and $E(G') = E(G) \cup \{xz, zy\} \cup \{zx_i : i \in [n]\}$. Without any possibility of confusion, we denote by G also the subgraph of G' induced by the vertices of the derived graph G.

Lemma 2.1. A set Ψ is an induced P_3 -packing of G if and only if $\Psi \cup \{(x, z, y)\}$ is a geodesic packing of the derived graph G'.

Proof. Note that all maximal geodesics in G' are of length 2. In particular, the path P: xzy is a maximal geodesic, and every induced path P_3 in G is a maximal geodesic in G'. The statement of the lemma now follows.

From Lemma 2.1 we also infer that $\operatorname{gpack}(G') = 1 + \operatorname{pack}_{ind}^3(G)$, where we denote by $\operatorname{pack}_{ind}^k(G)$ the maximum size of an induced P_k -packing in G. Now, turning back our attention to the decision versions of the problem, it is easy to see that an instance (G,k) of the Maxinduced P_3 -Packing Problem, where G is a bipartite graph with maximum degree 3, reduces to an instance (G',k+1) of the Geodesic Packing Problem.

Theorem 2.2. Geodesic Packing Problem is NP-complete.

By Theorem 2.2 it is of interest to bound the geodesic packing number and to determine it for specific families of graphs. The following straightforward upper bound is useful.

Lemma 2.3. Let d be the length of a shortest maximal geodesic of a graph G. Then, $\operatorname{gpack}(G) \leq \lfloor n(G)/(d+1) \rfloor$.

Given a set of vertex-disjoint maximal geodesics, each geodetic transversal clearly hits each of the paths by at least one private vertex of the path. This fact in particular implies the following upper bound.

Lemma 2.4. If G is a graph, then $gpack(G) \leq gt(G)$.

It is clear that $\operatorname{gpack}(P_n) = 1 = \operatorname{gt}(P_n)$ as well as $\operatorname{gpack}(K_{1,n}) = 1 = \operatorname{gt}(K_{1,n})$, hence the bound of Lemma 2.4 is sharp. On the other hand, the value $\operatorname{gt}(G)$ can be arbitrarily bigger than $\operatorname{gpack}(G)$. For instance, $\operatorname{gpack}(K_n) = \lfloor \frac{n}{2} \rfloor$ and $\operatorname{gt}(K_n) = n-1$. Observe also that in $K_{n,n}$, $n \geq 2$, every maximal geodesic is of length 2, hence $\operatorname{gpack}(K_{n,n}) = \lfloor \frac{2n}{3} \rfloor$, while on the other hand $\operatorname{gt}(K_{n,n}) = n$. However, we do not know whether the ratio of the two invariants is bounded and pose this as a problem.

Problem 2.5. Is there an absolute constant C such that $\frac{\operatorname{gt}(G)}{\operatorname{gpack}(G)} \leq C$, for all graphs G?

The example of complete graphs shows that if the constant C in Problem 2.5 exists, it cannot be smaller than 2. To show that it actually cannot be smaller than 3, consider the rook's graphs [13] that can be described as the Cartesian product of two complete graphs or, equivalently, as the line graphs of complete bipartite graphs [12].

Proposition 2.6. If $n \ge 1$, then $gt(K_n \square K_n) = n^2 - 2n + 2$.

Proof. Set $R_n = K_n \square K_n$, and note that vertices of R_n can be presented in the Cartesian $n \times n$ grid such that two vertices are adjacent if and only if they belong to the same row or the same column.

For n = 1, the statement is clear, so let $n \ge 2$. Note that maximal geodesics P in R_n are of length 2 and consist of three vertices, which can be described as follows: $(g,h) \in V(P)$, and there is a vertex $(g',h) \in V(P)$ in the same column as (g,h) and a vertex $(g,h') \in V(P)$ that is in the same row as (g,h). Let S be the complement of a (smallest) gt-set of R_n . Hence S contains no maximal geodesics as just described.

First, we prove that $|S| \leq 2n - 2$. Let S_i be the set of vertices in S that belong to the i^{th} row of R_n . Due to symmetry, we may assume that rows are ordered in such a way that $|S_1| \geq \cdots \geq |S_n|$. Note that $|S_1| = 1$, implies $|S| \leq n$ and we are done. Hence, let $|S_1| \geq 2$. Note that in the column in which there is a vertex of S_1 there are no other vertices of S, and the same holds for every row S_i having more than one vertex in S. Let $k \geq 1$ be the number of rows in which there are at least two vertices in S, That is, in S_i , $i \in [k]$, we have $|S_i| \geq 2$, but if $|S_j| > 0$, where j > k, then $|S_j| = 1$. Let C be the set of columns in which there are vertices from the sets S_i , where $i \in [k]$. Note that there are |C| vertices of S in these columns.

