Graphs whose mixed metric dimension is equal to their order

Ali Ghalavand^a Sandi Klavžar^{b,c,d} Mostafa Tavakoli^{a,*}

^a Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran m_tavakoli@um.ac.ir alighalavand@grad.kashanu.ac.ir

^b Faculty of Mathematics and Physics, University of Ljubljana, Slovenia sandi.klavzar@fmf.uni-lj.si

^c Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

^d Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Abstract

The mixed metric dimension $\operatorname{mdim}(G)$ of a graph G is the cardinality of a smallest set of vertices that (metrically) resolves each pair of elements from $V(G) \cup E(G)$. We say that G is a max-mdim graph if $\operatorname{mdim}(G) = n(G)$. It is proved that a max-mdim graph G with $n(G) \geq 7$ contains a vertex of degree at least 5. Using the strong product of graphs and amalgamations large families of max-mdim graphs are constructed. The mixed metric dimension of graphs with at least one universal vertex is determined. The mixed metric dimension of graphs G with cut vertices is bounded from the above and the mixed metric dimension of block graphs computed.

Key words: resolving set; mixed resolving set; strong product of graphs; cut vertex; chemical graphs; block graphs

^{*}Corresponding author

1 Introduction

The metric dimension is an extremely prolific and at the same time interesting area of graph theory, for several reasons. The main reason is certainly that the theory is extremely useful in other areas of science, for instance in computer science, chemistry, social networks, and biology, see respective papers [8, 4, 18, 17]. For more information on the metric dimension and its applications see the recent survey [16]. On the other hand, various applications also give rise to certain modifications of the basic concept, which leads to further intensive research to obtain additional insight into the classical topic and between variants. For more information on this point of view of the metric dimension see the other recent survey [7]. We also refer to a recent application of the local metric dimension to delivery services from [6].

A very interesting version of the metric dimension was introduced in 2017 by Kelenc, Kuziak, Taranenko, and Yero [5], namely mixed metric dimension, as follows. Let G = (V(G), E(G)) be a graph. Then two elements $x, y \in V(G) \cup E(G)$ are resolved by a vertex $v \in V(G)$ if $d_G(x, v) \neq d_G(y, v)$, where d_G stands either for the shortest-path distance between vertices, or the distance between an edge and a vertex. The latter distance is, for an edge x = ww' and a vertex v, defined by $d_G(x, v) = \min\{d_G(w, v), d_G(w', v)\}$. A set of vertices $W \subset V(G)$ is a mixed resolving set for G if any two elements (vertices or edges) $x, y \in V(G) \cup E(G)$ are resolved by a vertex of W. We note that V(G) is always a mixed resolving set for G. A mixed resolving set of the smallest cardinality is a mixed metric basis, its cardinality is the mixed metric dimension mdim(G). After the seminal paper, the mixed metric dimension was investigated in many papers, cf. [2, 9, 10, 11, 12, 14, 15].

Let G be a graph and $x \in V(G)$. Then a neighbor y of x is a maximal neighbor of x if y is adjacent to all neighbors of x. Denoting the order of G by n(G), we recall the following result which is the main source of inspiration for this article.

Theorem 1.1 [5, Theorem 3.8] If G is a graph, then mdim(G) = n(G) if and only if every vertex of G has a maximal neighbour.

Let us say that a graph G with $\operatorname{mdim}(G) = n(G)$ is a max-mdim graph. Theorem 1.1 thus characterizes max-mdim graphs. The main purpose of this article is to take a closer look at this class of graphs. We proceed as follows. In the rest of the introduction some further definitions are listed and a result is stated to be used later on. In the next section we prove that a max-mdim graph G with $n(G) \ge 7$ contains a vertex of degree at least 5. This implies that if G is a chemical graph, then $\operatorname{mdim}(G) \le n(G) - 1$. Afterwards we apply the strong product and amalgamations to construct large families of max-mdim graphs. In particular, the strong products $P_n \boxtimes K_2$ are max-mdim graphs with $\Delta = 5$ where Δ is the largest number of neighbors of a vertex in $P_n \boxtimes K_2$. We also determine the mixed metric dimension for graphs with universal vertices. In the concluding section we consider the mixed metric dimension of graphs G with cut vertices and prove an upper bound on their mixed metric dimension. As a consequence we determine the mixed metric dimension of block graphs.