Since in the remaining columns there are at most n-k vertices from S (because in each of the remaining rows there is at most one vertex in S), we altogether get $|S| \leq |C| + n - k$. Now, if |C| = n, then |S| = n and we are done. Otherwise, $|S| \leq |C| + n - k \leq (n-1) + (n-1) = 2n - 2$. To see that |S| = 2n - 2, take k = 1 with $|S_1| = n - 1$, and add n - 1 vertices in the last column to S.

In the proof of Proposition 2.6 we have reduced the search for the minimum geodesic transversal of rook's graphs to its complement. The latter is equivalent to searching for the largest number of 1-entries in a 0-1 matrix of order n, such that the matrix does not contain any of the four 2×2 submatrices with three 1-entries. As one of the reviewers pointed out, this is known to be 2n-2, however, we were not able to find a reference for it (as this reviewer has also failed to find). Say, in [8], which is one of the seminal papers on forbidden submatrices, the authors consider 0-1 matrices with four 1-entries and only have Corollary 2.4(1) on matrices with three 1-entries. We also add that the case when 2×2 submatrices with four 1-entries are forbidden is (a special case of) the Zarankiewitz's problem [20] which is a notorious open problem. Interestingly, it was very recently observed in [4, Corollary 3.7], that the latter problem is equivalent to determine the so-called mutual-visibility number [5] of the rook's graphs.

Since all maximal geodesics in $K_n \square K_n$ are of length 2, Lemma 2.3 implies that $\operatorname{gpack}(K_n \square K_n) \leq \frac{n^2}{3}$. We can thus estimate as follows:

$$\frac{\operatorname{gt}(K_n \square K_n)}{\operatorname{gpack}(K_n \square K_n)} \ge \frac{3(n^2 - 2n + 2)}{n^2} = 3\left(1 - \frac{2}{n} + \frac{2}{n^2}\right).$$

Letting n to infinity we have shown that in case the constant C from Problem 2.5 exists, it cannot be smaller than 3.

In rook's graphs $K_n \square K_n$, $n \ge 2$, every maximal geodesic is of length $2 = \operatorname{diam}(K_n \square K_n)$. More generally, a graph G is uniform geodesic if every maximal geodesic in G is of length $\operatorname{diam}(G)$. Complete graphs, cycles, and paths are simple additional families of uniform geodesic graphs. The fact that rook's graphs are uniform geodesic generalizes as follows.

Proposition 2.7. If G_1, \ldots, G_r , $r \ge 1$, are uniform geodesic graphs, then the product $G_1 \square \cdots \square G_r$ is also a uniform geodesic graph.

Proof. The result clearly holds for r=1. Moreover, by the associativity of the Cartesian product, it suffices to prove the lemma for two factors. Let hence P be an arbitrary maximal geodesic in $G \square H$. Then the projections P_G and P_H of P on G and on H are geodesics in G and H, respectively. If P_G is not maximal in G, then P_G can be extended to a longer geodesic in G, but then also P can be extended to a longer geodesic in $G \square H$, a contradiction. So P_G and P_H are maximal geodesics in G and G and G are provided in G and G and G are provided in G and G are provided in G and G are provided in G and G and G are provided in G and G are provided in G and G and G are provided in G and G are provided in

 $\operatorname{diam}(G)$ and $\operatorname{diam}(H)$, respectively. As the distance function is additive in Cartesian products, it follows that the length of P is $\operatorname{diam}(G) + \operatorname{diam}(H) = \operatorname{diam}(G \square H)$. \square

Proposition 2.7, Lemma 2.3, and the fact that the diameter is also additive on Cartesian products, yield the following result.

Corollary 2.8. If G_1, \ldots, G_r , $r \geq 1$, are uniform geodesic graphs, then

$$\operatorname{gpack}(G_1 \square \cdots \square G_r) \leq \left\lfloor \frac{n(G_1) \cdots n(G_r)}{\operatorname{diam}(G_1) + \cdots + \operatorname{diam}(G_r) + 1} \right\rfloor.$$

3 Trees

In this section we derive an efficient algorithm to obtain the geodesic packing number of an arbitrary tree. The approach used is in part similar to the approach from [16] to determine the geodetic transversal number of a tree.