In this paper, we consider finite, simple and connected graphs. Let G be a graph. The degree of $v \in V(G)$ will be denoted by $\deg_G(v)$. The (open) neighborhood of v will be denoted by $N_G(v)$. Then $\deg_G(v) = |N_G(v)|$. A pendant vertex of G is a vertex with degree one. A vertex of degree n(G) - 1 is a universal vertex. The minimum and the maximum degree of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. The number of cut vertices of a G is denoted by $\zeta(G)$ and the set of all cut vertices by CV(G), so that $|CV(G)| = \zeta(G)$. A block of a graph is a nonseparable maximal subgraph of the graph. A graph is 2-connected if it has no cut vertices. Note that if G is 2-connected, then $\zeta(G) = 0$. G is a block graph if each block of G is complete. To conclude this article introduction, we state the following result which implicitly follows from [5, Theorem 3.8].

Lemma 1.2 If W is a mixed resolving set for a graph G, and $v \in V(G)$ has a maximal neighbor, then $v \in W$.

2 Classes of max-mdim graphs and a maximum degree bound

In this section we prove that max-mdim graphs necessarily contain a vertex of degree at least 5 as soon as their order is at least 7. Then we use the strong product and amalgamations to construct large families of max-mdim graphs. In particular, the strong products $P_n \boxtimes K_2$ are max-mdim graphs with $\Delta = 5$. We also determine the mixed metric dimension for graphs with universal vertices.

Since every vertex of a complete graph has a maximal neighbour, by Theorem 1.1, complete graphs are max-mdim graphs. Moreover, if G is obtained from a complete graph by removing a matching, then G is also a max-mdim graph provided that it contains at least two universal vertices. In particular, $K_4 - e$ is a max-mdim graph. Another small example is shown in Fig 1.

In our first theorem we prove that a max-mdim graph contains a vertex of degree at least 5 as soon as it is not very small.

Theorem 2.1 If G is a (connected) max-mdim graph with $n(G) \ge 7$, then $\Delta(G) \ge 5$.

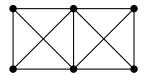


Figure 1: A max-mdim graph

Proof. Let G be a max-mdim graph with $n(G) \ge 7$ and set $\Delta = \Delta(G)$ for the rest of the proof.

Suppose $\Delta = 2$. If $\delta(G) = 1$, the support vertex of a pendant vertex does not admit a maximal neighbor. Otherwise G is a cycle which is not a max-mdim graph. Suppose $\Delta = 3$. Let x be a vertex of G with deg(x) = 3 and let $N_G(x) =$ $\{x_1, x_2, x_3\}$. Without loss generality assume that x_1 is a maximal neighbor of x, so that $x_1x_2, x_1x_3 \in E(G)$ and deg $_G(x_1) = 3$. As $n(G) \geq 7$ and G is connected, x_2 or x_3 is also of degree 3. Assume without loss of generality that $N_G(x_2) = \{x, x_1, x'_2\}$. Now, no matter which neighbor of x_2 is its maximal neighbor, we get deg $_G(x) \geq 4$ or deg $_G(x_1) \geq 4$, which is not possible.

Suppose $\Delta = 4$. Let x be a vertex of G with $\deg(x) = 4$ and let y be its maximal neighbor. Then $N_G(x) = N_G(y)$, say $N_G(x) = N_G(y) = \{x_1, x_2, x_3\}$. As $n(G) \ge 7$, there is another vertex of G, without loss of generality assume it is adjacent to x_1 , denote it by x'_1 . Consider now a maximal neighbor of x_1 . It cannot be x or y because then x or y would be adjacent to x'_1 and hence x or y would be of degree at least 5. For the same reason, a maximal neighbor of x_1 cannot be x'_1 . If there were another neighbor x''_1 of x_1 , it also cannot be a maximal neighbor of x_1 . So x_1 must have a maximal neighbor among the already introduced vertices and thus the fourth neighbor of x_1 is from $\{x_2, x_3\}$. If $x_1x_3 \in E(G)$, then x_3 is the only candidate for a maximal neighbor of x_1 and therefore $x'_1x_3 \in E(G)$, while if $x_1x_2 \in E(G)$, then we must have $x'_1x_2 \in E(G)$. But in both cases we get isomorphic graphs, see Fig. 2, where the labeling presented is with respect to the second case.