In this section we apply the "smoothing" operation on vertices of degree 2, which is formally defined as follows. Let xuy be a path of length 2 in G such that the degree of vertex u in G is 2. Then a new graph SM(G) is obtained from G by removing the vertex u and adding the edge xy. When there exist two adjacent 2-degree vertices in G, this operation is carried out sequentially one after another. Let further SM(G) denote the graph obtained from G by smoothing all the vertices of G of degree 2. In the smoothing operation, a path xuy is replaced by an edge xy when deg(u) = 2 and thus $deg_{SM(G)}(v) = deg_G(v)$ for every vertex v in G'. Since the smoothing operation preserves the degree of vertices, SM(G) is well-defined, that is, unique up to isomorphism. It was proved in [16, Lemma 4.2] that gt(T) = gt(SM(T)) in any tree T. We prove a similar result for the packing invariant.

Lemma 3.1. If T is a tree, then
$$gpack(T) = gpack(SM(T))$$
.

Proof. Note that each maximal geodesic in a tree connects two leaves of the tree. Let Ψ_T be a largest geodesic packing in T. Its elements can thus be represented by pairs of leaves that are endvertices of the corresponding geodesics. Note that a maximal geodesic in Ψ_T from which we remove all vertices of degree 2 becomes a maximal geodesic in SM(T). Thus the same pairs of leaves can be used in SM(T) to represent the maximal geodesics by its end-vertices. We denote by $SM(\Psi_T)$ the resulting set of maximal geodesics in SM(T). Since any two geodesics $g_1, g_2 \in \Psi_T$ are disjoint, so are also the corresponding geodesics in $SM(\Psi_T)$. This implies that $gpack(T) \leq gpack(SM(T))$. The reversed inequality can be proved in a similar way. Notably, since the maximal geodesics in SM(T) have two leaves of SM(T) as its end-vertices, the same two leaves are end-vertices of a maximal geodesic in T. It is clear that the resulting maximal geodesics in T are also mutually vertex-disjoint,

and thus together form a geodesic packing in T of cardinality gpack(SM(T)). Thus, gpack(T) \geq gpack(SM(T)).

Lemma 3.1 does not hold for an arbitrary graph G. See Fig. 1, where a graph G is shown for which we have $\operatorname{gpack}(G) = 4$ and $\operatorname{gpack}(\operatorname{SM}(G)) = 3$. Pairs of endvertices of maximal geodesics are marked by distinct colors.

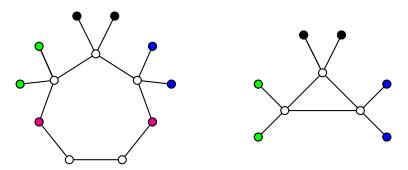


Figure 1: A graph G with gpack(G) = 4, and SM(G) with gpack(SM(G)) = 3.

A support vertex in a tree is a vertex adjacent to a leaf. An end support vertex is a support vertex that has at most one non-leaf neighbor. It is easy to see that an end support vertex does not lie between two end support vertices. In addition, every tree on at least two vertices contains an end support vertex (see, for instance, [16]). In [16, Lemma 4.3] the following result was proved.

Lemma 3.2. [16] Let T be a tree with no vertices of degree 2. Let u be an end support vertex of T and u_1, \ldots, u_s the leaves adjacent to u. Then $gt(T) = gt(T - \{u, u_1, \ldots, u_s\}) + 1$. Moreover, there exists a gt-set S of T such that $u \in S$.

We prove a result parallel to Lemma 3.2 concerning the geodesic packing number.

Lemma 3.3. Let T be a tree with no vertices of degree 2. Let u be an end support vertex of T and u_1, \ldots, u_s the leaves adjacent to u. Then $\operatorname{gpack}(T) = \operatorname{gpack}(T - \{u, u_1, \ldots, u_s\}) + 1$. Moreover, there exists a $\operatorname{gpack-set} \Psi$ of T such that $u_1uu_2 \in \Psi$.

Proof. Since T has no vertices of degree 2, the end support vertex u is adjacent to at least two leaves, that is, $s \geq 2$. If T is a star, and hence u being the center of it, then the assertion of the lemma is clear. In the rest of the proof we may thus assume that u has at least one non-leaf neighbor, and since u is an end support vertex, it has only one non-leaf neighbor. We denote the latter vertex by w, and let T' be the component of T - u that contains the vertex w.

Let Ψ' be a gpack-set of T'. Since u_1uu_2 is a maximal geodesic in T, and every maximal geodesic in T' is a maximal geodesic also in T, we infer that $\Psi' \cup \{u_1uu_2\}$ is a geodesic packing of T. Hence $\operatorname{gpack}(T) \geq \operatorname{gpack}(T') + 1$.

Note that there can be at most one maximal geodesic in a geodesic packing of T that contains vertex u. In addition, there is at least one geodesic that contains u if a geodesic packing of T is of maximum cardinality (for otherwise, one could add the geodesic u_1uu_2 and make it of larger cardinality, which is a contradiction). Now, let Ψ be a gpack-set of T and let $P \in \Psi$ be the geodesic that contains u. It is easy to see that all maximal geodesics in $\Psi \setminus \{P\}$ belong to T' and are also pairwise vertex-disjoint maximal geodesics of T'. Hence $\operatorname{gpack}(T') \geq \operatorname{gpack}(T) - 1$, and we are done.

Combining the facts that $\operatorname{gpack}(K_2) = 1 = \operatorname{gt}(K_2)$, that in any tree T we have $\operatorname{gt}(T) = \operatorname{gt}(\operatorname{SM}(T))$ and $\operatorname{gpack}(T) = \operatorname{gpack}(\operatorname{SM}(T))$, and using Lemmas 3.2 and 3.3, we deduce the following result.

Theorem 3.4. If T is a tree, then gpack(T) = gt(T).

Algorithm 1: gpack-set of a tree

Using the lemmas from this section, we can now present an algorithm that constructs a gpack-set of an arbitrary tree T. Note that a gpack-set of T is uniquely determined by pairs of endvertices of its maximal geodesics, and the outcome of the algorithm is the set of such (ordered) pairs.

```
Input: A tree T.

Output: A gpack-set \Psi, represented by pairs of end-vertices.
```

```
1 \Psi = \emptyset

2 T = \mathrm{SM}(T)

3 while n(T) \geq 3 do

4 | identify an end support vertex p of \mathrm{SM}(T), and its leaf-neighbors u_1, u_2

5 | \Psi = \Psi \cup \{(u_1, u_2)\}

6 | T = T - \{p, u_1, \dots, u_t\}, where u_1, \dots, u_t are the leaf neighbors of p

7 | T = \mathrm{SM}(T)

8 if n(T) = 2 then

9 | \Psi = \Psi \cup V(T)
```

Theorem 3.5. Given a tree T, Algorithm 1 returns the set of pairs of end vertices of maximal geodesics of a gpack-set of T in linear time.

The correctness of Algorithm 1 follows from Lemmas 3.1 and 3.3. The time complexity of the algorithm is clearly linear. For the running time of the algorithm, in Step 7, there is nothing to be done if T is a star. Otherwise, the unique non-leaf neighbor of the vertex p selected in Step 4 is the only vertex for which we need to check whether the smoothing operation is required.

4 Diagonal grids

Diagonal grids are strong products of paths [12]. If a diagonal grid is the strong product of r paths, then it is called an r-dimensional diagonal grid. By definition, the r-dimensional grid $P_{d_1} \square \cdots \square P_{d_r}$ is a spanning subgraph of $P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$, cf. Fig. 2. The edges of $P_{d_1} \square \cdots \square P_{d_r}$ (considered as a subgraph of $P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$) are called Cartesian edges of $P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$, the other edges are diagonal edges. We say that a geodesic consisting of only Cartesian edges is a Cartesian geodesic of $P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$. In the rest we will assume that the vertices of a path on r vertices are integers $1, \ldots, r$, and if $x \in V(P_{d_1} \boxtimes \cdots \boxtimes P_{d_r})$, then we will use the notation $x = (x_1, \ldots, x_r)$.

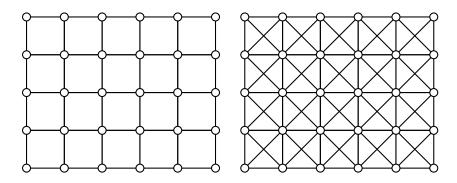


Figure 2: (a) A 2-dimensional grid $P_6 \square P_5$ and (b) a 2-dimensional diagonal grid $P_6 \boxtimes P_5$

Lemma 4.1. If P is a maximal geodesic in $P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$, where $r \geq 2$, and $d_1, \ldots, d_r \geq 2$, then $n(P) \in \{d_1, \ldots, d_r\}$.