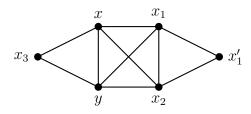


Figure 2: The graph G_6

Hence, if $\Delta = 4$, then G necessary contains the graph G_6 from Fig. 2 as an induced subgraph. Since $n(G) \geq 7$, there is another vertex, say x''_1 , and we may assume without loss of generality that $x'_1x''_1 \in E(G)$. We now infer that none of the vertices x_1, x_2, x''_1 , or a possible fourth new neighbor of x'_1 can be a maximal neighbor of x'_1 . So the only possibility is that $x'_1x_3 \in E(G)$ so that x_3 would be a maximal neighbor of x'_1 . But then $x_3x_1, x_2x_1 \in E(G)$ which means that x_1 would be of degree at least 5.

Note that the proof of Theorem 2.1 implies that the graph G_6 from Fig. 2 is the unique max-mdim graph with n(G) = 6 and $\Delta(G) = 4$.

In view of the applicability of the mixed metric dimension in chemistry [14], we recall that a graph G is called a *chemical graph* if $\Delta(G) \leq 4$. Theorem 2.1 cleary has the following application.

Corollary 2.2 If G is a chemical graph with $n(G) \ge 7$, then $mdim(G) \le n(G) - 1$.

The graphs in Figs. 1 and 2 motivate us to recall the definition of the strong product $G \boxtimes H$ of graphs G and H. Its vertex set is $V(G \boxtimes H) = V(G) \times V(H)$, and vertices (g_1, h_1) and (g_2, h_2) are adjacent if $h_1 = h_2$ and g_1 is adjacent to g_2 , or $g_1 = g_2$ and h_1 is adjacent to h_2 , or g_1 is adjacent to g_2 and h_1 is adjacent to h_2 . A standard reference for the strong product is the book [3]. The metric dimension of strong products was investigated in [13], and the local metric dimension of strong products in [1]. We now use this graph operation to significantly increase the variety of max-mdim graphs.

Proposition 2.3 If G is a graph, then $G \boxtimes K_2$ is a max-mdim graph.

Proof. Let n = n(G), let $V(G) = \{v_1, \ldots, v_n\}$ and $V(K_2) = \{0, 1\}$. Then by the definition of the strong product, $(v_i, 1)$ is a maximal neighbor of $(v_i, 0)$, and $(v_i, 0)$ is a maximal neighbor of $(v_i, 1)$ for each $i \in [n]$. Therefore, $G \boxtimes K_2$ is a max-mdim graph by Theorem 1.1.

The special case $P_n \boxtimes K_2$ of Proposition 2.3 gives an infinite family of max-mdim graphs G with $\Delta(G) = 5$. Hence Theorem 2.1 cannot be improved in general.

Another source for max-mdim graphs is the following construction. Let G and H be disjoint graphs, $e_G \in E(G)$ and $e_H \in E(H)$. Then the graph $A(G, e_G; H, e_H)$ is obtained from the disjoint union of G and H by identifying the edges e_G and e_H . ("A" stands here for an amalgamation.) Actually, this identification can be done in two ways, but for our purposes any of these will do it.

Proposition 2.4 If G and H are max-mdim graphs, and e_G and e_H are edges whose endpoints are maximal neighbors for each other, then $A(G, e_G; H, e_H)$ is a max-mdim graph.

Proof. Set $A = A(G, e_G; H, e_H)$. Let $e_G = gg'$ and $e_H = hh'$. If $x \in V(G) \setminus \{g, g'\}$, then its maximal neighbor in G is also a maximal neighbor of x in A. Similarly, a maximal neighbor of $y \in V(H) \setminus \{h, h'\}$ is a maximal neighbor of y in A. Finally, g = h is a maximal neighbor of g' = h' in A, and g' = h' is a maximal neighbor of g = h.