Proof. Let P be an arbitrary geodesic of $G = P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$ of length $\ell \geq 2$, so that $n(P) = \ell + 1$. Let xx' and yy' be the first and the last edge of P, where x and y' are the first and the last vertex of P, respectively. It is possible that x' = y. Then $\ell = d_G(x, y') = 1 + d_G(x', y) + 1$. (Note that if x' = y, then $d_G(x', y) = 0$.)

Since $d_G(x, y') = \max\{|x_1 - y_1'|, \dots, |x_r - y_r'|\}$, we may without loss of generality assume (having in mind that the strong product operation is commutative) that $\ell = d_G(x, y') = |x_1 - y_1'|$. We now claim that $y_1 \neq y_1'$ and suppose on the contrary that $y_1 = y_1'$. Using the facts that $d_G(x', y) = \max\{|x_1' - y_1|, \dots, |x_r' - y_r|\}$, $|x_1 - y_1'| = \ell$, $|x_1 - x_1'| \leq 1$, and $y_1 = y_1'$, we get that $|x_1' - y_1| \geq \ell - 1$. Consequently, $d_G(x', y) \geq \ell - 1$, which in turn implies that

$$\ell = d_G(x, y') = 1 + d_G(x', y) + 1 \ge 1 + (\ell - 1) + 1 = \ell + 1,$$

a contradiction. We have thus proved that if $d_G(x, y') = |x_1 - y_1'|$, then $y_1 \neq y_1'$. Let us emphasize that P was assumed to be an arbitrary geodesic.

Let now P be a maximal geodesic in G and use the same notation as above. Assume again wlog that $\ell = d_G(x, y') = |x_1 - y_1'|$. If uv is an arbitrary edge of P which is different from xx', then the above claim asserts that $u_1 \neq v_1$. Since $\ell = d_G(x, y') = |x_1 - y_1'|$ it follows that the first coordinates of the vertices of P are $\ell + 1$ consecutive integers $i, i + 1, \ldots, i + \ell$. If i > 1, then adding the edge between x and the vertex $(i - 1, x_2, \ldots, x_r)$ yields a geodesic which strictly contains P, a contradiction. Hence i = 1. By a parallel argument we get that $i + \ell = d_1$. We conclude that $n(P) = d_1$.

From the proof of Lemma 4.1 we can deduce also the following.

Lemma 4.2. Let $G = P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$, where $r \geq 2$ and $d_i \geq 2$ for $i \in [r]$. If $x = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots x_r)$ and $y = (y_1, \ldots, y_{i-1}, d_i, y_{i+1}, \ldots y_r)$ are vertices of G with $d_G(x, y) = d_i - 1$, then there exists a maximal x, y-geodesic in G of length $d_i - 1$.

We are now in position to determine the geodesic packing number of diagonal grids.

Theorem 4.3. If $r \geq 2$ and $2 \leq d_1 \leq \min\{d_2, ..., d_r\}$, then

$$\operatorname{gpack}(P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}) = d_2 \cdot d_3 \cdots d_r$$
.

Proof. Set $G = P_{d_1} \boxtimes \cdots \boxtimes P_{d_r}$. For each vector (i_2, \ldots, i_r) , where $i_j \in [d_j]$, $j \in \{2, \ldots, r\}$, let P_{i_2, \ldots, i_r} be the path

$$(1, i_2, \ldots, i_r)(2, i_2, \ldots, i_r) \ldots (d_1, i_2, \ldots, i_r).$$

By Lemma 4.2, P_{i_2,\dots,i_r} is a maximal geodesic of G. Hence the set

$$\{P_{i_2,\dots,i_r}: i_j \in [d_j], j \in \{2,\dots,r\}\}$$

is a geodesic packing of G. Its size is $d_2 \cdot d_3 \cdot \cdot \cdot \cdot d_r$ which means that hence $\operatorname{gpack}(G) \ge d_2 \cdot d_3 \cdot \cdot \cdot \cdot d_r$.

From Lemma 4.1 we know that a shortest maximal geodesic of G is of length $d_1 - 1$. This implies, by using Lemma 2.3, that $\operatorname{gpack}(G) \leq n(G)/d_1 = d_2 \cdot d_3 \cdots d_r$ and we are done.