Using Proposition 2.4, we can state the following result.

Theorem 2.5 If $n \ge t \ge 5$, then there exists a max-mdim graph G with n(G) = nand $\Delta(G) = t$.

Proof. Let $H_r = P_r \boxtimes K_2$, $r \ge 4$, and let H_r^- be the graph obtained from H_r by removing a vertex of degree 3. Let $\Lambda_{k,r} = A(H_r, e; K_k, f)$, where e is an edge of H_r both of its endpoints are of degree 3, and f is an arbitrary (but fixed) edge of K_k . The graph $\Lambda_{k,r}^- = A(H_r^-, e; K_k, f)$ is defined analogously. See Fig. 3 where the graphs H_5 , H_5^- , $\Lambda_{5,5}$, and $\Lambda_{5,5}^-$ are presented.

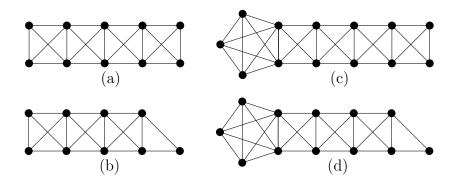


Figure 3: Graphs H_5 (a), H_5^- (b), $\Lambda_{5,5}$ (c), and $\Lambda_{5,5}^-$ (d)

By Proposition 2.4, each of the graphs H_r , H_r^- , $\Lambda_{k,r}$, and $\Lambda_{k,r}^-$ is a max-mdim graph. If n - t + 1 is even, then the graph $\Lambda_{t-1,\frac{n-t+3}{2}}$ is of maximum degree t, while if n - t + 1 is odd, the graph $\Lambda_{t-1,\frac{n-t+4}{2}}^-$ is of maximum degree t.

To complete the argument note that $n(\Lambda_{t-1,\frac{n-t+3}{2}}) = (t-1) + 2\left(\frac{n-t+3}{2} - 1\right) = n$ and $n(\Lambda_{t-1,\frac{n-t+4}{2}}) = (t-1) + 2\left(\frac{n-t+4}{2} - 1\right) - 1 = n$. At the beginning of the section we have observed that a graph obtained from a complete graph by removing a matching is a max-mdim graph provided that it contains at least two universal vertices. This fact generalizes as follows.

Proposition 2.6 If G is a graph, then the following holds.

(i) If G has at least two universal vertices, then G is a max-mdim graph.

(ii) If G has exactly one universal vertex, then mdim(G) = n(G) - 1.

Proof. (i) Let x and y be arbitrary universal vertices of G. If $u \in V(G) \setminus \{x, y\}$, then x (or y for that matter) is a maximal neighbor of u. Moreover, x is a maximal neighbor of y, and y is a maximal neighbor of x. By Theorem 1.1, G is a max-mdim graph.

(ii) Assume now that x is the unique universal vertex of G. Let W be a mixed resolving set for G. By Lemma 1.2 we get $\operatorname{mdim}(G) \ge n(G) - 1$. To complete the argument we claim that $V(G) \setminus \{x\}$ is a mixed resolving set for G. To do this, let $\{a, b\} \subseteq V(G) \cup E(G)$. If $a \in V(G) \setminus \{x\}$ and b = ax, then $\deg_G(a) \le n(G) - 2$. Thus $V(G) \setminus N_G[a] \ne \emptyset$ and $2 = d_G(a, v) \ne d_G(ax, v) = 1$ for each $v \in V(G) \setminus N_G[a]$. Otherwise, there exists $u \in V(G) \setminus \{x\}$ such that $d_G(a, u) = 0$ and $d_G(b, u) \ge 1$, or $d_G(a, u) \ge 1$ and $d_G(b, u) = 0$. Therefore, $V(G) \setminus \{x\}$ is a mixed resolving set for G. \Box

3 Graphs with cut vertices

In this section we consider the mixed metric dimension of graphs G with cut vertices and bound from the above their mixed metric dimension by $n(G) - \zeta(G)$. This of course implies (as we already know) that no graph with a cut vertex is a max-mdim graph. As a consequence we determine the mixed metric dimension of block graphs.