5 Conclusions

We have introduced the geodesic packing problem which is a min-max dual invariant to the earlier studied geodesic transversal problem. We have settled the

complexity status of the geodesic packing problem for general graphs and arbitrary trees, and determined the geodesic packing number for several classes of graphs. We have proved that gpack(T) = gt(T) for arbitrary trees T. It is not known that gpack(G) = gt(G) when G is a cactus graph or block graphs. There are numerous open problems that are left for future investigation. One open problem is explicitly stated in Problem 2.5. Other natural extensions of our research would be to study the geodesic packing number for general strong products or other graph products and the general packing number for intersection graphs such as interval graphs, circular arc graphs or chordal graphs.

Acknowledgments

This work was supported and funded by Kuwait University, Research Project No. (FI01/22).

Conflict of interest

The authors declare that they have no conflict of interest.

References

- [1] J. Akiyama, V. Chvátal, Packing paths perfectly, Discrete Math. 85 (1990) 247–255.
- [2] Y. Azar, N. Buchbinder, H. Chan, S. Chen, I. Cohen, A. Gupta, Z. Huang, N. Kang, V. Nagarajan, J. Naor, D. Panigrahi, Online algorithms for covering and packing problems with convex objectives. 57th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2016, 148–157, IEEE Computer Soc., Los Alamitos, CA, 2016.
- [3] K. Casel, H. Fernau, M. Khosravian Ghadikolaei, J. Monnot, F. Sikora, Extension of vertex cover and independent set in some classes of graphs, Lecture Notes in Comput. Sci. 11485 (2019) 124–136.
- [4] S. Cicerone, G. Di Stefano, S. Klavžar, On the mutual-visibility in Cartesian products and in triangle-free graphs, Appl. Math. Comput. 438 (2023) 127619.
- [5] G. Di Stefano, Mutual visibility in graphs, Appl. Math. Comput. 419 (2022) 126850.

- [6] J. Dreier, J. Fuchs, T.A. Hartmann, P. Kuinke, P. Rossmanith, B. Tauer H.-L. Wang, The complexity of packing edge-disjoint paths. 14th International Symposium on Parameterized and Exact Computation, Art. No. 10, 16 pp., Leibniz Int. Proc. Inform., 148, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019.
- [7] D. C. Fisher, S.L. Fitzpatrick, The isometric number of a graph, J. Combin. Math. Combin. Comput. 38 (2001) 97–110.
- [8] Z. Füredi, P. Hajnal, Davenport-Schinzel theory of matrices, Discrete Math. 103 (1992) 233–251.
- [9] T. Gallai, Über extreme Punkt- und Kantenmengen, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 2 (1959) 133–138.
- [10] M. Ghorbani, S. Klavžar, H.R. Maimani, M. Momeni, F. Rahimi-Mahid, G. Rus, The general position problem on Kneser graphs and on some graph operations, Discuss. Math. Graph Theory 41 (2021) 1199–1213.
- [11] P. Hansen, M. Labbé, D. Schindl, Set covering and packing formulations of graph coloring: Algorithms and first polyhedral results, Discrete Opt. 6 (2009) 135–147.
- [12] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs, Second Edition, CRC Press, Boca Raton, FL, 2011.
- [13] A.J. Hoffman, On the line graph of the complete bipartite graph, Ann. Math. Statist. 35 (1964) 883–885.
- [14] M. Jiang, G. Xia, Y. Zhang, Edge-disjoint packing of stars and cycles, Theoret. Comput. Sci. 640 (2016) 61–69.
- [15] P. Manuel, On the isometric path partition problem, Discuss. Math. Graph Theory 41 (2021) 1077–1089.
- [16] P. Manuel, B. Brešar, S. Klavžar, The geodesic-transversal problem, Appl. Math. Comput. 413 (2022) 126621.
- [17] J. Monnot, S. Toulouse, The path partition problem and related problems in bipartite graphs, Oper. Res. Lett. 35 (2007) 677–684.
- [18] I. Peterin, G. Semanišin, On the maximal shortest path cover number, Mathematics 9 (2021) 1592.
- [19] D. Wagner, Simple algorithms for Steiner trees and paths packing problems in planar graphs, CWI Quarterly 6 (1993) 219–240.

 $[20]\,$ K. Zarankiewicz, Problem~P~101, Colloq. Math. ${\bf 2}~(1951)~301.$