Theorem 3.1 If W is a mixed resolving set of a graph G, then the following holds.

- (i) If v is a cut vertex of G, then $W \setminus \{v\}$ is a mixed resolving set of G.
- (i) $\operatorname{mdim}(G) \leq n \zeta(G)$. Moreover, equality holds if and only if each vertex from $V(G) \setminus \operatorname{CV}(G)$ has a maximal neighbor in G.

Proof. (i) If $v \notin W$, then we have nothing to prove, hence assume in the remainder that $v \in W$. Let $G_1, \ldots, G_k, k \geq 2$, be the components of G - v, and for each $i \in [k]$ select a neighbor v_i of v in G_i . Since v is a cut vertex, $d_G(vv_i, x) = d_G(v, x)$ for

each $x \in V(G) \setminus V(G_i)$. As also $d_G(vv_i, v) = d_G(v, v) = 0$, there must be a vertex in $W \cap V(G_i)$ that distinguishes vv_i and v_i . For each $i \in [k]$ select such a vertex w_i .

Consider now two arbitrary elements a and b from $V(G) \cap E(G)$. Assume first that a and b belong to some G_i . If $d_G(a, v) = d_G(b, v)$, then a and b are necessarily resolved by some vertex from $W \cap V(G_i)$. On the other hand, if $d_G(a, v) \neq d_G(b, v)$, then a and b are resolved by each w_j , where $j \neq i$. Assume next that a lies in G_i and b belongs to G_j , where $i \neq j$. If $d_G(a, v) = d_G(b, v)$, then a and b are resolved by some vertex from $W \setminus \{v\}$. Assume that $d_G(a, v) \neq d_G(b, v)$, let without loss of generality $d_G(a, v) > d_G(b, v)$ holds. Then we claim that a and b are resolved by w_j . Indeed, suppose on the contrary that $d_G(a, w_j) = d_G(b, w_j)$. Then

$$d_G(a, w_j) = d_G(a, v) + d_G(v, w_j) = d_G(b, w_j) \le d_G(b, v) + d_G(v, w_j),$$

which in turn implies that $d_G(a, v) \leq d_G(b, v)$, a contradiction. We have thus proved that each pair of elements from $V(G) \cap E(G)$ is resolved by some vertex from $W \setminus \{v\}$, hence (i) holds.

(ii) Since a mixed metric basis is a mixed resolving set of smallest cardinality, the inequality $\operatorname{mdim}(G) \leq n - \zeta(G)$ follows immediately from (i). To prove the equality part, suppose first that that each vertex from $V(G) \setminus \operatorname{CV}(G)$ has a maximal neighbor. Then Lemma 1.2 together with the already proved inequality $\operatorname{mdim}(G) \leq n - \zeta(G)$ yields $\operatorname{mdim}(G) = n - \zeta(G)$. Conversely, suppose that $\operatorname{mdim}(G) = n - \zeta(G)$ and suppose on the contrary that $v \in V(G) \setminus \operatorname{CV}(G)$ has no maximal neighbor in G. Then we claim that $V(G) \setminus (\operatorname{CV}(G) \cup \{v\})$ is a mixed resolving set for G. Indeed, we already know that $V(G) \setminus \operatorname{CV}(G)$ is a a mixed resolving set, so the only problem could be that a neighbor u of v would not be distinguished from the edge uv, because G - v is a connected graph and by (i), $V(G) \setminus (\operatorname{CV}(G) \cup \{v\})$ is a mixed resolving set for it. However, since v has no maximal neighbor, there exists $x \in N_G(v)$ such that $ux \notin E(G)$. But then $d_G(x, uv) = 1$ and $d_G(x, u) = 2$. Hence $V(G) \setminus (\operatorname{CV}(G) \cup \{v\})$ is a mixed resolving set, a contradiction to the assumption that $\operatorname{mdim}(G) = n - \zeta(G)$. \Box

Clearly, no cut vertex can have a maximal neighbor. Hence the equality part of Theorem 3.1(ii) can be rephrased by saying that mdim(G) = n(G) if and only if every vertex of the graph G has a maximal neighbor, which is of course Theorem 1.1. Theorem 3.1 also implies the following.

Corollary 3.2 If G is a block graph, then $mdim(G) = n - \zeta(G)$.

Proof. Just observe that if $v \in V(G) \setminus CV(G)$, then v is a simplicial vertex and hence clearly has a maximal neighbor in G. The result then follows from Theorem 3.1(ii). \Box

Corollary 3.2 in turn implies that if T is a tree, then mdim(T) is the number of the leaves of T, a result first proved in [5, Theorem 4.3].

Acknowledgments

Sand Klavžar was supported by the Slovenian Research Agency (ARRS) under the grants P1-0297, J1-2452, N1-0285.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

References

- [1] G.A. Barragán-Ramírez, J.A. Rodríguez-Velázquez, The local metric dimension of strong product graphs, Graphs Combin. 32 (2016) 1263–1278.
- [2] A. Ghalavand, S. Klavžar, M. Tavakoli, I.G. Yero (2023): On mixed metric dimension in subdivision, middle, and total graphs, Quaestiones Mathematicae, DOI: 10.2989/16073606.2023.2169206.
- [3] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs, Second Edition, CRC Press, Boca Raton, FL, 2011.
- [4] M. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, J. Biopharm. Statist. 3 (1993) 203–236.
- [5] A. Kelenc, D. Kuziak, A. Taranenko, I.G. Yero, Mixed metric dimension of graphs, Appl. Math. Comput. 314 (2017) 429–438.
- [6] S. Klavžar, M. Tavakoli, Local metric dimension of graphs: Generalized hierarchical products and some applications, Appl. Math. Comp. 364 (2021) 124676.

- [7] D. Kuziak, I.G. Yero, Metric dimension related parameters in graphs: A survey on combinatorial, computational and applied results, arXiv:2107.04877 [math.CO] (10 Jul 2021).
- [8] R.A. Melter, I. Tomescu, Metric bases in digital geometry, Comput. Vision Graphics Image Process. 25 (1984) 113–121.
- [9] M. Milivojević Danas, The mixed metric dimension of flower snarks and wheels, Open Math. 19 (2021) 629–640.
- [10] M. Milivojević Danas, J. Kratica, A. Savić, Z.Lj. Maksimović, Some new general lower bounds for mixed metric dimension of graphs, Filomat 35 (2021) 4275– 4285.
- [11] J. Qu, N. Cao, Edge metric dimension and mixed metric dimension of planar graph Q_n , Discrete Appl. Math. 320 (2022) 462–475.
- [12] H. Raza, J.B. Liu, S.J. Qu, On mixed metric dimension of rotationally symmetric graphs, IEEE ACCESS 8 (2020) 11560–11569.
- [13] J.A. Rodríguez-Velázquez, D. Kuziak, I.G. Yero, J.M. Sigarreta, The metric dimension of strong product graphs, Carpathian J. Math. 31 (2015) 261–268.
- [14] S.K. Sharma, V.K. Bhat, H. Raza, S. Sharma, On mixed metric dimension of polycyclic aromatic hydrocarbon networks, Chem. Papers 76 (2022) 4115–4128.
- [15] J. Sedlar, R. Škrekovski, Mixed metric dimension of graphs with edge disjoint cycles, Discrete Appl. Math. 300 (2021) 1–8.
- [16] R.C. Tillquist, R.M. Frongillo, M.E. Lladser, Getting the lay of the land in discrete space: A survey of metric dimension and its applications, arXiv:2104.07201 [math.CO] (15 Apr 2021).
- [17] R.C. Tillquist, M.E. Lladser, Low-dimensional representation of genomic sequences, J. Math. Biol. 79 (2019) 1–29.
- [18] R. Trujillo-Rasúa, I.G. Yero, k-metric antidimension: A privacy measure for social graphs, Inform. Sci. 328 (2016) 403–417.